A Computational Method for Singularly Perturbed Differential Equations with both Spatial and Temporal Delays

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Salale University

College of Natural Sciences
Department of Mathematics

M.Sc. Thesis

By: Embiale Azane Afire

Advisor: Genanew Gofe (PhD, Associate Professor)

Co-Advisor: Imiru Takele (PhD)

May, 2024 G.C

Fitche, Ethiopia.

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A Thesis Submitted to the Department of Mathematics, Salale University in the Partial Fulfilment of the Requirement for the Degree of Master of Science in Mathematics (Numerical Analysis)

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SALALE UNIVERSITY

SCHOOL OF GRADUATE STUDIES

P.O.Box: 245, FITCHE, ETHIOPIA

APPROVAL SHEET FOR SUBMITTING FINAL THESIS

This is to certify that the research thesis entitled **"A Computational Method for Singularly Perturbed Differential Equations with both Spatial and Temporal Delays"**, submitted in partial fulfillment of the requirement for the degree of Masters with specialization in Numerical Analysis , the graduate program of Applied Mathematics, and has been carried out by **Embiale Azane Afire**, under my supervision. Therefore, I recommend that the student has fulfilled the requirements and hence here by can submit the thesis to the department for defence.

Advisor Name	Signature	Date
Co-Advisor Name	Signature	Date

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List of Abbreviations and Symbols

Abbreviations

BVP	:	Boundary Value Problem
FDM	:	Finite Difference Method
FEM	:	Finite Element Method
FMM	:	Fitted Mesh Method
FOM	:	Fitted Operator Method
FVM	:	Finite volume Method
1D	:	One Dimensional
ODE	:	Ordinary Differential Equation
SPP	:	Singularly Perturbed Problem
SPDDE	:	Singularly Perturbed Delay Differential Equation
SPDODE	:	Singularly Perturbed Delay Ordinary Differential Equation

Symbols

δ	:	Delay Parameter
D	:	Domain of the problem
$O(.)$	:	Order symbol
ε	:	Singular Perturbation Parameter
$\overline{D}$	:	$D \cup \partial D$

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Abstract

This Thesis is attempted to introduce the numerical method to solve singularly perturbed differential equations involving spatial and temporal delays. These singularly perturbed problems are described by differential equations in which the highest order derivative is multiplied by an arbitrary small parameter (say) ε known as singular perturbation parameter and containing negative shift parameters in both space and time directions. This leads to the existence of thin boundary/ interior layer(s) in the solutions as $\varepsilon \to 0$. Due to the appearance of the thin boundary/interior layer(s), it is a thought task to solve these problems analytically. Consequently, a direct or analytical method for singularly perturbed differential equations involving spatial and temporal delays is still lacking, so one has to rely on suitable numerical methods to solve such types of problems. Therefore, it is note worthy to develop $\varepsilon-$ uniform convergent numerical methods for these problems. The main aim of this study is to develop and analyze a parameter uniform numerical method based on fitted techniques for solving singularly perturbed differential equations involving spatial and temporal delays. I used for time direction implicit Euler method and for space direction Cubic spline intension method.

Chapter 1

Introduction

1.1 Background of the Study

Many scientific problems are formulated to describe relations between causes and their effects.This becomes very interesting when the cause is small and the effect is large. The study of this relation in the field of the theory of perturbations has long history Van Dyke (1994). Despite this long history the subject is still in a state of vigorous development and it is known as the theory of singular perturbation problems (SPPs), meaning: a small perturbation causing a large impact is to be made clear. The SPPs involving an arbitrary small parameter (say ε) , where $0 < \varepsilon \ll 1$ attract the attention of mathematicians as well as physicists.

The partial differential equations in which the highest order derivative is multiplied by a small positive parameter and containing at least delay(s) parameter (s) is known as singularly perturbed delay partial differential equation (SPDPDE). Such types of problems arises frequently in various modeling of biosciences, vibrational models in control theory, physiological process, diseases, economics, engineering, and so on. A few application fields are the mathematical modeling of: population dynamics (Kuang, 1993), immune response (Mayer et al., 1995), variational problem in control theory (Glizer, 2000,0), model of HIV infection (Culshaw and Ruan, 2000; Nelson and Perelson, 2002; Wilbur and Rinzel, 1982), activation of neuronal variability (Wilbur and Rinzel, 1982; Doolan et al., 1980), the modelling of biological oscillators (Murray, 2007), mathematical ecology (Kot, 2001), models

for physiological process (Mackey and Glass, 1977; Mallet-Paret and Nussbaum, 1989), evolutionary biology (Wazewska-Czyzewska and Lasota, 1976), the neuronal variability (Stein, 1967) and more others.

Singularly perturbed differential equations relate an unknown function to its derivatives evaluated at the same instance. Whereas, singularly perturbed delay differential equations model physical problems for which the evaluation depends on the present state of the system and also on the past history. Also, the difference between the non-delay and delay singularly perturbed problems is that sometimes delay problems exhibit extra interior layers (Lange and Miura, 1982). Apart from this, the solutions to delay problems have points of discontinuities. Due to the presence of singular perturbation(ε) the solutions of SPPs exhibit multi-scale phenomena. Careful examination of the results from the classical numerical methods (FDM, FEM, FVM, Spline method) and other methods on uniform meshes as $\varepsilon \to 0$ fails to provide satisfactory numerical solution and the truncation error becomes unbounded, unless a large number of mesh points is used in the approximation process (Shishkin et al., 2012). This shows that the numerical method is computationally inefficient. Sometimes, the increase in mesh points also causes the resulting systems of algebraic equations to be ill-conditioned. This drawback motivates the researchers to develop the concept of a robust numerical methods for SPPs. In this context, the fitted operator method (FOM) and fitted mesh method (FMM) are popular techniques to overcome the drawbacks of classical numerical methods. For more details about the FOMs and FMMs, one can refer the books and articles (Doolan et al., 1980; Miller et al., 1996; Roos et al., 2008; Shishkin et al., 2012) and the references therein. This proposal will consider both techniques (FOM and FMM) to construct the robust numerical method for the problems under consideration.

1.2 Models Depicting Singular Perturbation of Partial Differential Problems

Several real life problems are described by SPDPPDEs. For instance,

1. Neuronal Variability (Musila and Lánskỳ, 1991):

$$-\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}+(\mu D-\frac{x}{\tau})\frac{\partial u}{\partial x}+\lambda_s u(x+a_s,t)+w_s u(x+i_s,t)-(\lambda_s+w_s)u(x,t), \quad (1.2.1)$$

where σ and μ_D are diffusion moments of Wiener process characterizing the influence of dendritic synapses on the cell excitability. The membrane potential decays exponentially to the resting level with a membrane time constant τ . The first derivative term is obtained because of the exponential decay between two consecutive jumps caused by the input processes. The reaction terms correspond to the superposition of excitatory and inhibitory inputs, and we can assume that they are Poissonian Daba and Duressa (2020).This model makes available time evolution of the trajectories of the membrane potential. The excitatory input contributes to the membrane potential by an amplitude a_s with intensity θ_s and similarly, the inhibitory input contributes by an amplitude i_s with intensity ω_s.

2. Control system (Wang, 1963)

$$\left(\frac{\partial u}{\partial t}=\varepsilon\frac{\partial^2 u}{\partial x^2}+v\left(g\left(u(x,t-\tau)\right)\right)\frac{\partial u}{\partial x}\right)(x,t)+c\left[f\left(u(x,t-\tau)\right)-u\left(x,t\right)\right], \quad (1.2.2)$$

which models a furnace used to process metal sheets, where u represents the distributions of temperature in a metal sheet, v is the instantaneous material strip velocity depending on a prescribed spatial average of the time-delayed temperature distribution $u(x,t-\tau)$,and f represents a distributed temperature source function depending on $u(x,t-\tau)$.

For more on a class of SPP models, interested readers are referred to (Das et al., 2007; Miller et al., 1996; Li et al., 2007; Verhulst, 2005; Roos et al., 2008; Tuckwell, 1988,9; Kadalbajoo and Gupta, 2010; Stein, 1967) and references therein. The models (1.2.2) and (1.2.1) are an excellent example of SPDPDEs arising frequently in control system theory and computational neuroscience respectively, one can hardly derive their exact solutions. Thus, to simulate these models, one has to land on numerical methods. Recently, different

researchers turn to these equations since they can be used to describe various natural phenomena accurately. However, there is a few work on the model (1.2.1). This motivated the researchers to consider such types real life problems in this study.

1.3 Statement of the Problem

Recently, there has been a growing interest in the numerical treatment of SPDDEs, due to their wide applicability in modeling of processes in various application fields. Finding the solution to these problems has a significant role to capture the behavior of the physical phenomena of the problems. Analytical methods for solving such problems are still lacking, so one has to go for numerical methods. But, the classical numerical methods for SPDDEs are fail to provide accurate results as ε approaches to zero (Shishkin et al., 2012).Therefore, it is very important to develop numerical schemes that converges independent of ε.

However, in the past decades, singularly perturbed ordinary delay differential equations were widely investigated by using different numerical and asymptotic methods. In the past and recent years, different scholars developed various parameter uniform numerical schemes for SPDPDEs with shifts in space and time variable. Nevertheless, from the literature review and to the best of the researchers' knowledge, there are only a few $\varepsilon-$uniform numerical methods to solve the considered problems in Eq. (1.5.1) and it needs a lot of investigations. This motivate us to develop and analyze a parameter uniform numerical scheme for solving singularly perturbed parabolic problem with spatio-temporal delays.

1.4 Objectives of the study

1.4.1 General Objective of the Study

The main objective of this study is to construct a parameter uniform numerical scheme which are stable and give more accurate result for reasonable mesh size for solving one dimensional singularly perturbed parabolic problem with spatio-temporal delays.

1.4.2 Specific Objectives of the Study

The specific objectives of this study are the following:

1. The developed $\varepsilon-$uniform numerical scheme for solving singularly perturbed parabolic problem with spatio-temporal delays.

2. The establish and the stability of the numerical scheme.

3. The analyze and the convergence of the numerical scheme.

4. The examine, the effectiveness of the numerical scheme.

1.5 Delimitation of the Study

Singularly perturbed parabolic PDEs are huge subject (convection-diffusion without delay, reaction-diffusion without delay, convection-diffusion with spatial mixed shifts (*i.e.*, delay and advance) as well as time delay, reaction-diffusion with spatial as well as time delay, etc.). This study was delimited to focus on the following singularly perturbed parabolic problem with spatio-temporal delays

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)u(x,t) + \alpha(x)u(x-1,t-\tau) = f(x,t), (x,t) \in D \\ u(x,t) = u_0(x,t), (x,t) \in D_0 = \{(x,t) : x \in [0,2], t \in [-\tau, 0]\} \\ u(x,t) = \phi(x,t), (x,t) \in D_L = \{(x,t) : -1 \leq x \leq 0, t \in (0,T]\} \\ u(2,t) = \psi(2,t), (x,t) \in D_R = \{(2,t) : t \in (0,T]\} \end{cases} \tag{1.5.1}$$

where the domain $D = \Omega_x \times \Omega_t = (0,2) \times (0,T]$, $0 < \varepsilon \ll 1$, a temporal shift $\tau > 0$ of $o(\varepsilon)$ and a finite time T. The functions $a(x), \alpha(x), f(x,t)$ $u_0(x,t), \phi(x,t)$ and $\psi(x,t)$ on D, are sufficiently smooth and satisfying the conditions $\alpha(x) \geq a^* > 0, a(x) + \alpha(x) \geq \gamma > 0$, on $\overline{\Omega}_x$.

1.6 Significance of the Study

A singularly perturbed parabolic problem with spatio-temporal delays has been studied because of its numerous applications in many mathematical models and their use has spread

into the theoretical study of neuronal activity, optically bistable devices, description of human pupil reflex, a variety of models for physiological processes or diseases and others. To investigate the prediction of the physical phenomena of these problems, it is often necessary to approximate their solution numerically. Though, it is very hard to solve singularly perturbed parabolic problem with spatio-temporal delays analytically due to the presence of a multi-scale character in the solution. So, it is necessary to develop $\varepsilon-$ uniform convergent numerical methods that solve the problems under consideration successfully. Developing a robust numerical method for such problems is always a desirable task Das (2013). The main contribution of this thesis is the development of a $\varepsilon-$ uniform numerical scheme for solving singularly perturbed parabolic problem with spatio-temporal delays . Additionally, the developed numerical scheme can be used to fill the gap observed, would be significantly helpful to researchers working in this area to construct their new numerical schemes, and used as an alternative scheme for solving singularly perturbed parabolic problem with spatio-temporal delays.

Chapter 2

Review of Related Literature

A singularly perturbed delay parabolic PDE is a PDE in which the highest order derivative is multiplied by a small positive parameter $\varepsilon(0 < \varepsilon \ll 1)$ and having delay in space/and time direction. Designing a $\varepsilon-$ uniform convergent numerical schemes for the singularly perturbed delay parabolic PDEs are very challenging task and the developed parameter uniform numerical schemes for these problems are surveyed as follows.

The researchers Ramesh and Kadalbajoo (2008); Woldaregay and Duressa (2019); Daba and Duressa (2020); Gupta et al. (2018); Woldaregay and Duressa (2022); Daba and Duressa (2021); Woldaregay and Duressa (2020); Rao and Chakravarthy (2019); Ramesh and Priyanga (2019) and the references there in considered the following SPPPDE of convection-diffusion and reaction-diffusion type with shifts in space variable:

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + \alpha(x)u(x-\delta,t) + \beta(x)u(x,t) + \omega(x)u(x+\eta,t) = f(x,t), (x,t) \in D, \\ u(x,0) = u_0(x), x \in D_0 = \{(x,0) : x \in \Omega\}, \\ u(x,t) = \phi(x,t), \forall (x,t) \in D_L = \{(x,t) : -\delta \leq x \leq 0, t \in (0,T]\}, \\ u(x,t) = \psi(x,t), \forall (x,t) \in D_R = \{(x,t) : 1 \leq x \leq 1+\eta, t \in (0,T]\}, \end{cases} \tag{2.0.1}$$

where the domain $D = \Omega_x \times \Omega_t = (0,1) \times (0,T]$, $0 < \varepsilon \ll 1$ and δ and η to be order of $o(\varepsilon)$. The functions $a(x), \alpha(x), \beta(x), \omega(x), f(x,t)$ $u_0(x), \phi(x,t)$ and $\psi(x,t)$ on D, are sufficiently smooth and satisfying the conditions $a(x) \geq a^* > 0, \alpha(x) + \beta(x) + \omega(x) \geq \gamma > 0$, on $\overline{\Omega}_x$. They had suggested different parameter uniform numerical methods for solving Eq. (2.0.1).

The scholars Sharma (2017); Yadav et al. (2020); Peiraviminaei and Ghoreishi (2014);

Bashier and Patidar (2011); Kumar and Kumar (2014); Gowrisankar and Natesan (2014); Rao and Chakravarthy (2014); Wang et al. (2017); Singh et al. (2018); Wang (2011); Gowrisankar and Natesan (2017); Gelu and Duressa (2021); Das and Natesan (2015); Salama and Al-Amery (2017); Kumar and Kumari (2019); Das and Natesan (2015); Govindarao and Mohapatra (2019b,0); Kumar and Kumari (2019); Kaushik et al. (2010); Podila and Kumar (2020) and the references there in suggested various numerical methods for diffusion-reaction problems with time lag of the form: $\Gamma_b = [0,1] \times [-\tau, 0]$:

$$\begin{cases} \left(\frac{\partial}{\partial t} + L_\varepsilon\right) u(x,t) = -c(x,t)u(x,t-\tau) + f(x,t), (x,t) \in D, \\ u(x,t) = \phi_b(x,t), (x,t) \in \Gamma_b, \\ u(0,t) = \phi_l(t), \text{on } \Gamma_l = \{(0,t) : 0 \le t \le T\}, \\ u(1,t) = \phi_r(t), \text{on } \Gamma_r = \{(1,t) : 0 \le t \le T\}, \end{cases} \tag{2.0.2}$$

where $L_\varepsilon u(x,t) = -\varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} - a(x)\frac{\partial u(x,t)}{\partial x} + \alpha(x,t)u(x,t)$, $0 < \varepsilon \ll 1$ and $\tau > 0$ are given constants, $a(x,t), \alpha(x,t), c(x,t), f(x,t)$ on D, and $\phi_b(x,t), \phi_l(t), \phi_r(t)$ on Γ, are sufficiently smooth and bounded functions that satisfy

$$a(x) \ge \beta > 0, \alpha(x,t) \ge \gamma > 0, \text{ on } \overline{D}.$$

Ejere et al. (2023) have proposed and analyzed robust numerical scheme for singularly perturbed differential equations with both spatial and temporal delays of the form in Eq. (1.5.1). The proposed a numerical scheme is developed using the implicit Euler method in the temporal variable and Cubic spline intension method in the spatial variable on uniform meshes.

This study indicates that the development of the solution methodologies to solve the model problem in Eq. (1.5.1) is still very few and needs a lot of study. This gap motivates the researchers to consider this problem. Therefore, the primary goal of this thesis was to developed and analyze a parameter numerical scheme using fitting approaches for the families of the problem under consideration.

Chapter 3

Materials and Methods

3.1 Study Design

This study was employed an intensive up to date document review and numerical experimentation.

3.2 Source of Information

The relevant source of information for this study are books, published articles on international reputable journals and related studies from different university websites and so on.

3.3 Mathematical Procedures

In order to achieve the stated objectives, the study was followed the following mathematical procedures:

1. Defining/selecting model problem for the study,
2. Analyzing the properties of the continuous solution,
3. Developing numerical scheme for the problem,
4. Establishing the stability and convergence of the developed numerical scheme,
5. Developing an algorithm and writing MATLAB code for the presented scheme,

6. Validating the scheme using numerical examples,

7. Presenting the results using appropriate presentation (using tables, graphs),

8. Discussing and providing conclusion.

Chapter 4

Description of the Numerical Method, Results and Discussions

4.1 Description of the Problem

Consider the following Singularly Perturbed Differential Equations with both Spatial and Temporal Delays

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)u(x,t) + \alpha(x)u(x-1,t-\tau) = f(x,t), (x,t) \in D \\ u(x,t) = u_0(x,t), (x,t) \in D_0 = \{(x,t) : x \in [0,2], t \in [-\tau, 0]\} \\ u(x,t) = \phi(x,t), (x,t) \in D_L = \{(x,t) : -1 \leq x \leq 0, t \in (0,T]\} \\ u(2,t) = \psi(2,t), (x,t) \in D_R = \{(2,t) : t \in (0,T]\}. \end{cases} \tag{4.1.1}$$

where the domain $D = \Omega_x \times \Omega_t = (0,2) \times (0,T]$, $0 < \varepsilon \ll 1$ and δ to be order of $o(\varepsilon)$. The functions $a(x), \alpha(x), f(x,t)$ $u_0(x,t), \phi(x,t)$ and $\psi(x,t)$ on D are smooth enough and bounded on the considered domain. Moreover, to avoid oscillation in the solution for arbitrary positive constant λ, the coefficient functions $a(x)$ and $\alpha(x)$ satisfy the conditions Ejere et al. (2023):

$$a(x) + \alpha(x) \geq 2\lambda > 0 \text{ and } \alpha(x) < 0, x \in [1,2]. \tag{4.1.2}$$

4.1.1 Properties of Continuous Solution

Since $\tau < \epsilon$, the use of Taylor series expansion for the term containing lag is valid Tian (2002). Thus, on applying Taylor's series expansion on the terms with $u(x-1,t-\tau)$, we have

$$u(x-1,t-\tau) = u(x-1,t) - \tau \frac{\partial u(x-1,t)}{\partial t} + O(\tau^2). \tag{4.1.3}$$

Substituting equation (4.1.3) into equation (4.1.1), we obtain

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + a(x)u(x,t) + \alpha(x)[u(x-1,t) - \tau\frac{\partial u(x-1,t)}{\partial t}] = f(x,t), \\ u(x,0) = u_0(x,t), (x,t) \in \bar{D}, \\ u(x,t) = \psi(x,t), (x,t) \in \bar{D}_L = (x,t) : x \in [-1,0], t \in [0,T], \\ u(2,t) = \psi(t), (2,t) \in D_R = (2,t), t \in [0,T]. \end{cases} \tag{4.1.4}$$

Equation (4.1.1) can be rewritten as

$$\begin{cases} \pounds_\varepsilon u(x,t) = \gamma(x,t), (x,t) \in \bar{D}, \\ u(x,0) = u_0(x,t), (x,t) \in \bar{D}, \\ u(x,t) = \psi(x,t), (x,t) \in \bar{D}_L = (x,t) : x \in [-1,0], t \in [0,T], \\ u(2,t) = \psi(t), (2,t) \in D_R = (2,t), t \in [0,T], \end{cases} \tag{4.1.5}$$

where

$$\pounds_\varepsilon u(x,t) = \frac{\partial u(x,t)}{\partial t} - \varepsilon\frac{\partial^2 u(x,t)}{\partial x^2}) + \alpha(x)u(x,t),$$

$$\gamma(x,t) = \begin{cases} f(x,t) - \alpha(x)\phi(x-1,t) + \tau\alpha(x)\frac{\partial\phi(x-1,t)}{\partial t}, x \in (0,1], \\ f(x,t) - \alpha(x)\phi(x-1,t) + \tau\alpha(x)\frac{\partial u(x-1,t)}{\partial t}, x \in (1,2). \end{cases}$$

Now, we consider the maximum principle and stability of the solution u in Eq. (4.1.4), which is important for the proposed numerical analysis (Das, 2019),

Lemma 4.1.1. *Suppose z(x, t) is a smooth function in $\bar{D}$. If Z(x, t) ≥ 0, (x, t) ∈ ∂D and $\pounds_\epsilon$ and $\pounds_\epsilon z(x,t) \geq 0, (x,t) \in D, then \pounds_\epsilon z(x,t) \geq 0, (x,t) \in \bar{D}$*

Proof. For $(\bar{x},\bar{t}) \in \bar{D}$, suppose that $z(\bar{x},\bar{t}) = min_{\bar{D}} z(\bar{x},\bar{t}) < 0$. From the considered hypothesis $(\bar{x},\bar{t}) \notin \partial D$. By extreme value theorem, we have $\partial z(\hat{x},\hat{t}) = 0, \partial z(\hat{x},\hat{t}) = 0$, and $\partial z(\hat{x},\hat{t}) > 0$. Then,
Case 1: For $s \in (0,1]$,

$$\pounds_\epsilon(\bar{x},\bar{t}) = \frac{\partial z(\bar{x},\bar{t}}{\partial t}) - \epsilon\frac{\partial^2 z(\bar{x},\bar{t})}{\partial x^2} + a(\bar{x},\bar{t}) < 0.$$

Case 2: For $s \in (1,2)$,

$$\pounds_\varepsilon z(\bar{x},\bar{t}) = \frac{\partial z(\bar{x},\bar{t}}{\partial t}) - \epsilon\frac{\partial^2 z(\bar{x},\bar{t})}{\partial x^2} + a(\bar{x},\bar{t}) + \alpha(\bar{x}, z(\bar{x}-1,\bar{t}))$$

$$-\tau\alpha(\bar{x})\frac{\partial z(\bar{x}-1,\bar{t})}{\partial t} - \varepsilon\frac{\partial^2 z(\bar{x},\bar{t})}{\partial t^2} + [a(\bar{x}) + \beta(\bar{x})z(\bar{x},\bar{t})] - \tau\alpha(\bar{x})\frac{\partial z(\bar{x},\bar{t})}{\partial t} < 0.$$

Thus $\pounds_\epsilon z(\bar{x},\bar{t}) < 0$, which contracts the given conditions. Therefore, our supposition fails, so that $z(\bar{x},\bar{t}) \geq 0$, which implies that $z(x,t) \geq (x,t) \in \bar{D}$.. □

Lemma 4.1.2. *The solution is estimated as follows:*

$$|u(x,t)| \leq \frac{\|\gamma\|}{\lambda} + \max|\partial D|$$

.

Proof. Let us define $z^{\pm} = \frac{\|\gamma\|}{\lambda} + \max\{|\partial D|\} \pm u(x,t)$.

Then, we have $z^{\pm}(0,t) \geq 0$. Moreover, For $x \in (0,1], t \in [0,T]$,

$$\pounds_{\epsilon} z^{\pm}(x,t) = (\frac{\partial}{\partial t} - \varepsilon\frac{\partial}{\partial x^2} + \gamma(x))z^{\pm}(x,t) = \gamma(x)(\frac{\gamma}{\lambda} + \max\{|\partial D|\}) \pm \gamma(x,t) \geq 0.$$

For $s \in (1,2), t \in [0,T]$,

$$\pounds_{\epsilon} z^{\pm}(x,t) = (\frac{\partial}{\partial t} - \varepsilon\frac{\partial}{\partial x^2} + a(x))z^{\pm}(x,t) + a(x)z^{\pm}(x-1,t) - \tau a(x)(\frac{\partial z^{\pm}(x-1,t)}{\partial t} +$$

$$[a(x)\alpha(x)]\max\{|\partial D|\}) \geq 0.$$

Thus $L_{\epsilon} z^{\pm}(x,t) \geq 0$ and by lemma (4.1.1) we have $z^{\pm}(x,t) \geq 0, (x,t) \in \bar{D}$, which yields the stability estimate. □

Lemma 4.1.3. *The derivative of the solution $U(x,t)$ is bounded as follows:*

$$\left|\frac{\partial^{k+1}(x,t)U}{\partial x^k \partial t^t}\right| \leq \begin{cases} C(1+\varepsilon^{\frac{-k}{2}}[e^{-x\sqrt{\frac{\lambda}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\lambda}{\varepsilon}}}]), (x,t) \in D_h, \\ C(1+\varepsilon^{\frac{-k}{2}}[e^{-(x-1)\sqrt{\frac{\lambda}{\varepsilon}}} + e^{-(2-x)\sqrt{\frac{\lambda}{\varepsilon}}}]]), (x,t) \in D_R \end{cases}$$

for all nonnegative integer k,h such that $0 \leq k + 2h \leq 4$.

Proof. For k = 0 and h = 0 , it is to show the bound of the solution $U(x,t)$, which is lemma (4.1.2). For $k = 0, h \neq 0$, we show the bound of derivative of U(x,t) with respect to t. For $k \neq 0$ and $h = 0$, it is to show that the bound of derivatives of the solution U(x,t) with respect to x and for the case when $(h,k) \neq 0$, we determine the bound of its derivatives of the solution U(x,t) with respect to x and t. Let $q_{\varepsilon}, h_{(x)} = e^{-x\sqrt{\frac{\lambda}{\epsilon}}} + e^{-(x-1)\sqrt{\frac{\lambda}{\epsilon}}}, x \in (0,1]$ and $q_{\varepsilon}, 2(x) = e^{-x\sqrt{\frac{\lambda}{\epsilon}}} + e^{-(2-x)\sqrt{\frac{\lambda}{\epsilon}}}, x \in (1,2)$. For a fixed value of λ, following the approaches of we can get that $q_{\varepsilon,h(x)} \neq c$ and $h = 1, 2$. For the case $k = 0$, equation (6) becomes

$$\frac{\partial U(x,t)}{\partial t} = \varepsilon\frac{\partial^2 U(x,t)}{\partial x^2}) - a(x)U(x,t) + \begin{cases} f(x,t) - \alpha(x)\phi(x-1,t) + \tau\alpha(x)\frac{\partial\phi(x-1,t)}{\partial t}, x \in (0,1], \\ f(x,t) - \alpha(x)\phi(x-1,t) + \tau\alpha(x)\frac{\partial U(x-1,t)}{\partial t}, x \in (1,2) \end{cases} \tag{4.1.6}$$

Along the sides t =0, we have U = 0, which implies that $\frac{\partial^2 U(x,t)}{\partial x^2} = 0$, we get

$$\partial\frac{U(x,0)}{\partial t} = \varepsilon\frac{\partial^2 U(x,0)}{\partial x^2}) - a(x)U(x,t) + \begin{cases} f(x,0) - \alpha(x)\phi(x-1,0) + \tau\alpha(x)\frac{\partial\phi(x-1,0)}{\partial t}, x \in (0,1], \\ f(x,0) - \alpha(x)\phi(x-1,0) + \tau\alpha(x)\frac{\partial U(x-1,0)}{\partial t}, x \in (1,2) \end{cases} \tag{4.1.7}$$

Without loss of generality, assuming that all the values considered in equation (4.1.5) are zero, we have $U(x-1,0) = \psi(x-1,1=0) x \in [0,1]$and $u(x-1,1) = U_o(x-1,1)0, x \in (1,2]$.

Then equation becomes $\frac{\partial U(x,0)}{\partial t} = f(x,0)$. Since f is smooth function, it implies $\left|\frac{\partial U(x,0)}{\partial t}\right| \leq C$, for sufficiently chosen C on ∂D. Applying the operator L_ε given in the equation (4.1.3) on $\frac{\partial U(x,0)}{\partial t}$ we obtain $|L_\varepsilon| U_t(x,t) = |\theta(x,t)| = C$ on D. thus applying Lemma 1 gives $\left|\frac{\partial U(x,t)}{\partial t}\right| \leq C$ on $\bar{D}$. For h = 2, differentiating Equation with respect to t gives

$$\frac{\partial^2 U(x,t)}{\partial t^2} = \varepsilon \frac{\partial^3 U(x,t)}{\partial x^2 \partial t} - a(x)\frac{\partial U(x,t)}{\partial t} + \begin{cases} \frac{\partial f(x,t)}{\partial t} - \alpha(x)\frac{\partial \phi(x-1,t)}{\partial t} + \tau\alpha(x)\frac{\partial^2 \phi(x-1,t)}{\partial t^2}, x \in (0,1], \\ \frac{\partial f(x,t)}{\partial t} - \alpha(x)\frac{\partial \phi(x-1,t)}{\partial t} + \tau\alpha(x)\frac{\partial^2 \phi(x-1,t)}{\partial t^2}, x \in (0,2] \end{cases} \tag{4.1.8}$$

Along x = 0 and x = 2, we have $\frac{\partial^2 U(x,t)}{\partial t^2} = 0$ and along t= 0, we have w = 0 and $\frac{\partial^2 U(x,t)}{\partial x^2} = 0$. From $\frac{\partial U(x,0)}{\partial t} = f(x,0)$ we have $\frac{\partial^3 U(x,0)}{\partial x^2 \partial t} = \frac{\partial^2 f(x,0)}{\partial x^2}$. Assuming that the initial and boundary conditions are identically zero, we have $U_t(x-1,0) = \phi_t(x-1,0) = 0, x \in (0,1]$ and $U_t(x-1,0) = U_0(t)(x-1,0) = 0, x \in (1,2]$. Then from equation (4.1.9), we get

$$\frac{\partial^2 U(x,0)}{\partial t^2} = \varepsilon \frac{\partial^2 f(x,0)}{\partial x^2} - a(x)\frac{\partial f(x,0)}{\partial t} + \begin{cases} \frac{\partial f(x,0)}{\partial t}, x \in (0,1] \\ \frac{\partial f(x,0)}{\partial t} + \tau\alpha(x)\frac{\partial^2 u(x-1,0)}{\partial t^2}, x \in (1,2) \end{cases} \tag{4.1.9}$$

from Equation(4.1.10), along t = 0, we obtain that $\left|\frac{\partial^2 u(x,0)}{\partial t^2}\right| \leq 0$ on ∂D and the operator $\pounds_\varepsilon$ implies $\left|\pounds_\varepsilon \frac{\partial^2 u(x,t)}{\partial t^2}\right| = \left|\frac{\partial^2 \gamma(x,t)}{\partial t^2}\right| \leq C$ on D and applying Lemma 1 yields $\left|\frac{\partial^2 U(x,t)}{\partial t^2}\right| \leq C$ on $\bar{D}$. The bound of derivatives of the solution U(x,t) for the cases $k \neq 0$ can be determined by similar procedures as mentioned earlier. For the case k = 1, let $x \in D$ and consider a neighborhood $R = (e, e+\sqrt{\varepsilon}), \forall x \in R$. For some $q \in \bar{R}$ and $t \in (0,T]$, the mean value theorem gives

$$\left|\frac{\partial U(q,t)}{\partial x}\right| = \frac{1}{\sqrt{\varepsilon}} \left|u(e+\sqrt{\varepsilon},t) - U(e,t)\right| \leq 2\frac{1}{\sqrt{\varepsilon}} \|u\| \tag{4.1.10}$$

For $x \in \bar{R}$, we can get

$\frac{\partial U(x,t)}{\partial t} = \frac{\partial U(q,t)}{\partial x} + \frac{\partial U(x,t)}{\partial x} - \frac{\partial U(q,t)}{\partial x} = \frac{\partial U(q,t)}{\partial x} + \int_q^x \frac{\partial U(q,t)}{\partial x} dx = \frac{\partial U(q,t)}{\partial x} + \varepsilon^{-1}\int_q^x (\frac{\partial U(q,t)}{\partial x} + a(x)U(x,t) + \alpha(x)U(x-1,t) - f(x,t))dx.$

$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq \left|\frac{\partial u(q,t)}{\partial x}\right| + C\varepsilon^{-1}\int_q^x (\|U\| + \|f\|)dx = \left|\frac{\partial U(q,t)}{\partial x}\right| + C\varepsilon^{-1}(\|u\| + \|f\|)\sqrt{\varepsilon}$

Putting equation (4.1.10) into the aforementioned result, we obtain $\left|\frac{\partial U(x,t)}{\partial x}\right| \leq C(1 + \frac{1}{\sqrt{\varepsilon}} q_{\varepsilon,t}), (x,t) \in \bar{D}$ for sufficiently large chosen C. For k =2, by rearranging Equation(4.1.1) we get

$$\left|\frac{\partial U(x,t)}{\partial x}\right| = \varepsilon^{-1}\left[\left|\frac{\partial U(x,t)}{\partial x}\right| + a(x)U(x,t) + \alpha(x)U(x-1,t) - Y(x,t)\right] \tag{4.1.11}$$

For a fixed time $t \in [0,T]$ and $x \in [0,2]$, *since* $|U| \leq C$,
$\left|\frac{\partial U(x,t)}{\partial x}\right| \leq C$, and f is a smooth function, $\left|\frac{\partial U(x,t)}{\partial x}\right| \leq C(1 + \varepsilon^{-1} q_{\varepsilon,t})$. The derivative of equation with respect to t gives

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \varepsilon^{-1}(\frac{\partial^2 U(x,t)}{\partial t^2} + a(x)\frac{\partial U(x,t)}{\partial t}\beta(x)\frac{\partial u(x-1,t)}{\partial t} - \frac{\partial Y(x,t)}{\partial t})$$

. Since $\left|\frac{\partial U(x,t)}{\partial t}\right| \leq C$, $\left|\frac{\partial u(x,t)}{\partial t}\right| \leq C$, and $|f_t(x,t)| \leq C$, we get

$$\left|\frac{\partial^3 U(x,t)}{\partial x^2 \partial t}\right| \leq C(1+\varepsilon^{-1}q_{\varepsilon,t}).$$

In a similar procedure, the bounds on the derivatives of the solution can be easily determined for the remaining values of k and h with $0 \leq k+2h \leq 4$. □

4.2 Formulation of the Numerical method

4.2.1 Temporal Semi-discretization

We define the uniform mesh for the domain Ω_t as:

$$\Omega_t = D_{\Delta t}^M = \{t_j = j\Delta t, \forall_j = 0,2,3,...M, \Delta t = \frac{1}{M}\},$$

and we approximated the temporal derivative term by employing the $\theta-$method (when $\theta = 1$) method, which gives

$$\pounds_\varepsilon^M u^{j+1}(x) = \gamma^{j+1}(x), \tag{4.2.1}$$

where

$$\pounds_\varepsilon^M u^{j+1}(x) = \begin{cases} -\varepsilon\frac{d^2u^{j+1}(x)}{dx^2} + (\frac{1}{\Delta t} + a(x))u^{j+1}(x), 0 < x \leq 1, \\ -\varepsilon\frac{d^2u^{j+1}(x)}{dx^2} + (\frac{1}{\Delta t} + a(x))u^{j+1}(x) + (1-\frac{\tau}{\Delta t})\beta(x)u^{j+1}(x-1), 1 < x \leq 2 \end{cases}$$

and

$$\gamma^{j+1}(x) = \begin{cases} f^{j+1}(x) + \frac{u^j(x)}{\Delta t} - (1-\frac{\tau}{\Delta t})\alpha(x)\psi^{j+1}(x-1) - \frac{\tau\alpha(x)}{\delta t}\psi^j(x-1), x \in (0,1], \\ f^{j+1}(x) + \frac{u^j(x)}{\Delta t} - \frac{\tau\beta(x)}{\Delta t}u^j(x-1), x \in (1,2) \end{cases}$$

with the initial and boundary conditions

$$\begin{cases} u^0(x) = u_0(x), x \in [0,2] \\ u^{j+1}(x) = \phi^{j+1}(x), x \in (0,1]), \\ u^{j+1}(2) = \psi^{j+1}(2), x \in (1,2) \end{cases} \tag{4.2.2}$$

As a result for the operator $\pounds_\varepsilon^M$, we have

$$\left\|(\pounds_\varepsilon^M)^{-1}\right\| \leq (1+\lambda\Delta t)^{-1} \tag{4.2.3}$$

which is used in estimating the truncation error.

The local error committed in the semi-discrete scheme is the difference between the exact solution $u(x,t^{j+1})$ and the approximate solution $u(^{j+1})(x)$ of Eq.(4.2.1). That is, $e^{j+1(x)} = u(x,t^{j+1}) - u^{j+1}(x)$ and the global error E^{j+1} is the contribution of the local error up to the $(j+1)^{th}$ time level. The bound of error for the semi-discrete scheme is estimated as follows.

Lemma 4.2.1. *Suppose that* $\left|\frac{\partial^k u(x,t)}{\partial t^k}\right| \leq C, (x,t) \in \bar{D}, k = 0,1,2$. *The local error is estimated as* $\|e_{i+1}\| \leq C(\Delta t)^2$, *and with this condition, the global error is estimated as* $\|E_{i+1}\| \leq C(\Delta t), j+1 = 1,2,3\cdots M$.

Proof. From the Taylor's series expansion, we have $u^{j+1}(x) = u^j(x) + \Delta t \frac{\partial u(x,t_j)}{\partial t} + O((\Delta t)^2)$ and from this we obtain

$$\frac{u^{j+1}(x) - u^j(x)}{\Delta t} = \frac{\partial u_t(x,t)}{\partial t} + O((\Delta t)^2). \tag{4.2.4}$$

Using equation (4.1.5) into 4.2.4

$$\pounds_\varepsilon^M u^{j+1}(x) \quad + \quad O((\Delta t)^2) \quad = \quad \gamma^{j+1}(x). \tag{4.2.5}$$

From equation 4.2.1,4.2.5, we can obtain the boundary value problem of the form

$$\pounds_\varepsilon^M e^{j+1}(x) = O((\Delta t)^2), e^{j+1}(0) = 0 = e^{j+1}(2). \tag{4.2.6}$$

Using equation 4.2.3 in to 4.2.6 yields $\|e^{j+1}\| \leq C(\Delta t)^2$. Using the estimation of e^{j+1}, we have

$$\begin{aligned} \|E_j\| = \left\| \sum_{i_1}^{j} e^i \right\|_\infty, j(\Delta t) &\leq T \\ &= \left\|e^1 + e^2 + ... + e^j\right\| \\ &\leq \left\|e^1\right\| + \left\|e^2\right\| + ... + \left\|e^{j+1}\right\| \\ &\leq CT(\Delta t) \leq C(\Delta t), j+1 = 1,2,...M. \end{aligned}$$

Thus, the semi-discrete scheme is convergent of order one in time. □

4.2.2 Spatial Discretization

We define the uniform mesh for the space domain $[0, 2]$ as:

$$\Omega_x = \{x_i = ih, \forall i = 1, 2, 3, ...N, x_0 = 0, x_N = 2, h = \frac{2}{N}\}.$$

For the sake of simplicity, we can write Eq. (4.1.1) as

$$-\varepsilon\frac{\partial^2 u^{j+1}(x)}{\partial x^2} + p(x)u^{j+1}(x) = g(x)^{j+1}, \tag{4.2.7}$$

where $p(x) = (\frac{1}{\triangle t} + a(x)), g^{j+1}(x) = f(x)^{j+1} + (b(x) - \frac{b(x)\tau}{\triangle t})u(x-1)^{j+1}\frac{u(x)^j}{\triangle t} - \frac{b(x)\tau}{\triangle t}u(x-1)^j$. The approximation solution u_i^{j+1} of Eq.(4.2.7) obtained by the segment $S_\Delta(x)$ passing through the points (x_i, u_i^{j+1}) and (x_{i+1}, u_{i+1}^{j+1}). Each mixed spline method $S_\Delta^{j+1}(x)$ has the form :

$$S_\Delta^{j+1}(x) = A_i e^{\gamma(x-x_i)} + B_i e^{-\gamma(x-x_i)} + C_i(x - x_i) + D_i, 0 \le i \le N, \tag{4.2.8}$$

where A_i, B_i, C_i and D_i are constants and $\tau \neq 0$ is a free parameter that will be used to enhance the accuracy of the method. To find the values of A_i, B_i, C_i and D $_i$ in Eq.(4.2.8), let us denote

$$S_\Delta^{j+1}(x_1) = u_i^{j+1}, S_\Delta^{j+1}(x_{i+1}) = u_{i+1}^{j+1}$$

$$\frac{d^2 S_\Delta^{j+1}(x)}{dx^2} = \Xi_i, \frac{d^2 S_\Delta^{j+1}(x)}{dx^2} = \Xi_{i+1}. \tag{4.2.9}$$

Differentiating twice both sides of Eq.(4.2.8) we get

$$\frac{\partial^2 u_i^{j+1}}{\partial t^2} = \tau^2(A_i\Xi_i + B_i\Xi_{i-1}) \tag{4.2.10}$$

$\frac{\partial^2 u_i^{j+1}}{\partial x^2} = \Xi\tau^2(A_i + B_i)$ and yields

$$A_i = \frac{h^2\Xi_i}{\varrho^2} - B_i. \tag{4.2.11}$$

Let $\tau h = \varrho$

$A_i = \frac{h^2\Xi}{\tau h} - B_i, i = 0, 1, 2, 3....N$. Again, $x = x_{i+1}$ we obtain
$\frac{\partial^2 u_i^{j+1}}{\partial t^2} = \Xi\tau^2(A_i e^{\varrho} + B_i e^{-\varrho})$ and it gives

$$A_i = e^{-\varrho}\left(\frac{h^2\Xi_{i+1}}{\varrho^2}\right) - B_i e^{-2\varrho}. \tag{4.2.12}$$

we have

$$A_i = \frac{h^2(\Xi_{i+1} - \Xi_i e^{-\varrho})}{2\varrho^2 sinh(\varrho)}$$

$$B_i = \frac{h^2(\Xi_i \varrho^2 - \Xi_{i+1})}{2\varrho^2 sinh(\varrho)} \tag{4.2.13}$$

$$S_\Delta^{j+1} = u_i^{j+1} = A_i + B_i + D_i, and$$

$$S_{i+1}^{j+1} = u_{i+1}^{j+1} = A_i \varrho^\varrho + B_i \varrho^{-\varrho} + C_i h + D_i$$

and which implies

$$C_i = \frac{(u_{i+1}^{j+1} - u_i^{j+1})}{h} - \frac{h(\Xi_{i+1} - \Xi_i)}{\varrho^2}, and$$

$$D_i = u_i^{j+1} - \left(\frac{h^2}{\varrho^2}\Xi_i\right) \tag{4.2.14}$$

The first derivative continuity conditions at $x = x_i$ that is $S_{\Delta-1}(x_i) = S_\Delta(x_i)$ gives

$$\beta_1 \Xi_{i+1} + 2\beta_2 \Xi_i + \beta_1 \Xi_{i+1}) = \frac{u_{i-1}^{j+1} - \partial u_i^{j+1} + u_{i+1}^{j+1}}{h^2}, i = 1, 2, ...N - 1, \tag{4.2.15}$$

where

$$\alpha_1 = \left(\frac{sinh(\varrho) - \varrho}{\varrho^2 sinh(\varrho)}\right) and$$

$$\alpha_2 = \left(\frac{2\varrho cosh(\varrho) - 2sinh(\varrho))}{\varrho^2 sinh(\varrho)}\right).$$

Now from Eq.(4.2.7), we get

$$\begin{cases} \varepsilon\Xi_{i-1} = p_{i-1}u_{i-1}^{j+1} - g_{i-1}^{j+1}, \\ \varepsilon\Xi_{i} = p_{i}u_{i}^{j+1} - g_{i}^{j+1}, \\ \varepsilon\Xi_{i+1} = p_{i+1}u_{i+1}^{j+1} - g_{i+1}^{j+1}, \end{cases} \tag{4.2.16}$$

Substituting Eq.(4.2.16) into Eq.(4.2.15), we have

$$\begin{aligned} &\tfrac{1}{\varepsilon}[\beta_1(p_{i-1})u_{i-1}^{j+1} + \beta_2(p_i u_i^{j+1} + g_i^{j+1}) + \beta_1(p_{i+1}u_{i+1}^{j+1} + g_{i+1}^{j+1})] = \tfrac{u_{i-1}^{j+1} - u_i^{j+1} + u_{i+1}^{j+1}}{h^2} \\ &\tfrac{\varepsilon}{h^2}[u_{i+1}^{j+1} - u_i^{j+1} + u_{i+1}^{j+1}] = (\beta_1 p_{i-1})u_{i-1}^{j+1} + \beta_2 p_i u_i^{j+1} + \beta_1 p_{i+1}u_{i+1}^{j+1} + \beta_1 g_{i-1}^{j+1} + \beta 1 g_i^{j+1} + \beta_1 g_{i+1}^{j+1} \end{aligned} \tag{4.2.17}$$

To handle the effect of singular perturbation parameter "ε" on the solution profile we multiply the highest derivative by fitting factor "$\delta(\rho)$"

$$\begin{aligned} &\left(\tfrac{\delta(\rho)\varepsilon}{h^2} - \beta_1 p_{i-1}\right) u_{i-1}^{j+1} + \left(-\tfrac{\delta(\rho)\varepsilon}{h^2} - \beta_2 p_i\right) u_i^{j+1} + \left(\tfrac{\delta(\rho)\varepsilon}{h^2} - \beta_1 p_{i+1}\right) u_{i+1}^{j+1} \\ &= \beta_1 g_{i-1}^{j+1} + \beta_2 g_i^{j+1} + \beta_1 g_{i+1}^{j+1}. \end{aligned} \tag{4.2.18}$$

This can be re-written as three term recurrence relation

$$A_i^- U_{i-1}^{j+1} + A_i^c U_i^{j+1} + A_i^+ U_{i+1}^{j+1} = H_i, \tag{4.2.19}$$

where

$$\begin{aligned} A_i^- &= \frac{\delta(\rho)\varepsilon}{h^2} - \beta_1 p_{i-1}, \\ A_i^c &- \frac{\delta(\rho)\varepsilon}{h^2} - \beta_2 p_i \\ A_i^+ &= \frac{\delta(\rho)\varepsilon}{h^2} - \beta_1 p_{i+1}, \\ H_i &= \beta_1 g_{i-1}^{j+1} + \beta_2 g_i^{j+1} + \beta_1 g_{i+1}^{j+1}. \end{aligned}$$

Here, equation (4.2.19) is the tridiagonal system of equations concerning the xdirection and the coefficients A_i^-, A_i^c , A_i^+ and the right-hand side H_i are given that they satisfy the conditions $|A_i^-| > 0, |A_i^c| > 0, |A_i^+| > 0$ with $|A_i^c| > |A_i^-| + |A_i^+|$ at each $(n+1)^{th}$ level. These situations guarantee that the system is diagonally dominant. Thus, (4.2.19) can be solved by Thomas algorithm and stable.

To design the fitting factor, let us take the singularly perturbed homogeneous differential equation (4.2.7)

$$\frac{\varepsilon d^2u^{j+1}(x)}{dx^2} - pu^{j+1}(x) = 0 \tag{4.2.20}$$

subject to the boundary conditions $u^{j+1}(0) = q_0(t), u^{j+1}(1) = q_1(t)$ and its solution for constant C is

$$u^{j+1}(x) = C\exp(\sqrt{\frac{p(x)}{\varepsilon}}) \tag{4.2.21}$$

For ordinary differential equation case, representing the approximate solution $u^{j+1}(x)$ at the gird point x_i by u_i^{j+1} with the mesh size $h = \frac{1}{N}$, we have $x_i = ih, i = 0, 1, ..., N$; for N been positive integer. The central finite difference approximation for (Li et al., 2017; Smith, 1985) is

$$\frac{\varepsilon}{h^2}(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}) - pu_i^{j+1} = 0 \tag{4.2.22}$$

Introducing the fitting parameter σ on denoting $\sigma = \frac{h}{\sqrt{\varepsilon}}$ and evaluating limits on both sides of yields

$$\frac{\sigma}{\rho^2} = \frac{p\lim\limits_{h\to 0} u_i^{j+1}}{\lim\limits_{h\to 0}(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1})}$$

and considering on discreet domain of $\bar{\Omega}$, then we get the value of fitting parameter

$$\sigma = \frac{p\rho^2}{\exp\sqrt{p}\rho + \exp(-\sqrt{p}\rho) - 2}$$

which is equals to

$$\sigma = \frac{p\rho^2}{4}\left(csch(\frac{\sqrt{p\rho}}{4})\right)^2. \tag{4.2.23}$$

4.3 Stability and Consistency of the Method

The analysis of the proposed method is easily accomplished by the use of Fourier analysis. As authors of the books in (Li et al., 2017; Smith, 1985) provided detail reasons, the Von Neumann stability method is applied to investigate the stability of the developed scheme in (4.2.18), by assuming that its solution, at the grid point (x_i, t_{j+1}) is given by

$$u_i^j = \xi^j e^{nis}, \tag{4.3.1}$$

where $n = \sqrt{-1}$, s is the real number and ξ is the amplitude factor. Now, putting (4.3.1) into the homogeneous part of (4.2.18) yields the amplitude factor

$$\xi = \frac{\varepsilon\delta\left(\exp(ns) + \exp(-ns) - 2 + h^2 p_m^n\right)}{\varepsilon\delta(ns) + \exp(-ns - 2) - h^2 p_m^{n+1}}.$$

For sufficient small h, the condition of stability is $|\xi| \leq 1$ that can be satisfied,

$|-\varepsilon\delta\exp(ns) + \exp(-ns) - 2| \leq |\varepsilon\delta\exp(ns) + \exp(-ns) - 2|$.

Therefore, $|\xi| \leq 1$. Hence, the scheme given in (Li et al., 2017; Smith, 1985) is stable and, we can say formulated scheme is unconditionally stable.

To investigate the consistency of the method, then the truncation terms vanishes as $h \to 0$ and $\Delta t \to 0$. hence the scheme is constant with the order of $O(h^2 + \Delta t)$. Therefore, the constructed scheme is convergent by Laxs equivalence theorem (Li et al., 2017)

4.4 Numerical Examples, Results and Discussion

Some test examples have been presented to illustrate the efficiency of the proposed method. The exact solutions for these problems are unknown for comparison. Therefore, we use the double mesh principle to estimate the error as given in (Munyakazi and Patidar, 2013)

$$e_\varepsilon^{N,\tau} = \max_{(x_i,t_{j+1})} \in D^{N,M}\left(u^{N,\Delta t}(x_i, t_{j+1}) - u^{2N,\frac{\Delta t}{2}}(x_i, t_{j+1})\right) \tag{4.4.1}$$

where $u^{N,M}(x_i, t_{j+1})$ is the numerical solution with (N, M) mesh points and $u^{2N,\frac{\Delta t}{2}}(x_i, t_{j+1})$ is the numerical solution at the finer mesh with $(2N, \frac{\Delta t}{2})$ mesh points. The rate of convergence is calculated as ;

$$r_{\varepsilon}^{N,\Delta t} = \log_2 \left(\frac{e_{\varepsilon}^{N,\Delta t}}{e_{\varepsilon}^{2N,\frac{\Delta t}{2}}} \right) \tag{4.4.2}$$

For each N and M, the parameter uniform maximum absolute error and uniform rate of convergence are computed using

$$e^{N,\Delta t} = \max(e_{\varepsilon}^{N,\Delta t}), r^{N,\Delta t} = \log_2 \left(\frac{e^{N,\Delta t}}{e^{2N,\frac{\Delta t}{2}}} \right) \tag{4.4.3}$$

Example 4.4.1. Consider the following singularly perturbed parabolic partial differential equation with spatial and temporal delays:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + 6u(x,t) - 2u(x-1,t-\tau) = 1 (x,t) \in D \\ u(x,0) = 0, \\ u(0,t) = 0 \\ u(2,t) = 0 \end{cases}$$

Example 4.4.2. Consider the following singularly perturbed parabolic partial differential equation with spatial and temporal delays:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + (x+4)u(x,t) - (x^2+1)u(x-1,t-\tau) = 2 (x,t) \in D \\ u(x,0) = 0, \\ u(0,t) = 0 \\ u(2,t) = 0 \end{cases}$$

Example 4.4.3. Consider the following singularly perturbed parabolic partial differential equation with spatial and temporal delays:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + 5u(x,t) - 2u(x-1,t-\tau) = 2 (x,t) \in D \\ u(x,0) = \sin(\pi x), \\ u(0,t) = 0 \\ u(2,t) = 0 \end{cases}$$

Table 4.1: The maximum absolute error and uniform rate of convergence for Examples 4.4.1 when $\tau = \varepsilon/4, N = M$

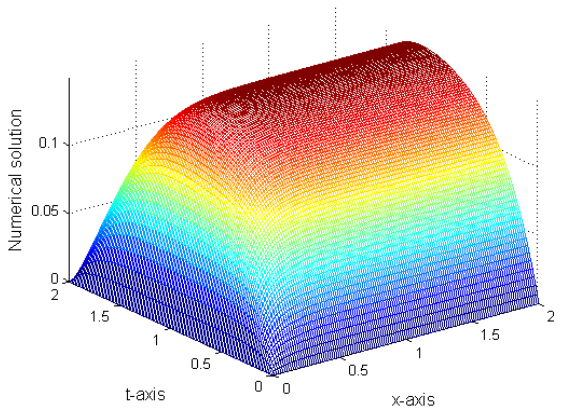
ε	N=16	N=32	N=64	N=128	N=256	N=512
2^{-2}	8.1596e-03	5.1663e-03	2.9275e-03	1.5664e-03	8.1319e-04	4.1482e-04
2^{-4}	9.0887e-03	5.2839e-03	2.9421e-03	1.5548e-03	7.9985e-04	4.0594e-04
2^{-6}	9.7565e-03	5.3189e-03	2.8388e-03	1.4886e-03	7.6275e-04	3.8615e-04
2^{-8}	9.9182e-03	6.6938e-03	3.2835e-03	1.6304e-03	8.3593e-04	4.2261e-04
2^{-18}	8.9867e-03	5.1804e-03	2.8020e-03	1.4615e-03	7.4711e-04	3.7809e-04
2^{-20}	8.9867e-03	5.1804e-03	2.8020e-03	1.4615e-03	7.4711e-04	3.7780e-04
E^N	9.9182e-03	6.6938e-03	3.2835e-03	1.6304e-03	8.3593e-04	4.2261e-04
r^N	5.6725e-01	1.0276	1.0100	9.6377	9.8406	

Table 4.2: The maximum absolute error and uniform rate of convergence for Examples 4.4.2 when $\tau = \varepsilon/4, N = M$

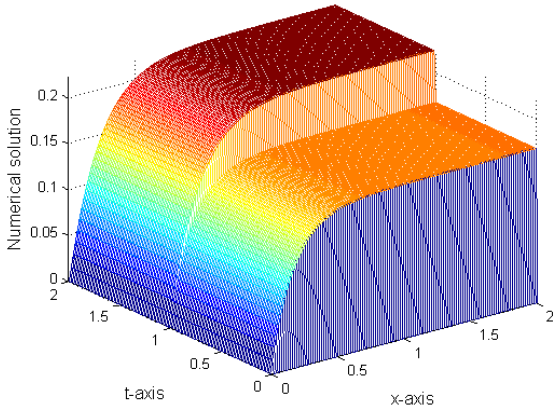
ε	N=16	N=32	N=64	N=128	N=256	N=512
2^{-2}	1.8454e-02	1.0996e-02	3.2026e-03	1.6529e-03	1.6529e-03	8.4031e-04
2^{-4}	2.0310e-02	1.1678e-02	6.3377e-03	3.3115e-03	1.6939e-03	8.5689e-04
2^{-6}	2.2359e-02	1.2004e-02	6.4049e-03	3.3168e-03	1.6888e-03	8.5220e-04
2^{-8}	2.5661e-02	1.6098e-02	8.1807e-03	4.7853e-03	2.4845e-03	1.2615e-03
2^{-18}	2.2675e-02	1.3005e-02	7.0389e-03	3.6734e-03	1.8782e-03	9.5094e-04
2^{-20}	2.2675e-02	1.3005e-02	7.0389e-03	3.6734e-03	1.8782e-03	9.4984e-04
E^N	2.2675e-02	1.3005e-02	7.0389e-03	3.6734e-03	1.8782e-03	9.4984e-04
r^N	6.7270e-01	9.7659e-01	7.7361e-01	9.4565e-01	9.7782e-01	

Table 4.3: The maximum absolute error and uniform rate of convergence for Examples 4.4.3 when $\tau = \varepsilon/4, N = M$

ε	N=16	N=32	N=64	N=128	N=256	N=512
2^{-2}	1.7906e-02	1.0869e-02	6.0353e-03	3.2092e-03	1.6587e-03	8.4421e-04
2^{-4}	1.9390e-02	1.1224e-02	6.1044e-03	3.1952e-03	1.6371e-03	8.2884e-04
2^{-6}	2.1450e-02	1.1322e-02	5.9247e-03	3.0788e-03	1.5702e-03	7.9306e-04
2^{-8}	2.3721e-02	1.5315e-02	7.4912e-03	4.0776e-03	2.1070e-03	1.0635e-03
2^{-18}	1.9272e-02	1.0935e-02	5.8407e-03	3.0239e-03	1.5400e-03	7.7809e-04
2^{-20}	1.9272e-02	1.0935e-02	5.8407e-03	3.0239e-03	1.5400e-03	7.7718e-04
E^N	2.3721e-02	1.5315e-02	7.4912e-03	4.0776e-03	2.1070e-03	1.0635e-03
r^N	6.3122e-01	1.0317	8.7748	9.5253	9.8637	

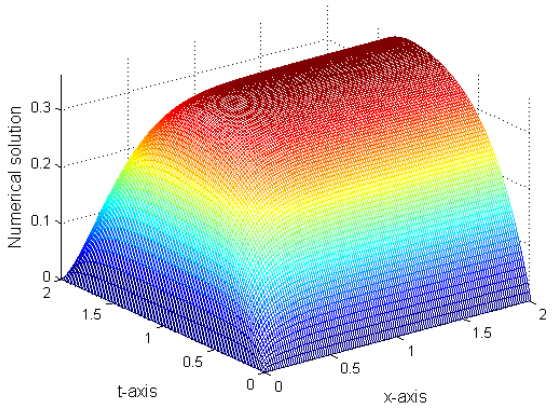


(a) $\varepsilon = 2^{-2}$

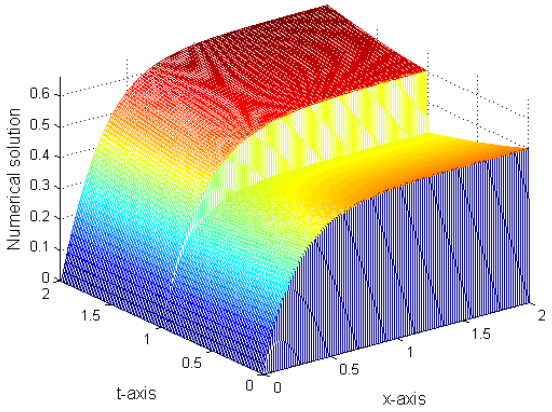


(b)$\varepsilon = 2^{-18}$

Figure 4.1: Numerical solution profile for Example 4.4.1 when $N = M = 128$.

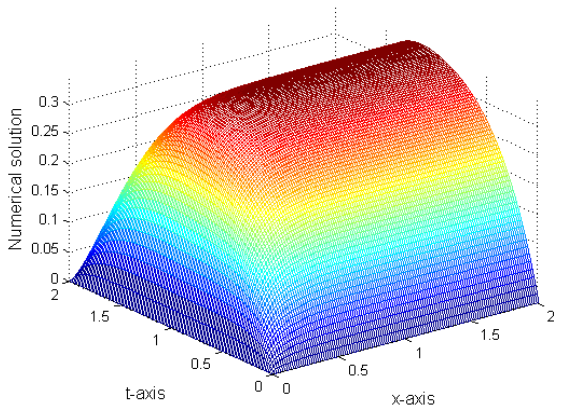


(a) $\varepsilon = 2^{-2}$

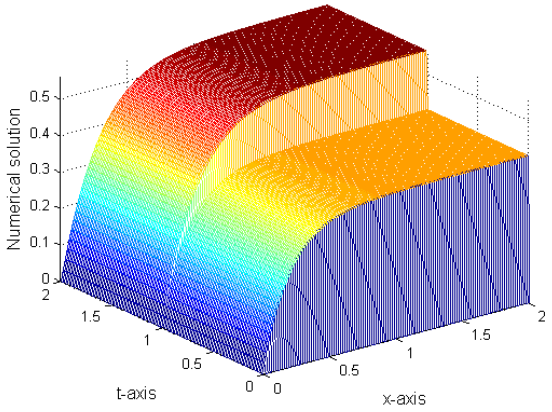


(b)$\varepsilon = 2^{-18}$

Figure 4.2: Numerical solution profile for Example 4.4.2 when $N = M = 128$.



(a) $\varepsilon = 2^{-2}$



(b)$\varepsilon = 2^{-18}$

Figure 4.3: Numerical solution profile for Example 4.4.3 when $N = M = 128$.

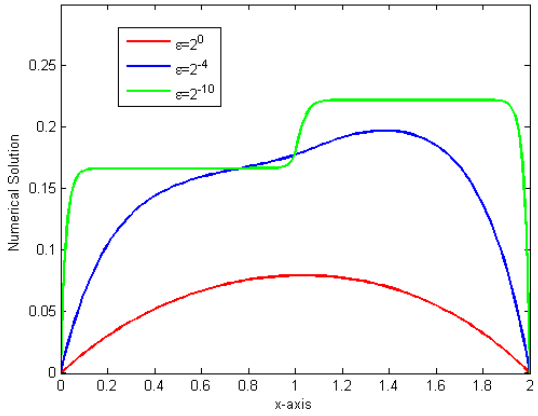


Figure 4.4: Effect of ε on solution profiles for Example 4.4.1 when $N = M = 128$.

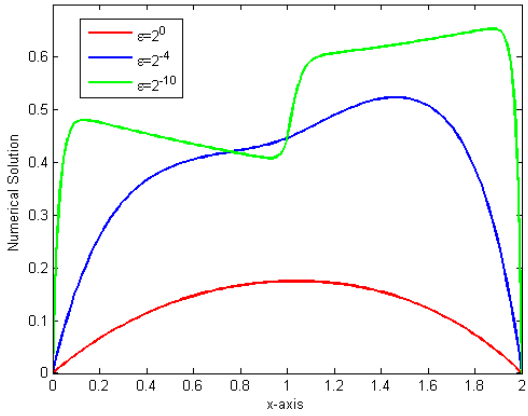


Figure 4.5: Effect of ε on solution profiles for Example 4.4.2 when $N = M = 128$.

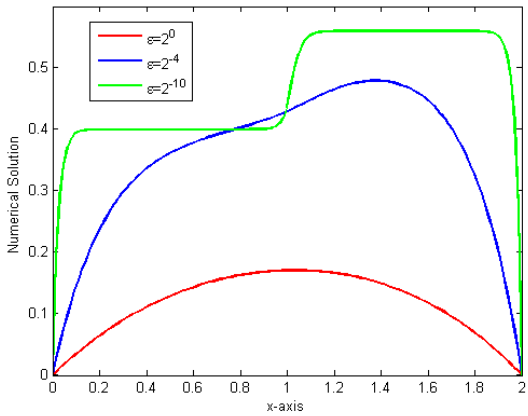


Figure 4.6: Effect of ε on solution profiles for Example 4.4.3 when $N = M = 128$.

Discussion

In each of the three examples, the solution has a boundary layer at the left side of the interval . Table 4.1 and 4.2 present the point-wise maximum error E^N and rate of convergence r^N for different values of ε and N. The results shown in the tables reveal that the proposed method is of first-order uniform convergence, as projected by the theoretical analysis. Figure 4.1 provides the plot of numerical solutions of Example 4.4.2 for $\varepsilon = 2^{-}18$ and N = M = 128. In figure 4.3, the plot of numerical solution of Example 4.4.3 for $\varepsilon = 2^{-}18$ and N = M = 128 is given.

Chapter 5

Conclusion and Recommendations

5.1 Conclusion

In this study, we proposed and analyzed a fitted numerical method for a singularly perturbed differential equation involving spatial and temporal delays in the reaction term. The solution varies abruptly in the layers due to the presence of the perturbation parameter. The rapidly changing behavior of the layers and the effect of the delays cause difficulties to find the analytical solution. To solve the problem, we proposed a fitted numerical method. The method is obtained by using the implicit Euler method in the temporal variable and the cubic spline intension in the spatial variable on uniform meshes. The effect of the temporal delay is handled by applying Taylors series approximation and the spatial delay is handled by choosing a special mesh so that the delay term lies on the mesh point $x_i = 1$. We investigated and proved that the proposed method is stable and uniformly convergent. We considered and solved three model examples to test the validity and applicability of the proposed method. The solutions and accuracy of the results of the examples are shown in graphical and tabular forms. From the theoretical and numerical findings discussed in the thesis, we can conclude that the proposed numerical method is uniformly convergent of order one in time and of order two in space.

5.2 Recommendations

In this thesis, NFDM for solving SPDDE has been presented. This work can be extended for singularly perturbed partial differential problems.

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