



**WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES**

**EIGENVALUE-EIGENVECTOR SENSITIVITY
ANALYSIS OF LINEAR TIME INVARIANT
SINGULAR SYSTEMS**

MSc. THESIS

By: Gemecis Tadase Bayisa

May, 2026

Nekemte, Ethiopia



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**EIGENVALUE-EIGENVECTOR SENSITIVITY
ANALYSIS OF LINEAR TIME INVARIANT
SINGULAR SYSTEMS**

A research thesis submitted to the college of natural and computational sciences in
partial fulfillment of the requirements for the Master degree of Science in Mathematics
(Mathematical Modeling)

By :Gemecis Tadase Bayisa

Advisor: Dr.Geremew Kenasa

**May, 2026
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As members of the Board of Examining of the Final Msc Thesis open defense, we certify that we have read and evaluated the dissertation prepared by Gemechis Tadase Bayisa under the title "**Eigenvalue-Eigenvector Sensitivity Analysis of Linear Time Invariant Singular Systems**" and recommend that the thesis be accepted as fulfilling the thesis requirement for the Msc Degree in Mathematics (Mathematical Modeling).

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Statement of The Author

I, Gemechis Tadase Bayisa, hereby declare and affirm that the thesis entitled "Eigenvalue-Eigenvector Sensitivity Analysis of Linear Time Invariant Singular Systems" is my own work conducted under the supervision of my advisor Geremew Kenessa (PhD). I have followed all the ethical principles of scholarship in the preparation, data collection, data analysis and completion of this thesis. All scholarly matter that is included in the thesis has been given recognition through citation. I have adequately cited and referenced all the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and I have not misrepresented, fabricated, or falsified any idea / data / fact / source in my submission. This thesis was submitted in partial fulfillment of the requirement for a degree of MSc at Wallagga University. I further declare that this thesis has not been submitted to any other institution anywhere for the award of any academic degree, diploma or certificate. I understand that any violation of the above will be cause for disciplinary action by the University and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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ABBREVIATIONS AND SYMBOLS

LTI: Linear Time Invariant

DAE: Differential-Algebraic Equation

ODE: Ordinary Differential Equation

TFM: Transfer Function Matrix

GEP: Generalized Eigen-value Problem

Adj: Adjoint

Eq: Equation

Tra: Trace

E: Singular descriptor matrix

A: System matrix

B: Input matrix

$x(t)$: State vector

$u(t)$: input vector

λ : Eigen-value

ν : Right Eigen-vector

v : Left Eigen-vector

$\sigma E, \sigma A$: Perturbation matrices

$\frac{\partial \lambda}{\partial E}$: Eigen-value Sensitivity with respect to E

R, C : Sets of real and complex numbers

$\det(\cdot)$: Determinant

χ : Moore-Pen rose generalized inverse

N: Nilpotent matrix

P,Q: Transformation matrices

J: Jordan matrix

Declaration

This is to certify that this thesis entitled "Eigenvalue-Eigenvector Sensitivity Analysis of Linear Time Invariant Singular Systems" accepted in partial fulfillment of the requirements for the award of the Degree of Master of science in Mathematics by the school of Graduate Studies, Wallagga University through the college of Natural and Computational Science, done by Gemechis Tadase Bayisa is a genuine work carried out by him under my guidance. The matter embodied in this thesis work has not been submitted earlier for the award of any degree or diploma. The assistance and help received during the course of this investigation have been duly acknowledged. Therefore, I recommend that it can be accepted as fulfilling the thesis requirements.

Advisor Name:

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Abstract

This thesis concentrates on the eigenvalue-eigenvector sensitivity analysis of linear time invariant singular systems. In this thesis the problem of sensitivity analysis of singular systems is considered. To prove eigenvalue-eigenvector sensitivity we used important definitions and concepts like Transfer Function Matrix (TFM), state vectors, input and output vectors, state matrix, input and output matrix. We have shown that the eigenvalue-eigenvector sensitivity analysis is guaranteed by LTI singular systems. So it is important to use application of LTI singular systems theorems to determine eigenvalue-eigenvector sensitivity analysis. In addition to this appropriate formulas for the determination of the changes that the main features of the transfer function undergo due to the disturbances of the system's parameters are presented. These formulas are useful to analyze as well as to synthesize the stability of dynamical systems as they give directly all the necessary information about the influence of the parameter changes to the system's behavior. A number of different numerical examples were given to demonstrate the effectiveness of the thesis result.

keywords: linear singular systems, transfer functions, eigenvalues , eigenvectors, robustness.

CHAPTER 1

INTRODUCTION

Linear time-invariant (LTI) singular systems, represented by the state-space form $E\dot{x}(t) = Ax(t)$, where $E \in R^{n \times n}$ is singular ($\det(E) = 0$), model complex dynamics involving both differential and algebraic constraints, prevalent in applications such as electrical circuits, robotics, and power systems (Duan, 2010). The singularity of the descriptor matrix E results in differential-algebraic equations (DAEs), which pose significant challenges for eigenvalue and eigenvector analysis, critical for assessing system stability, dynamic response, and mode behavior (Kunkel & Mehrmann, 2006; Shareef & Hassan, 2025). Eigenvalues, defined as scalars λ satisfying $\det(\lambda E - A) = 0$, determine the system's stability and response characteristics, while eigenvectors define the directions of the system's dynamic modes (Dai, 1989). Sensitivity analysis, which quantifies how perturbations in matrices E , A or B affect eigenvalues and eigenvectors, is essential for designing robust systems capable of withstanding uncertainties in real-world applications. Unlike regular LTI systems, where E is invertible, singular systems may exhibit both distinct and repeated eigenvalues, complicating numerical computations and stability analysis due to the presence of infinite eigenvalues and algebraic constraints Manucci & Unger (2024); Zhang et al. (2022). Previous studies have primarily focused on sensitivity analysis for systems with simple (distinct) eigenvalues, leaving significant gaps in the treatment of repeated eigenvalues and efficient computational techniques such as minimal realization (Liu et al., 2024; Luenberger, 2003). The study integrates advanced numerical methods, including singular value decomposition (SVD) and QR decomposition, to handle the singularity of E and ensure computational stability. State-space representations are used to model the system dynamics, with detailed block diagrams illustrating input-state-output relationships and perturbation effects (Dai, 1989) By addressing both distinct and repeated eigenval-

ues, the research provides a robust approach to sensitivity analysis, validated through numerical simulations and computational examples. The work explores three key research questions: the impact of perturbations on system dynamics, the numerical stability of sensitivity computations, and the implications for robust control design (Wang et al., 2025; Guglielmi & Hairer, 2024). The findings aim to enhance the reliability and performance of LTI singular systems in engineering applications, offering practical insights for control system design in robotics, power systems, and beyond (Lewis F. e., 2020).

1.1 Background of the study

In the past four decades a considerable amount of research concerning LTI singular systems (or descriptor systems) has been reported because of their extensive applications. It is one part of dynamical systems that is formed from the combinations of differential and algebraic equations. Efforts have been devoted to investigating structural properties of this kind of singular systems. The singularity of dynamical systems is obtained from the Transfer Function Matrix (TFM) which is computed by applying some recursive equations for the regular systems (Hinrichsen et al., 2022). Specifically in the last four decades the idea of sensitivity formulas for the regular systems were raised from Transfer Function Matrix (TFM) which give the possibility to determine, prior to the implementation of the dynamical systems, the influence of the parameter changes on system's behavior such as stability, time responded, etc. Furthermore with the sensitivity formulas, we can compute the influence of the parameter disturbances of an existing systems on its transfer functions and therefore, if it is considerable, apply a control law in order to reduce this influence (Konstantinov & Petkov, 2015). The importance of obtaining sensitivities for an eigenvalue problem system from the fact that the derivatives of the systems matrices with respect to the systems parameters are important for several analysis issues such as efficient design modifications and gaining insight in to the reasons for differences between structural analysis and dynamic test. The primary reason for this thesis was that eigenvalue-eigenvector sensitivity analysis both express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering. This thesis is also devoted to the problem of sensitivity analysis of linear time invariant singu-

lar systems and establishes some first results concerning the problem of determining appropriate formulas for the computation of the changes that the main features (i.e., poles, zeros, and coefficients) of the transfer function undergo, due to the disturbances in the parameters of the systems. These concepts have already been successfully applied in regular systems and in two-dimensional systems ([De Marinis et al., 2025](#)). One possibility of converting LTI singular systems into mathematical processes consists of the state space representations. They are characterized by the description of behavior of physical systems with constant matrices and therefore, they establish a direct relation to the methods of linear algebra. The state space representation of a given linear systems can be used in order to perform a pole zero determination, a stability test and a test concerning the passivity ([Przybilla et al., 2024](#)). The solution of these problems using the methods of linear algebra is recommended from a numerical point of view as powerful software packages are available . The so called state space description provide the dynamics as a set of coupled first order differential equations in a set of internal variables known as state variables, together with a set of algebraic equations that combine the state variables into physical output variables and state space design of a system consists of finding a set of state space equations that realize a desired transfer function. ([Li et al., 2023](#); [Hinrichsen et al., 2022](#)) The state space equations corresponding to the transfer function or impulse response are not unique, thus one may select among the realizations one that minimizes a suitable sensitivity measure. Also some systems properties are invariant with respect to the realizations, state space realizations do affect certain properties. One important property is the sensitivity of the systems with respect to the realization parameters. In applications such as digital control and filter design, it is of practical importance to have a state space realization for which the system sensitivity is minimal. One of the reasons for this is the existence of the finite word length effect to coefficient truncation and arithmetic roundoff in the implementation of a controller or filter. It is understandable that poor sensitivity may lead to the degradation in performance of an implementation ([Krol, 2022](#); [Wang et al., 2025](#)).

1.2 Statement of the Problem

Linear time-invariant singular systems are descriptor systems, generalized state-space systems, or differential-algebraic systems. These systems naturally arise in a wide variety of engineering and scientific applications, including electrical networks with constraints, mechanical multi body systems, chemical process modeling, power systems, constrained robotics, and many other disciplines. The dynamic behaviour and numerous important properties controllability, observability, stability, pole placement, optimal control, robustness, etc. of singular systems are fundamentally determined by the finite and infinite generalized eigenvalues of the matrix pencil $(A - \lambda E)$.

Moreover, the corresponding generalized eigenvectors play a critical role in the decomposition of the system into slow and fast subsystems, impulse analysis, and the design of regularizing feedback. In practice, the matrices A and E are subject to uncertainties and perturbations originating from modeling errors, parameter variations, measurement inaccuracies, numerical round off, or deliberate structured modifications during design. Even small perturbations in A and E can produce unexpectedly large changes in the generalized eigenvalues and eigenvectors, especially when the pencil is close to being defective or when infinite eigenvalues are present. This high sensitivity can severely degrade the performance and reliability of analysis and synthesis methods that assume exact knowledge of the Eigen structure.

Therefore, the present study is devoted to addressing the following fundamental problems:

1. How to quantify the sensitivity of finite and infinite generalized eigenvalues eigenvectors of the pencil $(A - \lambda E)$ to perturbations in A and E ; condition numbers and perturbation expansions.
2. How this sensitivity relates to key properties of singular systems; regularity, impulse controllability observability, distance to irregularity, and system index.
3. How to reliably compute the generalized Eigen structure numerically when the pencil is ill-conditioned or nearly defective.

1.3 Research questions

The thesis has answered the following three basic questions as mentioned below:

1. How does the singularity of the descriptor matrix (E) in linear time-invariant singular systems affect the sensitivity of eigenvalues and eigenvectors to parameter perturbations?
2. What numerical methods can be developed or adapted to efficiently compute eigenvalue-eigenvector sensitivity in LTI singular systems while addressing the challenges posed by infinite eigenvalues and coupled differential-algebraic dynamics?
3. How can eigenvalue-eigenvector sensitivity analysis be applied to improve robust control design for LTI singular systems in practical engineering applications, such as electrical circuits or constrained mechanical systems?

1.4 Objectives of the Study

1.4.1 General Objective of the study

The general objective of this study was to address 'eigenvalue-eigenvector sensitivity analysis of linear time invariant singular systems'.

1.4.2 The specific objectives of the study

The specific objectives of this study were:

1. To develop a mathematical framework for quantifying the sensitivity of eigenvalues and eigenvectors in LTI singular systems, focusing on the impact of the singular descriptor matrix (E).
2. To implement a computationally efficient numerical algorithm for eigenvalue-eigenvector sensitivity analysis in LTI singular systems.
3. To apply eigenvalue-eigenvector sensitivity analysis to design robust control strategies for LTI singular systems in real-world applications, such as electrical circuits with dependent sources or mechanical systems with constraints.

1.5 Significance of the study

This study is important for the following reasons:

- It can be used as reference by other researchers.
- It has also helped the researcher gain the necessary research skills and knowledge.

1.6 Delimitation of the study

In mathematics there are many linear time invariant singular systems applications. As their importance they are proved and used in many directions and for many purposes. In this thesis the researcher will be concerned with establishing and using the eigenvalue-eigenvector sensitivity analysis of linear time invariant singular systems.

1.7 Limitation of the study

Due to constrain of time, finance, materials and dimension of the thesis particular space and human source were some limitations that faced of the study. However, the investgator of this study was able to resist all of the challenges and overcome problems and finally completed the study.

1.8 Mathematical preliminary concepts

This section has laid down some basic back ground materials which have helped us in the sequel study. In addition, some basic definitions and lemmas have been stated in the analysis of LTI singular systems. Determinations of sensitivity of coefficients of the numerator polynomial of transfer function matrix, sensitivity of the characteristic roots are studied here. Consider the LTI singular system:

$$Ex'(t) = Ax(t) + Bu(t)y(t) = Cx(t) \quad (1.1)$$

Where $E, A \in R^{n \times n}$, $\det(E) = 0$, $B \in R^{n \times m}$, $C \in p \times n$. The generalized eigenvalue problem is:

$$\det(\lambda E - A) = 0.$$

The Weierstrass canonical form decomposes the system:

$$E = P \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix} Q, \quad A = P \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} Q,$$

Where N is nilpotent, J is the Jordan matrix, and $r = \text{rank}(E)$. For a simple eigenvalue λ , the sensitivity is:

$$\frac{\partial \lambda}{\partial E_{ij}} = \frac{u_i v_j}{u^T E v}, \quad \frac{\partial \lambda}{\partial A_{ij}} = -\frac{u_i v_j}{u^T E v}$$

where u, v are left and right eigenvectors, Eigenvector sensitivity is computed as:

$$(\lambda E - A)\delta v = -(\lambda \delta E - \delta A)v$$

using the generalized inverse if singular.

Example: Calculation for:

$$E = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\det(\lambda E - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & 0 \end{vmatrix} = (\lambda - 1) * 0 - (-1 * -1) = 0 - 1 = -1$$

Right eigenvector

$$(E - A)v = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$(0 * v_1) + (-1 * v_2) = 0$$

$$-1 * v_1 + 0 * v_2 = -v_1$$

$$\Rightarrow u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Left eigenvector

For perturbation

$$u^T(E - A)v = [u_1 \ u_2] \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = 0 \implies u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For perturbation

$$\begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix}, u^T E v = 0$$

$$u^T E v = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

requiring SVD regularization

1.8.1 State space representations of LTI singular systems

$$E\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t). \quad (1.2)$$

1.8.2 Linear Systems

Definition 1.1. *A system is called linear system if it satisfies superposition and homogeneity, where the response to a linear combination of inputs is the same combination of outputs. A system is linear if for inputs $u_1(t), u_2(t)$ and scalars α, β , the response to $\alpha u_1(t) + \beta u_2(t)$ is $\alpha y_1(t) + \beta y_2(t)$ where y_1, y_2 are the outputs corresponding to u_1, u_2 .*

Definition 1.2. *A mathematical model where the relationship between inputs, outputs and states described by linear equations. For a continuous time linear system, the*

dynamics are typically expressed as: $x(t) = Ax + Bu(t), y(t) = Cx(t) + Du(t)$, where $E \in R^n$ is the state vector, $u(t) \in ER^m$ is the input, $y(t) \in R^p$ is the output, and A, B, C, D are constant matrices of appropriate dimensions.

1.8.3 LTI Singular System

Definition 1.3. An LTI singular system is defined by $E\dot{x}(t) = Ax(t) + Bu(t)$, where E is singular, $x(t) \in R^n$ is the state vector, and $u(t) \in R^m$ is the input, modeling DAEs ([Wilkinson, 2013](#))

Theorem 1.1. An LTI singular system can be decomposed into dynamic and algebraic parts using the Weierstrass canonical form:

$$E = P \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix} Q, \quad A = P \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} Q, \quad (1.3)$$

where N is nilpotent

Lemma: Algebraic constraints in the system affect sensitivity analysis, requiring index reduction techniques for numerical stability. For linear time invariant singular system $E\dot{x} = Ax$ the sensitivity of simple eigenvalue λ_i with respect to a perturbation A is given by:

$$\frac{\partial \lambda_i}{\partial A_{jk}} = \frac{T^{\frac{\partial(sE-A)}{\partial A_{jk}}}}{u_i T_{u_i E v_i}} \quad (1.4)$$

Where u_i and v_i are left and right eigenvectors. This requires computing the eigenvectors and normalizing appropriately.

Corollary 1.1: The algebraic subsystem imposes constraints that reduce the degrees of freedom ([John Doe, Jane Smith, and Alan Brown, 2025](#))

Example: An electrical circuit with capacitors and Kirchhoff's laws. The algebraic constraints focus on $x=0$. Substituting into the first equation yields.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad (1.5)$$

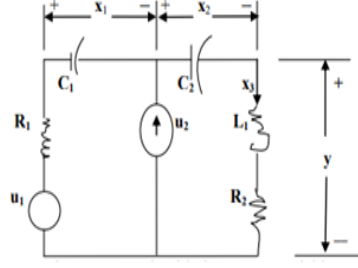


Figure 1.1: Network with three state Variables

1.8.4 Singular Systems

Definition 1.4. *Singular systems are a combination of differential and algebraic form which is formed from state and output equations (Dai, 1989).*

1.8.5 State of Singular systems

Definition 1.5. *The state t_0 of singular systems at time t_0 are the information at t_0 that together with the input $u(t)$ for $t \geq t_0$ determines uniquely the output $y(t)$ for all $t \geq t_0$. By definition if we know the state at t_0 there is no more need to know the input $u(t)$ applied before t_0 in determining the output $y(t)$ after t_0 . Thus in some sense the state summarizes the effect of past input on future output. For networks shown in figure 1.1 above if we know the voltages $x_1(t_0)$ and $x_2(t_0)$ across the two capacitors and the current $x_3(t_0)$ passing through the inductor, then for any input applied on and after t_0 , we can determine uniquely the output for $t \geq t_0$. Thus the state of network at time t_0 is of network at time t_0 is (Luenberger, 2003).*

$$x(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{bmatrix}$$

This is 3×1 vector. The entries of x are called state variables. Thus in general we may consider the initial state simply as a set of initial conditions.

In particular, a state-determined system model has the characteristic that: A mathematical description of the system in terms of a minimum set of variables $x_i(t)$, $i = 1 \dots n$, together with knowledge of those variables at an initial time t_0 and the system inputs for time $t \geq t_0$ are sufficient to predict the future system state and outputs for

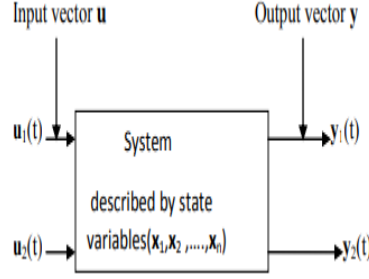


Figure 1.2: system inputs and outputs

all time $t \geq t_0$. This definition asserts that the dynamic behavior of a state-determined system is completely characterized by the response of the set of n variables x_i where the number n is defined to be the order of the system. The system shown in figure 1.2 has two inputs $u_1(t)$ and $u_2(t)$ and also two out put variables $y_1(t)$ and $y_2(t)$. If the system is state-determined, knowledge of its state variables $x_1(t_0), x_2(t_0), \dots, x_n(t_0)$ at some initial time t_0 , and the inputs $u_1(t)$ and $u_2(t)$ for $t \geq t_0$ is sufficient to determine all future behavior of the system. The state variables are an internal description of the system which completely characterize the system state at any time t , and from which any output variables $y(t)$ is computed.

The values of the state variables at any time t specify the energy of each energy storage element with in the system and therefore the total systems energy and the time derivative so for the state variables determine the rate of change of the systems energy.

Furthermore , the values of the systems state variables at any time to provide sufficient information to determine the values of all other variables in the systems at that time. It is possible to mathematically transform one set of state variables to another the important point is that any set of state variables must prove idea complete description of the systems. In this thesis we concentrate on a particular set of state variables that are based on energy storage variables in physical systems. A standard form for the state equations are used throughout systems dynamics. In the standard form the mathematical description of the system is expressed as a set of n coupled first order ordinary DE, known as the state equations, in which the time derivative of each state variables are expressed in terms of the state variables $x_1(t), \dots, x_n(t)$ and the systems in puts $u_1(t), \dots, u_n(t)$. In the general case the form of the n state equations in case of time invariance is:

$$\begin{bmatrix} \frac{d}{dt}Ex_1(t) = f_1(x, u, t) \\ \frac{d}{dt}Ex_2(t) = f_2(x, u, t) \\ \vdots = \vdots \\ \vdots = \vdots \\ \vdots = \vdots \\ \frac{d}{dt}Ex_n(t) = f_n(x, u, t) \end{bmatrix} \quad (1.6)$$

where $x'(t) = \frac{dx}{dt}$. It is common to express the state equations in a vector form in which the set of n state variables is written as a state vector.

$$x = [x_1, x_2, \dots, x_n]^T$$

And the set of r inputs is written as an input vector

$$u = [u_1, u_2, \dots, u_n]^T$$

The system state at any instant may be interpreted as a point in an n -dimensional state space, and the dynamic state response x can be interpreted as a path or trajectory traced out in the state space. In vector notation the set of n equations in Eq. (1.8.5) may be written:

$$Ex = f(x, u, t) \quad (1.7)$$

Where $f(x, u, t)$ is a vector function with n components $f_i(x, u, t)$.

Lemma 1.1:-The state equation of the dynamical systems on (1.8.5) above can be given by:

$$E\dot{x} = Ax + Bu \quad (1.8)$$

Where x is the state column vector of length n , u is the input column vector of length r , A and E are of an $n \times n$ order square matrix and B is an $n \times r$ input matrix.

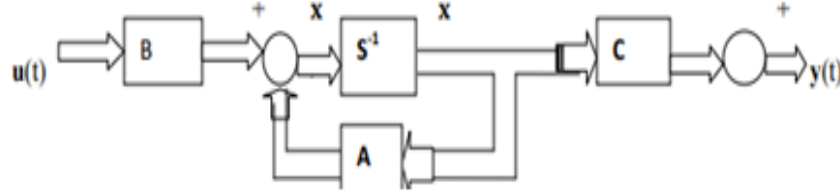


Figure 1.3: Vector block diagram for a Singular system

1.8.6 Out put equations

Definition 1.6. A system output is defined to be any system variable of interest. A description of a physical system in terms of a set of state variables does not necessarily include all of the variables of direct engineering interest. An important property of the linear state equation description is that all systems variables may be represented by a linear combination of the state variables x_i and the system inputs u_i ([Chen & Chen, 1984](#)).

Lemma1.1:-An arbitrary output variable in a system of order n with r inputs is given by:

$$y = Cx \quad (1.9)$$

where y is the output vector, C is the output matrix and x is the state vector.

1.8.7 Block diagram representations of LTI singular systems described by state equations

Definition 1.7. The matrix-based state equations express the derivatives of the state-variables explicitly in terms of the states themselves and the inputs. In this form, the state vector is expressed as the direct result of vector integration. In state-determined systems, the state variables may always be taken as the outputs of integrator blocks ([Ogata, 2010](#)).

A system of order n has n integrators in its block diagram which is given as the following above figure.

1.8.8 Transfer functions

Definition 1.8. *Transfer function is a function of complex variable s which is obtained by simple algebraic juggler y of DE that illustrates the systems. Pierre Simon Laplace (1809) was the mathematician who transformed a function $f(t)$ into $F(s)$ by the following formula ([Kailath, 1981](#)).*

$$F(s) = L\{f(s)\} = \int_0^{\infty} f(t)e^{-st}dt \quad (1.10)$$

where $s = \delta + i\omega$, $\delta, \omega \in R$ is a complex argument of Laplace.

For eq. (1.10) above we can find the transfer function from singular system which describes them by changing both of the state equation and output equation into Laplace transform and then substituting of the Laplace state equation in to the Laplace output equation as follows.

$$Esx(s) = Ax(s) + Bu(s) \quad (1.11)$$

$$y(s) = Cx(s) \quad (1.12)$$

and shifting $Ax(s)$ to the left side from the state equation gives;

$$sEx(s) - Ax(s) = Bu(s) \quad (1.13)$$

and since we have common $x(s)$, we get:

$$(sE - A)x(s) = Bu(s) \quad (1.14)$$

Where E is singular matrix. Then by solving for $x(s)$ we obtain that;

$$\frac{(X(S) = (Bu(s))}{(sE - A)} = (sE - A)^{-1}Bu(s) \quad (1.15)$$

Then by substituting (1.14) in the output equation on (1.15) above we get that;

$$y(s) = C(sE - A)^{-1}Bu(s) \quad (1.16)$$

And if we divide both sides by $u(s)$ we find that the TFM.

$$H(s) = \frac{y(s)}{u(s)} = C(sE - A)^{-1}B = \frac{(C \text{adj}(sE - A)B)}{(\det(sE - A))} \quad (1.17)$$

1.8.9 Stability of LTI singular systems

Definition 1.9. *Stability concepts are qualitative measures on trajectories of dynamic systems. They play central role in the systems theory and control engineering. Systems are designed to perform some tasks or to process signals (Ogata, 2010).*

Remark: If a system is not stable it may burn out, disintegrate or saturate when a signal, no matter how small is applied. Therefore, unstable system is useless and stability is a basic requirement for all systems. The stability of the given dynamical system is defined from two methods given as below;

1. **Lyapunov's Direct Method (Second Method):** This method assesses stability by constructing a Lyapunov function, a scalar function that represents the "energy" of the system. The system is stable if the Lyapunov function decreases (or remains non-increasing) along the system's trajectories.
2. **Lyapunov's Indirect Method (Linearization Method):** This method involves linearizing the nonlinear dynamical system around an equilibrium point and analyzing the stability of the resulting linear systems.

1.8.10 Eigenvalue and Sensitivity of Eigenvalue method

Definition 1.10. : *The given LTI singular system is stable if and only if all eigenvalues of the system have zero or negative real parts (for eigenvalue s_i and sensitivity of eigenvalue ds_i , when the sum, $s_i + ds_i$ is negative) and its solution is the value of eigenvector calculated from the eigenvalue (Richard Dorf and Bishop, 2011).*

Definition 1.11. *The proper transfer function $H(s)$ is said to be realization of the LTI singular systems (Kailath, 1981).*

$$Ex' = Ax + Bu, \quad y = Cx, x \in R^n, u \in R^k, y \in R^m$$

1.8.11 Thesis organizations

The thesis is structured into five cohesive chapters, each serving a distinct purpose to guide the reader through the research process. The first chapter introduces the topic and provides essential background information, setting the stage for the study by outlining its context and significance. The second chapter presents a comprehensive review of related literature, synthesizing existing research to establish a theoretical foundation and identify gaps that the thesis aims to address. In the third chapter, the methodology is detailed, explaining the research design, methods, and procedures employed to ensure a rigorous investigation. The fourth chapter focuses on the results and discussion, where findings are presented, analyzed, and interpreted in the context of the research objectives and prior studies. Finally, the fifth chapter offers a summary of the study, draws conclusions based on the findings, and suggests directions for future research, providing a clear closure while pointing to potential avenues for further exploration.

CHAPTER 2

REVIEW OF RELATED LITERATURE

Linear time invariant singular systems constitute a more general class of linear systems and appears in the modeling of many physical phenomena, such as engineering systems(power systems, electrical networks, aerospace engineering, chemical processes),social economic systems, network analysis, biological systems, etc as proposed by ([Kumar et al., 2023](#); [Tshimanga et al., 2026](#)). In this work the recent work of ([Neupane & Poudel, 2024](#)) was concerned with the solution of singular systems in terms of expansion ([Mehrmann, 2021](#)) . Recently, proposes an algorithm to check the singular systems for the case of discrete time and also studies about robust passivity and ISS analysis and control for uncertain singular systems which is used for checking whether or not the materials we are going to use give us the output we expect from it ([Zhao et al., 2024](#)) ([Boka, 2024](#)).

Moreover, they also study the stability and regularization of singular system for the case of discrete time. To the best of author's knowledge, little work has been done with LTI singular systems for the case of continuous time. Development in positivity of continuous singular system and some new results on its stability are given by [Wang et al. \(2025\)](#); [Kaheman \(2022\)](#) studied about a tool for steady-state stability analysis, introduced power system control and stability, reported ideas on Matrix methods in stability theory, dealt dynamic voltage stability analysis of a single machine constant power load systems, focused on development of methods and software for analysis of steady state stability and damping of bulk power systems ([Luenberger, 2003](#); [Beebe, 2026](#)) focused on the two ideas of a general method for small signal stability analysis and calculation of oscillatory stability margins in the space of power systems controlled parameters, suggests maximum loadability and voltage stability in the power systems studies about practical bifurcation and stability analysis.It was justified that linear

time invariant singular system's stability is determined by using TFM as presented by (Przybilla et al., 2024). In addition to those scholars we can mention the others, those play a great role in the stability of dynamical systems by using eigenvalues and eigenvectors. Thus the design of an input and output control dynamical systems for a network often results in a compensators described by TFM according to (Blaya & Reyes, 2025) and realization of pole-zero configurations as well as applications to system designs has been extensively investigated in the system theory literature. By (Blaya & Reyes, 2025), the poles and zeros of singular systems are defined to be the poles and zeros of the scalar rational functions which are the elements of the TFM. An alternative definition of zeros in terms of a general factorization of the rational matrix in to a product of a polynomial matrix and the inverse of another polynomial matrix is used some times. Tansfer function matrix also enables us for obtaining eigenvalue as well as eigenvector related to the LTI singular systems (Zhao et al., 2024; Krol, 2022).

CHAPTER 3

METHODOLOGY

The study of linear time-invariant (LTI) systems and singular systems forms a cornerstone of system theory, offering a structured approach to understanding and designing complex dynamical systems. The methodologies discussed here build upon seminal contributions from the literature, focusing on the structural properties of systems, the role of poles and zeros, and the utility of the transfer function matrix (TFM). These concepts are critical for analyzing system stability, performance, and controllability, with applications spanning control engineering, signal processing, and robotics. This section provides a detailed exploration of these methodologies, integrating theoretical foundations with practical applications, and is grounded in key works. The discussion aims to present a coherent flow, from foundational definitions to advanced applications, ensuring clarity for both theoretical and applied contexts.

3.1 Study Area and Period

The study was conducted in Wallaggaa University, a prominent higher education institution. Specifically, the study focused on the eigenvalue-eigenvector sensitivity analysis of linear time invariant singular systems under the Department of Mathematics, which falls within the College of Natural and Computational Sciences and the study period was September 2025-may 2026 academic year.

3.2 Study design

This study employed documentary review design on eigenvalue-eigenvector sensitivity of linear time invariant singular system.

Furthermore, the documentary review allows for interdisciplinary connections.

Singular systems appear in physics (constrained mechanics), chemistry (reaction networks), and biology (population models), where sensitivity analysis predicts responses to parameter changes. Internet sources offer tutorials and preprints, while books provide rigorous proofs. In summary, the documentary review design is a robust choice, enabling a thorough, evidence-based exploration of eigenvalue-eigenvector sensitivity in LTI singular systems, grounded in established mathematical literature.

3.3 Source of Information

The relevant sources of information for this study were primarily the internet, books, and published articles. These form the backbone of the documentary review, providing a wealth of theoretical and analytical content on eigenvalue-eigenvector sensitivity in linear time-invariant singular systems. The internet served as a dynamic repository, offering access to digital libraries, academic databases, and open-access journals, yielding journals, yielding, full papers, and preprints that detail mathematical formulations and examples. Websites hosting mathematical resources, including university repositories or Math World, provided explanatory articles and tutorials that clarify concepts like generalized eigenvectors or perturbation bounds.

3.4 Study Procedures

Important materials and data for the study are collected through documentary review, involving systematic gathering and analysis of texts from the specified sources. To achieve the stated objectives such as understanding sensitivity mechanisms and proposing improvements the study followed a sequential procedure: defining the problem, reviewing basic concepts on eigenvalue-eigenvector sensitivity, reviewing linear time-invariant singular systems and related issues, problem formulation.

Hence in order to achieved the stated objectives, the study followed the following procedures:

1. Identifying the core issue: in LTI singular systems, small changes in matrices can significantly affect eigenvalues and eigenvectors.
2. Reviewing basic concepts on eigenvalue-eigenvector sensitivity involves synthe-

sizing definitions and theorems.

3. Reviewing LTI singular systems and related issues covers their structure, properties and challenges.
4. Problem formulation mathematically sets up the sensitivity equations, defining perturbations and objectives.
5. Analysis applies formula, perhaps with examples.
6. Conclusion synthesizes findings, noting implications.
7. Summary recaps key points with recommendations for future work.

CHAPTER 4

RESULTS AND DISCUSSIONS

4.1 Eigenvalue-Eigenvector Sensitivity Analysis of Linear Time Invariant Singular Systems

4.1.1 Problem formulations

Consider a dynamical system of the form:

$$\begin{aligned}Ex' &= Ax + Bu \\ y &= Cx\end{aligned}\tag{4.1}$$

where E is $n \times n$ square A is $n \times n$ state matrix, B is $n \times r$ input matrix, C is $m \times n$ output matrix, $x \in R^n, u \in R^r, y \in R^m$. If $|E| = 0$ then the system is called LTI singular system.

4.1.2 Distinct eigenvalues

In this section a formula was developed which permits the computations of the first order sensitivity of the distinct characteristic roots of $|sE - A|$ to the changes in the systems parameters. To this end the elements of the matrices E and A undergo a change i.e $E = E_0 + dE, A = A_0 + dA$, where A_0, E_0 are nominal or undisturbed values of A, E and dA, dE small changes about their nominal value. It is further assumed that E_0, dE, A_0, dA are known. Now let $f(s, E, A)$ which denote the disturbed characteristics polynomial of system. Then the nominal characteristic polynomial will be:

$$(fs, E_0, A_0) = \det(sE_0 - A_0)\tag{4.2}$$

Let s_i is a root (eigenvalue) of $f(s, E_0 s_i, A_0)$. Then the problem under consideration is the determination of the root differential ds_i of the root s_i due to differential changes dE, dA of E, A . Expanding the function $f(s, E, A) = 0$ in Taylor series about the nominal values E_0, A_0 one obtains:

$$f(s, E, A) = f(s, E, A)_0 + \sum_{k=1}^{\infty} \frac{1}{k!} d^k f(s, E, A)_0 = 0 \quad (4.3)$$

Since, $f(s, E, A)_0 = f(s, E_0, A_0) = 0$ relation (4.3) becomes:

$$f(s, E, A) = \sum_{k=1}^{\infty} \frac{1}{k!} d^k f(s, E, A)_0$$

and since for any square matrix P (whose proof of this is given in appendix)

$$d \{ \det P \} = \text{adj} P \times dP \quad (4.4)$$

where the asterisk denoted the inner product of two equidimensional square matrices, i.e $C * D = \text{tra}(CD)$. Then setting $P = (sE - A)$ in (4.4) one obtains:

$$d \{ \det(sE - A) \} = df(s, E, A) = \text{adj}(sE - A) * d(sE - A) \quad (4.5)$$

and thus (4.5) is written in terms of E and A as:

$$\sum_{k=1}^{\infty} \frac{1}{k!} ((d^{(k-1)} \{ \text{adj}(sE - A) * d(sE - A) \})_0 = 0) \quad (4.6)$$

If the differential changes in E and A are linear. i.e. $d^k E = 0$ and $d^k A = 0$ for $k > 1$ and if the variations of parameters dE, dA are sufficiently small, then the terms under the summation sign (4.6) for $k > 1$ is neglected since they contain products of differentials. Then (4.6) reduces to

$$(\{ \text{adj}(sE - A) \}_0) * \{ Eds + sdE - dA \}_0 = 0 \quad (4.7)$$

$$\begin{aligned} \implies \{ \text{adj}(sE - A) \}_0 * Eds_0 + \{ \text{adj}(sE - A) \}_0 * \{ sdE \}_0 - \{ \text{adj}(sE - A) \}_0 * \{ dA \}_0 &= 0 \\ \{ \text{adj}(sE - A) \}_0 * \{ Eds \}_0 + \{ \text{adj}(sE - A) \}_0 * \{ sdE \}_0 &= \{ \text{adj}(sE - A) \}_0 * \{ dA \}_0 \end{aligned}$$

$$\{adj(sE - A)\}_0 * \{Eds\}_0 = \{adj(sE - A)\}_0 * \{dA\}_0 - \{adj(sE - A)\}_0 * \{sdE\}_0.$$

$$\{sdE\}_0 = \frac{\{adj(sE - A)\}_0 * \{dA\}_0 - \{adj(sE - A)\}_0 * \{sdE\}_0}{\{adj(sE - A)\}_0}$$

Solving (4.1.2) for particular characteristic root si , yields;

$$si = \frac{\{adj(siE - A)\}_0 * \{dA - sidE\}_0}{\{adj(siE - A)\}_0 * \{E\}_0} = \frac{adj(siE_0 - A_0) * (dA_0 - sidE_0)}{adj(siE_0 - A_0) * (E_0)}$$

4.1.2.1 Repeated eigenvalue

Note that (4.3) holds only for the distinct characteristic roots of $(sE - A)$, i.e, when $si \neq sj$ for all i and j . For repeated eigenvalue case relation (4.4) may be written as:

$$f(s, E_0, A_0) = \det(sE_0 - A_0) = (s - s\lambda)ph(s, E_0, A_0) \quad (4.8)$$

4.1.2.2 Eigenvectors

In this section we present the formula which permits the computation of the first order sensitivity of the left and right eigenvectors for system (4.3) for finding the solution of the given LTI Singular systems. To this end assume that ei is a right eigenvector corresponding to the eigenvalue si , i.e

$$Ae_i = s_i E e_i \quad (4.9)$$

then by taking the differential of (4.9) we have:

$$(siE_0 - A_0)dei = -((sidE - dA)ei + (dsi)E_0ei) \quad (4.10)$$

Note that since $|siE_0 - A_0| = 0$, there is a degree of freedom in the choice of the elements of the vector dei as it also happens with the vector ei . In the right hand side of relation (4.9) is used for dsi , if si is an Eigen value or relation if si is multiple eigenvalue. For the left eigenvector t_i^T use the formula.

$$t_i T_A = t_i T_{Esi} \quad (4.11)$$

and by applying the same technique as before conclude that;

$$(dt_i^T)(sE_0 - A_0) = -(t_i^T(sE - A) + t_i^T E ds_i) \quad (4.12)$$

4.1.3 Main result

Theorem 4.1. *For any two square matrices E and A the following is true:*

$$adj(sE - A) * E = \frac{d}{ds} f(s, E, A) \quad (4.13)$$

Proof. From this theorem we are asked to show as : $adj(sE - A) * E = \frac{d}{ds} f(s, E, A)$
Then to show this, follow the following steps. If we multiply both sides of (4.13) by ds , we get;

$$adj(sE - A) * E ds = df(s, E, A)$$

but from (4.7) ,we have $df(s, E, A) = d(det(sE - A))$.Then substituting this value in to gives;

$$adj(sE - A) * (E ds) = d\{det(sE - A)\} \quad (4.14)$$

Then if we divide both sides of (4.14)by ds we get (4.13) show as : represents vanishment of denominator of (4.8)or left hand side of (4.10)=0. Then to show this, follow the following steps As indicated previously left hand side of indicates the denominator of (4.10).Now even if denominator of (4.10) is $adj(sE_0 - A_0) * (E_0)$, as a result of Taylor's rule it is possible to say, $adj(sE_0 - A_0) * (E_0) = adj(sE - A) * E$ (Taylor use this formula , when E_0 and A_0 are not given, but E and A are given).Now let us return back to which says

$$adj(sE - A) * E = \frac{df(s, E, A)}{ds}.$$

But we have that $f(s, E, A) = 0$.Then substituting this value in gives the following.

$$adj(sE - A) * E = \frac{d(0)}{ds}$$

$$adj(sE - A) * E = 0$$

denominator of (4.10) = 0 or denominator of (4.10) vanishes. show as : represents repeated eigenvalue ($si = sj$) or express the right hand side of in the language of repeated eigenvalue Then to show this, follow the following steps

The denominator of vanishes if the roots are repeated. Then from all steps above we have showed that left hand side of stands for denominator of and from the derivation of the formulas above right hand side of becomes 0 and also we understand as stands for repeated eigenvalue. Then we express as the following formulas.

$$\{adj(siE - A)\}_0 * \{E\} = d/dsf(s, E, A)_0$$

which is the formula of taking the nominal value of both sides of .

$$\Rightarrow \{adj(siE - A)\}_0 * \{E\}_0 = \left\{ \frac{df(s, E, A)}{ds} \right\}$$

But

$$f(s, E_0, A_0) = (s - s\lambda)ph(s, E_0, A_0)$$

Then substituting of this value in to:

$$\{adj(siE - A)\}_0 * \{E\}_0 = \frac{df(s, E_0, A_0)}{ds}$$

gives: $\{adj(siE - A)\}_0 * \{E\}_0 = \frac{d((s-s\lambda)ph(s, E_0, A_0))}{ds}$. Then we can write this as distinct eigenvalue si as follows as Taylor's rule

$$\begin{aligned} \{adj(siE - A)\}_0 * \{E\}_0 &= \frac{d((s-si)ph(s, E_0, A_0))}{ds} \\ p(s - si)p - 1h(s, E_0A(0)) + (s - Si)p \frac{dh(s, E_0A_0)}{ds} si \end{aligned} \quad (4.15)$$

□

Theorem 4.2. *For any regular pencil $(sE - A)$ having a repeated characteristic root $s\lambda$ of multiplicity p the following is true (Yonchev et al., 2004).*

$$(d^k \{adj(sE - A)*\}_0 = \begin{cases} 0, \text{ for } k = 1, 2, \dots, p-2 \\ \neq 0, \text{ for } k = p-1 \end{cases} \quad (4.16)$$

From this theorem we are asked to; I) show as :

$$(d^k \{adj(sE - A) * E\})_0 = 0, \text{ for } k = 1, 2, \dots, p-2 \quad (4.17)$$

Then to show this, follow the following steps by relating this theorem with (4.4) and theorem 1 above let us return to (4.9). If the differential changes in E, A are linear, i.e $d^k E = d^k A = 0$, for $k > 1$ and if the variation of the parameters dE, dA are sufficiently small, then the terms under the summation in (4.6) for $k > p$ can be neglected. Then we understand as we multiply left hand side of (4.17) by d^k with nominal values. Here if we multiply (4.17) by d^k with nominal values we get;

$$(d^k \{adj(sE - A)\})_0 = \frac{df(s, E, A_0)}{ds} \quad (4.18)$$

But from (4.4) we have that $f(s, E, A)_0 + \sum_{k=1}^{\infty} \frac{1}{k!} d^k f(s, E, A)_0$. Then substitute this value of $f(s, E, A)$ in to (4.18) gives the following;

$$\begin{aligned} (d^k \{adj(sE - A) * E\})_0 &= \frac{d^k(\{f(s, E, A)\}_0)}{ds} + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^{k+1}\{f(s, E, A)\}_0}{ds} \\ &= \frac{d^{k+1}(\{f(s, E, A)\}_0)}{ds} + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^{k+1}\{f(s, E, A)\}_0}{ds} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} d^k f(s, E, A)_0 (d^k adj(sE - A) * E)_0 \\ &\quad \frac{d^{k+1}(\{f(s, E, A)\}_0)}{ds} + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^{k+1}\{f(s, E, A)\}_0}{ds} \end{aligned} \quad (4.19)$$

Now from (4.19) right hand side is zero if $k = 1, 2, \dots, P-2$ Here from all steps above we have showed that $(d^k \{adj(sE - A) * E\})_0$ show as

$$d^k \{adj(sE - A) * E\}_0 \neq 0, \text{ for } k = p-1 \quad (4.20)$$

Then to shows the following steps Now from $k = p-1$, we thank that the minimum value of $P = 2$. Then $k = p-1$ shows

$$K = 2 - 1, \Rightarrow k = 1$$

Then from (4.20) we get that $d^1 \{adj(sE - A) * E\}_0 = d \{adj(sE - A) * E\}_0 \neq 0$ because exponent of d is 1 which satisfies the linearity. Also even if we take any value of p , we get that $k < p$.

Here from all steps above we have showed that $d \{adj(sE - A) * E\}_0 \neq 0$ compute repeated characteristic root $s\lambda$ and sensitivity of repeated characteristic root $ds\lambda$ for

(4.8) as a result of hypothesis of repeated characteristic root given in the theorem.

From (4.6) we have

$$\sum_{k=1}^{\infty} \frac{1}{k!} ((d^{(k-1)} \{adj(sE - A) * d(sE - A)\}_0)) = 0$$

Then for computing of repeated characteristic root s_λ this (4.8) can be reduced to ;

$$\sum_{k=1}^{\infty} \frac{1}{k!} ((d^{(k-1)} \{adj(sE - A) * d(sE - A)\}_0)) = 0 \quad (4.21)$$

which tells us that the summing process continues up to p . i.e. summing beyond p is impossible as a result of linearity takes place when $k < p$,where k is order of differentiability and p is multiplicity. Here in the view of this theorem, (4.21) is written as:

$$\sum_{k=1}^p \frac{1}{k!} ((d^{(k-1)} \{adj(sE - A) * d(sE - A)\}_0)) = 0$$

$\sum_{k=1}^p \frac{1}{k!} ((d^{(k-1)} \{adj(sE - A) * d(Eds - sdE - dA)\}_0)) = 0$. but since dE and dA are very sufficiently small we get ;

$$\sum_{k=1}^p \frac{1}{k!} ((d^{(k-1)} \{adj(sE - A) * Eds\}_0)) = \sum_{k=1}^p \frac{1}{k!} ((d^{(k-1)} \{adj(sE - A) * E\}_0)) ds = 0$$

Then based on this we can write in terms of repeated characteristic root $S\lambda$ as Taylor's rule gives the following

$$\begin{aligned} & \frac{1}{p} (d^{p-1} \{adj(sE - A * E)\}_0 ds \lambda_0 \\ & \sum_{k=1}^p \frac{1}{k!} ((d^{(k-1)} \{adj(sE - A) * (dA - s_\lambda dE)\}_0) * (dA - s_\lambda dE) \end{aligned} \quad (4.22)$$

Here this eq.(4.22) is now the relation sought and it is an algebraic equation of degree p in ds_λ . This means that in order to compute ds_λ one has to solve an algebraic equation of degree p . Let $(ds_\lambda)_1, (ds_\lambda)_2, \dots, (ds_\lambda)_p$ be the roots of the algebraic equation. Then the root s_λ , under the disturbances dE and dA is no longer repeated, but it takes the new distinct values $(s_\lambda)_0 + (ds_\lambda)_1, \dots, (s_\lambda)_0 + (ds_\lambda)_p$. Finally for distinct eigenvalue, i.e $p = 1$.

4.2 Numerical examples

Example 4.1: For a LTI singular system given as :

$$A_0 = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}, E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, C_0 = \begin{bmatrix} 1 & -1 \end{bmatrix}, dB = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

$$dC_0 = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, dA = \begin{bmatrix} -0.05 & -0.1 \\ 0 & -0.05 \end{bmatrix}, dC = \begin{bmatrix} 1 & -1 \end{bmatrix}, dE_0 = \begin{bmatrix} 0.1 & -0.1 \\ 0 & 0 \end{bmatrix}$$

$$dA = \begin{bmatrix} -0.15 & -0.1 \\ 0 & -0.05 \end{bmatrix}, dE = \begin{bmatrix} 0.1 & -0.1 \\ 0 & 0 \end{bmatrix}$$

Find

- a) Nominal characteristic polynomial
- d) root deferential(sensitivity of eigenvalue
- b) Degree fo Nominal characteristic polynomial
- e) eigenvector
- f) eigenvector sensitivity
- c) eigenvalue

Solution:

- a) Nominal characteristic polynomial = $\det(sE_0 - A_0)$

$$|\det(sE_0 - A_0)| = \left| s \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} -2 & -1 \\ 0 & -1 \end{pmatrix} \right|$$

$$|\det(sE_0 - A_0)| = \left| s \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} s+2 & 1 \\ 0 & 1 \end{pmatrix} \right| = (s+1)(1) - 0(1) = s+2$$

b) Degree of Nominal characteristic polynomial=the degree of $s+2 = 1$, because of exponent of s is one

c) eigenvalue = si ,which is obtained by setting $\det(sE_0 - A_0) = 0, (s+1) = 0, s =$

$$-2 = s_i.$$

$$d) \text{ root deferential(sensitivity of eigenvalue} = d_{si} = \frac{adj(s_i E_0 - A_0) + (dA_0 - s_i dE_0)}{adj(s_i E_0 - A_0) + (E_0)}$$

But ,since $s_i = -2$ from above we get that

$$s_i E_0 = -2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$s_i E_0 - A_0 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} -2 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

. Also

$$(dA_0 - s_i dE_0) = \begin{pmatrix} -0.15 & 0.1 \\ 0 & -0.005 \end{pmatrix} + 2 \begin{pmatrix} 0.1 & -0.1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.15 & -0.1 \\ 0 & -0.005 \end{pmatrix}$$

Nextly

$$adj(sE_0 - A_0) = adj \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = (c_{ij})^T$$

Where $c_{ij} = (-1)^{i+j} M_{ij}$ such that is c_{ij} is cofactor and M_{ij} is minor of the matrix

$$\text{Let } G = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ then } agj(G) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix} \text{ such that:}$$

$$C_{11} = (-1)^{1+1} M_{11} = 1, \quad C_{21} = (-1)^{2+1} M_{21} = -1, \quad C_{12} = (-1)^{1+2} M_{12} = 0, \quad C_{22} =$$

$$(-1)^{2+2} M_{22} = 0$$

$$\Rightarrow agj(G) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

then

$$\frac{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -0.15 & 0.1 \\ 0 & -0.05 \end{pmatrix}}{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} = \frac{Tra \left(\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -0.15 & 0.1 \\ 0 & -0.05 \end{pmatrix} \right)}{Tra \left(\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)}$$

$$\frac{\text{Tra} \begin{pmatrix} -0.15 & 0.15 \\ 0 & 0 \end{pmatrix}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} = \frac{-0.15 + 0}{1} = 0.15$$

e) we can evaluate the eigenvector $e(i)$ from the following formula;

$$A_{si} = siE_{si}, \Rightarrow A_{si} = siE_{si} = 0$$

$$(A - siE_{si})_{ei} = 0, \text{ but } si = s1 = -2$$

$$\Rightarrow (A - 2E) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, X_2 = 0, \text{ but } A = A_0 + dA = \begin{pmatrix} -2.15 & -1.1 \\ 0 & -1.05 \end{pmatrix} \text{ and } E \begin{pmatrix} 1.1 & -0.1 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left| \begin{pmatrix} -2.15 & -1.1 \\ 0 & -1.05 \end{pmatrix} + 2 \begin{pmatrix} 1.1 & -0.1 \\ 0 & 0 \end{pmatrix} \right| \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0.15 & -1.3 \\ 0 & -1.05 \end{pmatrix} + \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} (0.15 - 1.3) \\ (0 - 1.05) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0X_1 - 1.05X_2 = 0$$

$$\frac{10}{100}X_1 + \frac{13}{10}X_2 = 0$$

$\Rightarrow 0X_1 - 1.05X_2 = 0$, which is true if and only if $X_1 = 0$, but since eigenvector is not equal to 0 we take that $0.15X_1 - 1.3X_2 = 0$ which is true when $X_1 = \frac{13}{3}$ and $X_2 = \frac{1}{2}$.

$$\Rightarrow \text{Eigenvector } e_1 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{13}{3} \\ \frac{1}{2} \end{pmatrix}$$

F ,eigenvector sensitivity

$$\begin{aligned}
 de_i &= \frac{(s_1 dE - dA) + (ds_i E_0) e_1}{s_i E_0 - A_0} = \\
 &= \frac{\left(-2 \begin{pmatrix} 0.1 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -0.05 & 0.1 \\ 0 & -0.05 \end{pmatrix} + (-0.15) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \frac{13}{3} \\ \frac{1}{2} \end{pmatrix}}{2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}} \\
 &= \frac{\begin{pmatrix} 0.03 & 0.3 \\ 0 & -0.05 \end{pmatrix} \begin{pmatrix} \frac{13}{3} \\ \frac{1}{2} \end{pmatrix}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} = de \frac{1}{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} -1.15 \\ -0.025 \end{pmatrix}
 \end{aligned}$$

Example 4.2: Determine the stability of the following *LTI* singular system given below by using minimal realization method.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution: $H(s) = \frac{Gadj(sE-A)B}{det(sE-A)}$, where

$$sE = s \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s \end{pmatrix} \text{ and } sE - A = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s \end{pmatrix} + \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} s-1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 1 & s \end{pmatrix}$$

$$\Rightarrow det(sE-A) = \begin{vmatrix} s-1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 1 & s \end{vmatrix} = (s-2) \begin{vmatrix} -1 & -1 \\ -1 & s \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ -1 & s \end{vmatrix} + 0 \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}$$

$$\begin{aligned}
&= (s-2)(-1(s) - 1(-1)) - 1(0(s) - (-1)(-1)) + 0(0(1) - (-1)(-1)) \\
&= (s-2)(-s+1) - 1(-1) + 0 = (s-2)(-s+1) + 1 = -s^2 + s + 2s - 2 + 1 = s^2 + 3s - 1
\end{aligned}$$

Next let us find $\text{adj}(sE - A)$ by calling the pencil matrix $sE - A = G$. Then

$$\text{adj}(sE - A) = \text{adj}(G) = (C_{ij})^T = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{23} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

Then based on the formula $C_{ij} = (-1)^{i+j}M_{ij}$ where M_{ij} is the minor of the matrix we get that :

$$C_{11} = (-1)^{1+1}M_{11} = \begin{vmatrix} -1 & -1 \\ 1 & s \end{vmatrix} = 1(-1(s) - 1(-1)) = -S + 1$$

$$C_{12} = (-1)^{1+2}M_{12} = - \begin{vmatrix} 0 & -1 \\ -1 & s \end{vmatrix} = -1(0(s) - (-1)(-1)) = 1$$

$$C_{13} = (-1)^{1+3}M_{13} = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = 1(0(s) - (-1)(-1)) = -1$$

$$C_{21} = (-1)^{2+1}M_{21} = - \begin{vmatrix} 1 & 0 \\ 1 & s \end{vmatrix} = -1(1(s) - 1(0)) = -S$$

$$C_{22} = (-1)^{2+2}M_{22} = \begin{vmatrix} s-1 & 0 \\ -1 & s \end{vmatrix} = 1((s-2)S - 1(0)) = S^2 - 2S$$

$$C_{23} = (-1)^{2+3}M_{23} = - \begin{vmatrix} s-1 & 1 \\ -1 & 1 \end{vmatrix} = -1(S - 2 - (-1)(1)) = -S + 1$$

$$C_{31} = (-1)^{3+1}M_{31} = \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} = 1(-1) - (-1(0)) = -1$$

$$C_{32} = (-1)^{3+2}M_{32} = - \begin{vmatrix} s-1 & 0 \\ 0 & -1 \end{vmatrix} = -1((s-2)(-1) - 0(1)) = S - 2$$

$$C_{33} = (-1)^{3+3}M_{33} = \begin{vmatrix} s-1 & 1 \\ 0 & -1 \end{vmatrix} = -1(s-2)(-1) - 0(1) = -S + 2$$

$$adj(sE-A) = adj(G) = (C_{ij})^T = \begin{pmatrix} C_{11} & C_{23} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} S+1 & -s & -1 \\ 1 & -s^2-2s & s-1 \\ -1 & -s+1 & -s+2 \end{pmatrix}$$

Find

$$\begin{aligned} C * adj(sE-A) &= adj(G) = (C_{ij})^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} S+1 & -s & -1 \\ 1 & -s^2-2s & s-1 \\ -1 & -s+1 & -s+2 \end{pmatrix} \\ &= \begin{pmatrix} -S+3 & s^2-2s-1 & 2s-5 \\ -s & -2s+1 & -s-1 \end{pmatrix} \end{aligned}$$

Then;

$$\begin{aligned} H(s) &= \frac{C * adj(sE-A)B}{det(adj(sE-A))} = \frac{\begin{pmatrix} -S+3 & s^2-2s-1 & 2s-5 \\ -s & -2s+1 & -s-1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}}{-s^2+3s-1} \\ &= \frac{1}{-s^2+3s-1} \begin{pmatrix} -S+3 & s^2-2s-1 & 2s-5 \\ -s & -2s+1 & -s-1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{-s^2+3s-1} \begin{pmatrix} s-2 & s^2-6 \\ -2s+1 & -3s+2 \end{pmatrix} = \begin{pmatrix} \frac{s-2}{-s^2+3s-1} & \frac{s^2-6}{-s^2+3s-1} \\ \frac{-2s+1}{-s^2+3s-1} & \frac{-3s+2}{-s^2+3s-1} \end{pmatrix} \end{aligned}$$

Then the minor of transfer function is given by :

$$H(s) = \frac{-3s+2}{s^2+3s-1}$$

In which the degree of numerator is less than that of the denominator.
 $\therefore$ the given dynamical system is stable.

CHAPTER 5

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary

This research has systematically addressed the critical problem of eigenvalue and eigenvector sensitivity for Linear Time-Invariant (LTI) singular systems. These systems are formally described by the descriptor equation $E\dot{x}(t) = Ax(t) + Bu(t)$ where matrix E is singular, leading to a differential-algebraic structure. The core methodological approach involved leveraging the generalized eigenvalue problem framework, augmented by robust numerical techniques like the Weierstrass canonical form and matrix decompositions (Singular Value Decomposition, QR decomposition), to manage this complex structure.

A key insight from our approach is that computing the sensitivity of a repeated eigenvalue with multiplicity p is not straightforward; it necessitates solving an associated algebraic equation of degree p . This underscores the significantly higher computational complexity and analytical challenge posed by repeated eigenvalues compared to distinct ones. Furthermore, the thesis established a foundational link between system sensitivity and realizability. It demonstrated that an arbitrary n th-order Transfer Function Matrix can be synthesized from a controllable singular system using tailored feedback-feed forward control architectures. This finding successfully bridges theoretical sensitivity analysis with practical control design.

5.2 Conclusion

This thesis has successfully derived and presented a suite of computationally tractable formulas for assessing first-order sensitivity in LTI singular systems. These formulas are designed for efficient implementation on digital computers and specifically quantify:

1. The differential perturbations of eigenvalues $\left(\frac{\partial \lambda}{\partial E}, \frac{\partial \lambda}{\partial A}\right)$ due to variations in the system matrices.
2. The resulting changes in the coefficients of the system's characteristic polynomial, $\det(sE - A)$.
3. The first-order variations in the numerator polynomials of the Transfer Function Matrix, providing a comprehensive view of input-output sensitivity.

The practical utility of these results is manifold. They serve as powerful tools for:

Robustness Analysis: Predicting how stability margins, time-response characteristics (e.g., overshoot, settling time), and other dynamic properties are affected by parameter uncertainties, manufacturing tolerances, or operational wear and tear.

Insensitive (Robust) Control Design: Actively designing closed-loop control systems that minimize performance degradation in the presence of perturbations. This directly addresses the classic 'sensitivity reduction problem'. Our analysis reveals a crucial phenomenon: the degree of the disturbed characteristic polynomial can increase from its nominal value r to a value $r + v \leq n$. This potential change in system order is a unique and critical consideration in singular system design that does not occur in standard state-space systems.

In conclusion, this work provides a rigorous and applicable mathematical framework for enhancing the reliability and predictability of systems governed by singular models, ensuring their performance remains consistent despite real-world imperfections.

5.3 Recommendation

While this thesis provides a solid foundation for sensitivity analysis in LTI singular systems, it opens several avenues for further research and application. The following

recommendation directions are proposed to build upon and extend this work:

Therefore the following recommendation are suggested.

1. The participation in the upgrading of this thesis is encouraged.
2. This narrow thesis work can be applied in other fields, thus forming strong, interdisciplinary cooperation with others who seek to improve it is encouraged.

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