

**SALALE UNIVERSITY**



**COLLEGE OF NATURAL SCIENCE  
DEPARTMENT OF MATHEMATICS**

**PARAMETER UNIFORM DISCRETIZATION OF  
SINGULARLY PERTURBED PARABOLIC DIFFERENTIAL  
EQUATIONS WITH DISCONTINUOUS COEFFICIENTS  
AND LARGE DELAY**

**MSc THESIS**

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**Parameter Uniform Discretization of Singularly Perturbed  
Parabolic Differential Equations with Discontinuous Coefficients  
and Large Delay**

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of Master's in Mathematics (Numerical Analysis)

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**M.Sc. THESIS APPROVAL SHEET**

We, the undersigned, member of the board of examiners of the final open defense by Kenna Diribi Abebe have read evaluated his/her thesis entitled "Parameter Uniform Discretization of Singularly Perturbed Parabolic Differential Equations with Discontinuous Coefficients and Large Delay" and examined the candidate. This is therefore to certify that the thesis has been accepted in partial fulfillment of the requirements for the degree Master of Science in Mathematics(Numerical Analysis).

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# Declaration

I here declare that the work which is being in this thesis "Parameter Uniform Discretization for Singularly perturbed Parabolic differential Equation with Discontinuiuos coefficient and Large Delay" in partial fulfillment of the requirement for the degree of masters of science in mathematics ,submitted to Salale University,department of mathematics is my original work and it has not been submitted for the award of any academic degree or the like in any other institution or university,and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Name:Kenna Diribi

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Date:.....

The work has been done under supervision of:

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# List of Abbreviations and Symbols

## **Abbreviations**

PDE	:	Partial Differential Equation
SPP	:	Singularly Perturbed Problem
SPDDE	:	Singularly Perturbed Delay Differential Equation
NSFDM	:	Non-standard Finite Difference Method

## **Symbols**

$\varepsilon$	:	Singular Perturbation Parameter
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# Abstract

This study effort to introduce numerical method for parameter uniform discretization for singularly perturbed parabolic differential equations with discontinuous coefficient and large delay that implicate governing equation with interior layers over domain. Due to the presence of interior layers appearing in the solutions, the classical methods are unable to provide an efficient numerical solution unless they are applied with very fine meshes inside the regions. The comparison errors also given in tabular form and the work has been illustrated through examples for different values of small parameter  $\varepsilon$ . For the discretization of time derivative, we used the implicit Euler method on a uniform mesh and for the spatial discretization. Finally, maximum absolute errors for each examples was shown both by tables and graphs with different perturbation parameters and mesh sizes which shows betterment of the present method.

# Chapter 1

## Introduction

### 1.1 Background of the study

Many scientific problems describe the relation between causes and the effects. Van Dyke (1994) studied of this relation in the subject of the perturbations theory has long history. Despite this long history, the topic is still in a state of irrepressible development and is termed as the theory of singular perturbation problems (SPPs). It is well known that singularly perturbed problem often have very thin boundary and internal layers where the solutions varies rapidly change, where as away from the layer, the solutions behaves regularly and varies slowly, so that the numerical treatment of singularly perturbed problems faces major difficulties Rai and Sharma (2015), Gadisa et al. (2018), Das and Natesan (2018).

Singularly perturbed partial differential equations often occur to the nature of certain physical phenomena such as small viscosity in the navier stokes equations. In a biology many singular perturbed diffusive models have been established to describe dynamics of some biological systems. In other words, past events explicitly affect future results. For this reason functional differential equations are more realistic and frequently appear applications in economics, the future behavior depends implicitly on the past. The convergence of the numerical approximations generated by standard numerical methods applied to such problems depends adversely on the singularly perturbation parameter. Singularly perturbed partial differential equations are the equations in which unknown function and its derivatives are evaluated at the same instance while in a singularly perturbed delay partial differential

equations the past history it also taken into consideration while evaluating the unknown functions and its derivatives. The differential equations in which higher order derivative is multiplied by small positive number named singular perturbation parameter and containing a shifting parameter are termed singularly perturbed delay differential equations. Classical numerical methods turn out to be inapplicable for singular perturbations problems, this happens because errors of the numerical solution depend on the perturbation parameter and become small only when the effective mesh-size in the layer is much less than the value of the parameter. Singh et al. (2018), Chakravarthy et al. (2015), Clavero and Gracia (2013).

The boundary layer occurs when a term containing the highest order derivative is multiplied by a singular perturbation parameter and the interior layer arises when there is discontinuity in the given data. The numerical scheme is presented based on the fitted operator method defined over a uniform mesh. In deed, it is exact finite difference scheme for the error functions associated with the discontinuity in the initial condition. In recent years, both mathematicians and physicists have devoted remarkable effort to the study of numerical solutions of singularly perturbed delay differential equations with discontinuous data and large delay parameter Daba and Duressa (2022), Kumar and Kumar (2014), Duressa et al. (2023), Kaushik et al. (2010) developed different numerical methods based on fitting technique for solving singularly perturbed delay parabolic partial difference equations and elucidated the effect of the shift arguments on the boundary layer behavior of the solution. Various scholars have been developing  $\varepsilon$ -uniform numerical methods for singularly perturbed delay ordinary and partial differential equations with shift parameters on the solution behavior. Wang (2011), Wang et al. (2017), Rai and Yadav (2020), Peiraviminaei and Ghoreishi (2014), Bashier and Patidar (2011b), Miller et al. (1996), Andargie and Reddy (2013a), Andargie and Reddy (2013b), Das and Natesan (2013).

The effect of the parameters on the boundary layer solutions are examined and presented in figures. The convergence of the constructed scheme is proposed of implicit Euler method for time discretization and non-standard finite difference method for the spatial discretization. For the theoretical analysis the global error is decomposed in two parts; first due to the time discretization and the second part due to the spatial discretization of semi-discrete problem obtained after the time discretization. Therefore, the aim of this thesis is to establish the stability and convergence of the proposed method.

## 1.2 Statement of the Problem

In the past and recent years of the studies much interest have been given to solve singular perturbed parabolic partial differential equations with discontinuous coefficients. Due to the presence of interior layers appearing solutions of processes in various application fields. The result of research solution to these problems has an important role to capture the behavior of the physical phenomena of the problems. Many of the published solved the singularly perturbed parabolic differential equation problem Kaushik and Sharma (2020), Van Dyke (1994), Kaushik and Sharma (2021), Kumar and Kumar (2014), Duressa and Reddy (2015), Gowrisankar and Natesan (2013) and references therein.

However, the methods suggested above for singularly parabolic differential equations, parameter uniform method and numerical methods on a uniform mesh fail to approximate the singularly perturbed differential equations. They require an unacceptably large number of mesh points to sustain the approximation because the mesh width depends on the perturbation parameter.

A few works studying with the parameter uniform numerical schemes for singularly perturbed parabolic differential equations exists Daba and Duressa (2022), Rai and Sharma (2015), Kadalbajoo and Patidar, 2002, Phaneendra et al., 2014, Miller et al. (1996) Nevertheless, from the thesis literature and to the best of the researchers' knowledge, the solution methodologies to solve the singularly perturbed differential equations is at infant stage and it needs a lot of studies. This motivates the research to develop and analyze a parameter uniform numerical schemes for solving the singularly perturbed differential equations.

## **1.3 Objectives of the Study**

### **1.3.1 General Objective of the Study**

The main objective of this thesis is to construct parameter-uniform discretization numerical scheme which are more accurate, stable and provides better result for reasonable mesh size for solving singularly perturbed parabolic differential equations with discontinuous and large delay.

### **1.3.2 Specific Objectives of the Study**

The specific objectives are:

1. To develop numerical scheme to solve singular perturbed parabolic delay partial differential problems with discontinuous coefficient and large delay.
2. To analyze the convergence of the proposed numerical scheme.
3. To establish the stability of the proposed method.

## 1.4 Significance of the Study

The future of behaviors of these problems described by their solutions. However, it is not easy to solve singularly perturbed parabolic differential equation analytically due to presence of boundary layer in the solution. The primary contribution of this thesis is the development of stable and parameter-uniform numerical scheme for solving singularly parabolic differential equation. The development of numerical scheme can be used to solving the problem under consideration. The result obtained in this thesis may serve as a reference material for scholars who works on this area.

1. Give an idea about the application of numerical methods in different parts of mathematics.
2. Help the graduate students to acquire scientific procedures.

## 1.5 Delimitation of the Study

This study delimited to solve singularly perturbed parabolic differential equations with discontinuous coefficients and large delay of the form:

$$L_\varepsilon U(x, t) = \varepsilon \frac{\partial^2 U(x, t)}{\partial x^2} + b(x) \frac{\partial U(x, t)}{\partial x} - \delta(x)U(x - 1, t) - \eta(x)U(x, t) - \frac{\partial U(x, t)}{\partial t} = \rho(x, t) \quad (1.5.1)$$

where  $\varepsilon$  is a perturbation parameter that satisfies  $0 < \varepsilon \ll 1$ ,  $\delta(x)$  and  $\eta(x)$  are sufficiently smooth functions.

## 1.6 Definitions Terms

**Definition 1.1** Singular perturbations problem in which the solutions becomes discontinuous in a narrow regions are solved using a perturbation technique called boundary layer. A singular perturbed problem is a problem containing a small positive parameter that cannot be approximated by setting the parameter value to zero.

**Definition 1.2** Temporal semi-discretization is a mathematical technique applied to transient problems that occur in the fields of applied physics and engineering. Transient



problems are often solved by conducting simulations using computer aided engineering packages, which required discrete the equations in both space and time.

Definition 1.3 Non-standard finite difference method is any discrete representation of differential equations that is constructed their rules:

- Discrete representation for derivatives must in general have nontrivial denominators functions.
- The order of the discrete derivatives should equal to the orders of the corresponding derivatives.

# Chapter 2

## Review of Related Literature

A singularly perturbed parabolic differential equation (SPPDE) is a non-linear partial differential equations in which the highest order derivative is multiplied by a small positive parameter. The analysis of higher-order methods for singularly perturbed parabolic partial differential equations with discontinuous data and degenerating convective terms has seen little development and lacks due attention.

Various researchers have been proposed different numerical methods to solve this problem. For instance Daba and Duressa (2022) developed computational method for singularly perturbed parabolic differential equations with discontinuous coefficients and large delay. The formulated method comprises the implicit and the cubic in compression methods for time and spatial dimensions. The result depict that the present method is more accurate than some some methods existing in the literature. The layer behavior of the solutions is presented using graphs and table. Kaushik and Sharma (2020) presented an adaptive difference scheme for the problem is bi-singular, where a classical singularity is twisted together with the singular nature of the problem. The proposed numerical method has been analyzed for stability and and convergence. The analysis of special methods for singularly perturbed parabolic partial functional differential equation with discontinuous data and degenerating convective terms has developed in the literature. A numerical scheme is presented based on the fitted operator method defined over a uniform mesh. Very recently Gobena and Duressa (2021) developed a parameter uniform numerical method for singularly perturbed

delay parabolic reaction-diffusion equations with integral boundary condition. It known as that standard numerical methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter is small. A parameter-uniform numerical method is constructed via non-standard finite difference method for the spatial direction and the back ward Euler method for the resulting system of initial value problems in temporal direction is used. The proposed method is shown to be parameter-uniformly convergent. Duressa et al. (2023) proposed on the solution methodology of singularly perturbed differential difference equations and developed new robust methods for solving the problems under consideration. This survey limits its coverage to singularly perturbation equations arising in the modeling of neuronal activity. In this singularly perturbed ordinary differential equations with small or large negative shifts and singularly perturbed delay partial differential -difference equations of the mixed type. Kaushik and Sharma (2021) presented s hybrid finite difference scheme to solve a singularly perturbed parabolic functional differential equation with discontinuous coefficient and source. The simultaneous presence of deviating argument with a discontinuous source and coefficient makes the problem stiff. The solution of the problem exhibits turning point behavior across discontinuity as  $\epsilon$  tends to zero. The hybrid scheme presented is a composition of central difference scheme and a midpoint upwind scheme on a specially generated mesh. At the same time, an implicit finite difference method is used to discretize the time variable.

In general, this thesis indicates that the development of the solution thesis to solve problem (4.1.1) has received very little attention from the research community. Developing a computational methods for solving statement problem equations (4.1.1) are still at the preliminarily stage and it needs a lot of investigations.

# Chapter 3

## Research Design and Methodology

### 3.1 Research Design

This study will employ an intensive up to date document research and numerical experimentation.

### 3.2 Mathematical Procedures

Mathematical procedures are the design of the anatomy of the frame work we follow in this thesis. The great of significance of materials for the relating to work will be collected by means of documentary research.

Therefore, The thesis will follow the following mathematical procedures:

1. Defining problem for the study,
2. Analyzing the properties of continuous solution,
3. Developing numerical scheme for the problem.
4. Developing an algorithm and writing a MATLAB code for the presented schemes.
5. Validating the scheme using numerical examples,
6. Discussing and providing conclusion.
7. Presenting the results using appropriate presentationusing tables,graphs.

# Chapter 4

## Description of the numerical methods, Discussion and Results

### 4.1 Description of the numerical methods

Singular perturbed partial differential equations (SPPDE) are perhaps arises in variety of mathematical and physical problems. However, this study is to singularly perturbed parabolic differential equations with discontinuous coefficients and large delay below the given form domain  $D = \omega^- U \omega^+ = (0, 1) \times (0, T) \cup (1, 2) \times (0, T)$ , of the form:

$$L_\varepsilon U(x, t) = \begin{cases} \varepsilon \frac{\partial^2 U(x, t)}{\partial x^2} + b(x) \frac{\partial U(x, t)}{\partial x} - \delta(x)U(x-1, t) - \eta(x)U(x, t) - \frac{\partial U(x, t)}{\partial t} = \rho(x, t), \\ U(x, t) = \zeta_0(x), x \in [0, 2] \\ U(x, t) = \zeta_1(x, t) \in [-1, 0] \times [0, T], \\ U(2, t) = \zeta_2(2, t), t \in [0, 2] \end{cases} \quad (4.1.1)$$

where  $0 < \varepsilon \ll 1$  is a perturbation parameter,  $x$  is distance,  $t$  is time,  $\delta(x)$  and  $\eta(x)$  are sufficiently smooth functions such that  $0 < \varpi \leq b(x)$ ,  $\delta(x) < 0$ ,  $\eta(x) > 0$  and  $\delta(x) + \eta(x) \geq 0$ ,  $\forall x \in [0, 2]$ .

we assume that:

$$b(x) = \begin{cases} b_1(x), \text{ if } 0 \leq x \leq 1 \\ b_2(x), \text{ if } 1 < x \leq 2 \end{cases} \quad (4.1.2)$$

$$\rho(x, t) = \begin{cases} \rho_1(x), \text{ if } (x, t) \in S^- \\ \rho_2(x), \text{ if } (x, t) \in S^+ \end{cases} \quad (4.1.3)$$

$$\zeta_1 < b_1 < \zeta_2 < 0, \zeta_2 > b_2 > \zeta_1 > 0 \quad (4.1.4)$$

$||U|| \leq C, ||\rho|| \leq C$ . where  $\zeta^* = \min(\zeta_1, \zeta_2)$  and  $\zeta_* = \max(\zeta_1^*, \zeta_2^*)$ .

The solution of equation(4.1.1) satisfies  $\hat{u} = 0$  and  $\hat{u}_x = 0$  at  $x = 1$ . Here,  $\hat{u}$  denotes the jump of  $u$  defined at the point discontinuity  $x = 1$  as  $\hat{u}(1, t) = \hat{u}(1^+, t) - \hat{u}(1^-, t)$  where  $u(1^\pm, t) = \lim_{x \rightarrow 1^\pm} u(x, t)$ . The functions  $\zeta_0, \zeta_1$  and  $\zeta_2$  are holders continues and compatibility conditions, hold at the corners of the domain.

The solution of equation(4.1.1) exhibits a strong interior layer at  $x = 1$  and weak boundary layer closer  $x = 0$  and  $x = 2$ . The preparation of  $\epsilon$  discontinuity and large delay in equation (4.1.1) make the problem tiresome to solve theoretically. This section consists of the properties of continuous solution of the problem under consideration formulation of the proposed method, uniform stability analysis and its convergence analysis.

Now consider the singular perturbed parabolic differential equations with discontinuous coefficients and large delay of the form given equation(4.1.1) and(4.1.2).

## 4.2 Properties of continuous solution

The differential operator for equation(4.1.1) is given by:

$$L_\epsilon U(x, t) = \begin{cases} L_\epsilon U(x, t) = \psi(x, t) & (4.2.1) \\ \left\{ \begin{array}{l} \epsilon \frac{\partial^2 U(x, t)}{\partial x^2} + b(x) \frac{\partial U(x, t)}{\partial x} - \eta(x)U(x, t) - \frac{\partial U(x, t)}{\partial t}, (x, t) \in S^- \\ \epsilon \frac{\partial^2 U(x, t)}{\partial x^2} + b(x) \frac{\partial U(x, t)}{\partial x} - \delta(x)U(x-1, t) - \eta(x)U(x, t) - \frac{\partial U(x, t)}{\partial t}, \\ (x, t) \in S^+ \\ U(x, t) = \zeta_0(x), x \in [0, 2] \\ U(x, t) = \zeta_1(x, t) \in [-1, 0] \times [0, T], \\ U(1^-, t) = U(1^+, t), U'(1^-, t) = U'(1^+, t) \\ U(2, t) = \zeta_2(2, t), t \in [0, 2] \end{array} \right. & (4.2.2) \end{cases}$$

$$\psi(x, t) = \begin{cases} \rho(x, t) + \delta(x)U(x-1, t), (x, t) \in S^- \\ \rho(x, t), (x, t) \in S^+ \end{cases} \quad (4.2.3)$$

The differential equation in equation(4.2.2) satisfies the following lemma 4.1.1

**Lemma 4.2.1.** (*Minimum principle*). If  $\phi \in A(0, 0)(D) \cap A^{1,0} \bar{D} \cap A^{2,0}(S^-) \cup (S^+)$  such that  $\phi(0, t) \geq 0, \phi(x, 0) \geq 0, \phi(2, t) \geq 0, [\phi'](1, t) = \phi'(1^+, t) - \phi'(1^-, t) \leq 0$  and  $L_\epsilon \phi(x, t) \leq 0$ , for all  $x \in$

$D$ , then  $\phi(x, t) \geq 0$ , for all  $(x, t) \in \bar{D}$ , Daba and Duressa (2022).

*Proof.* Assume that  $(x, t) \in \bar{D}$  such that  $\phi(x, t) = \min_{(x,t) \in D} \phi(x, t)$  and suppose  $\phi(x, t) < 0$ , then it follows that  $(x, t) \notin \Gamma$ . Consequently we have

$$\frac{\partial \phi(x, t)}{\partial x} = \frac{\partial \phi(x, t)}{\partial t} = 0 \text{ and } \frac{\partial^2 \phi(x, t)}{\partial x^2} > 0. \text{ To show } L_\epsilon^N > 0, \text{ we consider the following cases.}$$

Case 1. If  $(x, t) \in S^-$

$$L_\epsilon \phi(x, t) = \epsilon \frac{\partial^2 \phi(x, t)}{\partial x^2} + b_1(x) \frac{\partial \phi(x, t)}{\partial x} - \eta(x) \phi(x, t) - \frac{\partial \phi(x, t)}{\partial t} > 0$$

Case 2. If  $(x, t) \in S^+$

$$\begin{aligned} L_\epsilon \phi(x, t) &= \epsilon \frac{\partial^2 \phi(x, t)}{\partial x^2} + b_2(x) \frac{\partial \phi(x, t)}{\partial x} - \delta(x) \phi(x-1, t) - \eta(x) \phi(x, t) - \frac{\partial \phi(x, t)}{\partial t} \\ &= \epsilon \frac{\partial^2 \phi(x, t)}{\partial x^2} + b_2 \frac{\partial \phi(x, t)}{\partial x} - \delta(x) (\phi(x-1, t) - \phi(x, t) - \delta(x) \phi(x, t) - \eta(x) \phi(x, t) - \frac{\partial \phi(x, t)}{\partial t}) \\ &= \epsilon \frac{\partial^2 \phi(x, t)}{\partial x^2} + b_2(x) \frac{\partial \phi(x, t)}{\partial x} - \delta(x) (\phi(x-1, t)) - \phi(x, t) \phi(x, t) (\delta(x) + \eta(x) - \frac{\partial \phi(x, t)}{\partial t}) > 0 \end{aligned}$$

From case 1 and case 2, we obtain  $L_\epsilon \phi(x, t) \geq 0$ , that contradicts assumption done above  $L_\epsilon \phi(x, t) \leq 0$ , for all  $(x, t) \in D$ .

If  $x = 1$ , we have  $\phi'(1^-, t) \leq 0, \phi'(1^+, t) \geq 0$ , if and only if  $\phi'(1, t) \geq 0$ . This gives  $\phi(1, t) = \phi(1^+, t) = \phi(1^-, t)$  to the assumption that  $\phi'(1, t) \leq 0$ , we have already proved that  $\phi(x, t) \geq 0$ , for all  $(x, t) \in (S^- \cup S^+)$ . Therefore,  $\phi(x, t) \geq 0$  for all  $(x, t) \in \bar{D}$

**Lemma 4.2.2.** Let  $U(x, t)$  be the solution of (4.1.1),  $|U(x, t)|_\infty \cdot \bar{D} \leq \frac{1}{\vartheta} \|\phi\|$ , Kumar and Sharma (2008).

*Proof:* Consider

$$\sigma^\pm = \begin{cases} \|U\|_\infty, \Gamma + \frac{x}{\Omega} \|\psi\|_\infty \cdot \bar{D} \pm U, \text{ if } x \leq 1. \\ \|U\|_\infty, \Gamma + \frac{2-x}{\Omega} \|\psi\|_\infty \cdot \bar{D} \pm U, \text{ if } x \geq 1 \end{cases} \quad (4.2.4)$$

For  $(x, t) \in S^-$  follows that  $L_\epsilon \sigma^\pm(x, t) = \pm L_\epsilon U + \frac{b_1(x)}{\Omega} \|\psi\| - \eta(x) \|U\| - U_t \frac{x}{\Omega} \|\psi\| \leq 0$

since  $b_1(x)$  and  $\eta(x) > 0$ . Similarly for  $(x, t) \in S^+$ , it follows that  $L_\epsilon \sigma^\pm(x, t) = \pm L_\epsilon U + \frac{b_2(x)}{\Omega} \|\psi\| - \delta(x) \|U - 1\| - \eta(x) \|U\| - U_t(x) \frac{x}{\Omega} \leq 0$ , since  $b_2(x), -\delta(x) \leq 0$  and  $\eta(x) \leq 0$ , it is verify  $L_\epsilon \sigma^\pm(x, t) \leq 0$ .

The required result (4.1.4) follows from lemma 4.1.1

## 4.3 Numerical formulation of the proposed method

### 4.3.1 Temporal semi-discretization

The temporal semi-discretization is obtained by using implicit Euler method on the time variable of equation (4.1.1) and (4.1.2) with uniform step size  $\Delta t$  such that  $D_\Delta^N t = [t_j =$

$j\Delta t, \Delta t = \frac{T}{N}]$ results.

$$\begin{aligned} L_\epsilon^N U_x^{j+1} &= \psi_x^{j+1}. \\ U(x, 0) &= \zeta_0^{j+1}, x \in [0, 2] \\ U^{j+1}(x) &= \zeta_1^{j+1}(x), 0 \leq x \leq 1 \\ U_2^{j+1} &= \zeta_2, j = 0(1)N - 1 \end{aligned} \tag{4.3.1}$$

$$L_\epsilon^N U^{j+1}(x) = \epsilon \frac{\partial^2 U^{j+1}(x)}{\partial x^2} + b(x) \frac{\partial U^{j+1}}{\partial x} - c(x)U^{j+1}(x) - \delta(x)U^{j+1}_\epsilon(x-1) = \psi^{j+1}(x) - \frac{U^j(x)}{\Delta t} \tag{4.3.2}$$

Where,

$$L_\epsilon^N U_x^{j+1} = \begin{cases} \epsilon \frac{\partial^2 U^{j+1}(x)}{\partial x^2} + b(x) \frac{\partial U^{j+1}}{\partial x} - c(x)U^{j+1}(x), \text{ if } x \in [0, 1], j = 0 \dots N - 1. \\ \epsilon \frac{\partial^2 U^{j+1}(x)}{\partial x^2} + b(x) \frac{\partial U^{j+1}}{\partial x} - \delta(x)U^{j+1}_\epsilon(x-1) - c(x)U^{j+1}(x), 1 < x \leq 2. \end{cases} \tag{4.3.3}$$

$$\psi_x^{j+1} = \begin{cases} \Xi(x, t) + \delta(x)U^{j+1}_\epsilon(x-1), 0 \leq x \leq 1 \\ \Xi(x, t), 1 < x \leq 2, \end{cases} \tag{4.3.4}$$

with the following interval and boundary conditions:

$$\begin{aligned} U_0^{j+1} &= \zeta_0^{j+1}, x = 0, 0 \leq j \leq N - 1 \\ U_1^{j+1} &= \zeta_1^{j+1}, x = 1, 0 \leq j \leq N - 1 \\ U_2^{j+1} &= \zeta_2^{j+1}, x = 2, 0 \leq j \leq N - 1 \\ U_x^{j+1} &= \zeta_x^{j+1}, x \in (0, 2), j = 0, 1 \dots n \end{aligned} \tag{4.3.5}$$

where  $U_x^{j+1}$  is the solution of semi discretized problem (4.2.3) and (4.3.1) at  $(j+1)^{th}$ .

**Lemma 4.3.1.** (Semi-discrete stability estimate). Let  $\hat{U}_x^{j+1}$  be the solution of  $L_\epsilon^N U_x^{j+1} = \psi_x^{j+1}$  then  $|\hat{U}_x^{j+1}| \leq \max \left( |\zeta_0^{j+1}| \cdot \frac{\|\psi\|}{|\gamma|} \cdot |\zeta_2^{j+1}| \right)$ , for  $x \in [0, 2]$ , Kaushik et al. (2010).

proof: Defining the barrier functions as  $\gamma_x^{\pm j+1} = \max \left( |\zeta_0^{j+1}| \cdot \frac{\|\psi\|}{|\gamma|} \cdot |\zeta_2^{j+1}| \right) \pm \hat{U}_2^{j+1}$

Now,  $\gamma_x^{\pm j+1} \geq 0, \gamma_x^{\pm j+1} \leq 0$  and we have

Case:1,  $0 \leq x \leq 1$

$$L_\epsilon^N \gamma_x^{\pm j+1} = -c(x) \max \left( |\zeta_0^{j+1}| \cdot \frac{\|\psi\|}{|\gamma|} \cdot |\zeta_2^{j+1}| \right) \pm L_\epsilon^N \hat{U}_x^{j+1} \leq 0, \text{ since } -c(x) \leq 0.$$

Case:2,  $1 < x \leq 2$

$$L_\epsilon^N \gamma_x^{\pm j+1} = -c(x) \max \left( |\zeta_0^{j+1}| \cdot \frac{\|\psi\|}{|\gamma|} \cdot |\zeta_2^{j+1}| \right) \pm L_\epsilon^N \hat{U}_x^{j+1} \leq 0, \text{ since } \delta(x) + \eta(x) \geq 0.$$

Therefore, using Lemma 4.1.1, we obtain  $|\hat{U}_x^{j+1}| \leq \max \left( |\zeta_0^{j+1}| \cdot \frac{\|\psi\|}{|\gamma|} \cdot |\zeta_2^{j+1}| \right)$ , for  $x \in [0, 2]$ .



**Lemma 4.3.2.** Let  $\bar{e}_{j+1} = \bar{U}_x^{j+1} - \bar{U}_x^{j+1}$  be the local truncation at  $(K+1)^{th}$  time step. Then  $\|\bar{e}_{j+1}\|_\infty \leq K(\Delta t)^2$  for some constant  $K$ , Kaushik and Sharma (2021).

Moreover, if  $E_j = u^j(x) - U^j(x)$ , denotes the global error in the time direction. Then it follows that

$$\|E_{j+1}\|_\infty = \left\| \sum_{i=0}^j \bar{e}_i \right\|_\infty \leq \|\bar{e}_1\|_\infty + \|\bar{e}_2\|_\infty + \|\bar{e}_3\|_\infty + \dots \|\bar{e}_j\|_\infty \leq K \Delta t$$

. This in turns the uniform convergence of time semi-discretization process. Next, we obtained a priori estimate on the solution of semi-discretized of (4.3.2)

## 4.4 Spatial semi-discretization

In this section, the spatial semi-discretization under consideration via non-standard finite difference scheme for space derivative discretization along with implicit Euler method for time discretization.

On the space domain  $(0, 2)$  the equidistant meshes with uniform mesh length  $h$  such that  $x_i = ih, i = 1(1)N - 1, X_N = 2, h = \frac{2}{N}$ , where  $h$  is step size and  $N$  is the number of mesh points in the space direction. We use notation  $U(x_i, t_i) = U_i^j$ . Both the space and time discretization are on a uniform mesh, we rewrite equation (4.1.1) in the form

$$\varepsilon \frac{\partial^2 U^{j+1}(x)}{\partial x^2} + b(x) \frac{\partial U^{j+1}}{\partial x} - c(x) U^{j+1}(x) - \delta(x) U_{(x-1)}^{j+1} = \psi^{j+1}(x) - \frac{U^j(x)}{\Delta t} \quad (4.4.1)$$

From the theory of non-standard finite difference method the domain of  $h$  for stability in non-standard is larger of standard method and the denominator of functions are chosen in appropriate from non-standard methods produce better results, we can discretization (4.4.1) both in space and time directions concurrently to get the discrete problem in the form:

$$L_\varepsilon^N U_i^{j+1} = \varepsilon \left( \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{\omega^2} \right) - b_i \left( \frac{U_{i+1}^{j+1} - U_i^{j+1}}{h} \right) - c_i U_i^{j+1} = \psi_i^{j+1} + \delta_i U_{i-1}^{j+1} - \frac{U_i^j}{\Delta t}, \text{ for } i = 0(1)N \quad (4.4.2)$$

From (4.4.2), we take the homogeneous form of the constant coefficients as

$$\varepsilon \left( \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{\omega^2} \right) - b_i \left( \frac{U_{i+1}^{j+1} - U_i^{j+1}}{h} \right) - c_i U_i^{j+1} = 0, \text{ where } c_i = \eta(x) + \frac{1}{\Delta t} \quad (4.4.3)$$

(4.4.3) has two linearly independent analytical solutions, namely,  $\exp(\lambda_{1x})$  and  $\exp(\lambda_{2x})$ , where

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2 - 4\epsilon c}}{2\epsilon} \quad (4.4.4)$$

Following Mickne's we constant the second-order differential equation for (4.4.2) as follows

$$\begin{vmatrix} U_{i-1} & U_{1,i-1} & U_{2,i-1} \\ U_i & U_{1,i} & U_{2,i} \\ U_{i+1} & U_{1,i+1} & U_{2,i+1} \end{vmatrix} = \begin{vmatrix} U_{i-1} & \exp(\lambda_1, x_{i-1}) & \exp(\lambda_2, x_{i-1}) \\ U_i & \exp(\lambda_1, x_i) & \exp(\lambda_2, x_{i-1}) \\ U_{i+1} & \exp(\lambda_1, x_{i+1}) & \exp(\lambda_2, x_{i+1}) \end{vmatrix}$$

Simplifying the determinant, we get:

$$\exp\left(\frac{bh}{2\epsilon}\right)U_{i-1} - 2\cosh\left(h\frac{\sqrt{b^2 + 4\epsilon c}}{2\epsilon}\right)U_{i-1} + \exp\left(\frac{bh}{2\epsilon}\right)U_{i+1} = 0. \quad (4.4.5)$$

which is the exact scheme for (4.4.2) and (4.4.1) has the same general solutions.

$$U_i = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i) \quad (4.4.6)$$

The extraction of the denominator function from (4.4.5) is not straight forward. However, the fact the layers behaviors of the solution of the problem (4.3.4) and that of the problem (4.4.2) in the case when  $c=0$ . thus, for the latter case, that is  $c=0$ , we have the exact scheme of the form:

$$\exp\left(\frac{bh}{2\epsilon}\right)U_{i-1} + 2\cosh\left(\frac{bh}{2\epsilon}\right)U_i - \exp\left(\frac{bh}{2\epsilon}\right)U_{i+1} = 0. \quad (4.4.7)$$

multiplying both sides of (4.4.7) by  $\left(\frac{bh}{2\epsilon}\right)$ , and incorporating the term  $(U_{i+1}^{j+1} - U_i^{j+1})$  we obtain,

$$U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1} + (U_{i-1}^{j+1} - U_i^{j+1})\left(\exp\left(\frac{bh}{\epsilon}\right) - 1\right). \quad (4.4.8)$$

(4.3.4) can be written as

$$\epsilon \left( \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{\omega^2} \right) - b_i \left( \frac{U_{i+1}^{j+1} - U_i^{j+1}}{h} \right) = 0, \quad (4.4.9)$$

where the denominator functions is given by:

$$\omega^2 = \frac{h\epsilon}{bh} \left( \exp\left(\frac{bh}{\epsilon}\right) - 1 \right) \quad (4.4.10)$$

Thus ,we get the spatial semi-discretization of Eq.(4.3.2)

$$L_\epsilon^N U_i^{j+1} = \epsilon \left( \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{\omega^2} \right) - b_i \left( \frac{U_{i+1}^{j+1} - U_i^{j+1}}{h} \right) - c_i U_i^{j+1} = \psi_i^{j+1} + \delta_i U_{i-1}^{j+1} - \frac{U_i^j}{\Delta t}, \text{ where } ci = \eta_i - \frac{U_i^j}{\Delta t}$$

This can be written as the recurrence relative of the form:

$$AiU_{i-1}^{j+1} + FiU_i^{j+1} + GiU_{i+1}^{j+1} = Hi, i = 1, 2, \dots, N-1 \quad (4.4.11)$$

where  $Ai = \frac{\epsilon\epsilon}{\omega^2} - \delta_i$

$$Fi = \frac{-2\epsilon}{\omega^2} + \frac{bi}{h} - \eta_i - \frac{1}{\Delta t}, \quad Gi = \frac{\epsilon}{\omega^2} - \frac{bi}{h}, \quad Hi = \psi_i^{j+1} - \frac{U_i^j}{\Delta t}$$

**Lemma 4.4.1.** (Semi-discretization minimum principle) Let  $\Xi_i^{j+1}$  be smooth functions satisfies

$\Xi_i^{j+1} \geq 0$  for  $x = 0, 2$ .  $[\Xi']^{j+1}(1) = \Xi'(1)^{j+1} - \Xi'(1^-)^{j+1} \leq 0$  and  $L_\epsilon^N \leq 0$ , for all  $x \in (0, 2)$ . then  $\Xi_x^{j+1} \geq 0$ , for all  $x \in [0, 2]$ , Kaushik and Sharma (2021).

*Proof.* Assume  $x, t_{j+1} \in (x, t_{j+1}) : x \in [0, 2]$  be such that  $\Xi(x, t_{j+1}) = \min_{x \in [0, 2]} \Xi_x^{j+1}$  and suppose that  $\Xi_x^{j+1} < 0$ , as a result, we have  $\frac{\partial \Xi_x^{j+1}}{\partial x} = 0, \frac{\partial^2 \Xi_x^{j+1}}{\partial x^2} > 0$  and  $(x, t_{j+1}) \notin \Gamma$

Since  $\Xi(x, t_{j+1}^{j+1}) \geq 0$ , for  $x = 0, 2$ .

To show that  $L_\epsilon^N \Xi_x^{j+1} > 0$ , we consider the following cases

Case 1. If  $(x, t_{j+1}) \in S^-$

$$L_\epsilon^N \Xi(x, t_{j+1}) = \epsilon \frac{\partial^2 \Xi(x, t_{j+1})}{\partial x^2} + b_1(x) \frac{\partial \Xi(x, t_{j+1})}{\partial x} - c(x) \Xi(x, t_{j+1}) > 0$$

Case 2. If  $(x, t_{j+1}) \in S^+$

$$\begin{aligned} L_\epsilon^N \Xi(x, t_{j+1}) &= \epsilon \frac{\partial^2 \Xi(x, t_{j+1})}{\partial x^2} + b_2(x) \frac{\partial \Xi(x, t_{j+1})}{\partial x} - \delta(x) (\Xi_{x-1}^{j+1} - \Xi_x^{j+1}) - \delta(x) (\Xi_x^{j+1} - c(x) \Xi_x^{j+1}) \\ &= \epsilon \frac{\partial^2 \Xi(x, t_{j+1})}{\partial x^2} + b_2(x) \frac{\partial \Xi(x, t_{j+1})}{\partial x} - \delta(x) (\Xi_{x-1}^{j+1} - \Xi_x^{j+1}) - \Xi_x^{j+1} (\delta(x) + c(x)) > 0 \end{aligned}$$

From case 1 and 2, we obtain  $L_\epsilon^N \Xi_x^{j+1} > 0$ , that contradicts the assumption done above

$L_\epsilon^N \Xi_x^{j+1} < 0$ , for all  $(x, t_{j+1}) \in \bar{D}$ . If  $x = 1$ , we have  $\Xi'(1^+, t_{j+1}) \geq 0, \Xi'(1^-, t_{j+1}) \leq 0$ , implying  $\Xi'(1^+, t_{j+1}) \geq 0$ . This gives  $\Xi(1, t_{j+1}) = \Xi(1^+, t_{j+1}) = \Xi(1^-, t_{j+1}) = 0$ , to the assumption that  $\Xi(x, t_{j+1}) \leq 0$ . We have already proved that  $\Xi(x, t_{j+1}) \geq 0$ , for  $(x, t_{j+1}) \in (S^- \cup S^+)$

Therefore,  $\Xi(x, t_{j+1}) \geq 0, \forall (x, t_{j+1}) \in \bar{D}$ .

## 4.5 Uniform convergence analysis

The following lemma states the maximum principle for the semi-discretization finite difference operator  $L_\epsilon^N$  (4.3.3).

**Lemma 4.5.1.** *(Semi – discrete maximum principle). Assume for any mesh function  $\gamma^N(x_0) \geq 0, \gamma^N(x_N) \geq 0$  and  $L_\epsilon^N \gamma_n^N \geq 0$ , for  $0 \leq n \leq N$ , then  $\gamma_n^N \geq 0$ , for  $0 \leq n \leq N$ . Bashier and Patidar (2011a)*

*proof: Assume there exists  $n^*$  be such that  $\gamma^N(x_{n^*}) = \min \gamma^N(x_n) < 0$ ,  $0 \leq n \leq N$ . From the hypothesis, we have  $n^* \neq N$ .*

*Thus, we have  $L_\epsilon^N \gamma^N(x_{n^*}) = \epsilon \left( \frac{\gamma_{n^*+1}^N - 2\gamma_{n^*}^N + \gamma_{n^*-1}^N}{\omega_n^2} \right) - b(x) \left( \frac{\gamma_{n^*}^N - \gamma_{n^*-1}^N}{h} \right) - c(x_{n^*})\gamma_{n^*}^N < 0$ , which contradicts the assumption and hence,  $\gamma_n^N \geq 0$  for  $0 \leq n \leq N$ .*

*This leads to the following discrete uniform stability result. It plays central role in proving the parameter-uniform convergence of our proposed fitted operator finite difference method.*

## 4.6 Uniform Stability analysis

A discrete maximum principle can also be established directly, without appealing to the properties of the elements in the system matrix. Analogously to the continuous case, this is usually proved by the method of contradiction as we now show.

**Lemma 4.6.1.** *(Discrete maximum principle). Assume that mesh function  $\gamma_i^{j+1}$  satisfies  $\gamma_t^{j+1} \geq 0$  and  $\gamma_N^{j+1} \geq 0$ . Then  $(L_\epsilon^N) \gamma_i^{j+1}$ , for all  $xi \in D$ , implies that  $\gamma_i^{j+1} \geq 0$  for all  $xi \in \bar{D}$ . Kumar and Kumari (2019)*

*proof: Assume that there exists a positive integer  $t$  such that  $\gamma_t^{j+1} = \min_{0 \leq i \leq N} \gamma_i^{j+1}$  and assume  $\gamma_t^{j+1} < 0$  then, we have  $\gamma_0^{j+1} \geq 0$  and  $\gamma_N^{j+1} \geq 0$ , therefore it follows from the hypothesis that  $t \notin (0, N)$ .*

*Also  $\gamma_t^{j+1} - \gamma_{t-1}^{j+1} \leq 0$  and  $\gamma_{t+1}^{j+1} - \gamma_t^{j+1} \geq 0$  and note that  $L_\epsilon^N \gamma_t^{j+1} < 0$ . Since  $\gamma_t^{j+1} < 0$  by assumption  $b(xi) > \varsigma > 0$ , which is contradiction to our assumption. Therefore, the assumption  $\gamma_t^{j+1} < 0$  is wrong. Hence,  $\gamma_t^{j+1} \geq 0$  we have chosen  $t$  to be fixed and arbitrary, so  $\gamma_t^{j+1} \geq 0$  for all  $xi \in \bar{D}$ . As an immediate application of the maximum and minimum principle, we have the following uniform stability estimate.*

## 4.7 Numerical Experiments of Examples and Results.

To validate the applicability of the proposed scheme two numerical examples are presented. As exact solutions of these problems are not available, the maximum point wise absolute errors are calculated by using double mesh principle Doolan et al. (1980)

$$E_{\varepsilon}^{N,M} = \max_{1 \leq i,j \leq N-1, M-1} \left| U_{i,j}^{N,M} - U_{i,j}^{2N,2M} \right|$$

where  $U_{i,j}^{N,M}$  and  $U_{i,j}^{2N,2M}$  are computed numerical solutions obtained on the mesh  $D^{N,M} = S_x^N \chi S_t^M$  and  $D^{2N,2M} = S_x^{2N} \chi S_t^{2M}$  respectively.

The  $\varepsilon$ -uniform maximum absolute errors ( $E_{\varepsilon}^{N,M}$ ), order of convergence ( $R_{\varepsilon}^{N,M}$ ) and  $\varepsilon$ -uniform order of convergence ( $R^{N,M}$ ) are calculated using

$$E^{N,M} = \max E_{\varepsilon}^{N,M}, R_{\varepsilon}^{N,M} = \log_2 \left( \frac{E_{\varepsilon}^{N,M}}{E_{\varepsilon}^{2N,2M}} \right), R^{N,M} = \log_2 \left( \frac{E^{N,M}}{E^{2N,2M}} \right), \text{ respectively.}$$

**Example 4.7.1.** Consider the following problem of the form in (4.1.1).

$$\varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + b(x) \frac{\partial u(x,t)}{\partial x} + 2u(x-1,t) - 5u(x,t) - \frac{\partial u(x,t)}{\partial t} = \rho(x,t)$$

where

$$b(x) = \begin{cases} -(4+x^2), & \text{if } 0 \leq x \leq 1 \\ (8-x^2), & \text{if } 1 < x \leq 2 \end{cases}$$

$$\rho(x,t) = \begin{cases} 4xt^2 \exp(-t), & \text{if } (x,t) \in [0,1] \times [0,2] \\ 4(2-x)t^2 \exp(-t), & \text{if } (x,t) \in (1,2] \times [0,2] \end{cases}$$

**Example 4.7.2.** Consider the following problem of the form in (4.1.1).

$$\varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + b(x) \frac{\partial u(x,t)}{\partial x} + u(x-1,t) - 3u(x,t) - \frac{\partial u(x,t)}{\partial t} = \rho(x,t).$$

$$U(x, t) = 0, \text{ on } [0, 2] \times (t = 0)$$

$$U(x, t) = 0, \text{ on } [-1, 0] \times [0, 2]$$

$$U(x, t) = 0, \text{ on } x = 2 \times [0, 2]$$

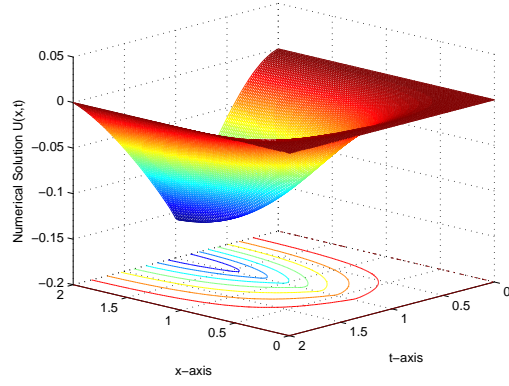
where

$$b(x) = \begin{cases} -(4 + x), & \text{if } 0 \leq x \leq 1 \\ (3 + x^2), & \text{if } 1 < x \leq 2 \end{cases}$$

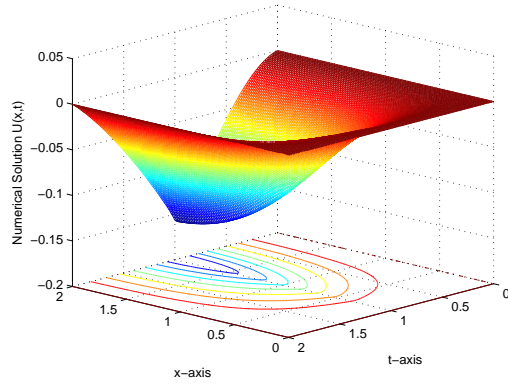
$$\rho(x, t) = \begin{cases} -1, & \text{if } (x, t) \in [0, 1] \times [0, 2] \\ 1, & \text{if } (x, t) \in (1, 2] \times [0, 2] \end{cases}$$

Table 4.1: Maximum absolute errors and order of convergence for Example (4.7.2) at different mesh point.

$\varepsilon \downarrow$	N=32	N=64	N=128	N=256
$10^{-2}$	9.8919e-03	6.8410e-03	5.3380e-03	3.1770e-03
	5.3207e-01	3.5795e-01	7.4863e-01	
$10^{-4}$	9.8919e-03	6.8410e-03	5.3380e-03	3.770e-03
	5.3207e-01	3.5795e-01	7.4863e-01	
$10^{-6}$	9.8919e-03	6.8442e-03	4.6558e-03	3.1770e-03
	5.3137e-01	5.5582e-01	5.5139e-01	
$10^{-8}$	9.8919e-03	6.8442e-03	4.6558e-03	3.1770e-03
	5.3137e-01	5.5582e-01	5.5139e-01	
$10^{-10}$	9.8919e-03	6.8442e-03	4.6558e-03	3.1770e-03
	5.3137e-01	5.5582e-01	5.5139e-01	
$E^{N,M}$	9.8919e-03	6.8442e-03	4.6558e-03	3.1770e-03
$R^{N,M}$	5.3137e-01	5.5582e-01	5.5139e-01	

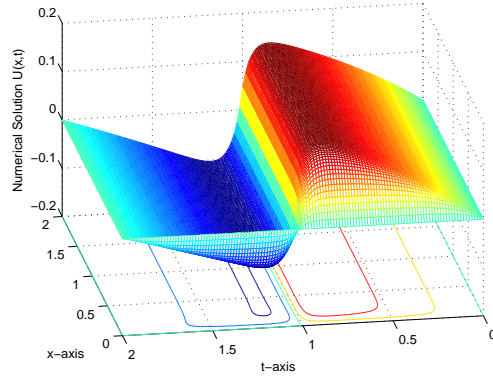


(a)  $\varepsilon = 2^{-2}$

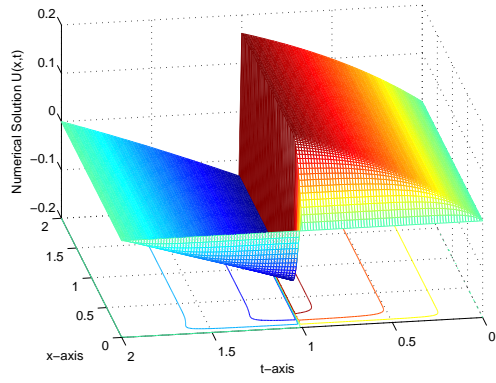


(b)  $\varepsilon = 2^{-10}$

Figure 4.1: Mesh plot to see the physical behaviour for Example (4.7.1) at  $N = 64 = M$ .



(a)  $\varepsilon = 2^{-2}$



(b)  $\varepsilon = 2^{-10}$

Figure 4.2: Mesh plot to see the physical behavior for Example (4.7.2) at  $N = 64 = M$ .



Table 4.2: Comparison absolute errors and order of convergence for Example (4.7.1) at different mesh point.

$\varepsilon \downarrow$	N=32	N=64	N=128	N=256
<i>proposed scheme</i>				
$2^{-2}$	2.0129e-03	1.0528e-03	5.3819e-04	2.7210e-04
	9.3501-01	9.6805-01	9.8397-01	
$2^{-4}$	2.0129e-03	1.0528e-03	5.3819e-04	2.7210e-04
	9.3501-01	9.6805-01	9.8397-01	
$2^{-6}$	2.0129e-03	1.0528e-03	5.3819e-04	2.7210e-04
	9.3501-01	9.6805-01	9.8397-01	
$2^{-8}$	2.0129e-03	1.0528e-03	5.3819e-04	2.7210e-04
	9.3501-01	9.6805-01	9.8397-01	
$2^{-10}$	2.0129e-03	1.0528e-03	5.3819e-04	2.7210e-04
	9.3501-01	9.6805-01	9.8397-01	
$E^{N,M}$	2.0129e-03	1.0528e-03	5.3819e-04	2.7210e-04
$R^{N,M}$	9.3501-01	9.6805-01	9.8397-01	
Results in Daba and Duressa (2022)				
$2^{-2}$	5.3283e-03	2.3275e-03	9.9391e-04	4.4560e-04
	1.1949e-00	1.2276e-00	1.1574e-00	
$2^{-6}$	2.0639e-03	1.0964e-03	5.6453e-04	2.8600e-04
	9.1260e-01	9.5765e-01	9.8104e-01	
$2^{-8}$	2.0649e-03	1.0663e-03	5.4158e-04	2.7295e-04
	9.5346e-01	9.7737e-01	9.8854e-01	
$2^{-10}$	2.0649e-03	1.0663e-03	5.4158e-04	2.7295e-04
	9.5346e-01	9.7737e-01	9.8854e-01	
$E^{N,M}$	5.3283e-03	2.3275e-03	9.9391e-04	4.4560e-04
$R^{N,M}$	1.1949e-00	1.2276e-00	1.1574e-00	

## 4.8 Discussion

In this thesis, singularly perturbed parabolic differential equations show attribute of turning point conduct across discontinuities and weak boundary layers is solved numerically. To fit the interior and boundary layer a suitable non-standard finite difference method on uniform mesh is constructed. The behavior of properties of continuous solution of the problem shown that is satisfies stability estimate. The numerical scheme is developed. To validate the applicability of the proposed scheme two numerical examples are presented. The numerical results are tabulated in terms of maximum absolute errors ,numerical rate of convergence and uniform errors(table 1-3) and compared with the results in Daba and Duressa (2022) is tabulated in table 3. From this table,one can conclude that the proposed scheme gives better

results than scheme in Daba and Duressa (2022).As expression in Figs(1-4),the numerical solutions of the condition examples posses singular behavior at the point  $x = 0, 1, 2$ . As a result of this singularity , the numerical solution expression exhibits an interior layer at  $x = 1$ .

# Chapter 5

## Conclusion and Scope for future work

### 5.1 Conclusion

In this thesis singularly perturbed parabolic differential equations show attribute of turning point conduct across discontinuities and weak boundary layers is solved numerically. The developed scheme constitutes the implicit Euler n time direction and non-standard finite difference method in the space direction on uniform size. The scheme has shown to be  $\varepsilon$ -uniformly convergent of first and second order in space directions. To validate the applicability of the proposed scheme two numerical examples are presented. The obtained numerical results and graphs are plotted by using MATHLAB2013a software package.

### 5.2 Scope for future work.

Most of work done till date is concentrated on problems having boundary layer in their solutions, but the problems with multiple delay and variable delay. Future work may enlarging; the developing numerical scheme for solving the above mentioned problems.

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