

SALALE UNIVERSITY



**COLLEGE OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS**

MSc. THESIS

**PARAMETER UNIFORM NUMERICAL METHOD
FOR SINGULARLY PERTURBED DIFFERENTIAL
EQUATION WITH A LARGE DELAY**

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**JUNE, 2023 G.C
FITCHE, ETHIOPIA**

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DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS
(NUMERICAL ANALYSIS)**

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SALALE UNIVERSITY

SCHOOL OF GRADUATE STUDIES

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We, the undersigned, member of the Board of Examiners of the final open defense by **Adugna Mengesha Wakjira** have read and evaluated his thesis entitled **parameter uniform numerical method for singularly perturbed differential equation with a large delay** and examined the candidate. This is therefore to certify that the thesis has been accepted in partial fulfillment for the degree Master of Science in **Mathematics (Numerical Analysis)**.

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Declarations

I declare that, this thesis entitled ” parameter uniform numerical method for singularly perturbed differential equation with a large delay” is my own original work and it has not been submitted to any institution elsewhere for the award of any academic degree and that all the sources i have used or quoted have been indicated and acknowledged as complete references.

Name: Adugna Mengesha Wakjira

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The work has been done under the supervision and the approval of the advisor:

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Abstract

This study attempted to introduce the finite difference method (FDM) for parameter-uniform numerical schemes for singularly perturbed differential equations with a large delay that involve one governing equation together with some boundary conditions over a domain. It is known that the classical numerical methods are not satisfactory when applied to solve singularly perturbed problems in delayed differential equations. The proposed scheme is analyzed for convergence. An approximate solution to differential equations that satisfies a given relationship between various of its derivatives in some given region of space along with some boundary conditions and along the edges of this domain, along with the ability to use a mesh of numerical analysis to accurately discretized domains of any size and shape, makes the numerical method a powerful tool for numerical analysis problems in these areas. These include improvements in methodology and shishk in meshes, as well as techniques to improve efficiency and estimate the error bound. Some aspects closely related to the finite difference method have also been investigated. The numerical results are tabulated in terms of maximum absolute errors and it observed that the present method is accurate results show the applicability and efficiency of the proposed scheme. Graphs plotted for the solution with varying shifts show the effect of small shifts on the boundary layer behavior of the solution. The numerical analyses of the method reveal that the method is able to produce uniformly convergent solutions with a quadratic convergence rate.

Chapter 1

Introduction

1.1 Background of the Study

There are a variety of physical and biological problems and processes in applied science and engineering in which boundary layer(s) arise in the solution for certain parameter ranges. These problems are generally characterized as singular perturbation problems, and the parameter is known as the perturbation parameter. A singularly perturbed problem is best defined as one in which the perturbation parameter tends to zero. The numerical and analytical treatments of such singularly perturbed problems are far from trivial due to the behavior of the boundary layer(s) of the solutions. The smoothness of the solutions deteriorates when the perturbation parameter tends to zero. A differential difference equation involving a delay (negative shift) term is recognized as a singularly perturbed differential-difference equation when its highest-order derivative term is multiplied with a small positive perturbation parameter. Problems of solving such singularly perturbed differential-difference equations with the interval and boundary conditions are the most ubiquitous and challenging tasks in the mathematical modeling of several physical and biological phenomena and processes, like oscillations in the human pupil light reflex with mixed and delayed feedback Longtin and Milton (1988), evolutionary biology (red cell system) Lasota and Wazewska (1976), the bifurcation gap in a hybrid optical system Derstine et al. (1982), epidemics and population dynamics Kuang (1993), tumor growth Villasana and Radunskaya (2003), and the oscillations and chaos in physiological control systems Mackey and Glass (1977), to mention but a few.

Any differential equation in which the highest order derivative is multiplied by a small positive parameter is called a singularly perturbed problem, and the parameter is known

as the perturbation parameter Wasow (1942). If the solution of the reduced problem (i.e., the problem that is obtained by putting $\epsilon = 0$ in the original problem) as the perturbation parameter tends to zero, the problem is known as regularly perturbed; otherwise, it is known as a singularly perturbed problem. Singularly perturbed problems often have very thin boundary and internal layers where the solution varies rapidly, whereas away from the layer, the solution behaves regularly and varies slowly, so that the numerical treatment of singularly perturbed problems faces major difficulties. Due to the variation in the width of the layer with respect to small perturbation parameters, several difficulties are experienced in solving singularly perturbed problems using the standard numerical Methods with a uniform mesh Kadalbajoo and Sharma (2004). Singular perturbation problems (SPPs) model convection diffusion processes in applied mathematics that arise in diverse areas, including the linearized Navier-Stokes equation at high Reynolds numbers, the drift diffusion equation of semiconductor device modeling, heat and mass transfer at high Péclet numbers, etc. Doolan et al. (1980) .

Amiraliyev and Cimen (2010) have given an exponentially fitted difference scheme on a uniform mesh for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term. Subburayan and Ramanujam (2013) presented an initial value technique to solve singularly perturbed boundary value problem for the second order ordinary differential equations of convectiondiffusion type with a delay.

LeVeque (2007), to describe the approximate solutions of differential equations, *i.e.*, to find a function (or some discrete approximation to this function) that satisfies a given relationship between various of its derivatives on some given region of spaces along with some boundary conditions along the edges of this domain. A finite difference method proceeds by replacing the derivatives in the differential equations by finite difference approximations. This gives a large algebraic system of equations to be solved in place of the differential equation.

In this paper we present an exponentially fitted finite difference scheme to solve singularly perturbed delay differential equation of second order with a large delay. For many singular perturbation problems a reduced problem is well defined and known a priori. A fitting factor is introduced in a finite difference scheme and is obtained from the theory of

singular perturbations. The proposed scheme is analyzed for convergence.

1.2 Statement of the Problem

The numerical analysis of SPPs has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the domain of the problem.

The treatment of SPPs presents severe difficulties that have to be addressed to ensure accurate numerical solutions, Doolan et al. (1980) and Kadalbajoo and Ramesh (2007), state that the accuracy of the problem increases by increasing the resolution of the grid, which might be impractical in some cases, like higher dimensions. The numerical methods whose accuracy does not depend on the parameter or that are uniformly convergent with respect to the parameter ϵ for solving parameter uniform numerical method for singularly perturbed differential equation with a large delay.

Due to this, the present research attempted to answer the following questions:

1. How does the present methods be described the parameter uniform numerical method for singularly perturbed differential equation with a large delay ?
2. To what extent the proposing method approximate the solutions ?
3. To what extent the proposing method convergent ?

1.3 Objectives of the Study

1.3.1 General Objective of the Study

The objective of this research is to develop a parameter uniform numerical method for singularly perturbed differential equation with a large delay

1.3.2 Specific Objectives

The specific objectives of the present research are:

1. To apply the numerical method for solving the parameter uniform numerical method for singularly perturbed differential equation with a large delay.

2. To apply the convergence of the research method.
3. To compare the convergence of the proposed method.

1.4 Significance of the Study

The outcomes of this research may have the following importance:

- For advancement of knowledge.
- Providing some background information for other researchers who work in this area.
- To introduce the application of numerical methods in different fields of study.
- Help graduate students acquire research skills and scientific procedures.

1.5 Delimitation of the Study

Singularly perturbed delay differential equations are vast topics and have many applications in the real world, so this research is delimiting parameter uniform numerical method for singularly perturbed differential equation with large delay of the form:

$$-\epsilon U''(x) - b(x) U'(x) + c(x) U(x-1) + d(x) U(x) = f(x), \quad x \in \Omega = (0, 2) \quad (1.5.1)$$

$$U(2) = 0, \quad (1.5.2)$$

$$U(x) = \Phi(x), \quad x \in (-1, 0] \quad (1.5.3)$$

where $0 < x \ll 1$, $b \geq \beta > 0$, $c \geq 0$, $d - \frac{b'}{2} - \frac{\|c\|_{L_\infty(1,2)}}{2} \geq \gamma > 0$.

, and $b(x)$, $c(x)$, $d(x)$, $f(x)$ are assumed to be sufficiently continuously differentiable function in the given domain.

Chapter 2

Review of Related Literature

The singular perturbation problem was first introduced by Prandtl (1905) during his talk on fluid motion with small friction in a seven-page report presented at the Third International Congress of Mathematicians in Heidelberg in 1904, in which he demonstrated that fluid flow past a body can be divided into two regions: a boundary layer and an outer region. However, the term singular perturbations was first used by O'Malley Jr (1968) in a paper presented at a seminar on non-linear vibrations at New York University. The solutions to singular perturbation problems typically contain layers. Prandtl (1905) originally introduced the term boundary layer, but this term became more general following the work of ?.

The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations that have a multi-scale character. However, most of the problems cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions of the singularly perturbed differential equations. Kadalbajoo and Gupta (2010) found that these singularly perturbed problems arise in the modeling of various modern complicated processes, such as fluid flow at high Reynolds numbers, water quality problems in river networks, convective heat transport problems with large numbers, the drift diffusion equation of semiconductor device modeling, electromagnetic field problems in moving media, financial modeling of option pricing, turbulence models, simulation of oil extraction from under-ground reservoirs, the theory of plates and shells, atmospheric pollution, groundwater transport, and chemical reactor theory.

Perturbation theory is a subject that studies the effect of small parameters in mathematical models of ordinary differential equations. In Mathematics, more precisely in perturbation theory, a singular perturbation problem is a problem containing a small parameter that cannot be approximated by setting the parameter value to zero.

During the last few years, much progress has been made in the theory and in the computer implementation of the numerical treatment of singular perturbation problems. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and slowly in some other parts. The main concern with singular perturbation problems is the rapid growth or decay of the solution in one or more narrow "layer region(s)".

In the modeling of these processes, characterized by dominant convection and/or intensive reactions, one can observe boundary and interior layers whose width, depending on the perturbation parameters, can be arbitrarily small. On the other hand, the domain itself, where the problem in question is considered, can be extremely large, even unbounded, compared to the available computational resources (especially in multidimensional problems for systems of equations). A complicated geometry of the domains and/or a lack of sufficient smoothness (or compatibility) of the problem data may result in singular solutions in which different parts have their own specific scales. Standard numerical methods applied to such multi-scale problems give unsatisfactorily large errors, which make these methods inapplicable for practical use. Thus, it is of considerable scientific interest to develop a solid mathematical theory and specific computational methods for singularly perturbed multi-scale problems and related problems arising from applications Kadalbajoo and Gupta (2010).

Delay differential equations arise widely in various application fields and are also described in technical devices like control circuits. Nowadays, delay differential equations are ubiquitous in various branches of bioscience and control theory: ecology, chemostat systems, epidemiology, immunology, compartmental studies, neural networks, navigational control of ships and aircraft (with respectively large and short lags), and more general control problems Lange and Miura (1994). Any system involving feedback control will almost always involve time delays. These arise because a finite amount of time is required to sense information and then react to it. Delay differential equations are of the retarded type if

the delay argument does not occur in the highest-order term. If we restrict this class to one in which the highest order term is multiplied by a small parameter, then we get singularly perturbed delay differential equations of the retarded type. The numerical study of the second-order singularly perturbed differential-difference equation with a small shift or delay has been given in Amiraliyev and Cimen (2010) and references therein.

Taylor series-based finite difference approximations are efficient numerical procedures for approximating the derivatives of a function at a reference mesh point by using the values of the function at neighboring mesh points. Automatic differentiation Corliss et al. (2002) is another efficient and accurate differentiation technique that can be used for functions that can be represented by computer code but cannot be used for inputs for which generating functions are not known, as is the case in most real-time applications. In contrast, finite difference approximations do not need to know the generating function of the data, although in some cases this information might be helpful in choosing the best approximation to use. In addition to differentiation, finite difference approximations can be used in finding the numerical solutions of differential and PDE equations Khan and Ohba (1999). For advection-diffusion problems, only a few cases with special initial conditions have analytical solutions. Therefore, the development of convergence, and efficient numerical methods for solving PDE is of vital importance.

As cited in Phaneendra et al. (2014), a numerical finite difference approach was presented to solve the boundary-value problem for a singularly perturbed differential-difference equation, which contains only a negative shift in the differentiated term. In this method, first approximate the shifted term by the Taylor series and then apply a fitted finite difference scheme. The effect of small shifts on the boundary layer solution of the problem has been given by considering several numerical experiments. Frequently, delay differential equations or differential-difference equations have been reduced to differential equations with coefficients that depend on the delay by means of first-order accurate Taylor series expansions of the terms that involve either delay, and the resulting differential equations have been solved either analytically when the coefficients of these equations are constant or numerically.

In general, The singular perturbation problem, introduced by Prandtl in 1904, divides fluid flow into boundary and outer regions. It is crucial in fields like fluid flow, water

quality, and chemical reactor theory. Numerical simulations are essential for understanding solutions, but standard methods often have large errors.

Advancements in theory and computer implementation have improved numerical treatment of problems involving rapid growth or decay of solutions in narrow "layer regions." Delay differential equations are studied using Taylor series-based finite difference approximations and automatic differentiation. Convergence and efficient numerical methods are crucial for solving advection-diffusion problems.

Chapter 3

Research Design Methodology

3.1 Study Area and Period

The research was being conducted at Salale University under the department of mathematics from January 2015 to June 2015 E.C. Conceptually, the study focused on parametric uniform numerical methods for solving parametric uniform numerical method for singularly perturbed differential equations with a large delay, particularly by second-order methods.

3.2 Study Design

This research employs mixed-design (documentary review design and experimental design) on the parameter uniform numerical method for singularly perturbed differential equations with large delay types.

3.3 Source of Information

The relevant sources of information for this study are books, published articles in international reputable journals, related studies from different university websites, and so on.

3.4 Study Procedures

Necessary materials and data for the study objectives were being collected by means of documentary review and algorithm development. Hence, in order to achieve the stated objectives, the study procedures followed were:

1. Defining problems for the study,
2. Discretization the domain/interval and formulating the methods,
3. Replacing the differential equation by the finite difference approximations and obtaining the systems of equations,
4. The systems of equations into tri-diagonal systems, which can be easily solved by the Thomas algorithm.
5. Developing an algorithm and writing MATLAB code for the presented scheme,
6. Establishing the stability and convergence of the methods,
7. Presenting the results using appropriate presentation (using tables and graphs),
8. Discussing and providing a conclusion

Chapter 4

Description of the Method, Results, and Discussion

4.1 Description of the method

Consider the parameter uniform numerical method for the singularly perturbed differential equation with a large delay of the form:

$$-\epsilon U''(x) - b(x) U'(x) + c(x) U(x-1) + d(x) U(x) = f(x), \quad x \in \Omega = (0, 2) \quad (4.1.1)$$

$$U(2) = 0, \quad (4.1.2)$$

$$U(x) = \Phi(x), \quad x \in (-1, 0] \quad (4.1.3)$$

where $0 < x \ll 1$, $b \geq \beta > 0$, $c \geq 0$, $d - \frac{b'}{2} - \frac{\|c\|_{L_\infty(1,2)}}{2} \geq \gamma > 0$.

, and ϵ is a small positive parameter (diffusion constant) $0 < x \ll 1$, $b(x)$, $c(x)$, $d(x)$, $f(x)$ and $\phi(x)$ are bounded smooth functions in $(0, 2)$ and the delay (negative shift) parameters.

Consider the following singularly perturbed problem of eq.(4.1.1) provides a good approximation to the solution of the equation of the form:

$$LU(x) \equiv -\epsilon U''(x) - b(x) U'(x) + c(x) U(x-1) + d(x) U(x) = f(x), \quad x \in \Omega = (0, 2)$$

Lemma 2.1. suppose $\pi(x)$ represents a smooth function satisfying the conditions $\pi(0) \geq 0$, $\pi(1) \geq 0$, then $L\pi(x) \leq 0$, $\forall(x) \in (0, 2)$ implies $\pi(x) \geq 0$, $\forall(x) \in [0, 2]$ Protter and Weinberger (2012).

Proof Let $x^* \in [0, 2]$ be such that $\pi(x^*) < 0$ and

$$\pi(x^*) = \min_{x \in [0, 2]} \pi(x),$$

since $\pi(0) \geq 0$ and $\pi(2) \geq 0$, therefore $\pi(x^*) \notin \{0, 2\}$. Thus $\pi(x^*) = 0$ and $\pi''(x^*) \geq 0$.

Hence, we obtain:

$$L\pi(x^*) = -\epsilon\pi''(x^*) - b(x)\pi'(x^*) + c(x)\pi(x^* - 1) + d(x) \geq 0,$$

which contradicts our assumptions. Hence, it is proved that $\pi(x^*) \geq 0$ and Since $x^* \in [0, 2]$ is chosen arbitrary, thus $\pi(x) \geq 0, \forall x \in [0, 2]$.

Lemma 2.2 Let $U(x)$ be the bounded solution of the problem eq.(4.1.1), then we have

$$\|U\| \leq \theta^{-1} \|b\| + \max(|\Phi_0|, |\Psi_2|), \text{ where } U(0) = \Phi(0) = \Phi_0 \text{ and } U(2) = \Psi_2 = \Psi_0$$

and

where $\|\cdot\|$ is the \mathbf{L}_∞ Norm given by

$$\|U\| = \min_{1 \leq i \leq 2} \|U(x)\|$$

Proof Let π^+ and π^- be two barrier functions defined by

$$\pi^\pm(x) = \theta^{-1}\|b\| + \max(|\Phi_0|, |\Psi_2|) \pm U(x)$$

Then this implies

$$\begin{aligned} \pi^\pm(0) &= \theta^{-1}\|b\| + \max(|\Phi_0|, |\Psi_2|) \pm U(0) \\ &= \theta^{-1}\|b\| + \max(|\Phi_0|, |\Psi_2|) \pm \Phi_0 \text{ since } U(0) = \Phi(0) = \Phi_0 \\ &\geq 0 \end{aligned}$$

$$\begin{aligned} \pi^\pm(2) &= \theta^{-1}\|b\| + \max(|\Phi_0|, |\Psi_2|) \pm U(2) \\ &= \theta^{-1}\|b\| + \max(|\Phi_0|, |\Psi_2|) \pm \Psi_2 \text{ since } U(2) = \Psi(2) = \Psi_0 \\ &\geq 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow L\pi^\pm(x) &= -\epsilon(\pi^\pm(x))'' - b(x)(\pi^\pm(x))' + c(x)\pi^\pm(x-1) + d(x) \geq 0 \\ &= k(x)[\theta^{-1}\|b\| + \max(|\Phi_0|, |\Psi_2|)] \pm L\pi^\pm(x) \\ &= k(x)[\theta^{-1}\|b\| + \max(|\Phi_0|, |\Psi_2|)] \pm b(x) \text{ using eq.(4.1.1)} \end{aligned}$$

Using the definition of continuous operator L and the inequality $k(x) \leq -\theta < 0$

implies $k(x)\theta^{-1} < 0$ and since $\|b\| \geq b(x)$, we have:

$$L\pi^\pm(x) \leq (-\|b\| \pm b(x) + k(x)\max(|\Phi_0|, |\Psi_2|)) \leq 0, \forall x \in [0, 2].$$

Thus, using the minimum principle, we obtain, $\pi^\pm(x) \geq 0, \forall x \in [0, 2]$. Now, for computing the error that has occurred in our numerical approximations, the derivative of the solution $U(x)$ should possess a boundedness that remains valid $\forall x \in [0, 2]$. Using Lemma 2.1, the required estimate is obtained Protter and Weinberger (2012).

4.1.1 Description of the Method for the Left-End Boundary Layer Problem

To describe the method for the left end boundary layer problems, we consider eq.(4.1.1) in $(0, 2)$.

In general, the solution of the problem eq.(4.1.1) exhibits boundary layer behavior at one end of the interval $x \in \Omega = (0, 2)$ depending on the sign of $b(x)$, $b(x) \geq N > 0$ and $d(x) \leq 0$ throughout the interval $(0, 2)$, where N is some constant.

Lemma 3.1. Let the solution of eq.(4.1.1) have its zeroth-order approximation as $U(x) = U_0 + V_0$, where U_0 denotes the zeroth-order approximation of the outer solution (i.e., the solution of the problem) and V_0 denotes the zeroth order approximation of the solution in the boundary layer region.

$$\lim_{h \rightarrow 0} U(ih) \approx U_0(0) + (\phi(0) - U_0(0))e^{-(b(0)i\rho)} \quad (4.1.4)$$

Where $\rho = \frac{h}{\epsilon}$

Proof Let $U_0(0)$ be the solution of the following reduced problem:

$c(x)U_0'(x) + d(x)U_0 = f(x)$; $\psi(2) = \psi_2$ and V_0'' be the solution of boundary value problem O'Malley Jr (1974).

$$V_0''(x) + c(x)V_0'(x) = 0, \quad V_0(0) = \Phi_0(0) - U_0(0), \quad V_0(\infty) = 0 \quad (4.1.5)$$

where $t = \frac{x}{\epsilon}$

Then, the zeroth-order asymptotic approximation to the solution of eq.(4.1.1) with eq.(4.1.2) is of the following form O'Malley Jr (1974).

$$U(x) \approx U_0(x) + \frac{c(0)}{c(x)} (\Phi(0) - U_0(0)) e^{-\int_0^x \left(\frac{c(x)}{\epsilon}\right) d(x)} \quad (4.1.6)$$

As we are considering the differential equations on sufficiently small subintervals, the coefficients could be assumed to be locally constant. Hence,

$$U(x) \approx U_0(x) + (\Phi(0) - U_0(0)) e^{-\left(\frac{b(0)}{\epsilon}\right)x} \quad (4.1.7)$$

Now we consider a uniform mesh Δ with nodal points x_i on $(0, 2)$ such that $\Delta: 0 = x_0 < x_1 < x_2 < \dots < x_N = 2$, where $x_i = ih$ and $h = \frac{1}{N}$, $i = 0, 1, 2, \dots, N$.

Substituting $x = x_i$ in eq.(4.1.7), we get:

$$U(ih) \approx U_0(ih) + (\Phi(0) - U_0(0)) e^{-\left(\frac{b(0)}{\epsilon}\right)ih} \text{ for } i = 0, 1, 2, \dots, N$$

Therefor, taking the limit as $h \rightarrow 0$, we obtain:

$$\lim_{h \rightarrow 0} U(ih) \approx U_0 + (\Phi(0) - U_0(0)) e^{-\left(\frac{b(0)}{\epsilon}\right)i\rho} \quad (4.1.8)$$

for $i = 0, 1, 2, \dots, N$ and where $\rho = \frac{h}{\epsilon}$

Let $U(x)$ be a smooth function in the interval $(0, 2)$. Then, by applying Taylor's series, we have:

$$U_{(xi+1)} = U_{(i+1)} = U_i + hU_i' + \frac{h^2}{2!}U_i'' + \frac{h^3}{3!}U_i''' + \frac{h^4}{4!}U_i^{(4)} + \frac{h^5}{5!}U_i^{(5)} + \frac{h^6}{6!}U_i^{(6)} + \frac{h^7}{7!}U_i^{(7)} + \frac{h^8}{8!}U_i^{(8)} + O(h^9) \quad (4.1.9)$$

$$U_{(xi-1)} = U_{(i-1)} = U_i - hU_i' + \frac{h^2}{2!}U_i'' - \frac{h^3}{3!}U_i''' + \frac{h^4}{4!}U_i^{(4)} - \frac{h^5}{5!}U_i^{(5)} + \frac{h^6}{6!}U_i^{(6)} - \frac{h^7}{7!}U_i^{(7)} + \frac{h^8}{8!}U_i^{(8)} - O(h^9) \quad (4.1.10)$$

Using the finite difference, we get

$$U_{i-1} - 2U_i + U_{i+1} = \frac{h^2}{2!}U_i'' + \frac{h^4}{4!}U_i^{(4)} + \frac{h^6}{6!}U_i^{(6)} + \frac{h^8}{8!}U_i^{(8)} + O(h^{10}) \quad (4.1.11)$$

and the relation

$$U_{i-1}'' - 2U_i'' + U_{i+1}'' = \frac{2h^2}{2!}U_i^{(4)} + \frac{2h^4}{4!}U_i^{(6)} + \frac{2h^6}{6!}U_i^{(8)} + \frac{2h^8}{8!}U_i^{(10)} + O(h^{12}) \quad (4.1.12)$$

Substituting $\frac{h^4}{12}U_i^{(6)}$ from the above eq.(4.1.11), we get

$$\begin{aligned} \frac{2h^4}{4!}U_i^{(6)} &= U_{i-1}'' - 2U_i'' + U_{i+1}'' - \frac{2h^2}{2!}U_i^{(4)} - \frac{2h^6}{6!}U_i^{(6)} - \frac{2h^8}{8!}U_i^{(8)} + O(h^{12}) \\ \Rightarrow U_{i-1} - 2U_i + U_{i+1} &= h^2U_i'' + \frac{2h^4}{4!}U_i^{(4)} + \frac{h^2}{30} \left(\frac{2h^4}{4!}U_i^{(6)} \right) + \frac{2h^8}{8!}U_i^{(8)} + O(h^{10}) \\ \Rightarrow U_{i-1} - 2U_i + U_{i+1} &= h^2U_i'' + \frac{2h^4}{4!}U_i^{(4)} + \frac{h^2}{30} \\ &\left(U_{i-1}'' - 2U_i'' + U_{i+1}'' - \frac{2h^2}{2!}U_i^{(4)} - \frac{2h^6}{6!}U_i^{(6)} - \frac{2h^8}{8!}U_i^{(8)} \right) - O(h^{12}) + \frac{2h^8}{8!}U_i^{(8)} + O(h^{10}) \\ \text{implies } U_{i-1} - 2U_i + U_{i+1} &= h^2U_i'' + \frac{h^2}{30}U_{i-1}'' - \frac{2h^2}{30}U_i'' + \frac{h^2}{30}U_{i+1}'' + \frac{2h^4}{30}U_i^{(4)} - \frac{h^4}{4!}U_i^{(4)} - \frac{2h^8}{30(6!)}U_i^{(8)} + \\ &\frac{2h^8}{8!}U_i^{(8)} + O(h^{10}) \\ U_{i-1} - 2U_i + U_{i+1} &= \frac{28h^2}{30}U_i'' + \frac{h^2}{30}U_{(i)}'' + \frac{2h^4}{4!}U_i^{(4)} - \frac{h^4}{4!}U_i^{(4)} - \frac{13h^8}{302400}U_i^{(8)} + O(h^{10}) \\ U_{i-1} - 2U_i + U_{i+1} &= \frac{h^2}{30} (U_{i-1}'' - 2U_i'' + U_{i+1}'') + \frac{h^4}{20}U_i^{(4)} - \frac{13h^8}{302400}U_i^{(8)} \\ U_{i-1} - 2U_i + U_{i+1} &= \frac{h^2}{30} (U_{i-1}'' - 2U_i'' + U_{i+1}'') + R. \end{aligned} \quad (4.1.13)$$

where $R = \frac{h^4}{20}U_i^{(4)} - \frac{13h^8}{302400}U_i^{(8)} + O(h^{10})$

Now equation eq.(4.1.1) can be written as:

$$\begin{cases} L\epsilon U(x) = -\epsilon U''(x) - b(x)U'(x) + c(x)U(x) + d(x)U(x) = f(x), \text{ for } x \in (0, 1) \\ L\epsilon U(x) = -\epsilon U''(x) - b(x)U'(x) + c(x)U(x-1) + d(x)U(x) = f(x), \text{ for } x \in (0, 2] \end{cases}$$

where $g(x) = b(x) - c(x)U(x-1)$

$$\begin{cases} L\epsilon U(x) = -\epsilon U''(x) - b(x)U'(x) + c(x)U(x) + d(x)U(x) = f(x), \text{ for } x \in (0, 1) \\ L\epsilon U(x) = -\epsilon U''(x) - b(x)U'(x) + d(x)U(x) = g(x), \text{ for } x \in (0, 2] \end{cases} \quad (4.1.14)$$

where $g(x) = f(x) - c(x)U(x-1)$

Eq.(4.1.14) can be written as:

$$\begin{cases} \epsilon U''_{i+1} = -b_{i+1}U'_{i+1} + d_{i+1}U_{i+1} - g_{i+1} \\ \epsilon U''_i = -b_iU'_i + d_iU_i - g_i \\ \epsilon U''_{i-1} = -b_{i-1}U'_{i-1} + d_{i-1}U_{i-1} - g_{i-1} \end{cases} \quad (4.1.15)$$

where we are using non-symmetric finite differences

$$\begin{cases} U'_i = \frac{U_{i+1}-U_{i-1}}{2h} + O(h^2) \\ U'_{i+1} = \frac{3U_{i+1}-4U_i+U_{i-1}}{2h} - hU''_i + O(h^2) \\ U'_{i-1} = \frac{-U_{i+1}+4U_i-3U_{i-1}}{2h} + hU''_i + O(h^2) \end{cases} \quad (4.1.16)$$

Taking eq.(4.1.16)into eq.(4.1.15), we obtain:

$$\begin{aligned} \epsilon U''_{i+1} &= -b_{i+1} \left(\frac{3U_{i+1}-4U_i+U_{i-1}}{2h} - hU''_i \right) + d_{i+1}U_{i+1} - g_{i+1} \\ U''_{i+1} &= \frac{1}{\epsilon} \left\{ -\frac{b_{i+1}}{2h}U_{i-1} + \frac{4b_{i+1}}{2h}U_i - \frac{3b_{i+1}}{2h}U_{i+1} + hb_{i+1}U''_i + d_{i+1}U_{i+1} - g_{i+1} \right\} \end{aligned} \quad (4.1.17)$$

$$\begin{aligned} \epsilon U''_i &= -b_i \left(\frac{U_{i+1}-U_{i-1}}{2h} + d_iU_i - g_i \right) \\ U''_i &= \frac{1}{\epsilon} \left\{ \frac{b_i}{2h}U_{i-1} - \frac{b_i}{2h}U_{i+1} + d_iU_i - g_i \right\} \end{aligned} \quad (4.1.18)$$

$$\begin{aligned} \epsilon U''_{i-1} &= -b_{i-1} \left(\frac{-U_{i+1}+4U_i-3U_{i-1}}{2h} + hb_{i-1}U''_i \right) + d_{i-1}U_{i-1} - g_{i-1} \\ U''_{i-1} &= \frac{1}{\epsilon} \left\{ \frac{3b_{i-1}}{2h}U_{i-1} - \frac{4b_{i-1}}{2h}U_i + \frac{b_{i-1}}{2h}U_{i+1} - hb_{i-1}U''_i + d_{i-1}U_{i-1} - g_{i-1} \right\} \end{aligned} \quad (4.1.19)$$

$$\begin{aligned} U_{i-1} - 2U_i + U_{i+1} &= \frac{1}{\epsilon} \times \frac{h^2}{30} \left\{ \left(\frac{3b_{i-1}}{2h}U_{i-1} - \frac{4b_{i-1}}{2h}U_i + \frac{b_{i-1}}{2h}U_{i+1} - hb_{i-1}U''_i + d_{i-1}U_{i-1} - g_{i-1} \right) \right. \\ &+ 28 \left(\frac{b_i}{2h}U_{i-1} - \frac{b_i}{2h}U_{i+1} + d_iU_i - g_i \right) \\ &\left. + \left(\frac{-3b_{i+1}}{2h}U_{i+1} + \frac{4b_{i+1}}{2h}U_i + \frac{b_{i+1}}{2h}U_{i-1} - hb_{i+1}U''_i + d_{i+1}U_{i+1} - g_{i+1} \right) \right\} \end{aligned} \quad (4.1.20)$$

$$\begin{aligned} \Rightarrow \epsilon \left(\frac{U_{i-1}-2U_i+U_{i+1}}{h^2} \right) &= \left(\frac{3b_{i-1}}{60h}U_{i-1} - \frac{4b_{i-1}}{60h}U_i + \frac{b_{i-1}}{60h}U_{i+1} - \frac{hb_{i-1}}{30}U''_i + \frac{d_{i-1}}{30}U_{i-1} - \frac{g_{i-1}}{30} \right) \\ &+ \left(\frac{28b_i}{60h}U_{i-1} - \frac{28b_i}{60h}U_{i+1} + \frac{28d_i}{30}U_i - \frac{28g_i}{30} \right) \\ &+ \left(-\frac{3b_{i+1}}{60h}U_{i+1} + \frac{4b_{i+1}}{60h}U_i - \frac{b_{i+1}}{60h}U_{i-1} + \frac{hb_{i+1}}{30}U''_i + \frac{d_{i+1}}{30}U_{i+1} - \frac{g_{i+1}}{30} \right) \end{aligned}$$

$$\begin{aligned}
& \Rightarrow \left(\epsilon U_i'' + \frac{hb_{i-1}}{30} U_i'' - \frac{hb_{i+1}}{30} U_i'' \right) = \left(\frac{3b_{i-1}}{60h} U_{i-1} - \frac{4b_{i-1}}{60h} U_i + \frac{b_{i-1}}{60h} U_{i+1} \right) \\
& + \left(\frac{28b_i}{60h} U_{i-1} - \frac{28b_i}{60h} U_{i+1} \right) \\
& + \left(-\frac{3b_{i+1}}{60h} U_{i-1} + \frac{4b_{i+1}}{60h} U_i - \frac{3b_{i+1}}{60h} U_{i+1} \right) + \frac{d_{i-1}}{30} U_{i-1} - \frac{28d_i}{30} U_i + \frac{d_{i+1}}{30} U_{i+1} - \frac{g_{i-1}}{30} - \frac{28g_i}{30} - \frac{g_{i+1}}{30} \\
& \Rightarrow \left(\epsilon + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) U_i'' = \frac{b_{i-1}}{60h} (3U_{i-1} - 4U_i + U_{i+1}) + \frac{28b_i}{60h} (U_{i-1} - U_{i+1}) + \frac{b_{i+1}}{60h} (-U_{i-1} + 4U_i - 3U_{i+1}) \\
& + \frac{d_{i-1}}{30} U_{i-1} - \frac{28d_i}{30} U_i + \frac{d_{i+1}}{30} U_{i+1} - \frac{1}{30} (g_{i-1} + 28g_i + g_{i+1}) \\
& \Rightarrow \left(\epsilon + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) \left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} \right) = \frac{b_{i-1}}{60h} (3U_{i-1} - 4U_i + U_{i+1}) + \frac{28b_i}{60h} (U_{i-1} - U_{i+1}) + \frac{b_{i+1}}{60h} (-U_{i-1} + 4U_i - 3U_{i+1}) \\
& + \frac{d_{i-1}}{30} U_{i-1} - \frac{28d_i}{30} U_i + \frac{d_{i+1}}{30} U_{i+1} - \frac{1}{30} (g_{i-1} + 28g_i + g_{i+1})
\end{aligned}$$

To handle the effect of ϵ on the solution of the above equation, we multiply the form with ϵ by $\delta(\rho)$

$$\begin{aligned}
& \Rightarrow \left(\epsilon \delta(\rho) + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) \left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} \right) = \frac{b_{i-1}}{60h} (3U_{i-1} - 4U_i + U_{i+1}) + \frac{28b_i}{60h} (U_{i-1} - U_{i+1}) + \frac{b_{i+1}}{60h} (-U_{i-1} + 4U_i - 3U_{i+1}) \\
& + \frac{d_{i-1}}{30} U_{i-1} - \frac{28d_i}{30} U_i + \frac{d_{i+1}}{30} U_{i+1} - \frac{1}{30} (g_{i-1} + 28g_i + g_{i+1}) \\
& \left(\epsilon \delta(\rho) + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) \left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} \right) \\
& + \frac{b_{i-1}}{60h} (-3U_{i-1} + 4U_i - U_{i+1}) + \frac{28b_i}{60h} (U_{i+1} - U_{i-1}) + \frac{b_{i+1}}{60h} (U_{i-1} - 4U_i + 3U_{i+1}) \\
& - \frac{d_{i-1}}{30} U_{i-1} + \frac{28d_i}{30} U_i - \frac{d_{i+1}}{30} U_{i+1} = -\frac{1}{30} (g_{i-1} + 28g_i + g_{i+1}) \quad (4.1.21)
\end{aligned}$$

Multiplying eq.(4.1.21) by h and taking the limits as $h \rightarrow 0$, we get:

$$\begin{aligned}
& \lim_{h \rightarrow 0} \epsilon \delta(\rho) \left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h} \right) + \frac{b_0}{60} \lim_{h \rightarrow 0} (-3U_{i-1} + 4U_i - U_{i+1}) + \\
& \frac{28b_0}{60} \lim_{h \rightarrow 0} (U_{i+1} - U_{i-1}) + \frac{b_0}{60} (U_{i-1} - 4U_i + 3U_{i+1}) = 0 \quad (4.1.22)
\end{aligned}$$

Now from the eq.(4.1.4), we get:

$$\begin{aligned}
& \lim_{h \rightarrow 0} (U(ih - h) - 2U(ih) + U(ih + h)) = (\phi - U_{(0)}(0)) e^{-b(0)i\rho} (e^{b(0)\rho} + e^{-b(0)\rho} - 2) \\
& \lim_{h \rightarrow 0} (-3U(ih - h) + 4U(ih) - U(ih + h)) = (\phi - U_{(0)}(0)) e^{-b(0)i\rho} (-3e^{b(0)\rho} - e^{-b(0)\rho} + 4) \\
& \lim_{h \rightarrow 0} (U(ih - h) - 4U(ih) + 3U(ih + h)) = (\phi - U_{(0)}(0)) e^{-b(0)i\rho} (e^{b(0)\rho} + 3e^{-b(0)\rho} - 4) \\
& \lim_{h \rightarrow 0} (U(ih + h) - U(ih - h)) = (\phi - U_0(0)) e^{-b(0)i\rho} (e^{-b(0)\rho} - e^{b(0)\rho})
\end{aligned}$$

Using the above equations in eq.(4.1.22), we obtain:

$$\frac{\delta(\rho)}{\rho} (e^{b(0)\rho} + e^{-b(0)\rho} - 2) = -\frac{b(0)}{60} (-30e^{-b(0)\rho} + 30e^{b(0)\rho})$$

Simplifying, we gate:

$$\delta(\rho) = \frac{\rho b(0)}{2} \coth \left(\frac{\rho b(0)}{2} \right) \quad (4.1.23)$$

As the required constant fitting factor $\delta(\rho)$

Finally, using eq.(4.1.21) and the values of $\delta(\rho)$ given by eq.(4.1.23), we obtain the proposed new exponentially fitted three-term scheme/recurrence relationship:

$$A_i U_{i-1} - B_i U_i + C_i U_{i+1} = D_i, \quad (i = 1, 2, 3, \dots, N-1) \quad (4.1.24)$$

where

$$\begin{aligned}
& \Rightarrow \left(\epsilon \delta(\rho) + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) \left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} \right) = \frac{b_{i-1}}{60h} (3U_{i-1} - 4U_i + U_{i+1}) + \frac{28b_i}{60h} (U_{i-1} - U_{i+1}) + \\
& \frac{b_{i+1}}{60h} (-U_{i-1} + 4U_i - 3U_{i+1}) + \frac{d_{i-1}}{30} U_{i-1} + \frac{28d_i}{30} U_i + \frac{d_{i+1}}{30} U_{i+1} - \frac{1}{30} (g_{i-1} + 28g_i + g_{i+1}) \\
& \Rightarrow \left\{ \frac{1}{h^2} \left(\epsilon \delta(\rho) + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) - \frac{3b_{i-1}}{60h} - \frac{28b_i}{60h} + \frac{b_{i+1}}{60h} - \frac{d_{i-1}}{30} U_{i-1} \right\} (U_{i-1} \\
& + \left\{ \frac{2}{h^2} \left(\epsilon \delta(\rho) + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) + \frac{4b_{i-1}}{60h} - \frac{4b_{i+1}}{60h} - \frac{28d_i}{30} U_i \right\} U_i \\
& + \left\{ \frac{1}{h^2} \left(\epsilon \delta(\rho) + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) - \frac{b_{i-1}}{60h} + \frac{28b_i}{60h} + \frac{3b_{i+1}}{60h} - \frac{d_{i+1}}{30} U_{i+1} \right\} U_{i+1} \\
& = -\frac{1}{30} (g_{i-1} + 28g_i + g_{i+1}) \\
& A_i = \frac{1}{h^2} \left(\epsilon \delta(\rho) + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) - \frac{3b_{i-1}}{60h} - \frac{28b_i}{60h} + \frac{b_{i+1}}{60h} - \frac{d_{i-1}}{30} \\
& B_i = \frac{2}{h^2} \left(\epsilon \delta(\rho) + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) + \frac{4b_{i-1}}{60h} - \frac{4b_{i+1}}{60h} - \frac{28d_i}{30} \\
& C_i = \frac{1}{h^2} \left(\epsilon \delta(\rho) + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) - \frac{b_{i-1}}{60h} + \frac{28b_i}{60h} + \frac{3b_{i+1}}{60h} - \frac{d_{i+1}}{30} \\
& D_i = -\frac{1}{30} (g_{i-1} + 28g_i + g_{i+1})
\end{aligned}$$

The relation eq.(4.1.24) represents a system of $(N - 1)$ equations with $(N + 1)$ unknowns U_0 to U_N . These $(N - 1)$ equations, together with the boundary conditions $U_0 = U(0) = \phi(0) = 0$, $U(2) = U_2 = \psi_2$ given by eq.(4.1.14), are sufficient to solve for the unknowns U_1 to U_{N-1} . The resulting matrix problem associated with eq.(4.1.24) is a tri-diagonal system of linear equations. The coefficient matrix of such a system of equations is non singular, if it is either strictly diagonally dominant or irreducibly diagonally dominant Nichols (1989). Moreover, if these conditions hold, the Thomas Algorithm (or Discrete Invariant Embedding Algorithm) described in Kadalbajoo and Sharma (2003) provides a numerically convergent algorithm for solving the system. By assuming $b(x) = b$ and $d(x) = d$ as constant functions in $(0, 2)$, where b and d are finite constants, we easily observe that the inequalities: $A_i > 0$, $B_i > 0$, $C_i > 0$, $B_i > (A_i + C_i)$ and $|A_i||C_i|$ are satisfied under the assumptions $b(x) = b > 0$, $d(x) = d < 0$, and $(\frac{b}{2h} + \frac{d}{30}) < (\frac{\delta\epsilon}{h^2})$. Thus, the coefficient matrix of the tri-diagonal system of eq.(4.2.24) with boundary conditions eq.(4.1.14) is irreducibly diagonally dominant and hence non-singular. We have solved the resulting tri-diagonal system of equations using the discrete invariant embedding algorithm, Kadalbajoo and Reddy (1986).

4.1.2 Description of the Method for Right-End Boundary Layer Study

To describe the method for right end boundary layer problems, we again consider the equation eq.(4.1.14) with the boundary conditions eq.(4.1.3) under the assumption that

$b(x) < 0, d(x) < 0$ in $(0, 2)$.

Lemma 3.2. Let the solution of eq.(4.1.14) has its zeroth order approximation as $U(x) = U_0 + V_0$, where U_0 denotes the zeroth order approximate of the outer solution (*i.e.*, the solution of the reduced problem) and V_0 denotes the zeroth order approximate of the solution in the boundary layer region. Then, for a fixed positive integer i ,

$$\lim_{h \rightarrow 0} U(ih) \approx U_0(0) + (\psi - U_0(1))e^{(b(1)(\frac{1}{\epsilon} - i\rho))}, \text{ where } \rho = \frac{h}{\epsilon} \quad (4.1.25)$$

Proof The proof is based on asymptotic analysis Doolan et al. (1980) and O'Malley Jr (1974) and is similar to the proof of Lemma 3.1.

Using eq.(4.1.25), we have

$$\begin{aligned} \lim_{h \rightarrow 0} (U(ih - h) - 2U(ih) + U(ih + h)) &= (\psi - U_0(1))e^{(b(1)(\frac{1}{\epsilon} - i\rho))} (e^{b(1)\rho} + e^{-b(1)\rho} - 2) \\ \lim_{h \rightarrow 0} (-3U(ih - h) + 4U(ih) - U(ih + h)) &= (\psi - U_0(1))e^{(b(1)(\frac{1}{\epsilon} - i\rho))} (-3e^{b(1)\rho} - e^{-b(1)\rho} + 4) \\ \lim_{h \rightarrow 0} (U(ih - h) - 4U(ih) + 3U(ih + h)) &= (\psi - U_0(0))e^{(b(1)(\frac{1}{\epsilon} - i\rho))} (e^{b(1)\rho} + 3e^{-b(1)\rho} - 4) \\ \lim_{h \rightarrow 0} (U(ih + h) - U(ih - h)) &= (\psi - U_0(0))e^{(b(1)(\frac{1}{\epsilon} - i\rho))} (e^{b(1)\rho} - e^{-b(1)\rho}) \end{aligned}$$

By using the above equations in the eq.(4.1.21),

$$\frac{\delta(\rho)}{\rho} (e^{b(1)\rho} + e^{-b(1)\rho} - 2) = -\frac{b(0)}{60} (-30e^{-b(1)\rho} + 30e^{b(1)\rho})$$

On simplifying, we get:

$$\delta(\rho) = \frac{\rho b(0)}{2} \coth \left(\frac{\rho b(1)}{2} \right) \quad (4.1.26)$$

which is the required value of the constant fitting factor $\delta(\rho)$ in this case of problems having boundary layers at the right end of the underlying interval. Using eq.(4.1.22) with $\delta(\rho)$ given by $U(0) = \phi(0) = \phi_0, U(2) = \psi(2) = \psi_0$, we obtain the proposed exponentially fitted three-term scheme in the form of eq.(4.1.25) for solving the problems with the right-end boundary layer. By assuming $b(x) = b$ and $d(x) = d$ as constant functions in $(0, 2)$, where b and d are finite negative constants, we can easily observe that the inequalities: $A_i > 0, B_i > 0, C_i > 0, B_i > (A_i + C_i)$ and $|A_i| \leq |C_i|$ are satisfied under the assumptions that $(b(x) = b) < 0, d(x) = d < 0$ with $(\frac{b}{2h} + \frac{d}{30}) < (\frac{-b}{2h} + \frac{d}{30})$ in $(0, 2)$. Thus, the coefficient matrix of the tri-diagonal system of eq.(4.1.25) with boundary condition-seq.(4.1.3) is irreducibly diagonally dominant and hence non-singular. We have solved the resulting tri-diagonal system of equations using the discrete invariant embedding algorithm Kadalbajoo and Reddy (1986).

4.2 Convergence Analysis

In this section, we discuss the convergence/error analysis of the method Kadalbajoo and Sharma (2006).

Definition 4.1 (Consistency): Let

$$\tau_i[U] \equiv L_h U(x_i) - L_\tau U(x_i), \quad i = 1, 2, \dots, N$$

where U represents a smooth function on $I = (0, 2)$ with L_h as the discrete difference operator. Then the difference problem (4.1.24)-(4.1.14) bears consistency with the differential problem eq.(4.1.14)-eq.(4.1.3) if

$$|\tau_i[U]| \rightarrow 0 \text{ as } h \rightarrow 0$$

The quantities $\tau_i[U]$, $i = 1, 2, \dots, N$, are said to be the local truncation (or local discretization) errors.

Definition 4.2 The difference problem eq.(4.1.25)-eq.(4.1.14) has local p^{th} order accuracy if, for sufficiently smooth data, a positive constant C exists independent of h and ϵ such that

$$\max_{1 \leq i \leq N} |\tau_i[U]| \leq Ch^p$$

The consistency of the difference problem eq.(4.1.25) with eq.(4.1.14) its locally second-order accuracy are demonstrated by the following lemma:

Lemma 4.1 If $U \in C^3(I)$ Protter and Weinberger (2012), then

$$|\tau_i| \leq \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{28bh^2}{180} |U^3(x)| \right) + O(h^3); 1, 2, \dots, N$$

Proof by definition

$$\begin{aligned} \tau_i &= \delta(\rho) \left\{ \left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} \right) - U_i'' \right\} + \frac{b_{i-1}}{30} \left\{ \left(\frac{-3U_{i-1} + 4U_i - U_{i+1}}{2h} + hU_i'' \right) - U_{i-1}' \right\} \\ &+ \frac{28b_i}{30} \left\{ \left(\frac{U_{i+1} - U_{i-1}}{2h} \right) - U_i' \right\} + \frac{b_{i+1}}{30} \left\{ \left(\frac{U_{i-1} - 4U_i + 3U_{i+1}}{2h} - hU_i'' \right) - U_{i+1}' \right\}; 1, 2, \dots, N \\ &\Rightarrow \tau_i = \delta(\rho) \left\{ \frac{h^2}{12} U_i^4 + \frac{h^4}{12} U_i^6 + \dots \right\} + \frac{b_{i-1}}{30} \left\{ hU_i'' - \frac{2h^2}{3} U_i^3 + \dots \right\} \\ &+ \frac{28b_i}{30} \left\{ \frac{h^2}{6} U_i^3 + \frac{h^4}{120} U_i^5 + \dots \right\} \\ &+ \frac{b_{i+1}}{30} \left\{ -hU_i'' - \frac{2h^2}{3} U_i^3 + \dots \right\} \end{aligned}$$

$$\Rightarrow |\tau_i| \leq \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{\delta h^2 \epsilon}{12} |U^4(x)| \right) + \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{bh^2}{30} |U''(x)| \right)$$

$$\begin{aligned}
& + \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{28bh^2}{180} |U^3(x)| \right) - \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{bh}{30} |U''(x)| \right) \\
\Rightarrow |\tau_i| & \leq \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{\delta h^2 \epsilon}{12} |U^4(x)| \right) + \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{28bh^2}{180} |U^3(x)| \right)
\end{aligned}$$

Using the relation eq.(4.1.21) with $M = \frac{b(0)}{2} \coth \left(\frac{b(0)\rho}{2} \right)$, we get :

$$\begin{aligned}
\Rightarrow |\tau_i| & \leq \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{Mh^3}{12} |U^4(x)| \right) + \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{28bh^2}{180} |U^3(x)| \right) \\
\Rightarrow |\tau_i| & \leq \max_{x_{i-1} \leq x \leq x_{i+1}} \left(\frac{28bh^2}{180} |U^3(x)| \right) + O(h^3)
\end{aligned}$$

$$\Rightarrow |\tau_i| \leq O(h^3), \quad i = 1, 2, \dots, N-1.$$

Thus, the desired result is obtained. Now, we will analyze the convergence of the proposed method over the whole interval range $0 < x < 2$. Following the procedure described in Sirisha et al. (2018), we write the tridiagonal system eq.(4.1.24) in matrix-vector form:

$$WU = D \quad (4.2.1)$$

$$-\epsilon U''(x) - b(x) U'(x) + c(x) U(x-1) + d(x) U(x) = f(x) \quad (4.2.2)$$

Formulation of the numerical method, equation eq.(4.2.2) can be written as:

$$-\epsilon U''(x) - b(x) U'(x) + d(x) U(x) = g(x) \quad (4.2.3)$$

where $g(x) = -c(x)U(x-1) + f(x)$

On applying novel FDM (special FDM) on eq.(4.2.3)

$$U_{i-1} - 2U_i + U_{i+1} = \frac{h^2}{30} \left(U''_{i-1} - 2U'_i + U''_{i+1} \right) + R \quad (4.2.4)$$

Rewriting eq.(4.2.3) as:

$$\begin{cases} \epsilon U''_{i-1} = -b_{i-1} U'_{i-1} + d_{i-1} U_{i-1} - g_{i-1} \\ \epsilon U''_i = -b_i U'_i + d_i U_i - g_i \\ \epsilon U''_{i+1} = -b_{i+1} U'_{i+1} + d_{i+1} U_{i+1} - g_{i+1} \end{cases} \quad (4.2.5)$$

where U'_i, U'_{i+1} and U'_{i-1} is approximated by using non-symmetric FDM:

$$\begin{cases} U'_i = \frac{U_{i+1}-U_{i-1}}{2h} + O(h^2) \\ U'_{i+1} = \frac{3U_{i+1}-4U_i+U_{i-1}}{2h} - hU''_i + O(h^2) \\ U'_{i-1} = \frac{-U_{i+1}+4U_i-3U_{i-1}}{2h} + hU''_i + O(h^2) \end{cases} \quad (4.2.6)$$

Plugging eq.(4.2.6) into eq.(4.2.5), we have:

$$\begin{aligned} \epsilon U''_{i+1} &= -b_{i+1} \left(\frac{3U_{i+1}-4U_i+U_{i-1}}{2h} - hU''_i \right) + d_{i+1}U_{i+1} - g_{i+1} \\ U''_{i+1} &= \frac{1}{\epsilon} \left\{ -\frac{b_{i+1}}{2h}U_{i-1} + \frac{4b_{i+1}}{2h}U_i - \frac{3b_{i+1}}{2h}U_{i+1} + hb_{i+1}U''_i + d_{i+1}U_{i+1} - g_{i+1} \right\} \end{aligned} \quad (4.2.7)$$

$$\begin{aligned} \epsilon U''_i &= -b_i \left(\frac{U_{i+1}-U_{i-1}}{2h} + d_iU_i - g_i \right) \\ U''_i &= \frac{1}{\epsilon} \left\{ \frac{b_i}{2h}U_{i-1} - \frac{b_i}{2h}U_{i+1} + d_iU_i - g_i \right\} \end{aligned} \quad (4.2.8)$$

$$\begin{aligned} \epsilon U''_{i-1} &= -b_{i-1} \left(\frac{-U_{i+1}+4U_i-3U_{i-1}}{2h} + hb_{i-1}U''_i \right) + d_{i-1}U_{i-1} - g_{i-1} \\ U''_{i-1} &= \frac{1}{\epsilon} \left\{ \frac{3b_{i-1}}{2h}U_{i-1} - \frac{4b_{i-1}}{2h}U_i + \frac{b_{i-1}}{2h}U_{i+1} - hb_{i-1}U''_i + d_{i-1}U_{i-1} - g_{i-1} \right\} \end{aligned} \quad (4.2.9)$$

Substituting eqs.(4.2.7),(4.2.8) and (4.2.9) into eq.(4.2.4), we get:

$$\begin{aligned} &\frac{\epsilon}{h^2} [U_{i-1} - 2U_i + U_{i+1}] = \frac{1}{30} \left\{ b_{i-1} \left(\frac{-U_{i+1}+4U_i-3U_{i-1}}{2h} + hU''_i \right) + d_{i-1}U_{i-1} - g_{i-1} \right\} \\ &+ \frac{28}{30} \left\{ -b_i \left(\frac{U_{i+1}-U_{i-1}}{2h} \right) + d_iU_i - g_i \right\} + \frac{1}{30} \left\{ -b_{i+1} \left(\frac{3U_{i+1}-4U_i+U_{i-1}}{2h} - hU''_i \right) + d_{i+1}U_{i+1} - g_{i+1} \right\} \\ &\Rightarrow \left(\epsilon + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) \left(\frac{U_{i-1}-2U_i+U_{i+1}}{h^2} \right) = \frac{b_{i-1}}{60h} (3U_{i-1} - 4U_i + U_{i+1}) + \frac{28b_i}{60h} (U_{i-1} - U_{i+1}) + \\ &\frac{b_{i+1}}{60h} (-U_{i-1} + 4U_i - 3U_{i+1}) - \frac{1}{30} (g_{i-1} + 28g_i + g_{i+1}) \\ &\left(\epsilon + \frac{hb_{i-1}}{30} - \frac{hb_{i+1}}{30} \right) \left(\frac{U_{i-1}-2U_i+U_{i+1}}{h^2} \right) = \left\{ \frac{3b_{i-1}}{60h} + \frac{28b_i}{60h} + \frac{d_{i-1}}{30} - \frac{3b_{i+1}}{60h} \right\} U_{i-1} \\ &+ \left\{ \frac{-4b_{i-1}}{60h} + \frac{28d_i}{30} + \frac{3b_{i+1}}{60h} \right\} U_i \\ &+ \left\{ \frac{b_{i-1}}{60h} + \frac{28b_i}{60h} - \frac{3b_{i+1}}{60h} + \frac{d_{i+1}}{30} \right\} U_{i+1} \\ &+ \frac{1}{30} \{-g_{i-1} - 28g_i - g_{i+1}\} \end{aligned} \quad (4.2.10)$$

By introducing $\delta(\rho)$ on eq.(4.2.10), we have:

$$\begin{aligned} &\left\{ \left(\frac{\epsilon\delta(\rho)}{h^2} + \frac{b_{i-1}}{30h} - \frac{b_{i+1}}{30h} \right) - \frac{3b_{i-1}}{60h} - \frac{28b_i}{60h} - \frac{d_{i-1}}{30} + \frac{b_{i+1}}{60h} \right\} U_{i-1} \\ &+ \left\{ \left(\frac{-2\epsilon\delta(\rho)}{h^2} - \frac{b_{i-1}}{30h} + \frac{b_{i+1}}{30h} \right) + \frac{4b_{i-1}}{60h} - \frac{28b_i}{60h} - \frac{d_i}{30} - \frac{3b_{i+1}}{60h} \right\} U_i \\ &+ \left\{ \left(\frac{\epsilon\delta(\rho)}{h^2} + \frac{b_{i-1}}{30h} - \frac{b_{i+1}}{30h} \right) - \frac{b_{i-1}}{60h} + \frac{28b_i}{60h} - \frac{d_{i-1}}{30} + \frac{3b_{i+1}}{60h} \right\} U_{i+1} = \frac{1}{30} (-g_{i-1} - 28g_i - g_{i+1}) \end{aligned}$$

where $A_{ij}, 1 \leq i, j \leq N-1$ is a tri-diagonal matrix of order N-1 with

$$A_{i,j-1} = \left(\frac{\epsilon\delta(\rho)}{h^2} + \frac{b_{i-1}}{30h} - \frac{b_{i+1}}{30h} \right) - \frac{3b_{i-1}}{60h} - \frac{28b_i}{60h} - \frac{d_{i-1}}{30} + \frac{b_{i+1}}{60h}$$

$$A_{i,j} = \left(\frac{-2\epsilon\delta(\rho)}{h^2} - \frac{b_{i-1}}{30h} + \frac{b_{i+1}}{30h} \right) + \frac{4b_{i-1}}{60h} - \frac{28d_i}{60h} - \frac{3b_{i+1}}{60h}$$

$$A_{i,j+1} = \left(\frac{\epsilon\delta(\rho)}{h^2} + \frac{b_{i-1}}{30h} - \frac{b_{i+1}}{30h} \right) - \frac{b_{i-1}}{60h} + \frac{28b_i}{60h} - \frac{d_{i-1}}{30} + \frac{3b_{i+1}}{60h}$$

and $D = (d_i)$ is a column vector with $d_i = -\frac{1}{30}(g_{i-1} + 28g_i + g_{i+1})$ for $i = 1, 2, \dots, N-1$, with local truncation error τ_i :

$$|\tau_i| \leq O(h^2) \quad (4.2.11)$$

we also have:

$$W\bar{U} - \tau(h) = D \quad (4.2.12)$$

where $\bar{U} = (\bar{U}_0, \bar{U}_1, \bar{U}_2, \dots, \bar{U}_N)^t$ and $\tau(h) = (\tau_1(h), \tau_2(h), \dots, \tau_N(h))^t$ denote the actual solution and the local truncation error respectively.

From eqs.(4.2.11) and (4.2.12), we have:

$$W(\bar{U} - U) = \tau(h) \quad (4.2.13)$$

Thus, the error equation is:

$$WE = \tau(h) \quad (4.2.14)$$

where $E = \bar{U} - U = (e_0, e_1, e_2, \dots, e_N)^t$

Let a_i^* be the sum of elements of i^{th} row of W , we have:

$$a_1^* = \sum_{j=1}^{N-1} A_{1,j} = \left(\frac{-\epsilon\delta}{h^2} + \frac{b_{i-1}}{60h} + \frac{b_{i+1}}{60h} + \frac{28d_i}{30} + \frac{d_{i+1}}{30} + \frac{28b_i}{60h} \right)$$

$$a_{N-1}^* = \sum_{j=1}^{N-1} A_{N-1,j} = \left(\frac{-\epsilon\delta}{h^2} - \frac{b_{i-1}}{60h} - \frac{b_{i+1}}{60h} + \frac{28d_i}{30} + \frac{d_{i+1}}{30} - \frac{28b_i}{60h} \right)$$

$$a_i^* = \sum_{j=1}^{N-1} A_{i,j} = \frac{1}{30}(-g_{i-1} - 28g_i - g_{i+1}) + O(h^2) = b_i = B_{i0}; i = 2(1)N-2$$

where $B_{i0} = b_i = \frac{1}{30}(-g_{i-1} - 28g_i - g_{i+1})$

Since $0 < \epsilon \ll 1$, for sufficiently small h the matrix W is irreducible and monotone. Then, it follows that W^{-1} exists and its elements are non-negative. Hence, from eq.(4.2.14), we have:

$$(\bar{U} - U) = W^{-1}\tau(h) \quad (4.2.15)$$

$$\|(\bar{U} - U)\| = \|W^{-1}\|\|\tau(h)\| \quad (4.2.16)$$

Let \bar{A}_{ki} be the $(ki)^{th}$ elements of W^{-1} . Since $\bar{A}_{ki} \geq 0$, from the operation of matrices, we have:

$$\sum_{j=1}^{N-1} \bar{A}_{ki} a_j^* = 1; k = 1, 2, \dots, N-1. \quad (4.2.17)$$

Therefore, it follows:

$$\sum_{j=1}^{N-1} \bar{A}_{ki} \leq \frac{1}{\min_{0 \leq i \leq N-1} a_i^*} = \frac{1}{B_{i0}} \leq \frac{1}{|B_{i0}|} \quad (4.2.18)$$

for some i_0 between 1 and $N-1$, and $B_{i0} = b_i$.

Therefore, from eqs.(4.2.11) and (4.2.17), we obtain:

$$e_j = \sum_{j=1}^{N-1} \bar{A}_{ki} \tau_i(h); j = 1(1)N - 1$$

which implies

$$e_j \leq \frac{O(h^2)}{b_i}; j = 1(1)N - 1 \quad (4.2.19)$$

Therefore, using the definitions and eq.(4.2.19) , we get:

$$||E|| = O(h^2)$$

This implies that the method presented has a quadratic convergence rate. By following a similar procedure, we can easily find that the scheme for right-end boundary value problems is one of quadratic convergence.

4.2.1 Numerical Examples, Results and Discussion

To demonstrate the accuracy and efficiency of the proposed method, we have solved four widely discussed example problems and presented the numerical results in terms of the maximum absolute errors (MAE) in the tables. These maximum absolute errors for different values of N and ϵ are obtained using the relation $\max ||U(xi) - Ui||$, where $U(xi)$ and Ui denote the exact and approximate solutions, respectively. Wherever exact solutions are not known, the MAE are calculated using the double mesh principle Doolan et al. (1980) given by

$$E_\epsilon^N = \max_{1 \leq i \leq N} |U_i^N - U_i^{2N}|,$$

where U_i^N and U_i^{2N} denote the numerical solutions obtained using N and $2N$ mesh intervals respectively.

Table 4.1: Comparison of maximum absolute error for Example 4.2.1 results obtained by the proposed scheme and results in Chakravarthy et al. (2017) for different value of ε

$\varepsilon \downarrow$	N=32	N=64	N=128	N=256	N=512	N=1024
Present Method						
2^{-5}	7.3647e-03	3.0748e-03	1.0792e-03	3.9753e-04	1.8189e-04	8.6311e-05
2^{-6}	7.7631e-03	3.6823e-03	1.5374e-03	5.3962e-04	1.9876e-04	9.0943e-05
2^{-7}	7.8116e-03	3.8815e-03	1.8412e-03	7.6870e-04	2.6981e-04	9.9381e-05
2^{-8}	7.8125e-03	3.9058e-03	1.9408e-03	9.2058e-04	3.8435e-04	1.3490e-04
2^{-9}	7.8125e-03	3.9062e-03	1.9529e-03	9.7038e-04	4.6029e-04	1.9218e-04
2^{-10}	7.8125e-03	3.9062e-03	1.9531e-03	9.7645e-04	4.8519e-04	2.3015e-04
2^{-11}	7.8125e-03	3.9062e-03	1.9531e-03	9.7656e-04	4.8823e-04	2.4260e-04
2^{-12}	7.8125e-03	3.9062e-03	1.9531e-03	9.7656e-04	4.8828e-04	2.4411e-04
2^{-13}	7.8125e-03	3.9062e-03	1.9531e-03	9.7656e-04	4.8828e-04	2.4414e-04
2^{-14}	7.8125e-03	3.9062e-03	1.9531e-03	9.7656e-04	4.8828e-04	2.4414e-04
Results in Chakravarthy et al. (2017)						
2^{-5}	2.0741e-02	5.4962e-03	1.3953e-03	3.5024e-04	8.7651e-05	2.1918e-05
2^{-6}	3.5344e-02	1.0725e-02	2.8528e-03	7.2423e-04	1.8177e-04	4.5489e-05
2^{-7}	4.5585e-02	1.8006e-02	5.4949e-03	1.4561e-03	3.6979e-04	9.2808e-05
2^{-8}	4.7271e-02	2.3160e-02	9.1484e-03	2.7805e-03	7.3682e-04	1.8705e-04
2^{-9}	4.7302e-02	2.4016e-02	1.1672e-02	4.6105e-03	1.3988e-03	3.7082e-04
2^{-10}	4.7302e-02	2.4033e-02	1.2104e-02	5.8590e-03	2.3143e-03	7.0214e-04
2^{-11}	4.7302e-02	2.4033e-02	1.2112e-02	6.0756e-03	2.9352e-03	1.1594e-03
2^{-12}	4.7302e-02	2.4033e-02	1.2112e-02	6.0797e-03	3.0438e-03	1.4691e-03
2^{-13}	4.7302e-02	2.4033e-02	1.2112e-02	6.0797e-03	3.0458e-03	1.5234e-03
2^{-14}	4.7302e-02	2.4033e-02	1.2112e-02	6.0797e-03	3.0458e-03	1.5244e-03

Example 4.2.1. Chakravarthy et al. (2017) . Consider the following singularly perturbed delay differential equation

$$\begin{aligned} \epsilon y''(x) - 2y'(x) + 5y(x-1) &= 0, \\ y(x) &= 1, \quad -1 \leq x \leq 0, \quad y(2) = 2. \end{aligned}$$

The numerical results are presented in Table 1 for different values of perturbation parameter ϵ .

Table 4.2: Comparison of maximum absolute error for Example 4.2.2 results obtained by the proposed scheme and results in Chakravarthy et al. (2017) for different value of ε

$\varepsilon \downarrow$	N=32	N=64	N=128	N=256	N=512	N=1024
Present Method						
2^{-5}	7.1001e-04	3.6867e-04	1.7588e-04	7.5632e-05	3.1564e-05	1.3590e-05
2^{-6}	6.9530e-04	3.5501e-04	1.8434e-04	8.7939e-05	3.7816e-05	1.5782e-05
2^{-7}	6.9445e-04	3.4765e-04	1.7750e-04	9.2168e-05	4.3969e-05	1.8908e-05
2^{-8}	6.9444e-04	3.4722e-04	1.7383e-04	8.8751e-05	4.6084e-05	2.1985e-05
2^{-9}	6.9444e-04	3.4722e-04	1.7361e-04	8.6913e-05	4.4376e-05	2.3042e-05
2^{-10}	6.9444e-04	3.4722e-04	1.7361e-04	8.6806e-05	4.3456e-05	2.2188e-05
2^{-11}	6.9444e-04	3.4722e-04	1.7361e-04	8.6806e-05	4.3403e-05	2.1728e-05
2^{-12}	6.9444e-04	3.4722e-04	1.7361e-04	8.6806e-05	4.3403e-05	2.1701e-05
2^{-13}	6.9444e-04	3.4722e-04	1.7361e-04	8.6806e-05	4.3403e-05	2.1701e-05
2^{-14}	6.9444e-04	3.4722e-04	1.7361e-04	8.6806e-05	4.3403e-05	2.1701e-05
Results in Chakravarthy et al. (2017)						
2^{-5}	5.1552e-04	1.4652e-04	3.7888e-05	9.5547e-06	2.3939e-06	5.9883e-07
2^{-6}	7.5922e-04	2.6637e-04	7.5145e-05	1.9455e-05	4.9061e-06	1.2292e-06
2^{-7}	8.3676e-04	3.8575e-04	1.3534e-04	3.8180e-05	9.8729e-06	2.4898e-06
2^{-8}	8.4092e-04	4.2513e-04	1.9441e-04	6.8209e-05	1.9242e-05	4.9774e-06
2^{-9}	8.4093e-04	4.2724e-04	2.1425e-04	9.7588e-05	3.4253e-05	9.6682e-06
2^{-10}	8.4093e-04	4.2725e-04	2.1532e-04	1.0755e-04	4.8915e-05	1.7177e-05
2^{-11}	8.4093e-04	4.2725e-04	2.1532e-04	1.0808e-04	5.3879e-05	2.4506e-05
2^{-12}	8.4093e-04	4.2725e-04	2.1532e-04	1.0808e-04	5.4147e-05	2.6966e-05
2^{-13}	8.4093e-04	4.2725e-04	2.1532e-04	1.0808e-04	5.4148e-05	2.7100e-05
2^{-14}	8.4093e-04	4.2725e-04	2.1532e-04	1.0808e-04	5.4148e-05	2.7100e-05

Example 4.2.2. Chakravarthy et al. (2017) . Consider the following singularly perturbed delay deferential equation

$$\begin{aligned} \varepsilon y''(x) - 3y'(x) + y(x-1) &= 0, \\ y(x) &= 1, \quad -1 \leq x \leq 0, \quad y(2) = 2. \end{aligned}$$

The numerical results are presented in Table 2 for different values of perturbation parameter ε .

Example 4.2.3. Chakravarthy et al. (2017) . Consider the following singularly perturbed delay deferential equation

$$\begin{aligned} \varepsilon y''(x) - 5y'(x) + \frac{1}{2}y(x-1) &= \begin{cases} -1; & 0 \leq x \leq 1 \\ 1; & 1 \leq x \leq 2 \end{cases} \\ y(x) &= 1, \quad -1 \leq x \leq 0, \quad y(2) = 2. \end{aligned}$$

The numerical results are presented in Table 3 for different values of perturbation parameter ε .

Table 4.3: Comparison of maximum absolute error for Example 4.2.3 results obtained by the proposed scheme and results in Chakravarthy et al. (2017) for different value of ε

$\varepsilon \downarrow$	N=32	N=64	N=128	N=256	N=512	N=1024
Present Method						
2^{-5}	2.8711e-04	5.5025e-04	7.0481e-04	5.3399e-04	3.1889e-04	1.7261e-04
2^{-6}	2.0887e-04	1.4355e-04	2.7512e-04	3.5241e-04	2.6700e-04	1.5945e-04
2^{-7}	2.0833e-04	1.0443e-04	7.1777e-05	1.3756e-04	1.7620e-04	1.3350e-04
2^{-8}	2.0833e-04	1.0417e-04	5.2217e-05	3.5889e-05	6.8781e-05	8.8101e-05
2^{-9}	2.0833e-04	1.0417e-04	5.2083e-05	2.6108e-05	1.7944e-05	3.4390e-05
2^{-10}	2.0833e-04	1.0417e-04	5.2083e-05	2.6042e-05	1.3054e-05	8.9722e-06
2^{-11}	2.0833e-04	1.0417e-04	5.2083e-05	2.6042e-05	1.3021e-05	6.5271e-06
2^{-13}	2.0833e-04	1.0417e-04	5.2083e-05	2.6042e-05	1.3021e-05	6.5104e-06
2^{-14}	2.0833e-04	1.0417e-04	5.2083e-05	2.6042e-05	1.3021e-05	6.5104e-06
Results in Chakravarthy et al. (2017)						
2^{-5}	1.9126e-04	6.2523e-05	1.7063e-05	4.3683e-06	1.0987e-06	2.7510e-07
2^{-6}	2.2400e-04	9.7185e-05	3.1769e-05	8.6781e-06	2.2223e-06	5.5895e-07
2^{-7}	2.2703e-04	1.1381e-04	4.8981e-05	1.6049e-05	4.3824e-06	1.1219e-06
2^{-8}	2.2705e-04	1.1535e-04	5.7355e-05	2.4657e-05	8.0719e-06	2.2029e-06
2^{-9}	2.2705e-04	1.1536e-04	5.8131e-05	2.8790e-05	1.2377e-05	4.0479e-06
2^{-10}	2.2705e-04	1.1536e-04	5.8136e-05	2.9180e-05	1.4423e-05	6.2006e-06
2^{-11}	2.2705e-04	1.1536e-04	5.8136e-05	2.9182e-05	1.4618e-05	7.2188e-06
2^{-12}	2.2705e-04	1.1536e-04	5.8136e-05	2.9182e-05	1.4620e-05	7.3164e-06
2^{-13}	2.2705e-04	1.1536e-04	5.8136e-05	2.9182e-05	1.4620e-05	7.3171e-06
2^{-14}	2.2705e-04	1.1536e-04	5.8136e-05	2.9182e-05	1.4620e-05	7.3171e-06

Example 4.2.4. Chakravarthy et al. (2017) . Consider the following singularly perturbed delay deferential equation

$$\begin{aligned} \varepsilon y''(x) - (x + 10) y'(x) + y(x - 1) &= -x, \\ y(x) &= x, \quad -1 \leq x \leq 0, \quad y(2) = 2. \end{aligned}$$

The numerical results are presented in Table 4 for different values of perturbation parameter ε .

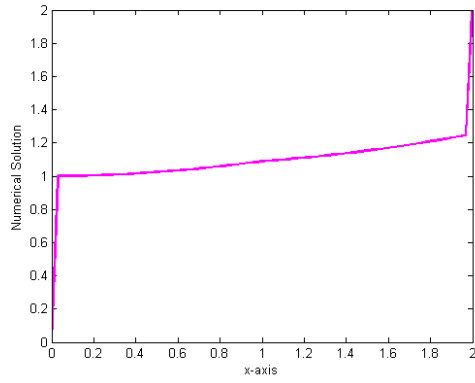
4.3 Discussion

We solved these considered example problems (1 to 4) (left and right layers) for the various values of N with $\delta = 0.5 * \varepsilon$ and presented the computational results in Tables 1, 2,3, and 4. Tables 1, 2,3, and 4 compare the results (MAE) obtained by the proposed scheme with the results presented in the article, Chakravarthy et al. (2017) exponentially, for the example problems 1, 2,3, and 4, respectively. Table 1 compares the results (MAE) with

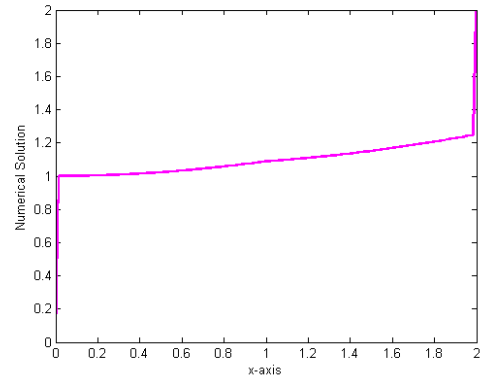
Table 4.4: Comparison of maximum absolute error for Example 4.2.4 results obtained by the proposed scheme and results in Chakravarthy et al. (2017) for different value of ε

$\varepsilon \downarrow$	N=32	N=64	N=128	N=256	N=512	N=1024
Present Method						
2^{-5}	4.1928e-03	2.1090e-03	9.5460e-04	3.4998e-04	1.1569e-04	1.3447e-04
2^{-6}	4.1929e-03	2.1249e-03	1.0615e-03	4.7901e-04	6.7450e-05	5.7967e-05
2^{-7}	4.1929e-03	2.1249e-03	1.0695e-03	5.3252e-04	1.7569e-04	8.8020e-05
2^{-8}	4.1929e-03	2.1249e-03	1.0696e-03	5.3654e-04	2.4027e-04	1.2032e-04
2^{-9}	4.1929e-03	2.1249e-03	1.0696e-03	5.3656e-04	2.6670e-04	1.3346e-04
2^{-10}	4.1929e-03	2.1249e-03	1.0696e-03	5.3656e-04	2.6871e-04	1.3447e-04
2^{-11}	4.1929e-03	2.1249e-03	1.0696e-03	5.3656e-04	2.6872e-04	1.3447e-04
2^{-12}	4.1929e-03	2.1249e-03	1.0696e-03	5.3656e-04	2.6872e-04	1.3447e-04
2^{-13}	4.1929e-03	2.1249e-03	1.0696e-03	5.3656e-04	2.6872e-04	1.3447e-04
2^{-14}	4.1929e-03	2.1249e-03	1.0696e-03	5.3656e-04	2.6872e-04	1.3447e-04
Results in Chakravarthy et al. (2017)						
2^{-5}	1.4272e-03	6.2958e-04	2.0947e-04	5.7659e-05	1.4802e-05	3.7252e-06
2^{-6}	1.4424e-03	7.2567e-04	3.1741e-04	1.0583e-04	2.9132e-05	7.4754e-06
2^{-7}	1.4425e-03	7.3344e-04	3.6586e-04	1.5976e-04	5.3246e-05	1.4646e-05
2^{-8}	1.4425e-03	7.3349e-04	3.6978e-04	1.8369e-04	8.0213e-05	2.6706e-05
2^{-9}	1.4425e-03	7.3349e-04	3.6981e-04	1.8566e-04	9.2033e-05	4.0190e-05
2^{-10}	1.4425e-03	7.3349e-04	3.6981e-04	1.8567e-04	9.3020e-05	4.6064e-05
2^{-11}	1.4425e-03	7.3349e-04	3.6981e-04	1.8567e-04	9.3026e-05	4.6558e-05
2^{-12}	1.4425e-03	7.3349e-04	3.6981e-04	1.8567e-04	9.3026e-05	4.6561e-05
2^{-13}	1.4425e-03	7.3349e-04	3.6981e-04	1.8567e-04	9.3026e-05	4.6561e-05
2^{-14}	1.4425e-03	7.3349e-04	3.6981e-04	1.8567e-04	9.3026e-05	4.6561e-05

the results presented in Chakravarthy et al. (2017) exponentially for example problem 1. These comparisons clearly show that the results obtained are much better than the results presented in Chakravarthy et al. (2017) . Furthermore, from the results presented in Tables 1 ,2,and 3 for different values of N and with $\delta = 0.5 * \epsilon$, we can easily observe that the maximum absolute errors decrease with the quadratic rate of convergence as the step size $h = 1/N$ tends to zero for all the values of the perturbation parameter. To show the effect of small shifts on the boundary layer behavior of the solutions, we solved all the three considered example problems for the varying values of δ with a fixed $\epsilon = 0.1$ and plotted the graphs of the solution in Fig. 1, represent the graphs of the solution for varying values of δ with fixed $\epsilon = 0.1$. These figures clearly show that there is no significant effect of the delay parameters on the boundary layer behavior of the solution to the problems having boundary layers at left-end points of the underlying interval (Fig. 1), whereas these parameters affect the solution in the case of the problems having boundary layers at right-end points of the



(a) $\varepsilon = 2^{-5}, N = 64$



(b) $\varepsilon = 2^{-11}, N = 128$

Figure 4.1: Numerical solution profile for different value of Example 4.2.4, for graph (a) $N = 64 = M, \varepsilon = 2^{-5}$ and $\delta = 0.5 * \epsilon$ also, for second graph (b) $\varepsilon = 2^{-11}, N = 128$, and, $\delta = 0.5 * \epsilon$

underlying interval. As the size of the delay parameter increases, the thickness of the layer increases.

Chapter 5

Conclusions and Recommendations

5.1 Conclusions

In this study, we present an exponentially fitted three-term finite difference scheme developed for the numerical solution of boundary value problems of the parameter uniform numerical method for solving singularly perturbed differential equations with a large delay of 2^{nd} order with mixed shifts, which have the boundary layers at the left or right end of the underlying interval. We solved test example problems for the various values of N with $\delta = 0.5^* \epsilon$ presented. The computational results in terms of maximum absolute errors and the numerical rates of convergence demonstrate the efficiency of the method. We easily observe from Tables 1, 2, 3, and 4 that the maximum absolute errors decrease with the quadratic convergence rate as the step size $h = 1/N$ tends to zero for all the values. The convergence analysis of the scheme has been derived, and it is found that the present method converges uniformly of order one with respect to the perturbation parameter ϵ . The numerical rate of convergence is also calculated, and it is found that the theoretical rate of convergence matches the numerical rate of convergence. We have implemented the present method on four linear examples with a left- and right-end boundary layer by taking different values of ϵ . Numerical results are presented in tables. From the results, it can be observed that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution. To show the layer behavior of the solution, the numerical solution has been plotted for Example 4. It has been observed that the thickness of the boundary layer reduces to zero as the perturbation parameter ϵ tends to zero. On the basis of the numerical results of a variety of examples, it is concluded that the present method offers a significant advantage over the parameter uniform numerical method for solving singularly

perturbed large-delay convection-diffusion equations.

5.2 Recommendation

In this thesis, FDM are constructing for solving singularly perturbed differential equations . Hence, the scheme proposed in this thesis can also be extended to solve singularly perturbed differential equation involving large delay .

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