

**A Parameter Uniform Numerical Scheme for Singularly  
Perturbed Parabolic Differential Equation with Discontinuous  
Coefficients and Negative Shift**



**Salale University  
College of Natural Sciences  
Department of Mathematics**

**MSc Thesis**

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**May, 2024  
Fitcha, Ethiopia**

# A Parameter Uniform Numerical Scheme for Singularly Perturbed Parabolic Differential Equation with Discontinuous Coefficients and Negative Shift



A Thesis Submitted to the Department of Mathematics, Salale  
University in the Partial Fulfilment of the Requirement for the  
Degree of Master of Science in Mathematics (Numerical  
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**APPROVAL SHEET FOR SUBMITTING FINAL THESIS**

This is to certify that the thesis entitled "**A Parameter Uniform Numerical Scheme for Singularly Perturbed Parabolic Differential Equation with Discontinuous Coefficients and Negative Shift**", submitted in partial fulfillment of the requirement for the degree of Masters with specialization in Numerical Analysis , the graduate program of Applied Mathematics, and has been carried out by **Ketema Negash Feyisa**, under my supervision. Therefore, I recommend that the student has fulfilled the requirements and hence hereby can submit the thesis to the department for defence.

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Advisor Name

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Co-Advisor Name

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Date

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# List of Abbreviations and Symbol

## Abbreviations

|         |   |   |
|---------|---|---|
| DE      | : | Differential Equation                                     |
| PDEs    | : | Partial Differential Equations                            |
| SPPs    | : | Singularly Perturbed Problems                             |
| SPDEs   | : | Singularly Perturbed Differential Equations               |
| SPDPDEs | : | Singularly Perturbed Delay Partial Differential Equations |
| SPDDEs  | : | Singularly Perturbed Differential-Difference Equations    |
| FDM     | : | Finite Difference Method                                  |
| FEM     | : | Finite Element Method                                     |
| FVM     | : | Finite Volume Method                                      |
| FOM     | : | Fitted Operator Method                                    |
| FMM     | : | Fitted Mesh Method  |
| GEE     | : | Global Error Estimate                                     |
| LTE     | : | Local Truncation Error                                    |

## Symbol

|               |   |                                 |
|---------------|---|---------------------------------|
| $\varepsilon$ | : | Singular Perturbation Parameter |
|---------------|---|---------------------------------|

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# Abstract

In this study, a parameter uniform numerical scheme for a singularly perturbed delay parabolic partial differential equation with discontinuous coefficients involving large negative shift is presented. Due to the presence of singular perturbation parameter and discontinuity in the problem, it is a difficult task to solve this problem analytically. Consequently, a direct or analytical method for singularly perturbed delay parabolic partial differential equation with discontinuous coefficients involving large negative shift is still lacking, so one has to rely on suitable numerical methods to solve such problems. Therefore, it is noteworthy to develop  $\varepsilon$ -uniform convergent numerical methods for such types of problems. The main aim of this study is to develop and analyze a parameter uniform numerical scheme based on fitted techniques for solving singularly perturbed delay parabolic partial differential equation with discontinuous coefficients involving large negative shift. The proposed method consists of the implicit Euler method for temporal discretization and special designed finite difference method for spatial discretization on a uniform mesh step size. The stability and parameter uniform convergence of the proposed method is carried out. The proposed scheme gives an accurate, stable and  $\epsilon$ -uniform numerical result. The error analysis of the proposed scheme was carried out and proved that the scheme is uniformly convergent of first order in temporal and second order in spatial variables. To demonstrate applicability of the method some numerical examples are considered. The maximum absolute errors and rate of convergence for various perturbation parameters and uniform mesh size values are tabulated for two model examples. Further, the numerical results depict that the present method is more convergent than some methods available in the literature.

# Chapter 1

## Introduction

### 1.1 Background of the Study

Differential equations are used to relate an unknown function to its derivatives evaluated at the same time. A differential equation (DE) is an equation which contains the derivatives of one or more dependent variables, with respect to one or more independent variables. A DE contains derivatives which are either ordinary or partial derivatives. If an equation contains only partial derivatives of one or more dependent variables with respect to one or more independent variable, it is said to be partial differential equation. Whereas if an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, it is said to be ordinary differential equation.

The model which contains small parameters is termed as a perturbed model, whereas the degenerate simplified model is called an unperturbed or reduced model. A singularly perturbed delay partial differential equation is one in which the highest order derivative term is multiplied by a small parameter  $\epsilon$  ( $0 < \epsilon \ll 1$ ) and at least one delay parameter is included.

Singularly perturbed parabolic delay differential equations, also known as time - dependent problems, are used in fields such as bioscience, control theory, economics, material science, medicine, and robotics (Kaushik and Sharma, 2020). Due to the dual presence of singular perturbation ( $\epsilon$ ) and shift arguments in the SPDDEs, it is very difficult to obtain oscillation-free solutions on a uniform mesh unless using specially designed meshes. A thoughtful examination of the results from the conventional numerical methods, such as the finite difference method (FDM), the finite element method (FEM), the finite volume

method (FVM), the spline method and other methods on uniform meshes as  $\epsilon$  tends to zero fails for a satisfactory numerical solution and the truncation error becomes unbounded unless a large number of mesh points or adaptive layer mesh is used in the approximation process.

There are some specific features in constructing special difference schemes in the case of moving boundary and interior layers. In the case of singularly perturbed problems with moving boundary and interior layers, investigation of necessary and sufficient conditions for the  $\epsilon$ -uniform convergence of numerical methods is very important.

Due to the presence of interior layers appearing in the solution, the classical numerical methods, when applied to these problems on a uniform mesh, are unable to provide an efficient numerical solution unless they are applied with very fine meshes inside the regions. Thus, to resolve this issue, the fitted operator and layer-adapted mesh methods are competitive computational techniques to overcome the limitations of the classical numerical methods.

In this study, we considered a class of singularly perturbed parabolic convection-diffusion problems with non-smooth data and a large negative shift. As a result, inspired by the work of (Kaushik and Sharma, 2020), we developed and analyzed an accurate and uniformly convergent numerical scheme for solving singularly perturbed delay parabolic differential equations with non-smooth coefficients and source terms. The presence of singular perturbation and delay parameters in the problems increases complexity to solve problems specially when the delay is large. We give attention to solve parabolic delay differential equation with discontinuous coefficients. The study of solution of these equations is of great significance due to the formation of sharp boundary or interior layers when the perturbation parameter approaches to zero. In this study, we suggested a parameter uniform numerical scheme for the class of singularly perturbed delay partial differential equations (SPDPDEs) with discontinuous coefficients and large negative shift.

## 1.2 Statement of the Problem

In the past and recent years studies much interest have been given to solve singularly perturbed parabolic delay differential equation problem with discontinuous coefficients and interior layers , due to their wide applicability in modeling of processes in various application fields. Finding the solution to these problems has a significant role to capture the behavior of the physical phenomena of the problems. Different numerical scheme were developed in (Elango et al., 2021; Gobena and Duressa, 2021; Kumar and Kumari, 2020) to study a class of singularly perturbed delay parabolic differential equations with smooth data and large delay in space variables. An adaptive generated grid is formulated using a FDM for SPDPDEs of reaction- diffusion type with a large negative shift in (Bansal and Sharma, 2018). (Kaushik and Sharma, 2020), presented adaptive layer numerical scheme based on finite difference method for solving singularly perturbed delay parabolic differential equations with non-smooth coefficients and source terms.

Nevertheless, from the literature survey a developed parameter uniform numerical scheme to solve the singularly perturbed parabolic delay differential equation problem with discontinuous coefficients and interior layers is at primary stage and it needs a lot of investigation. This limitation motivates the researchers to develop uniformly convergent numerical schemes for solving the singularly perturbed delay parabolic differential equation problem with discontinuous coefficients and interior layers. In this work, we suggested a parameter uniform numerical scheme for the class of singularly perturbed delay partial differential equations (SPDPDEs) with discontinuous coefficient term and large negative shift.

## 1.3 Objectives of the Study

### 1.3.1 General Objective

The general objective of this study is to formulate  $\varepsilon$ -uniformly convergent numerical method to solve SPDPDE with discontinuous coefficient term and interior layers.

### 1.3.2 Specific Objectives

The specific objectives of this study are the following:

1. To develop numerical method to solve SPDPDE with discontinuous coefficient term and interior layers.
2. To establish the stability of the proposed numerical scheme.
3. To analyze the convergence of the proposed numerical scheme.
4. To examine the effectiveness of the proposed numerical scheme.

## 1.4 Delimitation of the Study

This research is delimited to focus on the singularly perturbed delay parabolic differential equation with discontinuous coefficients and source terms on the domain  $D = S^- \cup S^+ = (0, 1) \times (1, T] \cup (1, 2) \times (0, T]$ , where  $S^- = (0, 1) \times (0, T]$ ,  $S^+ = (1, 2) \times (0, T]$ ,  $\partial D = \overline{D} \setminus D$  and  $T$  is some fixed positive time of the form:

$$\begin{cases} \epsilon \frac{\partial^2}{\partial x^2} u(x, t) + a(x) \frac{\partial}{\partial x} u(x, t) - b(x)u(x-1, t) - c(x)u(x, t) - \frac{\partial}{\partial t} u(x, t) = f(x, t), (x, t) \in D, \\ u(x, t) = p_0(x) \text{ on } [0, 2] \times t = 0, \\ u(x, t) = p_1(x, t) \text{ in } [-1, 0] \times [0, t], u(x, t) = p_2(t) \text{ on } x = 2 \times [0, T], \end{cases} \quad (1.4.1)$$

where  $\epsilon \ll 1$  is a small positive parameter,  $b$  and  $c$  are sufficiently smooth functions such that  $b(x) < 0$ ,  $c(x) > 0$  and  $b(x) + c(x) \geq 0$  for all  $x \in [0, 2]$ . Moreover,

$$a(x) = \begin{cases} a_1(x) & \text{if } 0 \leq x \leq 1 \\ a_2(x) & \text{if } 1 < x \leq 2 \end{cases} \quad (1.4.2)$$

$$f(x, t) = \begin{cases} f_1(x, t) & \text{if } (x, t) \in \overline{S}^- \\ f_2(x, t) & \text{if } (x, t) \in \overline{S}^+ \end{cases} \quad (1.4.3)$$

$$a_1(x) < -\gamma_1 < -2\gamma < 0, a_2(x) > \gamma_2 > 2\gamma > 0,$$

$$||a|| \leq C, ||f|| \leq C,$$

where  $\gamma = \min \{\gamma_1, \gamma_2\}$  and compatibility of the boundary data.

## 1.5 Significance of the Study

Singularly perturbed delay parabolic differential equation problem with discontinuous coefficients and interior layers has been studied because of its numerous applications in many

mathematical models and their use has spread into the theoretical study of neuronal activity, optically bistable devices, description of human pupil reflex, a variety of models for physiological processes or diseases and others. To investigate the prediction of the physical phenomena of these problems, it is often necessary to approximate their solution numerically. Though, it is very hard to solve singularly perturbed parabolic problem with discontinuous coefficients and interior layers delay analytically due to the presence of a multi-scale character in the solution. So, it is necessary to develop  $\varepsilon$ -uniform convergent numerical methods that solve the problems under consideration successfully. Developing a robust numerical method for such problems is always a desirable task (Das, 2013). The main contribution of this study is the development of a  $\varepsilon$ - uniform numerical scheme for solving singularly perturbed delay parabolic differential equation problem with discontinuous coefficients and interior layers. Additionally, the developed numerical scheme can be used to fill the gap observed, would be significantly helpful to researchers working in this area to construct their new numerical schemes, and used as an alternative scheme for solving singularly perturbed delay parabolic differential equation problem with discontinuous coefficients and interior layers.

# Chapter 2

## Review of Related Literature

### 2.1 Singular Perturbation Problem

Singular perturbation problem was first introduced by (Prandtl, 1904) in which he demonstrated that fluid flow past over a body can be divide in two regions, a boundary layer and outer region. However, the term singular perturbation was first used by (Friedrichs and Wasow, 1946) in a paper presented at a seminar on non-linear vibrations at new York University. Singular perturbation theory concerns the study of problems featuring parameter for which the solutions of the problem at a limiting value of the parameter are different in character from the limit of the solutions of the general problem. In contrast, for regular perturbation problems, the solutions of the general problem converge to the solutions of the limit-problem as the parameter approaches the limit value. The solution of SPPs typically contains layers.

If the solution of the problem can be approximated by setting the value of the perturbation parameter equal to zero, then the problem is called regular perturbation problem, otherwise it is called singular perturbation problem. That is, if it is impossible to approximate the solution by asymptotic expansion as the perturbation parameter tends to zero, then the problem is called singular perturbation problems. SPPs have always played prominent role in the theory of differential equation. In fact, any differential equations whose solution changes rapidly in some parts of the solution domain/interval is generally known as singular perturbation problem and also called boundary layer problems. In real life, we often encounter many problems which are described by parameter dependent differential equations. The behavior of the solution of these types of differential equation depends on



the magnitude of the parameter. (Miller et al., 1996) said that boundary layer is a region of the independent variable over which the dependent variable changes rapidly.

## 2.2 Solution Methodology Developed for Singularly Perturbed Delay Parabolic Differential Equation with Discontinuous coefficients and Source Terms

A class of singularly perturbed parabolic differential-difference PDE is a PDE in which the highest order derivative is multiplied by a small positive parameter  $\varepsilon$  ( $0 < \varepsilon \ll 1$ ) and having delay(s) in space directions. Various studies and surveys have been conducted by researchers on the development in SPDEs. Designing a  $\varepsilon$ -uniform convergent numerical schemes for the singularly perturbed delay parabolic PDEs are very challenging task and the developed parameter uniform numerical schemes for these problems are surveyed as follows.

Kaushik and Sharma, 2020 have considered the following singularly perturbed delay parabolic differential equation with discontinuous coefficients and source terms on the domain  $D = S^-US^+ = (0, 1) \times (1, T] \cup (1, 2) \times (0, T]$ , where  $S^- = (0, 1) \times (0, T]$ ,  $S^+ = (1, 2) \times (0, T]$ ,  $\partial D = \overline{D} \setminus D$  and  $T$  is some fixed positive time:

$$\begin{cases} \varepsilon \frac{\partial^2}{\partial x^2} u(x, t) + a(x) \frac{\partial}{\partial x} u(x, t) - b(x)u(x-1, t) - c(x)u(x, t) - \frac{\partial}{\partial t} u(x, t) = f(x, t), (x, t) \in D, \\ u(x, t) = p_0(x) \text{ on } [0, 2] \times t = 0, \\ u(x, t) = p_1(x, t) \text{ in } [-1, 0] \times [0, t], u(x, t) = p_2(t) \text{ on } x = 2 \times [0, T], \end{cases} \quad (2.2.1)$$

where

$$a(x) = \begin{cases} a_1(x) & \text{if } 0 \leq x \leq 1 \\ a_2(x) & \text{if } 1 < x \leq 2 \end{cases} \quad (2.2.2)$$

$$f(x, t) = \begin{cases} f_1(x, t) & \text{if } (x, t) \in \overline{S}^- \\ f_2(x, t) & \text{if } (x, t) \in \overline{S}^+ \end{cases} \quad (2.2.3)$$

$a_1(x) < -\gamma_1 < -2\gamma < 0$ ,  $a_2(x) > \gamma_2 > 2\gamma > 0$ ,  $|[a]| \leq C$ ,  $|[f]| \leq C$ ,  $\varepsilon \ll 1$  is a small positive parameter,  $b$  and  $c$  are sufficiently smooth functions such that  $b(x) < 0$ ,  $c(x) > 0$  and  $b(x) + c(x) \geq 0$  for all  $x \in [0, 2]$ . Moreover, we assume that  $\gamma = \min\{\gamma_1, \gamma_2\}$  and compatibility of the boundary data. They have developed an adaptive difference scheme

consists of the backward Euler method and upwind finite difference method for temporal and spatial discretization, respectively. The proposed scheme has been analysed for consistency, stability and convergence. A numerical scheme is presented based on the fitted operator method defined over a uniform mesh. It found that the method proposed is unconditionally stable, and the convergence obtained is parameter uniform.

Daba and Duressa (2022) have considered the above Eq.(2.2.1) and have suggested different numerical methods for solving singularly perturbed partial differential equations with discontinuous coefficients and large delay parameter using fitting techniques. The developed scheme constitutes the implicit Euler in the time direction and the cubic spline in compression method in the space direction on uniform step size. The scheme has shown to be  $\epsilon$ -uniformly convergent of first-order in time and second-order in space directions.

Hailu and Duressa (2022) have considered the above Eq.(2.2.1) and have suggested that the proposed scheme is based on the method of lines, a nonstandard finite difference method for space discretization, followed by the backward Euler method for the resulting system of initial value problems for time discretization. The stability and uniform convergence of the scheme are investigated and proved. Furthermore, we discovered that the proposed scheme outperforms the existing methods in the literature in terms of numerical accuracy. The developed scheme has the potential to be extended to solve higher-dimensional singularly perturbed parabolic problems.

Ayele et al. (2022) have considered the above Eq.(2.2.1) and the problem is discretized by a nonstandard finite difference scheme in the spatial variable and for the time derivative, they used Crank-Nicolson scheme. The error analysis of the proposed scheme was carried out and proved that the scheme is uniformly convergent of second order in both spatial and temporal variables. To enhance the order of convergence of the spatial variable, the Richardson extrapolation technique is used.

Sharma and Kaushik (2022) have considered the above Eq.(2.2.1) and have suggested that the hybrid scheme presented is a composition of a central difference scheme and a

midpoint upwind scheme on a specially generated mesh. Consistency, stability, and convergence of the presented numerical approach have been investigated.

In general, one can observe from the survey there is the room for developing  $\varepsilon$ - uniform numerical schemes singularly perturbed parabolic differential equation with discontinuous coefficients and negative shift. As we see in the above literature, most researchers try to find different numerical methods to find solution for singularly perturbed parabolic differential equation with discontinuous coefficients and negative shift.

In this thesis, we tried to develop another method which is more accurate and uniformly convergent numerical method for the problem under consideration. We used an implicit Euler method for temporal discretization and novel finite difference method for spacial discretization on a uniform mesh size for solving singularly perturbed parabolic differential equation with discontinuous coefficients and negative shift.

# Chapter 3

## Research Design and Methodology

### 3.1 Study Site and Period

This study was conducted at Salale University, College of Natural Sciences, Department of Mathematics from January 2024 to June 2024.

### 3.2 Study Design

This study employed an intensive upto date document review and numerical experimentation.

### 3.3 Source of Information

The relevant source of information for this study are books, published articles on international reputable journals and related studies from different university websites and so on.

### 3.4 Mathematical Procedures

In order to achieve the stated objectives, the study followed the following mathematical procedures:

1. Describing model problem for the study,
2. Analyzing the properties of the continuous solution,
3. Developing numerical schemes for the problem,

4. Establishing the stability and convergence of the developed numerical schemes,
5. Developing an algorithm and writing MATLAB code for the presented schemes,
6. Validating the schemes using numerical examples,
7. Presenting the results using appropriate presentation (using tables, graphs),
8. Discussing and providing conclusion.

# Chapter 4

## Description of the Numerical Scheme, Results and Discussions

### 4.1 Description of the Problem

Let  $D = S^- \cup S^+ = (0, 1) \times (0, T] \cap (1, 2) \times (0, T]$ , and consider the non homogeneous initial boundary value problem where  $S^- = (0, 1) \times (0, T]$ ,  $S^+ = (1, 2) \times (0, T]$ ,  $\partial D = \overline{D} \setminus D$  and  $T$  is some fixed positive time:

$$\begin{cases} \varepsilon \frac{\partial^2}{\partial x^2} u(x, t) + a(x) \frac{\partial}{\partial x} u(x, t) - b(x)u(x-1, t) - c(x)u(x, t) - \frac{\partial}{\partial t} u(x, t) = f(x, t), (x, t) \in D, \\ u(x, t) = p_0(x) \text{ on } [0, 2] \times \{t = 0\}, \\ u(x, t) = p_1(x, t) \text{ in } [-1, 0] \times [0, t], u(x, t) = p_2(t) \text{ on } \{x = 2\} \times [0, T], \end{cases} \quad (4.1.1)$$

where  $0 < \varepsilon \ll 1$  is a small positive parameter,  $b(x)$  and  $c(x)$  are sufficiently smooth functions such that  $b(x) < 0$ ,  $c(x) > 0$  and  $b(x) + c(x) \geq 0$  for all  $x \in [0, 2]$ . Moreover,

$$a(x) = \begin{cases} a_1(x) & \text{if } 0 \leq x \leq 1 \\ a_2(x) & \text{if } 1 < x \leq 2 \end{cases} \quad (4.1.2)$$

$$f(x, t) = \begin{cases} f_1(x, t) & \text{if } (x, t) \in \overline{S}^- \\ f_2(x, t) & \text{if } (x, t) \in \overline{S}^+ \end{cases} \quad (4.1.3)$$

$$a_1(x) < -\gamma_1 < -2\gamma < 0, a_2(x) > \gamma_2 > 2\gamma > 0,$$

$$||a|| \leq C, ||f|| \leq C,$$

where  $\gamma = \min \{\gamma_1, \gamma_2\}$  and compatibility of the boundary data. The solution of Eq.(4.1.1) satisfies  $[u] = 0$  and  $[u_x] = 0$  at  $x = 1$ . Here,  $[u]$  denotes the jump of  $u$  defined at the point of discontinuity  $x=1$  as  $[u](1, t) = u(1^+, t) - u(1^-, t)$  where,  $u(1^\pm, t) = \lim_{x \rightarrow 1^\pm} u(x, t)$ .

The functions  $p_0$ ,  $p_1$  and  $p_2$  are holder continuous and the compatibility conditions hold at the corners of the domain. The simultaneous presence of discontinuity and delay make the problem stiff.

## 4.2 Properties of Continuous Solution

**Lemma 4.2.1.** [*Minimum Principle*]. If  $P \in C^{(0,0)}(\overline{D}) \cap C^2(S^- \cup S^+)$  satisfies  $P(x, t) \geq 0$   $\forall (x, t) \in \Gamma := \overline{D} \setminus D$  and  $L_\varepsilon P(x, t) \leq 0$   $\forall (x, t) \in D$ . Then  $P(x, t) \geq 0$   $\forall (x, t) \in \overline{D}$ .

*Proof.* Assume that  $(x, t) \in D$  such that  $P(x, t) = \min_{(x,t) \in \overline{D}} P(x, t)$  and suppose  $P(x, t) < 0$ , then it follows that  $(x, t) \neq \partial D$ . Consequently, we have  $\frac{\partial P(x,t)}{\partial x} = \frac{\partial P(x,t)}{\partial t} = 0$ , and  $\frac{\partial^2 P(x,t)}{\partial x^2} > 0$ . To show  $L_\varepsilon^M P(x, t) > 0$ , we consider the following cases:

Case 1: If  $(x, t) \in S^-$ ,

$$L_\varepsilon P(x, t) = \varepsilon \frac{d^2 P(x, t)}{dx^2} + a_1(x) \frac{dP(x, t)}{dx} - \beta(x) P(x, t) > 0.$$

Case 2: If  $(x, t) \in S^+$ ,

$$\begin{aligned} L_\varepsilon P(x, t) &= \frac{d^2 P(x, t)}{dx^2} + a_2(x) \frac{dP(x, t)}{dx} - \alpha(x) P(x-1, t) - \beta(x) P(x, t) - \frac{dP(x, t)}{dt}, \\ &= \varepsilon \frac{d^2 P(x, t)}{dx^2} + a_2(x) \frac{dP(x, t)}{dx} - \alpha(x) (P(x-1, t) - P(x, t)) - \beta(x) (P(x, t)) - \delta(x) P(x, t) - \frac{dP(x, t)}{dt}, \\ &= \varepsilon \frac{d^2 P(x, t)}{dx^2} + a_2(x) \frac{dP(x, t)}{dx} - \alpha(x) (P(x-1, t) - P(x, t)) - P(x, t) (\beta(x) + \delta(x)) - \frac{dP(x, t)}{dt} > 0. \end{aligned}$$

Combining the above two cases, we obtain  $L_\varepsilon Y(x, t) > 0$ , that contradicts the assumption made above  $L_\varepsilon Y(x, t) \leq 0, \forall (x, t) \in D$ . We have  $P(x, t_{j+1}) \geq 0$  for  $(x, t_{j+1}) \in (S^- \cup S^+)$ . Therefore,  $P(x, t) \geq 0, \forall (x, t) \in \overline{D}$ .  $\square$

**Lemma 4.2.2.** [*Uniform Stability Estimate*]. Let  $u(x, t)$  be the solution of (4.1.1), then

$$\|u(x, t)\|_{\overline{D}} \leq C \max \{ \|u\|_{\Gamma_1}, \|u\|_{\Gamma_o}, \|L_\varepsilon u\|_D, \|u\|_{\Gamma_r} \}.$$

*Proof.* By defining barrier functions

$\phi(x, t)^\pm = C \max \{ \|u\|_{\Gamma_1}, \|u\|_{\Gamma_o}, \|L_\varepsilon u\|_D, \|u\|_{\Gamma_r} \} s(x, t) \pm u(x, t)$ , and using the minimum principle in Lemma 4.2.1 we can obtain the required estimate.  $\square$

Rewriting (4.1.1) - (4.1.3) as

$$L_\varepsilon u(x, t) = \begin{cases} \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x) \frac{\partial u}{\partial x} - \beta(x) u(x, t) - \frac{\partial u}{\partial t}, & 0 \leq x \leq 1 \\ \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x) \frac{\partial u}{\partial x} - \beta(x) u(x, t) - \delta(x) u(x-1, t) - \frac{\partial u}{\partial t}, & \end{cases} \quad (4.2.1)$$

$$f(x, t) = \begin{cases} f_1(x, t) + \delta(x) u(x-1, t) \\ f_2(x, t) \end{cases}.$$

## 4.3 Construction of the Numerical Scheme

In this section, we apply implicit Euler method for temporal discretization and novel finite difference method for spatial discretization to solve Eq. (4.2.1) on a uniform mesh sizes.

### 4.3.1 Temporal Semi-discretization

On applying the implicit Euler method to approximate the t-direction of Eq. (4.2.1) with the uniform mesh,  $D_{\Delta t}^M = j\Delta t, j = 0, 1, 2, \dots, M, \Delta t = \frac{T}{M}$ , where  $M$  is number of mesh points in t-direction in the interval  $[0, T]$ , we obtain:

$$\begin{cases} L_{\epsilon}^M U(x, t_{j+1}) = g(x, t_{j+1}), \\ U(x, 0) = p_0(x), x \in [0, 2], \\ U^{j+1}(x) = p_1(x, t_{j+1}), x \in [-1, 0], j = 0(1)M - 1, \\ u^{j+1}(2) = p_2(2, t_{j+1}), j = 0(1)M - 1, \end{cases} \quad (4.3.1)$$

where,

$$L_{\epsilon}^M U(x, t_{j+1}) = \begin{cases} \epsilon \frac{d^2 U(x, t_{j+1})}{dx^2} + a_1(x) \frac{dU(x, t_{j+1})}{dx} - \beta(x)U(x, t_{j+1}), \text{ if } x \in [0, 1], j = 0(1)M - 1, \\ \epsilon \frac{d^2 U(x, t_{j+1})}{dx^2} + a_2(x) \frac{dU(x, t_{j+1})}{dx} - \delta(x)U(x - 1, t_{j+1}) - (\beta(x) + \frac{1}{\Delta t})U(x, t_{j+1}), \\ \text{ if } x \in (1, 2], j = 0(1)M - 1, \end{cases}$$

$$g(x, t_{j+1}) = \begin{cases} f(x, t_{j+1}) - \frac{U(x, t_j)}{\Delta t} + \delta(x)U(x - 1, t_{j+1}), \text{ if } x \in [0, 1], j = 0(1)M - 1, \\ f(x, t_{j+1}) - \frac{U(x, t_j)}{\Delta t}, \text{ if } x \in (1, 2], j = 0(1)M - 1. \end{cases}$$

Equation (4.3.1) can be rewritten as

$$\epsilon \frac{\partial^2 U^{j+1}(x)}{\partial x^2} + a(x) \frac{\partial U^{j+1}(x)}{\partial x} - p(x)U^{j+1}(x) = G^{j+1}(x), \quad (4.3.2)$$

where  $p(x) = \beta(x) + \frac{1}{\Delta t}$ ,  $G^{j+1}(x) = f^{j+1}(x) - \frac{U^j(x)}{\Delta t} + \alpha(x)U^{j+1}(x - 1)$ .

**Lemma 4.3.1.** [Local Truncation Error (LTE)]. If  $\frac{|\partial^n u(x, t)|}{\partial t^n} \leq C, \forall (x, t) \in \overline{D}, n = 0, 1, 2, \dots$ , then the LTE in the temporal direction satisfies

$$\|(LTE)_{j+1}\|_{\infty} \leq C(\Delta t)^2,$$

where  $C$  is a positive constant independent of  $\epsilon$  and  $\Delta t$ .

*Proof.* Using Taylor's series expansion for  $u(x, t_{j+1})$  we have,

$$u(x, t_{j+1}) = u(x, t_j) - \Delta t u_t(x, t_j) + O((\Delta t)^2). \quad (4.3.3)$$

This implies

$$\frac{u(x, t_{j+1}) - u(x, t_j)}{\Delta t} = u_t(x, t_j) + O((\Delta t)). \quad (4.3.4)$$



This implies

Substituting Eq (4.3.1) into Eq (4.3.14) we have,

$$\begin{aligned} \frac{u(x, t_{j+1}) - u(x, t_j)}{\Delta t} &= \left( \left( \epsilon \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x} - \delta u \right) (x, t_j) \right) + c(x, t_j)u(x, t_{j-\lambda}) - f(x, t_j) + O((\Delta t)) \\ (-1 + \Delta t L_\epsilon^M)u(x, t_{j+1}) + \Delta t c(x, t_j)u(x, t_{j-\lambda}) - f(x, t_j) + \Delta t u(x, t_j) &= O((\Delta t)^2). \end{aligned} \quad (4.3.5)$$

The local truncation error  $(LTE)_{j+1} = u(x, t_{j+1}) - U(x, t_{j+1})$  at  $(j+1)^{th}$  is the solution of a boundary value problem

$$(-1 + \Delta t L_\epsilon^M)(LTE)_{j+1} = O((\Delta t)^2), (LTE)_{j+1}(0) = 0 = (LTE)_j(1), \quad (4.3.6)$$

where,  $U(x, t_j)$  is the solution of the boundary value problem Eq.(4.3.1). Hence, using the maximum principle on the operator provides:

$$\|(LTE)_{j+1}\|_\infty \leq C(\Delta t)^2. \quad \square$$

**Lemma 4.3.2.** *[Global Error Estimate(GEE)]. Under the hypothesis of Lemma 4.3.1, the GEE in the temporal direction is given by  $\|P_{j+1}\|_\infty \leq C(\Delta t)$ .*

*Proof.* Using Lemma (4.3.1) at  $(j+1)^{th}$  time step, we have

$$\begin{aligned} \|P_{j+1}\|_\infty &= \left\| \sum_{i=1}^j (LTE)_i \right\|_\infty, \quad j \leq \frac{T}{\Delta t} \leq \|(LTE)_1\|_\infty + \|(LTE)_2\|_\infty + \|(LTE)_3\|_\infty + \dots + \\ \|(LTE)_j\|_\infty &\leq c_1 j (\Delta t)^2 \leq c_1 (j) \Delta t (\Delta t) \leq c_1 T (\Delta t) (j \Delta t \leq T) \leq C(\Delta t), \end{aligned}$$

where  $c_1$  and  $C$  are the positive constants of  $\epsilon$  and  $\Delta t$ .  $\square$

### 4.3.2 Spatial Discretization

In this subsection, finite difference method is employed for the numerical solution of Eq. (4.3.2) with a uniform step size. Let we divide the interval  $[0, 1]$  into  $N$  equal parts, with constant mesh length  $h$ . Then, we have  $x_i = ih$ ,  $i = 0, 1, 2, \dots, N$ ,  $h = \frac{1}{N}$ .

For convenience, denote  $U^{j+1}(x_i)$  by  $U_i^{j+1}$  at the nodal point  $x_i$ . Then, applying Taylor series expansion on  $U^{j+1}(x_{i\pm 1})$  up to the term with order 9 and adding the two, we obtain;

$$\begin{aligned} U_{i-1}^{j+1} &= U_i^{j+1} - h \frac{dU_i^{j+1}}{dx} + \frac{h^2}{2!} \frac{d^2 U_i^{j+1}}{dx^2} - \frac{h^3}{3!} \frac{d^3 U_i^{j+1}}{dx^3} + \frac{h^4}{4!} \frac{d^4 U_i^{j+1}}{dx^4} - \frac{h^5}{5!} \frac{d^5 U_i^{j+1}}{dx^5} + \frac{h}{6!} \frac{d^6 U_i^{j+1}}{dx^6} \\ &\quad - \frac{h^7}{7!} \frac{d^7 U_i^{j+1}}{dx^7} + \frac{h^8}{8!} \frac{d^8 U_i^{j+1}}{dx^8} - o(h^9). \end{aligned} \quad (4.3.7)$$

$$U_{i+1}^{j+1} = U_i^{j+1} + h \frac{dU_i^{j+1}}{dx} + \frac{h^2}{2!} \frac{d^2 U_i^{j+1}}{dx^2} + \frac{h^3}{3!} \frac{d^3 U_i^{j+1}}{dx^3} + \frac{h^4}{4!} \frac{d^4 U_i^{j+1}}{dx^4} + \frac{h^5}{5!} \frac{d^5 U_i^{j+1}}{dx^5} + \frac{h}{6!} \frac{d^6 U_i^{j+1}}{dx^6} \\ + \frac{h^7}{7!} \frac{d^7 U_i^{j+1}}{dx^7} + \frac{h^8}{8!} \frac{d^8 U_i^{j+1}}{dx^8} + o(h^9). \quad (4.3.8)$$

$$U_{i+1}^{j+1} = U_i^{j+1} + h \frac{dU_i^{j+1}}{dx} + \frac{h^2 d^2 U_i^{j+1}}{2! dx^2} + \frac{h^3 d^3 U_i^{j+1}}{3! dx^3} + \frac{h^4 d^4 U_i^{j+1}}{4! dx^4} + \dots + O(h^{10}). \quad (4.3.9)$$

Then by adding them we have;

$$U_{i-1}^{j+1} + U_{i+1}^{j+1} = 2U_i^{j+1} + 2 \frac{h^2 d^2 U_i^{j+1}}{2! dx^2} + 2 \frac{h^4 d^4 U_i^{j+1}}{4! dx^4} + 2 \frac{h^6 d^6 U_i^{j+1}}{6! dx^6} + 2 \frac{h^8 d^8 U_i^{j+1}}{8! dx^8} + O(h^{10}).$$

$$U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1} = 2 \frac{h^2 d^2 U_i^{j+1}}{2! dx^2} + 2 \frac{h^4 d^4 U_i^{j+1}}{4! dx^4} + 2 \frac{h^6 d^6 U_i^{j+1}}{6! dx^6} + 2 \frac{h^8 d^8 U_i^{j+1}}{8! dx^8} + O(h^{10}) \quad (4.3.10)$$

and obtain

$$d^2 U_{i-1}^{j+1} - 2d^2 U_i^{j+1} + d^2 U_{i+1}^{j+1} = 2 \frac{h^2 d^4 U_i^{j+1}}{2! dx^4} + 2 \frac{h^4 d^6 U_i^{j+1}}{4! dx^6} + 2 \frac{h^6 d^8 U_i^{j+1}}{6! dx^8} + 2 \frac{h^8 d^{10} U_i^{j+1}}{8! dx^{10}} + O(h^{12}). \quad (4.3.11)$$

Further, From Eq.(4.3.9) and Eq.(4.3.10) ( i.e. by substituting  $\frac{h^4 d^6 U_i^{j+1}}{12 dx^6}$  from the above Eq.(4.3.10) into Eq.(4.3.9)) we obtain the relation;

$$U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1} = \frac{h^2}{30} \left( \frac{d^2 U_{i-1}^{j+1}}{dx^2} + 28 \frac{d^2 U_i^{j+1}}{dx^2} + \frac{d^2 U_{i+1}^{j+1}}{dx^2} \right) + R, \quad (4.3.12)$$

$$\text{where } R = \frac{h^4}{20} \frac{d^4 U_i^{j+1}}{dx^4} - \frac{13h^6}{302400} \frac{dx^8 U_i^{j+1}}{dx^8} + O(h^{10}).$$

Equation (4.3.2) can be rewritten as

$$\begin{cases} \varepsilon \frac{d^2 U_{i-1}^{j+1}}{dx^2} = G_{i-1}^{j+1} - a_{i-1} \frac{dU_{i-1}^{j+1}}{dx} + p_{i-1} U_{i-1}^{j+1} \\ \varepsilon \frac{d^2 U_i^{j+1}}{dx^2} = G_i^{j+1} - a_i \frac{dU_i^{j+1}}{dx} + p_i U_i^{j+1} \\ \varepsilon \frac{d^2 U_{i+1}^{j+1}}{dx^2} = G_{i+1}^{j+1} - a_{i+1} \frac{dU_{i+1}^{j+1}}{dx} + p_{i+1} U_{i+1}^{j+1}. \end{cases} \quad (4.3.13)$$

The first derivatives in Eq. (4.3.13) can be approximated by the following non-symmetric finite difference method :

$$\begin{cases} \frac{dU_{i-1}^{j+1}}{dx} = \frac{-U_{i+1}^{j+1} + 4U_i^{j+1} - 3U_{i-1}^{j+1}}{2h} + h \frac{d^2U_i^{j+1}}{dx^2} \\ \frac{dU_i^{j+1}}{dx} = \frac{U_{i+1}^{j+1} - U_{i-1}^{j+1}}{2h} \\ \frac{dU_{i+1}^{j+1}}{dx} = \frac{3U_{i+1}^{j+1} - 4U_i^{j+1} - U_{i-1}^{j+1}}{2h} - h \frac{d^2U_i^{j+1}}{dx^2}. \end{cases} \quad (4.3.14)$$

Substituting Equation (4.3.14) into Equation (4.3.13)

$$\begin{aligned} \frac{d^2U_{i-1}^{j+1}}{dx^2} &= \frac{1}{\varepsilon} \left[ G_{i-1}^{j+1} - a_{i-1} \left[ \frac{-U_{i+1}^{j+1} + 4U_i^{j+1} - 3U_{i-1}^{j+1}}{2h} + h \frac{d^2U_i^{j+1}}{dx^2} \right] + p_{i-1}U_{i-1}^{j+1} \right] \\ \frac{d^2U_i^{j+1}}{dx^2} &= \frac{1}{\varepsilon} \left[ G_i^{j+1} - a_i \left[ \frac{U_{i+1}^{j+1} - U_{i-1}^{j+1}}{2h} \right] + p_iU_i^{j+1} \right] \\ \frac{d^2U_{i+1}^{j+1}}{dx^2} &= \frac{1}{\varepsilon} \left[ G_{i+1}^{j+1} - a_{i+1} \left[ \frac{3U_{i+1}^{j+1} - 4U_i^{j+1} - U_{i-1}^{j+1}}{2h} - h \frac{d^2U_i^{j+1}}{dx^2} \right] + p_{i+1}U_{i+1}^{j+1} \right]. \end{aligned} \quad (4.3.15)$$

Taking Equation (4.3.14) into Equation (4.3.13), we have

$$\begin{aligned} \varepsilon \frac{[U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}]}{2h} &= \frac{1}{30} \left[ G_{i-1}^{j+1} - a_{i-1} \left[ \frac{-U_{i+1}^{j+1} + 4U_i^{j+1} - 3U_{i-1}^{j+1}}{2h} + h \frac{d^2U_i^{j+1}}{dx^2} \right] + p_{i-1}U_{i-1}^{j+1} \right] + \\ &\quad \frac{28}{30} \left[ G_i^{j+1} - a_i \left[ \frac{U_{i+1}^{j+1} - U_{i-1}^{j+1}}{2h} \right] + p_iU_i^{j+1} \right] + \\ &\quad \frac{1}{30} \left[ G_{i+1}^{j+1} - a_{i+1} \left[ \frac{3U_{i+1}^{j+1} - 4U_i^{j+1} - U_{i-1}^{j+1}}{2h} - h \frac{d^2U_i^{j+1}}{dx^2} \right] + p_{i+1}U_{i+1}^{j+1} \right]. \\ \left[ \varepsilon + \frac{a_{i-1}h}{30} - \frac{a_{i+1}h}{30} \right] \left( \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{h^2} \right) &= \left[ \frac{3a_{i-1}}{60h} - \frac{p_{i-1}}{30} + \frac{28a_i}{60h} - \frac{a_{i+1}}{60h} \right] U_{i-1}^{j+1} + \\ &\quad \left[ \frac{-4a_{i-1}}{60h} + \frac{28p_i}{30} + \frac{4a_{i+1}}{60h} \right] U_i^{j+1} + \left[ \frac{a_{i-1}}{60h} - \frac{28a_i}{60h} + \frac{3a_{i+1}}{60h} + \frac{p_{i+1}}{30} \right] U_{i+1}^{j+1} + \\ &\quad \frac{1}{30} G_{i-1}^{j+1} + \frac{28G_i^{j+1}}{30} + \frac{1}{30} G_{i+1}^{j+1}. \end{aligned} \quad (4.3.16)$$

This implies

$$\begin{aligned}
& \left[ \frac{\delta\varepsilon}{h^2} + \frac{a_{i-1}}{30h} - \frac{a_{i+1}}{30h} - \frac{3a_{i-1}}{60h} + \frac{p_{i-1}}{30} - \frac{28a_i}{60h} + \frac{a_{i+1}}{60h} \right] U_{i-1}^{j+1} + \\
& \quad \left[ \frac{\delta\varepsilon}{h^2} - \frac{2a_{i-1}}{30h} + \frac{2a_{i+1}}{30h} + \frac{4a_{i-1}}{60h} + \frac{28p_i}{30} - \frac{4a_{i+1}}{60h} \right] U_i^{j+1} + \\
& \quad \left[ \frac{\varepsilon}{h^2} + \frac{a_{i-1}}{30h} - \frac{a_{i+1}}{30h} - \frac{a_{i-1}}{60h} + \frac{28a_i}{60h} + \frac{3a_{i-1}}{60h} + \frac{p_{i+1}}{30} \right] U_{i+1}^{j+1} + \\
& \quad = \frac{1}{30} [G_{i-1}^{j+1} + G_{i+1}^{j+1}] + \frac{28}{30} G_i^{j+1}. \quad (4.3.17)
\end{aligned}$$

To handle the effect of the perturbation parameter ( $\varepsilon$ ), exponential fitting factor  $\delta(\rho)$  is multiplied (4.3.17) on the term containing the perturbation parameter as

$$\begin{aligned}
& \left( \delta(\rho)\varepsilon - \frac{a_{i+1}h}{30} + \frac{a_{i+1}h}{30} \right) \left( \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{h^2} \right) = \\
& \quad \left( \frac{a_{i-1}}{20h} + \frac{p_{i-1}}{30} + \frac{7a_i}{15h} + \frac{a_{i+1}}{60h} \right) U_{i-1}^{j+1} + \left( \frac{-a_{i-1}}{15h} + \frac{14p_i}{15h} + \frac{a_{i+1}}{15h} \right) U_i^{j+1} + \\
& \quad \left( \frac{a_{i-1}}{60h} - \frac{7a_i}{15h} + \frac{a_{i+1}}{20h} + \frac{p_{i+1}}{30} \right) U_{i+1}^{j+1} + \frac{1}{30} (G_{i-1}^{j+1} + 28G_i^{j+1} + G_{i+1}^{j+1}). \quad (4.3.18)
\end{aligned}$$

Multiplying (4.3.18) by  $\rho$  and taking the limit as  $h \rightarrow 0$ , we get

$$\lim_{h \rightarrow 0} \sigma(\rho) \left( \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{\rho} \right) = \frac{a(0)}{2} (U_{i+1}^{j+1} - U_{i-1}^{j+1}). \quad (4.3.19)$$

where,  $\rho = \frac{h}{2}$ ,

$$\frac{\sigma(\rho)}{\rho} (e^{a(0)\rho} - 2 + e^{-a(0)\rho}) = \frac{a(0)}{2} (e^{a(0)\rho} - e^{-a(0)\rho}).$$

By simplifying we get;

$$\sigma(\rho) = \frac{a(0)\rho}{2} \cot h \left( \frac{a(1)\rho}{2} \right) \quad (4.3.20)$$

which is the required value of the constant fitting factor  $\sigma(\rho)$ .

Finally, from Equations (4.3.18) and (4.3.20), we obtain

$$L^{N,M} U_i^{j+1} = \begin{cases} g_i^- U_{i-1}^{j+1} - g_i^c U_i^{j+1} + g_i^+ U_{i+1}^{j+1} = H_i^{j+1}, i = 1, 2, \dots, N-1, j = 0, 1, \dots, M-1, \\ U^{j+1}(0) = p_1^{j+1}, U^{j+1}(1) = p_2^{j+1}, 0 < j < M-1, \end{cases} \quad (4.3.21)$$

where,

$$\begin{cases} g_i^- = \frac{1}{h^2} \left( \sigma(\rho)\varepsilon - \frac{a_{i+1}h}{30} + \frac{a_{i-1}h}{30} \right) + \left( -\frac{a_{i-1}}{20h} - \frac{p_{i-1}}{30} - \frac{7a_i}{15h} + \frac{a_{i+1}}{60h} \right), \\ g_i^c = \frac{2}{h^2} \left( \sigma(\rho)\varepsilon - \frac{a_{i+1}h}{30} + \frac{a_{i-1}h}{30} \right) - \left( \frac{a_{i-1}}{15h} - \frac{14p_i}{15h} - \frac{a_{i+1}}{15h} \right), \\ g_i^+ = \frac{1}{h^2} \left( \sigma(\rho)\varepsilon - \frac{a_{i+1}h}{30} + \frac{a_{i-1}h}{30} \right) + \left( -\frac{a_{i-1}}{60h} + \frac{7a_i}{15h} + \frac{a_{i+1}}{20h} - \frac{p_{i+1}}{30} \right), \\ H_i^{j+1} = \frac{1}{30} (G_{i-1}^{j+1} + 28G_i^{j+1} + G_{i+1}^{j+1}). \end{cases}$$

For small mesh sizes, the above matrix is  $|g_i^c| \geq |g_i^-| + |g_i^+|$  (i.e, the matrix is diagonally dominant) and nonsingular. Hence, the matrix  $g$  is M-matrix and the system of equations can be solved by matrix inverse with the given boundary conditions.

## 4.4 Convergence Analysis

**Lemma 4.4.1.** *The matrix associated with the discrete scheme (4.3.21) is M-matrix.*

*Proof.* By assuming that  $a(x) = A$  and  $p(x) = B$  are constant functions in  $[0,1]$ , where  $A$  and  $B$  are arbitrary constants, one can easily see that the inequalities  $g_i^- > 0$ ,  $g_i^+ > 0$ ,  $g_i^c > 0$ ,  $g_i^c > g_i^- + g_i^+$ , and  $|g_i^-| \leq |g_i^+|$  are satisfied under the assumptions that  $(a(x) = A) > 0$ ,  $(p(x) = B) > 0$ , and  $(A/2\Delta + B/30) < \sigma(\rho)\epsilon/\Delta h^2$ . Therefore, the matrix associated with the discrete scheme is M-matrix. □

**Lemma 4.4.2.** *If  $U \in C^3(I)$ , then the LTE in space discretization is written as*  
 $|T_i| \leq \max_{X_{i-1} \leq X \leq X_{i+1}} \left\{ \frac{28a\Delta h^2}{180} \left| \frac{d^3 U^{j+1}(x)}{dx^3} \right| \right\} + O(\Delta h^3), \quad i=1,2,\dots, N-1.$

*Proof.* By definition

$$\begin{aligned}
T_i &= \sigma\epsilon \left\{ \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{h^2} - \frac{d^2 U_i^{j+1}}{dx^2} \right\} \\
&\quad + \frac{a_{i-1}}{30} \left\{ \left( \frac{-3U_{i-1}^{j+1} + 4U_i^{j+1} - U_{i+1}^{j+1}}{2h} + h \frac{d^2 U_i^{j+1}}{dx^2} \right) - \frac{dU_{i-1}^{j+1}}{dx} \right\} \\
&\quad + \frac{28a_i}{30} \left\{ \frac{U_{i+1}^{j+1} - U_{i-1}^{j+1}}{2h} - \frac{dU_i^{j+1}}{dx} \right\} \\
&\quad + \frac{a_{i+1}}{30} \left\{ \left( \frac{U_{i+1}^{j+1} - 4U_i^{j+1} + 3U_{i-1}^{j+1}}{2h} - h \frac{d^2 U_i^{j+1}}{dx^2} \right) - \frac{U_{i+1}^{j+1}}{dx} \right\}, i = 1(1)N-1, \\
T_i &= \sigma\epsilon \left\{ \frac{h^2 d^4 U_i^{j+1}}{12dx^4} + \frac{h^4 d^6 U_i^{j+1}}{360dx^6} + \dots \right\} \\
&\quad + \frac{a_{i-1}}{30} \left\{ \frac{hd^2 U_i^{j+1}}{dx^2} - \frac{2h^2 d^3 U_i^{j+1}}{3dx^3} + \dots \right\} \\
&\quad + \frac{28a_i}{30} \left\{ \frac{h^2 d^3 U_i^{j+1}}{6dx^3} + \frac{h^4 d^5 U_i^{j+1}}{120dx^5} + \dots \right\} \\
&\quad + \frac{a_{i+1}}{30} \left\{ -h \frac{d^2 U_i^{j+1}}{dx^2} - \frac{2h^2 d^3 U_i^{j+1}}{3dx^3} + \dots \right\}, \\
|T_i| &\leq \max_{X_{i-1} \leq X \leq X_{i+1}} \left\{ \frac{\sigma h^2 \epsilon}{12} \left| \frac{d^4 U^{j+1}(x)}{dx^4} \right| \right\} + \max_{X_{i-1} \leq X \leq X_{i+1}} \left\{ \frac{28}{180} a h^2 \left| \frac{d^3 U^{j+1}(x)}{dx^3} \right| \right\} \\
|T_i| &\leq \max_{X_{i-1} \leq X \leq X_{i+1}} \left\{ \frac{U h^3}{12} \left| \frac{d^4 U^{j+1}(x)}{dx^4} \right| \right\} + \max_{X_{i-1} \leq X \leq X_{i+1}} \left\{ \frac{28}{180} a h^2 \left| \frac{d^3 U^{j+1}(x)}{dx^3} \right| \right\} \\
|T_i| &\leq \max_{X_{i-1} \leq X \leq X_{i+1}} \left\{ \frac{28}{180} a h^2 \left| \frac{d^3 U^{j+1}(x)}{dx^3} \right| \right\} + O(h^3) \\
|T_i| &\leq O(h^2), i = 1, 2, \dots, N-1.
\end{aligned}$$

Thus, the desired result is obtained.  $\square$

**Lemma 4.4.3.** Let  $U(x_i, t_{j+1})$  be the solution of problem Eq. (4.3.2) and  $U_i^{j+1}$  be the solution of the discrete problem (4.3.21). Then, the following estimate is obtained:

$$|U(x_i, t_{j+1}) - U_i^{j+1}| \leq O(h^2). \quad (4.4.1)$$

*Proof.* Rewrite Eq.(4.3.21) in matrix vector form as  $YU = H$ , where  $Y = (g_{i,j}), 0 \leq j \leq M-1, 1 \leq i \leq N-1$  is a tridiagonal matrix with

$$\begin{cases} g_{i-1,j+1} = \frac{\sigma(\rho)\epsilon}{h^2} - \frac{a_{i-1}}{60h} - \frac{28a_i}{60h} - \frac{a_{i+1}}{60h} - \frac{p_{i-1}}{30}, \\ g_{i,j+1} = \frac{-2\sigma(\rho)\epsilon}{h^2} - \frac{28p_i}{30}, \\ g_{i+1,j+1} = \frac{\sigma(\rho)\epsilon}{h^2} + \frac{a_{i-1}}{60h} - \frac{a_{i+1}}{60h} - \frac{2a_i}{60h} - \frac{p_{i+1}}{30}, \end{cases} \quad (4.4.2)$$

and  $H = (L_i^{j+1})$  is a column vector with  $(L_i^{j+1}) = (\frac{1}{30})(G_{i-1}^{j+1} + 28G_i^{j+1} + G_{i+1}^{j+1})$  for  $i=1,2,\dots,N-1$ , with local truncation error  $e_i : |T_i| \leq C(h^2)$ . We also have

$$Y\bar{U} - T(h) = H. \quad (4.4.3)$$

where,  $\bar{U} = (\bar{U}_0, \bar{U}_2, \dots, \bar{U}_N)^t$  and  $T(h) = (T_1(h), T_2(h), T_3(h), \dots, T_N(h))^t$  denote the actual solution and the local truncation error, respectively. We get

$$Y(\bar{U} - U) = T(h). \quad (4.4.4)$$

Thus, the error equation is

$$YP = T(h) \quad (4.4.5)$$

where  $P = \bar{U} - U = (T_0, T_1, T_2, \dots, T_N)^t$ . Let  $L$  be the sum of elements of the  $i^{th}$  row of  $Y$ ; then, we have

$$D_1 = \sum_{j=1}^{N-1} g_{1,j} = \frac{\sigma\epsilon}{h^2} + \frac{a_{i+1}}{60h} + \frac{a_{i-1}}{60h} + \frac{28p_i}{30} + \frac{p_{i+1}}{30} + \frac{28a_i}{60h},$$

$$D_{N-1} = \sum_{j=1}^{N-1} g_{N-1,j} = \frac{\sigma\epsilon}{h^2} - \frac{a_{i+1}}{60h} - \frac{a_{i-1}}{60h} + \frac{28p_i}{30} + \frac{p_{i-1}}{30} - \frac{28a_i}{60h},$$

$$D_i = \sum_{j=1}^{N-1} g_{i,j} = \frac{1}{30} (G_{i-1}^{j+1} + 28G_i^{j+1} + G_{i+1}^{j+1}) = D_i + O(h^2) = B_{i0}, i = 2(1)N - 2,$$

where  $B_{i0} = D_i = (\frac{1}{30}) (G_{i-1}^{j+1} + 28G_i^{j+1} + G_{i+1}^{j+1})$ .

Since  $0 < \epsilon \ll 1$ , for sufficiently small  $h$ , the matrix  $U$  is irreducible and monotone. Then, it follows that  $Y^{-1}$  exists, and its elements are nonnegative. Hence, from Equation (4.2.1), we obtain

$$P = Y^{-1}T(h), \quad (4.4.6)$$

$$\|P\| \leq \|Y^{-1}\| \|T(h)\|. \quad (4.4.7)$$

Let  $\bar{g}_{ki}$  be the  $(ki)^{th}$  elements of  $Y^{-1}$ . Since  $\bar{g}_{ki} \geq 0$  by the definition of multiplication of matrices with its inverses, we have

$$\sum_{i=1}^{N-1} \bar{g}_{ki} D_i = 1, k = 1, 2, \dots, N - 1.$$

Therefore, it follows that

$$\sum_{i=1}^{N-1} \bar{g}_{ki} \leq \frac{1}{\min_{0 \leq i \leq N-1} D_i} = \frac{1}{B_{i,0}} \leq \frac{1}{|B_{i,0}|}, \quad (4.4.8)$$

for some  $i_0$  between 1 and  $N-1$ , and  $B_{i0} = D_i$  we obtain

$$P_i = \sum_{i=1}^{N-1} \bar{g}_{ki} T(h), i = 1(1)N - 1, \quad (4.4.9)$$

which implies

$$P_i \leq \frac{C(h^2)}{|D_i|} i = 1(1)N - 1. \quad (4.4.10)$$

Therefore,

$$||P|| \leq C(h^2). \quad (4.4.11)$$

□

**Theorem 4.4.4.** *Let  $u(x,t)$  be the solution of the problem (4.1.1) and  $U_i^j$  be the numerical solution obtained by the proposed scheme (4.3.21). Then, for sufficiently small  $h$ , the error estimate for the totally discrete scheme is given by*

$$\sup_{0 < \epsilon < 1} \max_{x_i, t_j} |u(x_i, t_j) - U_i^j| \leq C(\Delta t + (h)^2). \quad (4.4.12)$$

*Proof.* By combining the result of Lemma (4.3.2) and Lemma (4.2.1) the required bound is obtained. □

## 4.5 Numerical Examples, Results and Discussions

Some numerical examples are presented to show the applicability of the proposed numerical scheme. Since the analytical solutions of the considered problems are not available, we used double mesh principle to compute the maximum principle.

$$E_{\epsilon}^{N,M} = \max_{(x_i, t_{j+1}) \in D^{N,M}} |U^{N,M}(x_i, t_{j+1}) - U^{2N,2M}(x_i, t_{j+1})|, \quad (4.5.1)$$

where,  $U^{N,M}(x_i)$  are computed numerical solutions obtained on the mesh  $D^{N,M} = D_x^N \times D_t^M$  with  $N$  and  $M$  mesh intervals in the spacial and temporal directions, respectively, whereas  $U^{2N,2M}(x_i, t_{j+1})$  are computed numerical solutions on the mesh  $D^{2N,2M} = D_x^{2N} \times D_t^{2M}$  by adding the mid point  $X_{i+1/2} = (X_{i+1} + X_i)/2$  and  $t_{j+1/2} = (t_{j+1} + t_j)/2$  into the mesh points. The corresponding rate of convergence for the proposed scheme is determined by

$$r_{\epsilon}^{N,M} = \log_2 \left( \frac{E_{\epsilon}^{N,M}}{E_{\epsilon}^{2N,2M}} \right). \quad (4.5.2)$$

The parameter uniform maximum absolute error ( $E^{N,M}$ ) and uniform order of convergence ( $r^{N,M}$ ) are calculated using

$$E^{N,M} = \max_{\epsilon} \{E_{\epsilon}^{N,M}\}, r^{N,M} = \frac{\log(E^{N,M}) - \log(E^{2N,2M})}{\log(2)}, \quad (4.5.3)$$

respectively. The maximum absolute errors and rate of convergence for various perturbation parameters and uniform mesh size values are tabulated for two model examples.



**Example 4.5.1.** Consider the following singularly perturbed problem of the form in Eq.(2.2.1):

$$\epsilon \frac{\partial^2 u(x, t)}{\partial x^2} + a(x) \frac{\partial u(x, t)}{\partial x} - 5u(x, t) + 2u(x - 1, t) - \frac{\partial u(x, t)}{\partial t} = f(x, t)$$

where

$$a(x) = \begin{cases} -(4 + x^2), & \text{if } 0 \leq x \leq 1 \\ (8 - x^2), & \text{if } 1 < x \leq 2 \end{cases}$$

$$f(x, t) = \begin{cases} 4xt^2 \exp(-t), & \text{if } 0 \leq x \leq 1 \\ 4(2 - x)t^2 \exp(-t), & \text{if } 1 < x \leq 2 \end{cases}$$

**Example 4.5.2.** Consider the following singularly perturbed problem :

$$\epsilon \frac{\partial^2 u(x, t)}{\partial x^2} + a(x) \frac{\partial u(x, t)}{\partial x} - 3u(x, t) - u(x - 1, t) - \frac{\partial u(x, t)}{\partial t} = f(x, t)$$

where

$$a(x) = \begin{cases} -(4 + x), & \text{if } 0 \leq x \leq 1 \\ 3 + x^2, & \text{if } 1 < x \leq 2 \end{cases}$$

$$f(x, t) = \begin{cases} -1, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } 1 < x \leq 2 \end{cases}$$

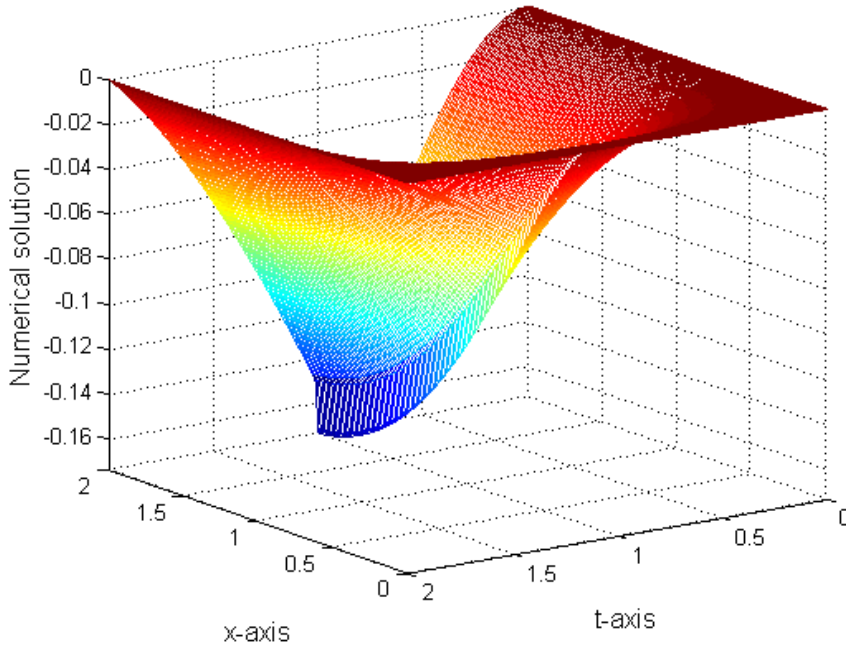


Figure 4.1: Numerical solution profile for Example 4.5.1 at  $\epsilon = 2^{-8}$  and  $N = M = 128$ .

The validation of the theoretical findings is carried out by considering two examples whose  $E_\epsilon^{N,M}$ ,  $r_\epsilon^{N,M}$ ,  $E^{N,M}$ , and  $r^{N,M}$  are plotted in Tables 4.1 and 4.2. From these tables,

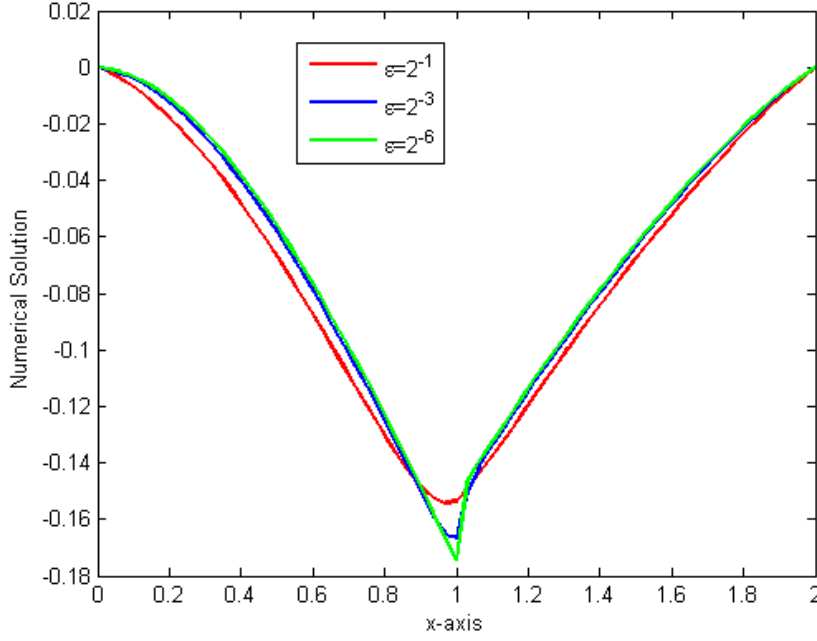


Figure 4.2: Numerical solution profile for Example 4.5.1 at  $\varepsilon = 2^{-8}$  and  $N = M = 128$ .

one can see that as  $\epsilon \rightarrow 0$  and the error goes constant. Moreover, as the mesh size decreases, the order of convergence goes to one, and the maximum absolute error decreases. These reveal that the solution of the proposed method converges parameter uniformly with the order of convergence in good agreement with the theoretical findings. These Figures depict that as  $\varepsilon \rightarrow 0$  an interior layer appears at  $x=1$ . It is observed that singularity at point  $x=0$  on the spatial domain is propagated at the point  $x=1$  due to presence of the negative shift in the reaction term. The scheme has shown to be  $\varepsilon$ -uniformly convergent of first-order in time and second-order in space directions.

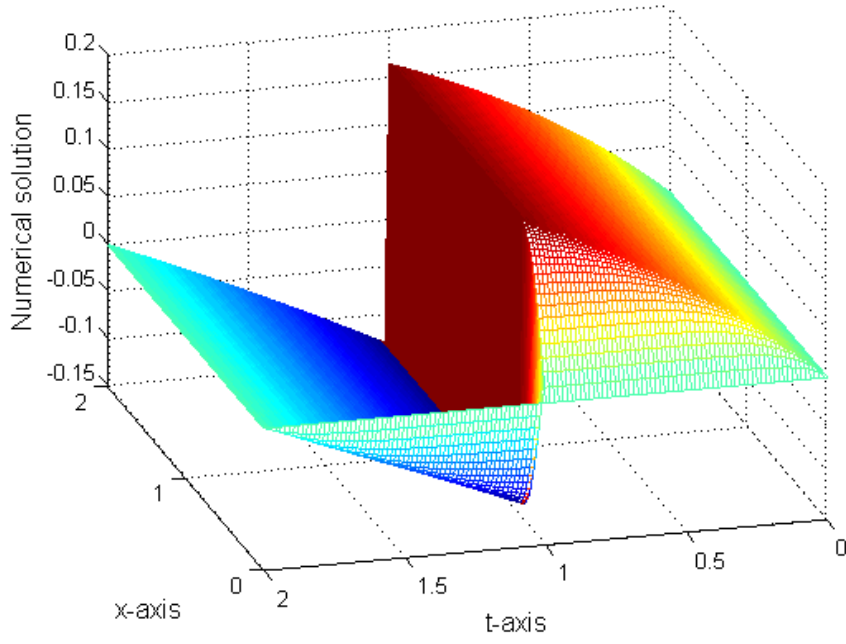


Figure 4.3: Numerical solution profile for Example 4.5.2 at  $\varepsilon = 2^{-8}$  and  $N = M = 128$ .

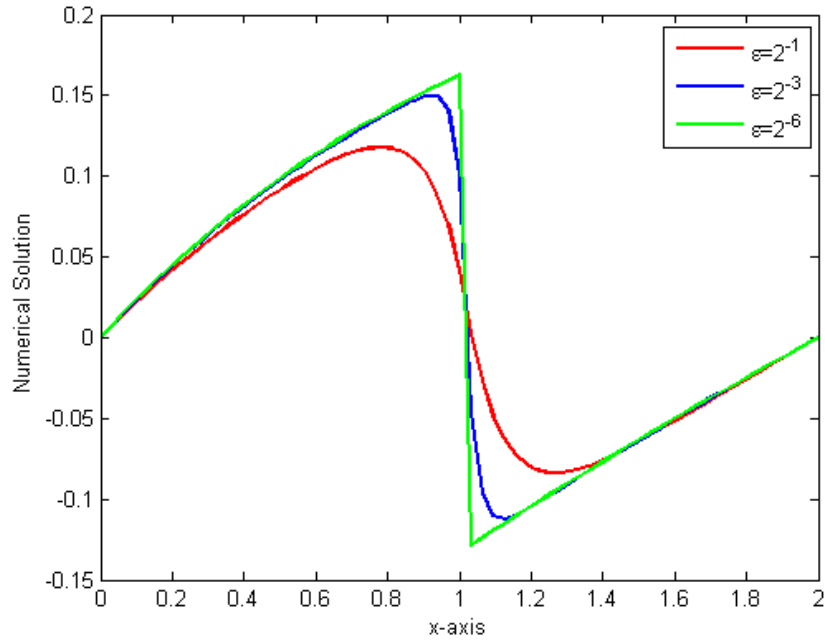


Figure 4.4: Numerical solution profile for Example 4.5.2 at  $\varepsilon = 2^{-8}$  and  $N = M = 128$ .

Table 4.1: Numerical results of Example 4.5.1 : maximum errors,  $\varepsilon$  uniform maximum errors, and uniform convergence rate for different values of  $\varepsilon$  and  $N = M$

| $\varepsilon$                        | N=32       | N=64       | N=128      | N=256      | N=512      |
|--------------------------------------|------------|------------|------------|------------|------------|
| <b>Proposed Scheme</b>               |            |            |            |            |            |
| $2^0$                                | 9.3168e-04 | 3.8931e-04 | 2.0485e-04 | 1.0523e-04 | 5.3420e-05 |
| $2^{-2}$                             | 5.2726e-03 | 2.1296e-03 | 8.4479e-04 | 3.5838e-04 | 1.6238e-04 |
| $2^{-4}$                             | 4.1339e-03 | 2.0330e-03 | 3.5739e-04 | 1.8536e-04 | 8.9156e-04 |
| $2^{-8}$                             | 2.1529e-03 | 1.1111e-03 | 5.6408e-04 | 2.0310e-04 | 1.6233e-04 |
| $2^{-10}$                            | 2.1529e-03 | 1.1111e-03 | 5.6437e-04 | 2.8444e-04 | 1.4272e-04 |
| $2^{-12}$                            | 2.1529e-03 | 1.1111e-03 | 5.6437e-04 | 2.8444e-04 | 1.4272e-04 |
| $E^{N,M}$                            | 5.2726e-03 | 2.1296e-03 | 8.4479e-04 | 3.5838e-04 | 1.6238e-04 |
| $r^{N,M}$                            | 1.3079     | 1.3339     | 1.2371     | 1.1421     |            |
| Results in Kaushik and Sharma (2020) |            |            |            |            |            |
| $2^0$                                | 3.389e-03  | 1.924e-03  | 9.94 e-04  | 5.15e-04   | 2.65e-04   |
| $2^{-2}$                             | 3.854e-03  | 2.060e-03  | 1.101e-03  | 5.82 e-04  | 3.06 e-04  |
| $2^{-4}$                             | 3.972e-03  | 2.126e-03  | 1.43e-03   | 6.07e-04   | 3.20e-04   |
| $2^{-8}$                             | 5.892e-03  | 3.196e-03  | 1.736e-03  | 9.38e-04   | 5.01e-04   |
| $2^{-10}$                            | 5.894e-03  | 3.214e-03  | 1.746e-03  | 9.46e-04   | 5.06e-04   |
| $E^{N,M}$                            | 5.894e-03  | 3.214e-03  | 1.746e-03  | 9.46e-04   | 5.06e-04   |
| $r^{N,M}$                            | 0.8749     | 0.8863     | 0.8841     | 0.9027     |            |

Table 4.2: Numerical results of Example 4.5.2 : maximum errors and convergence rates for different values of  $\varepsilon$  and  $N = M$

| $\varepsilon$ | N=32       | N=64       | N=128      | N=256      | N=512      |
|---------------|------------|------------|------------|------------|------------|
| $2^{-8}$      | 9.9900e-03 | 6.9328e-03 | 4.7191e-03 | 3.2047e-03 | 1.9245e-02 |
| $2^{-10}$     | 9.9900e-03 | 6.9328e-03 | 4.7195e-03 | 3.2227e-03 | 2.2148e-03 |
| $2^{-12}$     | 9.9900e-03 | 6.9328e-03 | 4.7195e-03 | 3.2227e-03 | 2.2150e-03 |
| $\vdots$      | $\vdots$   | $\vdots$   | $\vdots$   | $\vdots$   | $\vdots$   |
| $2^{-20}$     | 9.9900e-03 | 6.9328e-03 | 4.7195e-03 | 3.2227e-03 | 2.2150e-03 |
| $E^{N,M}$     | 9.9900e-03 | 6.9328e-03 | 4.7195e-03 | 3.2227e-03 | 2.2150e-03 |
| $r^{N,M}$     | 0.5270     | 0.5548     | 0.5504     | 0.5410     |            |

# Chapter 5

## Conclusion and Recommendation

### 5.1 Conclusion

A parameter uniform convergent numerical scheme for singularly perturbed parabolic problem with discontinuous coefficients and negative shift is presented. The proposed numerical scheme comprises the implicit Euler method and novel finite difference method in the time and space directions, respectively. Parameter uniform convergence analysis of the scheme is investigated theoretically as well as numerically. Numerical examples have been presented to validate the applicability of the scheme and the theoretical findings. Due to the presence of large delay and discontinuity in coefficient and source term, the problem exhibits strong interior layers at  $x=1$  and weak boundary layer at  $x=2$ . The obtained results show that the presented scheme gives better accuracy than the existing schemes. The presented scheme is accurate and convergent with the order of convergence  $O(\Delta t + h^2)$ . Furthermore, we discovered that the proposed scheme outperforms the existing methods in the literature in terms of numerical accuracy. The developed scheme has the potential to be extended to solve higher-dimensional singularly perturbed parabolic problems.

### 5.2 Recommendation

We believe that more than third-order spline techniques for SPDDEs with discontinuous coefficients and negative shift are one of the possible directions of future research work. One can try to extend the techniques used for solving SPDDEs with discontinuous coefficients and negative shift to develop robust numerical schemes for multiple turning point problems, non-linear problems, higher-order problems, and so on.

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