Numerical Treatment of Singularly Perturbed Differential Equation Having Small Delays

School of Graduate Studies

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Department of Mathematics

M.Sc. Thesis

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May, 2024

Fitche, Ethiopia

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A Thesis Submitted to the Department of Mathematics, Salale University in the Partial Fulfilment of the Requirement for the Degree of Master of Science in Mathematics (Numerical Analysis)

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APPROVAL SHEET FOR SUBMITTING FINAL THESIS

This is to certify that the research proposal entitled **"Numerical Treatment of Singularly Perturbed Differential Equation Having Small Delays"**, submitted in partial fulfillment of the requirement for the degree of Masters with specialization in Numerical Analysis , the graduate program of Applied Mathematics, and has been carried out by **Tsige Assefa Faye**, under my supervision. Therefore, I recommend that the student has fulfilled the requirements and hence hereby can submit the thesis to the department for defence.

_____	_____	_____
Advisor Name	Signature	Date
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Co-Advisor Name	Signature	Date

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List of Abbreviations and Symbols

Abbreviations

BVP	:	Boundary Value Problem
FDM	:	Finite Difference Method
FEM	:	Finite Element Method
FMM	:	Fitted Mesh Method
FOM	:	Fitted Operator Method
FVM	:	Finite volume Method
1D	:	One Dimensional
ODE	:	Ordinary Differential Equation
SPP	:	Singularly Perturbed Problem
SPDDE	:	Singularly Perturbed Delay Differential Equation
SPDODE	:	Singularly Perturbed Delay Ordinary Differential Equation

Symbols

δ	:	Delay Parameter
D	:	Domain of the problem
$O(.)$	:	Order symbol
ε	:	Singular Perturbation Parameter
$\overline{D}$	:	$D \cup \partial D$

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Abstract

This thesis deals with the numerical treatment of singularly perturbed delay differential equations. These singularly perturbed problems are described by differential equations in which the highest order derivative is multiplied by an arbitrarily small parameter (say) ε known as the singular perturbation parameter and containing a negative shift parameter. The presence of the perturbation parameter exhibits strong boundary layers in the solution. The abruptly changing behaviors of the solution in the layers make it very hard to solve the problem analytically. Standard numerical methods do not give satisfactory results unless a large mesh number is considered, which needs an enormous computational cost. So, one has to rely on suitable numerical methods to solve such types of problems. Therefore, it is important to develop robust numerical schemes for these problems. The main aim of this study is to develop and analyze a robust numerical method based on fitted techniques for solving singularly perturbed delay differential equations. The terms involving the delay are approximated using Taylors series approximation. Then, the resulting equation is solved by the fitted non-polynomial cubic spline method. It is shown that the method converges uniformly concerning the perturbation parameter. Numerical experiments have been carried out to corroborate the theoretical results. The layer behavior of the solutions is presented using tables and graphs and observed to agree with the existing theories. The error analysis of the scheme is done and observed that the proposed method is parameter uniform convergent with the order of convergence (h^2). To validate the applicability of the proposed method the obtained results were compared to some methods that appear in the literature. The comparison shows that the proposed method provides better results than some results available in the literature.

Chapter 1

Introduction

In this section, we gave a general description of singularly perturbation problems, some models of singularly perturbed differential equations, statement of the problem, objective of the study, significance of the study, and delimitation of the study.

1.1 Background of the Study

A variety of models in physics, chemical kinetics, mathematical biology, and many other fields are quite naturally formulated interns of differential equations. If the model is formulated interns of a differential equation, taking into account the additional small parameter will often lead to the appearance of some small factors(parameters) multiplying some of the terms inappropriately non-dimensionalized equations. Then the equations on the influence of the small parameter on the process reduce to the study of the dependence of solution of such differential equation on small parameters. The term containing small parameters is called perturbations. The extended model is perturbed and the simplified model (that does not contain the small parameter) is unperturbed (Vasil'Eva et al., 1995).

Singularly perturbation problem is a problem containing a small parameter ε that can not be approximated by setting the parameter value to zero. this is in contrast to regular perturbation problems for which an approximation can be obtained by simply setting the small parameter to zero. It means the solution can not be uniformly approximated be an asymptotic expression as $\varepsilon \longrightarrow 0$.

The birth of singular perturbation problems occurred at the Third International Congress of Mathematicians in Heidelberg in 1904 by Prandtl's seven-page report was published in the proceedings (Prandtl, 1905). In his innovative work on the subject of boundary layer theory, he explained that how a quality as small as the viscosity of common fluids such as water and air could play a key role in determining their flow. This boundary layer theory became the foundation stone for modern fluid dynamics. However, The term "singularly perturbations" was first used by Friedrichs and Wasow(1946), a paper presented at a seminar on non linear vibrations at New York University. The solution of singular perturbation problems typically contains layers (Roos, 2008). Though Prandtl introduced the terminology boundary layer in this conference it got much greater generality in the substantial work of (Wasow, 1942). A boundary layer is defined to be a small region of the independent variable over which the solution changes very rapidly, whereas the solution changes slowly in the outer region.

Several works have been done on the asymptotic and numerical treatment of various classes of SPPs. Nevertheless, early works focus only on asymptotic analysis of such problems, whereas the numerical treatment of SPPs has flourished since the mid-1960s. Numerous good books have appeared in this area which dealt with asymptotic or/and numerical approaches, to cite few (Miller et al., 1996; Shishkin and Shishkina, 2008; Johnson, 1983; Roos, 2008; Nayfeh, 2011; O'malley et al., 1991; Robert Jr et al., 2012; Wasow, 1942; Kevorkian and Cole, 2012; Bender and Orszag, 2013) references therein.

The class of differential equations which have characteristics of both classes, i.e. delay and singularly perturbed behavior are known as singularly perturbed delay differential equations. The expression negative shift are also used for delay parameter. In general, the ordinary differential equations in which the highest order derivative is multiplied by a small positive parameter and containing at least delay parameter is known as singularly

perturbed delay ordinary differential equation (SPDODE). The various application of SPDODE arises in the modeling of various modern complicated processes such as: the first exit time problem in the modeling of the activation of neuronal variability (Lange and Miura, 1994), in the study of bistable devices (Derstine et al., 1982), evolutionary biology (Mackey, 1997), in a variety of models for physiological processes or diseases (Mackey and Glass, 1977), to describe the human pupil-light reflex (Longtin and Milton, 1988) and variational problems in control theory (Glizer, 2003), fluid dynamics, plasticity, chemical reactor theory, nuclear reactor theory, plasma physics aerodynamics, meteorology, oceanography, rare field gas dynamic, diffraction theory, reaction-diffusion process, non equilibrium (File and Edosa, 2017) and other domains of the great world of fluid dynamics.

Singularly perturbed differential equations relate an unknown function to its derivatives evaluated at the same instance. Whereas, singularly perturbed delay differential equations model physical problems for which the evaluation depends on the present state of the system and also on the past history. Also, the difference between the non-delay and delay singularly perturbed problems is that sometimes delay problems exhibit extra interior layers (Lange and Miura, 1994). Apart from this, the solutions to delay problems have points of discontinuities.

The numerical treatment of SPPs has attracted the attention of many researchers in the past and recent years. The solutions of SPPs are non-smooth with singularities related to boundary layers. When the perturbation parameter is decreased, even the most contemporary numerical methods fail to be robust, that is, they do not behave uniformly well for all values of the perturbation parameter (ε) and the mesh length. Careful examination of the results from classical numerical methods (Finite difference Method (FDM, Finite element Method (FEM), Finite Volume Method (FVM), Collocation method, and Spectral Method) on uniform meshes fails to provide a fairly accurate approximate solution of the exact solution and the truncation error becomes unbounded as the singular perturbation parameter

tends to zero unless a large number of mesh points is used in the approximation process (Miller et al., 1996). However, this shows that the numerical method is computationally inefficient. Sometimes, the increase in mesh points also causes the resulting systems of algebraic equations to be ill-conditioned. This is due to the presence of the so-called boundary or interior layer(s) exhibited in the solution.

In the past few decades, a great deal of research has been done on the qualitative and quantitative analysis of various classes of singularly perturbed ordinary differential equations(SPODEs). However, the development of competitive computational methods for solving SPDODEs with shift arguments is still in an open direction. Therefore, in this thesis, we have developed a fitted non-polynomial cubic spline method to find the solutions of singularly perturbed differential equations having small delays. In addition to that, we analyze the uniform convergence of the scheme.

1.2 Statement of the Problem

Developing a higher-order uniformly convergent numerical scheme is an active research area. Different authors developed a numerical scheme that converges independent of the perturbation parameter to solve SPDODE. However, the computation of its solution has been a great challenge and has been of great importance due to the versatility of such equations in the mathematical modeling of processes in various application fields, where they provide the best simulation of observed phenomena and hence the numerical approximation of such equations has been growing more and more.

The increasing desire for numerical solutions to mathematical problems, which are more difficult or impossible to solve analytically, has become present-day scientific research. This time it sounds more appropriate to find an approximate solution to a more complicated model. Numerical methods can efficiently give approximate solutions when ordinary analytic methods fail. However, it is well known that some of the standard discretization

methods for solving singularly perturbation problems are unstable and fail to give accurate results. Obtaining accurate and fast numerical solutions for singularly perturbed delay differential problems has great importance due to its wide applications in scientific and engineering research. Therefore, it is important to develop simple, more accurate, stable, and parameter uniform for solving singularly perturbed delay differential equations with small delays. Owing to this, we developed a fitted-non-polynomial cubic spline method to find the solutions to singularly perturbed delay differential equation problems.

1.3 Objectives of the Study

1.3.1 General Objective

The main objective of this study is to construct and analyze robust numerical scheme, which are more accurate, stable, and convergent independent of the mesh size and perturbation parameter and give good results for reasonable mesh size for solving singularly perturbed differential equations having small delays.

1.3.2 Specific Objectives

The specific objectives of this study are

1. To develop $\varepsilon-$uniform numerical method for solving singularly perturbed differential equation having small delays.
2. To establish the stability of the proposed numerical method.
3. To analyze the convergence of the proposed numerical method.
4. To examine the effectiveness of the proposed numerical method.

1.4 Significance of the Study

A singularly perturbed differential equation having small delays has been studied because of its frequent applications in many mathematical models. The future behaviors of these problems are assumed to be described by their solutions. However, it is very hard to solve

non-linear delay differential equations analytically due to the presence of a multi-scale character in the solution. So, it is necessary to develop $\varepsilon-$uniform convergent numerical methods that solve the problems under consideration successfully. Developing a parameter numerical method for such problems is always an anticipated task (Robert Jr et al., 2012). The main contribution of this study is the development of a parameter numerical method for solving singularly perturbed differential equation having small delays. Additionally, the developed numerical scheme can be used to fill the gap observed, will be significantly helpful to researchers working in this area to construct their new numerical schemes, and used as an alternative scheme for solving the problem under consideration.

1.5 Delimitation of the Study

The solutions of singularly perturbed delay differential problems are investigated by different scholars. But this study has been delimited to solve singularly perturbed ordinary differential equation having a delay on the convection and reaction terms of the form:

$$-\varepsilon\, u^{"}(x) + a(x)u^{'}(x-\delta) + \beta u(x) + \alpha(x)u(x-\delta) = f(x),\ x \in \Omega = (0,1), \tag{1.5.1}$$

subject to boundary condition

$$u(x) = \phi(x), \qquad x \in \Omega_L = [-\delta, 0], \qquad u(1) = \gamma, \tag{1.5.2}$$

where ε, $0 < \varepsilon \ll 1$ is singularly perturbation parameter and δ is delay parameter satisfying $\delta < \varepsilon$ the function $a(x), \beta(x), \alpha(x)$ and $f(x)$ are sufficiently smooth and bounded for the existence of its solution. The value of $\phi(x)$ and γ are finite constant. We assume also the coefficient of non-derivative term α and β satisfy

$$\alpha(x) + \beta(x) \geq\ q^*\ > 0, \forall x \in \Omega$$

for some constant q^*.

Chapter 2

Review of Related Literature

A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involves at least one delay term. In the past, less attention had been paid to the numerical solution of singularly perturbed delay differential equations. However, in recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such types of differential equations in the mathematical modeling of processes in various application fields. The works done on the singularly perturbed delay ordinary differential equations are presented in chronological order as follows:

In Amiraliyev and Cimen (2010), author studied the boundary value problem (BVP) for the delay differential equation (DDE):

$$\varepsilon u^{"}(x) + a(x)u^{'}(x) + b(x)u(x-r) = f(x), x \in \Omega,$$

subject to the interval and boundary conditions

$$u(x) = \varphi(x), x \in \Omega_0; u(l) = B,$$

where, $\Omega = \Omega_1 \cup \Omega_2, \Omega_1 = (0, r], \Omega_2 = (r, 1), \bar{\Omega} = [0, 1], \Omega_0 = [-r, 0]$ and $0 < \varepsilon \leq 1$ is the perturbation parameter, $a(x) \geq \alpha \geq 0$, $b(x)$, $f(x)$ *and* $\phi(x)$ and are given sufficiently smooth functions satisfying certain regularity conditions to be specified and r(l¡2r) is a constant delay, which is independent of ε *and* B is a given constant. For small values of ε the function $u(x)$ has a boundary layer near $x = 0$. The numerical method presented here comprises a fitted difference scheme on a uniform mesh.The numerical method presented

here comprises a fitted difference scheme on a uniform mesh. This approach has been derived based on the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. This results in a local truncation error containing only first-order derivatives of the exact solution and hence facilitates examination of the convergence. In this paper, the author states some important properties of the exact solution and discretizes the domain by the finite difference method. The error analysis for the approximate solution is presented. Uniform convergence is proved in the discrete maximum norm. The authors also formulated the iterative algorithm for solving the discrete problem and presented numerical results that validate the theoretical analysis computationally.

File et al. (2013) have presented a computational method to solve these equations with negative shifts whose solution has a boundary layer. In this scheme, authors reduced the second-order SPDDE to a first-order equation and then employed numerical integration and interpolations. Authors claim that the available asymptotic expansion methods for solving singularly perturbed problems are difficult to apply as it is not easy to find appropriate asymptotic expansions in the inner and outer regions and matching of the coefficients of the inner and outer solution expansions is also a process that need skills. The proposed method overcame the above problems. The left-ended and right ended problem was treated separately. Taylor series expansion of $y(x)$ in the neighborhood of $x = 0$ and $x = 1$ is used to reduce the second-order singularly perturbed delay differential equation in the first-order delay differential equation. The interval $[0, 1]$ is divided into equal subparts of constant length $1/N$. The new equation is integrated over an interval and solved by the Trapezoidal Rule and Taylor series. The authors discussed the discrete invariant embedding algorithm to solve the term recurrence relation.

A fourth order finite difference scheme presented by (File and Edosa, 2017) for numerical

solution of SPODDE. The authors considered Reaction-Diffusion Equation,

$$\varepsilon^2 y^{"}(x) + a(x)y^{'}(x-\delta) + b(x)y(x) = f(x), 0 \leq x \leq 1$$

with $y(x) = \phi(x),\ -\delta \leq x \leq 0\ and\ y(1) = \beta$ and uniform mesh with constant mesh length is considered. A fourth-order finite difference scheme is derived by using Taylor series expansion and the resultant tri-diagonal system is solved by the method of discrete invariant embedding algorithm. A stability analysis of the above scheme was carried out and found that the method is of fourth order uniformly convergent.

Fitted fourth order numerical method for solving singularly perturbed delay convection-diffusion equations has been presented by (Gadissa and File, 2019). The authors considered convection-diffusion equation

$$\varepsilon y^{"}(x) + a(x)y^{'}(x-\delta) + b(x)y(x) = f(x), for x \in [0,1],$$

with the interval and boundary condition,

$$y(x) = a(x),\ -\delta \leq x \leq 0\ and\ y(1) = \beta.$$

and uniform mesh with constant mesh length is considered. To demonstrate the efficiency of the method, four model examples with constant and variables coefficients have been considered for different values of the perturbation parameter ε and delay parameter.
Further,it is significant that all of the maximum absolute errors decrease rapidly as N increases, which in turn shows the convergence of the computed solution. The convergence analysis of the present method is investigated. The results presented in article confirmed that the computational rate of convergence, as well as theoretical estimates, indicates that the present method is of fourth order convergence. Accordingly, depending on the sign of the coefficient of delay term one can see that as a delay parameter δ increases the width of the left boundary layer decreases. When the coefficient of the delay term in the problem is of O(1) and delay increases, the thickness of the right boundary layer decreases, but when the coefficient of the delay term of $O(1)$ and delay increases, the thickness of the right

boundary layer increases.

The authors in Subburayan and Ramanujam (2013) and Sekar and Tamilselvan (2019) considered singular perturbation problem with derivative depending on large delay ($\tau = 1$) variable, such as

$$\varepsilon y^{"}(t) + a(t)y^{'}(t) + b(t)y(t) + c(t)y(t-1) = f(t),$$

has been developed various numerical schemes are finite and hybrid difference method (Subburayan and Ramanujam, 2013), iterative scheme (Selvi and Ramanujam, 2016), finite element method (Nicaise and Xenophontos, 2013) and (Shah et al., 2020).The study in (Sekar and Tamilselvan, 2019) proposed solving singularly perturbed delay differential equation with integral boundary condition using finite difference method. Throughout the literature, the researcher concentrates on solving the singular perturbation problem with a small delay or mixed small delay or large delay using finite or hybrid or finite element methods on uniform meshes or non uniform mesh.

Uniformly convergent fitted operator method for singularly perturbed delay differential equations has been presented by (Woldaregay et al., 2022). The authors considered numerical treatment of singularly perturbed delay differential equations

$$-\varepsilon u^{"}(x) + a(x)u^{'}(x-\delta) + b(x)u(x) = f(x), x \in \Omega = (0,1),$$

with interval boundary conditions

$$u(x) = \phi(x),\ -\delta \leq x \leq 0,\ u(1) = \psi(1).$$

The solution of the considered problem exhibits boundary layer behavior. Using Taylor's series approximation for the delay term asymptotically equivalent singularly perturbed boundary value problem obtained. The numerical schemes are developed using the exponentially fitted finite difference method. The stability of the schemes is investigated using the barrier function for the solution bound and discrete maximum principle is used for

the existence of the unique discrete solution. The uniform convergence of the schemes is proved. The proposed scheme was investigated by considering four test examples exhibiting boundary layers. The effect of the perturbation parameter and the delay parameter on the solution is shown using figures and tables. The developed scheme is accurate and uniformly convergent with the rate of convergence one.

Fitted numerical scheme for singularly perturbed differential equation having small delay has been presented by (Woldaregay and Duressa, 2022). The author considered a class of singularly perturbed differential equation having delay on the convection and reaction terms of the form

$$-\varepsilon u^{"}(x) + a(x)u^{'}(x-\delta) + \beta u(x) + \omega u(x-\delta) = f(x),\ \times \epsilon \Omega = (0,1), \tag{2.0.1}$$

with interval-boundary conditions

$$u(x) = \phi(x), \qquad x \in \Omega_L = [-\delta, 0], \qquad u(1) = \gamma.$$

The solution of the problem have boundary layer on left or right side of the domain depending on the sign of the convective term. The analytical properties of the solution is discussed. Uniformly convergent numerical scheme is developed using exponentially fitted upwind finite difference method. The developed scheme satisfies the discrete maximum principle and uniform stability estimate. The stability and the parameter uniform convergence of the scheme theoretically investigated.

Adilaxmi et al. (2019) approximate the problem to equivalent BVPs and solved using non standard FDM using exponential fitting factor. The authors in (Kadalbajoo and Ramesh, 2007; Kumar and Kadalbajoo, 2012; Woldaregay and Duressa, 2021; Angasu et al., 2021) considered the same problem Eq. 2.0.1 and presented various numerical methods which include upwind, midpoint upwind and hybrid of midpoint upwind on regular region and central finite difference on boundary layer region using piecewise uniform Shishkin mesh,

B-spline collocation method on shishkin mesh, non-standard finite difference method, exponentially fitted finite difference method.

Our survey indicates that the solution methodologies of SPDODEs still need improvement and designing more accurate, stable, and higher-order schemes is an active research area. This motivates us to develop a simple, accurate, and uniformly convergent numerical scheme for treating singularly perturbed differential equations having small delays.

Chapter 3

Methodology

This chapter consists: study area and period, study design, source of information, mathematical procedures.

3.1 Study Site and Period

This study is conducted at Salale University, College of Natural Sciences, department of Mathematics from January 2024 to May 2024.

3.2 Study Design

This study employed both documentary review and numerical experimentation on singularly perturbed ordinary differential equation with small delay.

3.3 Source of Information

The relevant sources of information for this study are books, published articles, and related studies from Internet.

3.4 Study Procedure

In order to achieve the stated objectives, the study followed the following mathematical procedures:

1. Defining model problem for the study,

2. Analyzing the properties of the continuous solution,

3. Developing numerical method for the problem,

4. Establishing the stability and convergence of the developed numerical method,

5. Developing an algorithm and writing MATLAB code for the presented method,

6. Validating the method using numerical examples,

7. Presenting the results using appropriate presentation (using tables, graphs),

8. Discussing and providing conclusion.

Chapter 4

Description and Analysis of the Numerical Method

4.1 Description of the Problem

In this study, we consider a class of singularly differential equation perturbed having delay on the convection and reaction terms of the form:

$$-\varepsilon u^{"}(x) + a(x)u^{'}(x-\delta) + \beta(x)u(x) + \alpha(x)u(x-\delta) = f(x),\ x \in \Omega = (0,1), \qquad (4.1.1)$$

subject to boundary condition

$$u(x) = \phi(x), \qquad x \in \Omega_L = [-\delta, 0], \qquad u(1) = \gamma, \qquad (4.1.2)$$

where ε $(0 < \varepsilon \ll 1)$ is singularly perturbation parameter and δ is delay parameter satisfying $\delta < \varepsilon$, the function $a(x), \beta(x), \alpha(x)$ and $f(x)$ are assumed to be sufficiently smooth and bounded. the value of $\phi(x)$ *and* γ are assume finite constant. We assume also the coefficient of non-derivative term α *and* β satisfy, $\alpha(x)+\beta(x) \geq\ q^*\ > 0, \forall x \in \Omega$, for some constant q^*. This condition ensures that the solution (3) and (4) exhibits boundary layer in the neighborhood of $x = 0$ or $x = 1$ depending on the sign of the convective term $a(x)$.

When the delay parameter is zero (i.e, $\delta = 0$) the problem reduced to singularly perturbed BVP, for small ε the problem exhibits boundary layer depending up on the value of the convective term coefficient $a(x)$. When $a(x) < 0$ regular boundary layer appears in the neighborhood of $x = 0$ and $\alpha(x) > 0$ corresponds to existence of boundary layer in the

neighborhood of $x = 1$. If $a(x)$ change sign, shock layer will appear on the middle of the domain(Woldaregay and Duressa, 2019). the layer is maintained for $\delta \neq 0$ but sufficiently small.

4.2 Properties of the Analytical Solution

When the shift parameter δ is smaller than the singular perturbation parameter ε, the use of Taylor's series expansion for the terms containing shift arguments is valid (Tian, 2002). In this thesis, we considered the case when $\delta < \varepsilon$. Thus, to approximate the terms with delay parameter, we apply Taylor's series expansion as follows:

$$u^{'}(x-\delta) \approx u^{'}(x) - \delta u(x)^{"} + O(\delta^2), \tag{4.2.1}$$

$$u(x-\delta) \approx u(x) - \delta u^{'}(x) + \frac{\delta^2}{2} u^{"}(x) + O(\delta^3). \tag{4.2.2}$$

Substituting Eq. (4.2.1) and (4.2.2) into Eq. (4.1.1) gives a singular perturbed BVP:

$$-c_{\varepsilon,\delta}(x)u^{"}(x) + p(x)u^{'}(x) + q(x)u(x) = f(x),\ x \in \Omega = (0,1), \tag{4.2.3}$$

with the boundary condition

$$u(0) = \phi(0), \qquad u(1) = \gamma, \tag{4.2.4}$$

where

$$c_{\varepsilon,\delta}(x) = \varepsilon + \delta a(x) - \frac{\delta^2}{2}\alpha(x),\ p(x) = a(x) - \delta\alpha(x)$$

and

$$q(x) = \beta(x) + \alpha(x) \text{ for some value of } \varepsilon.$$

For small δ, Eq. (4.1.1) and Eq. (4.2.3) are asymptotically equivalent, because the difference between the two equations is the order of $O(\delta^2, \delta^3)$. We assume $0 < c_{\varepsilon,\delta}(x) \leq \varepsilon - \delta K_1 - \delta^2 K_2 = c_{\varepsilon,\delta}$, where $a(x) \geq K_1$ and $a(x) \geq 2K_2$ for K_1 and K_2 are constants.

Let us consider first the case $p(x) \leq p^* < 0$ which imply occurrence of boundary layer on the left side of the domain, the other case $p(x) \geq p^* > 0$ imply the occurrence of boundary

layer on the right side of the domain.

Let $L_{\varepsilon,\delta}$ be denoted for differential operator $L_{\varepsilon,\delta}u(x) = -c_{\varepsilon,\delta}u^{"}(x) + p(x)u^{'}(x) + q(x)u(x)$ in Eqs. (4.2.3) and (4.2.4) satisfies the following maximum principle.

Lemma 4.2.1. *(Maximum Principle) Let z be a sufficiently smooth function defined on Ω which satisfies $z(0) \geq 0$ and $z(1) \geq 0$, then $L_{\varepsilon,\delta}z(x) > 0, \forall x \in \Omega$ implies that $z(x) \geq 0, \forall x \in \bar{\Omega}$.*

Proof. let x^* be such that $z(x^*) = \min_x z(x), x \in \bar{\Omega}$ and suppose that $z(x^*) < 0$. It is clear that $x^* \notin \{0, 1\}$. Since $z(x^*) = \min_{x\in\bar{\Omega}} z(x)$ we have $z^{'}(x^*) = 0$ and $z^{"}(x^*) \geq 0$ implies that $L_{\varepsilon,\delta}z(x^*) < 0$, which is contradiction to the assumption that made above $L_{\varepsilon,\delta}z(x^*) > 0, \forall x \in \Omega$. Therefore, $z(x) \geq 0, \forall x \in \bar{\Omega}$. □

Lemma 4.2.2. *(Stability Estimate) Let $u(x)$ be the solution of Eqs.* (4.2.3) *and* (4.2.4)*, then we obtain the bound*

$$|u| \leq \frac{\|L_{\varepsilon,\delta}u\|}{q^*} + \max\{\phi(0), \gamma\},$$

$$\textit{where } q^* > 0 \quad \textit{is lower bound of } q(x).$$

Proof. By defining barrier function $\vartheta^{\pm}(x)$ as

$$\vartheta^{\pm}(x) = \frac{\|L_{\varepsilon,\delta}u\|}{q^*} + \max\{\phi(0), \gamma\} \pm u(x),$$

at the boundary point, we obtain

$$\begin{aligned}
\vartheta^{\pm}(0) &= \frac{\|L_{\varepsilon,\delta}u\|}{q^*} + \max\{\phi(0), \gamma\} \pm u(0) \geq 0,\\
\vartheta^{\pm}(1) &= \frac{\|L_{\varepsilon,\delta}u\|}{q^*} + \max\{\phi(0), \gamma\} \pm u(1) \geq 0.
\end{aligned}$$

And on the differential operator

$$\begin{aligned}
L_{\varepsilon,\delta}\vartheta^{\pm}(x) =& - c_{\varepsilon,\delta}\vartheta^{"}_{\pm} + p(x)\vartheta^{'}_{\pm}(x) + q(x)\vartheta_{\pm}(x)\\
=& \mp c_{\varepsilon,\delta}u^{'}(x) + q(x)(\frac{\|L_{\varepsilon,\delta}u\|}{q^*} + \max\{\phi(0), \gamma\} \pm u(0))\\
=& q(x)(\frac{\|L_{\varepsilon,\delta}u\|}{q^*} + \max\{\phi(0), \gamma\}) \pm f(x)\\
\geq& 0, \ \textit{since } q^* > 0 \text{ is lower bound of } q(x),
\end{aligned}$$

which implies, $L_{\varepsilon,\delta}\vartheta^{\pm}(x) \geq 0$, hence using maximum principle in Lemma 4.2.1 we obtain $\vartheta^{\pm}(x) \geq 0, \forall x \in \bar{\Omega}$. □

Lemma 4.2.3. *The derivative of the solution $u(x)$ of the problem in* (4.2.3) *and* (4.2.4) *is bounded as*

$$\left|u^{(k)}(x)\right| \leq C\left(1 + c_{\varepsilon,\delta}^{-k}\exp^{(\frac{-p^*x}{c_{\varepsilon,\delta}})}\right), x \in \bar{\Omega}, \text{ for left layer}$$

$$\left|u^{(k)}(x)\right| \leq C\left(1 + c_{\varepsilon,\delta}^{-k}\exp^{(\frac{-p^*(1-x)}{c_{\varepsilon,\delta}})}\right), x \in \bar{\Omega}, \text{ for right layer}$$

for $0 \leq k \leq 4$ *, where* $p(x) \leq p^* < 0$ *for left boundary layer case.*

Proof. For the detail proofs of the lemma one can see See Kellogg and Tsan (1978) and Miller et al. (1996). □

4.3 Formulation of the Numerical Method

We divide the domain $[0, 1]$ in to equal subinterval with step size $h = \frac{1}{N}$. Let $0 = x_0, x_1, ..., x_N = 1$ be the mesh points then we have: $x_i = ih, i = 0, 1, 2, ..., N$. For each segment $[x_i, x_{i+1}], i = 1, 2, ..., N-1$ the non-polynomial cubic spline $E_\Delta(x)$ has the following form

$$E_\Delta(x) = b_i + c_i(x - x_i) + d_i(e^{v(x-x_i)} + e^{-v(x-x_i)}) + g_i(e^{v(x-x_i)} - e^{-v(x-x_i)}), \tag{4.3.1}$$

where b_i, c_i, d_i *and* g_i are unknown coefficients, and $v \neq 0$ arbitrary parameter which will be used to increase the accuracy of the method. To determine the unknown coefficient in Eq. (4.3.1) in terms of u_i, u_{i+1}, M_i, and M_{i+1} first we define

$$\begin{cases} E_\Delta(x_i) = u_i \ , & E_\Delta(x_{i+1}) = u_{i+1} \\ E_\Delta^{"}(x_i) = M_i \ , & E_\Delta^{"}(x_{i+1}) = M_{i+1}. \end{cases} \tag{4.3.2}$$

The coefficients in Eq. (4.3.1) are determined as

$$\begin{cases} b_i = u_i - \dfrac{M_i}{v^2}, \\ c_i = \dfrac{u_{i+1} - u_i}{h} + \dfrac{M_i - M_{i+1}}{v\theta}, \\ d_i = \dfrac{M_{i+1}}{v^2(e^\theta - e^{-\theta})} - \dfrac{M_i(e^\theta + e^{-\theta})}{2v^2(e^\theta - e^{-\theta})}, \\ g_i = \dfrac{M_i}{2v^2}, \end{cases} \tag{4.3.3}$$

where $\theta = vh$.

Using the continuity condition of the first derivative at $x_i, E'_{\Delta-1}(x_i) = E'_{\Delta}(x_i)$, we have

$$c_{i-1} + vc_{i-1}(e^{\theta} + e^{-\theta}) + vg_{i-1}(e^{\theta} - e^{-\theta}) = c_i + 2vd_i. \tag{4.3.4}$$

Reducing indices of Eq. (4.3.3) by one and substituting in to Eq. (4.3.4), we get

$$\frac{u_{i+1} - u_i}{h} + \frac{M_i - M_{i+1}}{v\theta} + v(\frac{2M_i - (e^{\theta} + e^{-\theta})M_{i-1}}{2v^2(e^{\theta} + e^{-\theta})}) = \frac{u_{i+1} - u_i}{h} + \frac{M_i - M_{i+1}}{v\theta}$$
$$+2v(\frac{M_{i+1}}{v^2(e^{\theta} - e^{-\theta})} - \frac{M_i(e^{\theta} + e^{-\theta})}{2v^2(e^{\theta} - e^{-\theta})}).$$

The above equation can be written as

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = \lambda_1 M_{i-1} + 2\lambda_2 M_i + \lambda_1 M_{i+1}, \tag{4.3.5}$$

where

$$\lambda_1 = \frac{1}{\theta^2}(1 - \frac{2\theta}{e^{\theta} - e^{-\theta}}), \qquad \lambda_2 = \frac{1}{\theta^2}(\frac{\theta(e^{\theta} + e^{-\theta})}{e^{\theta} - e^{-\theta}} - 1).$$

If $h \to 0$, then $\theta = hk \to 0$ in Eq. (4.3.5), we get $\lambda_1 + \lambda_2 = \frac{1}{2}$. Using L'hopitals rule we have $\lim_{h\to 0} \lambda_1 = \frac{1}{6}$ *and* $\lim_{h\to 0} \lambda_2 = \frac{1}{3}$.

Using $E''_{\Delta}(x_i) = u''_i = M_i$ in to Eq. (4.2.3), we get

$$\begin{cases} (c_{\varepsilon,\delta})_i M_i = f_i - p_i u'_i - q_i u_i \\ (c_{\varepsilon,\delta})_i M_{i-1} = f_{i-1} - p_{i-1} u'_{i-1} - q_{i-1} u_{i-1} \\ (c_{\varepsilon,\delta})_i M_{i+1} = f_{i+1} - p_{i+1} u'_{i+1} - q_{i+1} u_{i+1}. \end{cases} \tag{4.3.6}$$

Using Taylor's series expansion of u_{i-1}, u_i and u'_{i+1}, simplifying we have

$$\begin{cases} u'_i = \frac{u_{i+1} - u_{i-1}}{2h} + T_1, \\ u'_{i-1} = \frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h} + T_2, \\ u'_{i+1} = \frac{3u_{i+1} - 4u_i + u_{i-1}}{2h} + T_2, \end{cases} \tag{4.3.7}$$

where $T_1 = -\frac{h^2}{6}u'''(\xi)$ and $T_2 = -\frac{h^2}{12}u'''(\xi)$, for $\xi \in (x_{i-1}, x_i)$.

Substituting Eq. (4.3.7) in to Eq. (4.3.6), we get

$$\begin{cases} M_i = \frac{1}{(c_{\varepsilon,\delta})_i}\left\{f_i - p_i(\frac{u_{i+1} - u_{i-1}}{2h}) - q_i u_i\right\}, \\ M_{i-1} = \frac{1}{(c_{\varepsilon,\delta})_i}\left\{f_{i-1} - p_{i-1}(\frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h}) - q_{i-1} u_{i-1}\right\}, \\ M_{i+1} = \frac{1}{(c_{\varepsilon,\delta})_i}\left\{f_{i+1} - p_{i+1}(\frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}) - q_{i+1} u_{i+1}\right\}. \end{cases} \tag{4.3.8}$$

Plugging Eq. (4.3.8) in to Eq. (4.3.5), we get

$$\begin{aligned}(c_{\varepsilon,\delta})_i\left(\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}\right) &= \lambda_1\left(f_{i-1}-p_{i-1}(\frac{-u_{i+1}+4u_i-3u_{i-1}}{2h})-q_{i-1}u_{i-1}\right)\\ &+2\lambda_2\left(f_i-p_i(\frac{u_{i+1}-u_{i-1}}{2h})-q_iu_i\right)\\ &+\lambda_1\left(f_{i+1}-p_{i+1}(\frac{3u_{i+1}-4u_i+u_{i-1}}{2h})-q_{i+1}u_{i+1}\right).\end{aligned}$$

Rearranging the above equation, we have

$$\begin{aligned}(c_{\varepsilon,\delta})_i\left(\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}\right)+\frac{\lambda_1p_{i-1}}{2h}\left(-u_{i+1}-4u_i-3u_{i-1}\right)+\frac{2\lambda_2p_i}{2h}\left(u_{i+1}-u_{i-1}\right)\\ +\frac{2\lambda_1p_{i+1}}{2h}\left(3u_{i+1}-3u_i+u_{i-1}\right)=\lambda_1\left(-q_{i-1}u_{i-1}-q_{i+1}u_{i+1}\right)+\lambda_2q_iu_i\\ +\lambda_1f_{i-1}+2\lambda_2f_i+\lambda_1f_{i+1}+T,\end{aligned}\tag{4.3.9}$$

where $T=(4\lambda_2p_i-\lambda_1p_{i-1}-\lambda_1p_{i+1})\frac{h^2}{12}u'''(\xi)$ is the local truncation error.

From the theory of singularly perturbations describe by O'malley et al. (1991) and Taylor's series expansion of $u(x)$ about the point '0' in the asymptotic solution of the problem in Eq. (4.2.3), we have

$$u(x_i)\approx u_0(x_i)+(\phi_0-u_0(0))e^{-p(0)\frac{ih}{c_{\varepsilon,\delta}(x)}}$$

and letting , $\rho=\frac{h}{c_{\varepsilon,\delta}(x)}$ we get,

$$\lim_{h\to0}u(ih)\approx u_0(ih)+(\phi_0-u_0(0))e^{-p(0)i\rho},$$

since $x_i=x_0+ih$.

Now, introducing fitting factor $\sigma(\rho)$ on Eq. (4.3.9) to control the effect of singular perturbation parameter on solution behavior, we get

$$\begin{aligned}\sigma(\rho)\left(c_{\varepsilon,\delta}\right)_i\left(\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}\right)+\frac{\lambda_1p_{i-1}}{2h}\left(-u_{i+1}-4u_i-3u_{i-1}\right)+\frac{2\lambda_2p_i}{2h}\left(u_{i+1}-u_{i-1}\right)\\ +\frac{2\lambda_1p_{i+1}}{2h}\left(3u_{i+1}-3u_i+u_{i-1}\right)=\lambda_1\left(-q_{i-1}u_{i-1}-q_{i+1}u_{i+1}\right)+\lambda_2q_iu_i\\ +\lambda_1f_{i-1}+2\lambda_2f_i+\lambda_1f_{i+1}+T.\end{aligned}\tag{4.3.10}$$

Multiplying Eq. (4.3.10) by h and taking a limit as $h\to0$, we get

$$\frac{\sigma}{\rho}\lim_{h\to 0}(u_{i-1}-2u_i+u_{i+1})+\frac{\lambda_1 p_{i-1}}{2}\lim_{h\to 0}(-u_{i+1}-4u_i-3u_{i-1})+\lambda_2 p_i\lim_{h\to 0}(u_{i+1}-u_{i-1})$$
$$+\frac{2\lambda_1 p_{i+1}}{2}\lim_{h\to 0}(3u_{i+1}-3u_i+u_{i-1})=0. \tag{4.3.11}$$

Thus, in this thesis we consider the right-end boundary layer ($p(x)<0$) and we have

$$\begin{cases}\lim_{h\to 0}(u_{i-1}-2u_i+u_{i+1})=(\gamma-u_0(1))e^{-p(1)i\rho}\left(e^{p(1)\rho}+e^{-p(1)\rho}-2\right)\\ \lim_{h\to 0}(-u_{i+1}-4u_i-3u_{i-1})=(\gamma-u_0(1))e^{-p(1)i\rho}\left(-3e^{p(1)\rho}-e^{-p(1)\rho}+4\right)\\ \lim_{h\to 0}(u_{i+1}-u_{i-1})=(\gamma-u_0(1))e^{-p(1)i\rho}\left(e^{p(1)\rho}+3e^{-p(1)\rho}-4\right)\\ \lim_{h\to 0}(3u_{i+1}-3u_i+u_{i-1})=(\gamma-u_0(1))e^{-p(1)i\rho}\left(e^{-p(1)\rho}-3e^{-p(1)\rho}\right)\end{cases} \tag{4.3.12}$$

Taking Eq. (4.3.12) into Eq. (4.3.12) obtained the required fitting factor

$$\sigma=\rho_i p(1)(\lambda_1+\lambda_2)\coth(\frac{p(1)\rho}{2}), \tag{4.3.13}$$

where $\rho=\frac{h}{c_{\varepsilon,\delta}}$.

In general, we take a variable fitting parameter as

$$\sigma_i=\rho_i p(x_i)(\lambda_1+\lambda_2)\coth(\frac{p(x_i)\rho}{2}). \tag{4.3.14}$$

Thus, Eq. (4.3.9) can be written as

$$\begin{aligned}&\left(\frac{\sigma_i\left(c_{\varepsilon,\delta}\right)_i}{h^2}-\frac{3\lambda_1 p_{i-1}}{2h}+\lambda_1 q_{i-1}-\frac{\lambda_2 p_i}{h}+\frac{\lambda_1 p_{i+1}}{2h}\right)u_{i-1}\\ &-\left(\frac{2\sigma_i\left(c_{\varepsilon,\delta}\right)_i}{h^2}-\frac{2\lambda_1 p_{i-1}}{h}-2\lambda_2 q_i+\frac{2\lambda_1 p_{i+1}}{h}\right)u_i\\ &+\left(\frac{\sigma_i\left(c_{\varepsilon,\delta}\right)_i}{h^2}-\frac{\lambda_1 p_{i-1}}{2h}+\frac{\lambda_2 p_i}{h}+\frac{3\lambda_1 p_{i+1}}{2h}+\lambda_1 q_{i+1}\right)u_{i+1}\\ &=\lambda_1 f_{i-1}+2\lambda_2 f_i+\lambda_1 f_{i+1}+T.\end{aligned} \tag{4.3.15}$$

From the above Eq. (4.3.15), we get the tridiagonal system of the equation of the form

$$L^N_{c_{\varepsilon,\delta}}U_i\equiv F_i^- u_{i-1}-F_i^c u_i+F_i^+ u_{i+1}=H_i\ ,\qquad i=1,2,...N-1, \tag{4.3.16}$$

where

$$\begin{cases} F_i^- = \dfrac{\sigma_i \left(c_{\varepsilon,\delta}\right)_i}{h^2} - \dfrac{3\lambda_1 p_{i-1}}{2h} + \lambda_1 q_{i-1} - \dfrac{\lambda_2 p_i}{h} + \dfrac{\lambda_1 p_{i+1}}{2h}, \\ F_i^c = \dfrac{2\sigma_i \left(c_{\varepsilon,\delta}\right)_i}{h^2} - \dfrac{2\lambda_1 p_{i-1}}{h} - 2\lambda_2 q_i + \dfrac{2\lambda_1 p_{i+1}}{h}, \\ F_i^+ = \dfrac{\sigma_i \left(c_{\varepsilon,\delta}\right)_i}{h^2} - \dfrac{\lambda_1 p_{i-1}}{2h} + \dfrac{\lambda_2 p_i}{h} + \dfrac{3\lambda_1 p_{i+1}}{2h} + \lambda_1 q_{i+1}, \\ H_i = \lambda_1 f_{i-1} + 2\lambda_2 f_i + \lambda_1 f_{i+1} + T. \end{cases}$$

For sufficiently small mesh sizes the above matrix is non-singular and $|F_i^c| \geq \left|F_i^-\right| + \left|F_i^+\right|$ (*i.e.,* the matrix are diagonally dominant). Hence, by Kadalbajoo and Reddy (1989), the matrix F is M-matrix and have an inverse. Therefore, the system of equations can be solved by matrix inverse with the given boundary conditions.

4.4 Convergence Analysis

In this section, we give the convergence analysis for the scheme in (4.3.15).

Lemma 4.4.1. *(Discrete Maximum Principle) Assume that,the mesh function $z(x_i)$ satisfies $z(x_0) \geq 0$, and $z(1) \geq 0$. If $L^N_{c_{\varepsilon,\delta}} z(x_i) \geq 0, \forall x \in \Omega$, then $z(x_i) \geq 0, \forall x \in \bar{\Omega}$.*

Proof. Let choose k such that $z(x_k) = \min_{x_i} z(x_i),\ 1 \leq i \leq N-1$. If $z(x_k) \geq 0$, the proof completed. We can see that $z(x_{k+1}) - z(x_k) \geq 0$ and $z(x_k) - z(x_{k-1}) \leq 0$. Now from Eq. (4.3.15), we obtain $L^N_{c_{\varepsilon,\delta}} z(x_k) < 0$, which contradicts $L^N_{c_{\varepsilon,\delta}} z(x_k) \geq 0$. Hence, the assumption is wrong, We conclude that $z(x_i) \geq 0,\ \forall_i, 0 \leq i \leq N$. □

Lemma 4.4.2. *(Discrete Stability Estimate):The solution u_i of the discrete scheme in Eq.* (4.3.15) *satisfy the following bound*

$$|u_i| \leq \frac{\left\| L^N_{c_{\varepsilon,\delta}} u_i \right\|}{q^*} + \max\{u_0, u_N\}. \tag{4.4.1}$$

Proof. Let $Q = \dfrac{\left\| L^N_{c_{\varepsilon,\delta}} u_i \right\|}{q^*} + \max\{u_0, u_N\}$ and define the barrier function $\vartheta_i^{\pm}$ by $\vartheta_i^{\pm} = Q \pm U_i$. On the boundary points, we obtain

$$\vartheta_0^{\pm} = Q \pm u_0 = \frac{\left\| L^N_{c_{\varepsilon,\delta}} u_i \right\|}{q^*} + \max\{u_0, u_N\} \pm \phi(0) \geq 0$$

$$\vartheta_N^{\pm} = Q \pm u_N = \frac{\left\| L^N_{c_{\varepsilon,\delta}} u_i \right\|}{q^*} + \max\{u_0, u_N\} \pm \gamma \geq 0.$$

On the discretized spatial domain $x_i, 1 < i < N-1$, we obtain

$$\begin{aligned}
L^N_{c_{\varepsilon,\delta}}\vartheta_i^{\pm} =&(\frac{\sigma_i c_{\varepsilon,\delta}(x)}{h^2} - \frac{3\lambda_1 p_{i-1}}{2h} + \lambda_1 q_{i-1} - \frac{\lambda_2 p_i}{h} + \frac{\lambda_1 p_{i+1}}{2h})(Q \pm u_{i-1})\\
&-(\frac{2\sigma_i c_{\varepsilon,\delta}(x)}{h^2} - \frac{2\lambda_1 p_{i-1}}{h} - 2\lambda_2 q_i + \frac{2\lambda_1 p_{i+1}}{h})(Q \pm u_{i-1})\\
&+(\frac{\sigma_i c_{\varepsilon,\delta}(x)}{h^2} - \frac{\lambda_1 p_{i-1}}{2h} + \frac{\lambda_2 p_i}{h} + \frac{3\lambda_1 p_{i+1}}{2h} + \lambda_1 q_{i+1})(Q \pm u_{i+1})\\
=&(\frac{\sigma_i c_{\varepsilon,\delta}(x)}{h^2} - \frac{3\lambda_1 p_{i-1}}{2h} - \frac{\lambda_2 p_i}{h} + \frac{\lambda_1 p_{i+1}}{2h})(Q \pm u_{i-1}) + (\lambda_1 q_{i-1})(Q \pm u_{i-1})\\
&-(\frac{2\sigma_i c_{\varepsilon,\delta}(x)}{h^2} - \frac{2\lambda_1 p_{i-1}}{h} + \frac{2\lambda_1 p_{i+1}}{h})(Q \pm u_i) + (2\lambda_2 q_i)(Q \pm u_i)\\
&+(\frac{\sigma_i c_{\varepsilon,\delta}(x)}{h^2} - \frac{\lambda_1 p_{i-1}}{2h} + \frac{\lambda_2 p_i}{h} + \frac{3\lambda_1 p_{i+1}}{2h})(Q \pm u_{i+1}) + (\lambda_1 q_{i+1})(Q \pm u_{i+1})\\
=&\pm \frac{\sigma_i c_{\varepsilon,\delta}(x)}{h^2}(u_{i-1} - 2u_i + u_{i+1}) \pm \frac{\lambda_1 p_{i-1}}{2h}(-u_{i+1} + 4u_i - 3u_{i-1}) \pm \frac{\lambda_2 p_i}{h}(u_{i+1} - u_{i-1})\\
&\pm \frac{\lambda_1 p_{i+1}}{2h}(3u_{i+1} - 4u_i + u_{i-1}) \pm (\lambda_1 q_{i-1}u_{i-1} \pm 2\lambda_2 q_i u_i \pm \lambda_1 q_{i+1}u_{i+1})\\
&+(\lambda_1 q_{i-1} + 2\lambda_2 q_i + \lambda_1 q_{i+1})Q\\
=&(\lambda_1 q_{i-1} + 2\lambda_2 q_i + \lambda_1 q_{i+1})\left(\frac{\left\|L^N_{c_{\varepsilon,\delta}}u_i\right\|}{q^*} + \max\{u_0, u_N\}\right) \pm \lambda_1(f_{i-1} + f_{i+1}) + 2\lambda_2 f_i\\
\geq& 0, \ \textit{since } q(x_i) \geq q^* > 0.
\end{aligned}$$

By discrete maximum principle in Lemma 4.4.1 we obtain $\vartheta_i^{\pm} \geq 0, \forall x_i \in \bar{\Omega}^N$. Hence the required bound is obtained. □

Theorem 4.4.3. *Let $u(x_i)$ be the analytical solution of the problem in Eqs.* (4.1.1) *and* (4.2.1) *and u_i be the numerical solution of the discretized problem of Eq.* (4.3.15)*. Then,*

$$\|u(x_i) - u_i\| \leq ch^2$$

for sufficiently small h and c is a positive constant.

Proof. Multiplying both side of Eq. (4.3.15) by $-\frac{h^2}{c_{\varepsilon,\delta}(x)\sigma_i}$ and simplifying, we obtain

$$(-1 + k_i)u_{i-1} + (2 + l_i)u_i + (-1 + m_i)u_{i+1} + G_i + T_1 = 0, \tag{4.4.2}$$

where

$$\begin{cases}
k_i = \dfrac{1}{c_{\varepsilon,\delta}(x)\sigma_i}(\dfrac{3\lambda_1 h p_{i-1}}{2} - \lambda_1 h^2 q_{i-1} + \lambda_2 h p_i - \dfrac{\lambda_1 h p_{i+1}}{2})\\
l_i = \dfrac{2}{c_{\varepsilon,\delta}(x)\sigma_i}(\lambda_1 h p_{i+1} - \lambda_1 h p_{i-1} - \lambda_2 h^2 q_i)\\
m_i = \dfrac{1}{c_{\varepsilon,\delta}(x)\sigma_i}(\dfrac{\lambda_1 h p_{i-1}}{2} - \lambda_2 h p_i - \dfrac{3\lambda_1 h p_{i+1}}{2} - \lambda_1 h^2 q_{i+1})\\
G_i = -\dfrac{h^2}{c_{\varepsilon,\delta}(x)\sigma_i}(\lambda_1(f_{i-1} + f_{i+1}) + 2\lambda_2 f_i) \qquad \textit{and}\\
\qquad T_i = (\lambda_1(p_{i-1} + p_{i+1}) - 4\lambda_2 p_i)\dfrac{h^4 u'''(\xi)}{12 c_{\varepsilon,\delta}(x)\sigma_i}\\
\qquad \text{is a local truncation error for} i = 1, 2, ..., N-1
\end{cases}$$

Incorporating the boundary condition $u_0 = \phi_0,\ u_N = \gamma$ in Eq. (4.4.2), we get the system of equation of the form

$$(D+P)U + M + T(h) = 0, \tag{4.4.3}$$

where,

$$D = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & - & - & & 0 \\ \vdots & & & & -1 \\ 0 & - & - & -1 & 2 \end{pmatrix}, \; P = \begin{pmatrix} l_1 & m_1 & 0 & \cdots & 0 \\ k_2 & l_2 & m_2 & \cdots & 0 \\ 0 & - & - & & 0 \\ \vdots & & & & m_{N-2} \\ 0 & - & - & k_{N-1} & l_{N-1} \end{pmatrix}$$

are tri-diagonal matrices of order N-1, $M = [G_1+(-1+k_1)\phi_0, G_2, ..., G_{N-1}+(-1+m_{N-1})_\gamma]^T$, $T(h) = O(h^4)\ and\ U = [u_1, u_2, ..., u_{N-1}]^T$, $T(h) = [T_1, T_2, ..., T_{N-1}]^T$, $\bar{0} = [0, 0, ..., 0]^T$ are associated vector of Eq. (4.4.3).
Let $U^N = [u_1^N, ..., u_{N-1}^N]^T \cong U$ be the solution which satisfies Eq. (4.4.3), We have

$$(D+P)U^N + M + T(h) = \bar{0}. \tag{4.4.4}$$

Let $e_i = U_i - U_i^N\ for\ i = 1, 2, ..., N-1$ be the discretization error then $U_i - U_i^N = [e_1, e_2, ..., e_{N-1}]^T$ substituting Eq. (4.4.3) in to Eq. (4.4.4) we get

$$(D+P)(U^N - U) = T(h) \tag{4.4.5}$$

Let $|p_{i-1}| \le \eta_1, |p_i| \le \eta_2, |p_{i+1}| \le \eta_3, |q_{i-1}| \le \varphi_1, |q_i| \le \varphi_2, |q_{i+1}| \le \varphi_3$ and $t_{i,j}$ be the$(i,j)^{th}$ element of the matrix P, Then

$$\begin{cases} |t_{i,i+1}| = |m_i| \le \dfrac{h}{c_{\varepsilon,\delta}(x)\sigma_i}(\dfrac{3\lambda_1\eta_1}{2} + \lambda_1 h\eta_2 + 3\lambda_1\eta_3 + \lambda_1\varphi_3) \\ |t_{i,i-1}| = |k_i| \le \dfrac{h}{c_{\varepsilon,\delta}(x)\sigma_i}(\dfrac{3\lambda_1\eta_1}{2} + \lambda_1 h\varphi_2 + \lambda_2\eta_2 + \dfrac{\lambda_1\eta_3}{2}), \qquad i = 1, 2, ..., N-1 \end{cases}$$

Thus, for sufficiently small h, we have

$$\begin{cases} -1 + |t_{i,i+1}| \ne 0, \qquad i = 1, 2, ..., N-2 \\ -1 + |t_{i,i-1}| \ne 0, \qquad i = 2, 3, ..., N-1 \end{cases}$$

Hence, the matrix (D+P) is irreducible(Varga, 1962).
Let s_i be the sum of the element of the i^{th} row of the matrix $(D+P)$, then

$$s_i = 1 + l_i + m_i = 1 + \frac{2h}{c_{\varepsilon,\delta}(x)\sigma_i}(\lambda_1 p_{i+1} - \lambda_1 p_{i-1} + \frac{\lambda_1 p_{i-1}}{4} - \frac{\lambda_2 p_i}{2} - \frac{3\lambda_1 p_{i+1}}{4}) + O(h^2), \quad i = 1$$

$$s_i = k_i + l_i + m_i = \frac{h^2}{c_{\varepsilon,\delta}(x)\sigma_i}(-\lambda_1 q_{i-1} - \lambda_2 q_i - \lambda_1 q_{i+1}), \quad i = N-2$$

$$s_i = 1 + k_i + l_i = 1 + \frac{2h}{c_{\varepsilon,\delta}(x)\sigma_i}(\frac{3\lambda_1 p_{i+1}}{4} - \frac{\lambda_1 p_{i-1}}{4} + \frac{\lambda_2 p_i}{2}) + O(h^2), \quad i = N-2$$

Let

$$\begin{cases} r_1 = \min\limits_{1\le i\le N-1} \dfrac{1}{c_{\varepsilon,\delta}(x)\sigma_i}(-\lambda_1 q_{i-1} - 2\lambda_2 q_i - \lambda_1 q_{i+1}) \\ r_2 = \max\limits_{1\le i\le N-1} \dfrac{1}{c_{\varepsilon,\delta}(x)\sigma_i}(-\lambda_1 q_{i-1} - 2\lambda_2 q_i - \lambda_1 q_{i+1}), \end{cases}$$

Then, $0 \leq r_1 \leq r_2$.
For sufficient small h, $(D+P)$ is monotone(Varga, 1962), Hence, $(D+P)^{-1}$ exists and $(D+P)^{-1} \geq 0$.
From the error in Eq. (4.4.5), we have

$$\|U - U^N\| \leq \|(D+P)^{-1}\|\|T(h)\|. \tag{4.4.6}$$

For sufficiently small h, we have $s_i > h^2 r_1, \ for \ i = 1, 2, ..., N-1$,
where

$$r_1 = \min_{1 \leq i \leq N-1} \frac{1}{c_{\varepsilon,\delta}(x)\sigma_i}(-\lambda_1 q_{i-1} - 2\lambda_2 q_i - \lambda_1 q_{i+1}).$$

Let $(D+P)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(D+P)^{-1}$ and we define

$$\begin{aligned} \|(D+P)^{-1}\| &= \max_{1 \leq i \leq N-1} \sum_{k=1}^{N-1} (D+P)^{-1}_{i,k} \qquad and \\ \|T(h)\| &= \max_{1 \leq i \leq N-1} |T_i|. \end{aligned} \tag{4.4.7}$$

Since $(D+P)^{-1}_{i,k} \geq 0$ from the theory of matrices, we have:
$\sum_{k=1}^{N-1} (D+P)^{-1}_{i,k}.s_k = 1 \ for \ i = 1, 2, ..., N-1$.
Hence,

$$\sum_{k=1}^{N-1} (D+P)^{-1}_{i,k} s_k \leq \frac{1}{\min_{1 \leq i \leq N-1} s_k} \leq \frac{1}{h^2 r_1} \tag{4.4.8}$$

Now, from Eqs. (4.4.5)- (4.4.8), we get

$$\|(D+P)^{-1}\| \leq \frac{1}{h^2 r_1}\left(\left|(\lambda_1(p_{i-1}+p_{i+1}) - 4\lambda_2 p_i)\frac{h^4 u^{'''}(\xi)}{12 c_{\varepsilon,\delta}(x)\sigma_i}\right|\right) \tag{4.4.9}$$

$$\leq \left(\frac{u^{'''}(\xi)(4\lambda_2 p_i + \lambda_1(p_{i-1}+p_{i+1}))}{12 r_1 \sigma_i}\right) h^2 \tag{4.4.10}$$

$$= Ch^2,$$

where $C = \frac{u^{'''}(\xi)(4\lambda_2 p_i + \lambda_1(p_{i-1}+p_{i+1}))}{12 r_1 \sigma_i}$ which is independent of mesh size$h$. This establishes that the method is second order uniform convergence. □

4.5 Numerical Results and Discussion

In this section, two examples are given to illustrate the numerical method discussed above.The exact solutions of the test problems are not known. Therefore, the maximum absolute errors are computed by the double mesh principle (Phaneendra and Lalu, 2019):

$$E^N_{\varepsilon,\delta} = \max_{0 \leq i \leq 2N} |u^N_i - u^{2N}_{2i}|,$$

where u_i^N is the numerical solution with N mesh points and u_{2i}^{2N} is the numerical solution at the finer mesh with $2N$ mesh points. The numerical rate of convergence is calculated as

$$r_{\varepsilon,\delta}^N = \log_2 \left(E_{\varepsilon,\delta}^N / E_{\varepsilon,\delta}^{2N} \right).$$

For each N, the parameter uniform maximum absolute errors are computed using

$$E^N = \max(E_{\varepsilon,\delta}^N).$$

Example 4.5.1. Consider the following problem

$$-\varepsilon u''(x) + (1+x)u'(x-\delta) - \exp^{-2x} u(x-\delta) + \exp^{-x} u(x) = 0$$

with the interval-boundary condition $u(x) = 1,\ -\delta \leq x < 0$ and $u(1) = -1$

Example 4.5.2. Consider the following problem

$$-\varepsilon u''(x) + (1+x)u'(x-\delta) - \exp^{-2x} u(x-\delta) + \exp^{-x} u(x) = \exp^{x-1}$$

with the interval-boundary condition $u(x) = 1,\ -\delta \leq x < 0$ and $u(1) = -1$

Table 4.1: Comparison of maximum absolute errors of proposed scheme and results in Kadalbajoo and Ramesh (2007), (Kumar and Kadalbajoo, 2012) and (Angasu et al., 2021) for Example 4.5.1 with $\delta = 0:3 \times \varepsilon$.

$\varepsilon \downarrow$	N=32	64	128	256	512	1024
Present Method						
2^{-0}	1.6733e-06	4.1921e-7	1.0484e-7	2.6211e-8	6.5545e-9	1.6375e-09
2^{-4}	7.7227e-05	1.9467e-05	4.8768e-06	1.2198e-06	3.0499e-07	7.6249e-08
2^{-8}	9.5491e-04	3.6796e-04	1.0835e-04	2.8393e-05	7.1855e-06	1.8019e-06
2^{-12}	1.0257e-03	5.4045e-04	2.7738e-04	1.4014e-04	6.5819e-05	2.4223e-05
2^{-16}	1.0257e-03	5.4046e-04	2.7738e-04	1.4052e-04	7.0720e-05	3.5476e-05
2^{-20}	1.0257e-03	5.4046e-04	2.7738e-04	1.4052e-04	7.0720e-05	3.5476e-05
E^N	1.0257e-03	5.4046e-04	2.7738e-04	1.4052e-04	7.0720e-05	3.5476e-05
Results in (Kadalbajoo and Ramesh, 2007)				Upwind		
2^{-4}	3.72e-2	2.18e-2	1.27e-2	7.28e-3	4.12e-3	2.30e-3
2^{-8}	4.22e-2	2.56e-2	1.49e-2	8.31e-3	4.53e-3	2.45e-3
2^{-12}	3.93e-2	2.36e-2	1.38e-2	7.89e-3	4.49e-3	4.49e-3
2^{-16}	3.90e-2	2.33e-2	1.35e-2	7.66e-3	4.30e-3	2.39e-3
2^{-20}	3.90e-2	2.33e-2	1.35e-2	7.65e-3	4.28e-3	2.37e-3
E^N	3.90e-2	2.33e-2	1.35e-2	7.65e-3	4.28e-3	2.37e-3
Results in (Kumar and Kadalbajoo, 2012)				B-spline		
2^{-4}	1.1491e-02	2.7950e-03	6.9407e-04	1.7323e-04	4.3289e-05	1.0821e-05
2^{-8}	3.7010e-02	1.2611e-02	4.2314e-03	1.4187e-03	4.6812e-04	1.5501e-04
2^{-12}	3.6669e-02	1.2541e-02	4.2449e-03	1.4298e-03	4.7508e-04	1.5924e-04
2^{-16}	3.6701e-02	1.2463e-02	4.1666e-03	1.4086e-03	4.7513e-04	1.5946e-04
2^{-20}	3.6709e-02	1.2478e-02	4.1763e-03	1.3898e-03	4.5369e-04	1.5379e-04
E^N	3.6709e-02	1.2478e-02	4.1763e-03	1.3898e-03	4.5369e-04	1.5379e-04
Results in (Angasu et al., 2021)				Finite Difference Method		
E^N	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05

The maximum absolute errors of Examples 4.5.1 and 4.5.2 are presented in Tables 4.1-4.4 for different values of perturbation parameter ε and N. These tables show that as $\varepsilon \to 0$, the maximum absolute errors goes to constant, which indicates that the proposed scheme converges uniformly independent of the effect of ε. As one see in Table 4.4 the scheme have second order uniform rate of convergence. The comparison of the maximum absolute errors using the proposed scheme with the results in (Kadalbajoo and Ramesh, 2007), (Kumar and Kadalbajoo, 2012) and Angasu et al. (2021) are give in In Table 4.1. This table shows that the proposed scheme provides more accurate than the result given in Kadalbajoo and Ramesh (2007), Kumar and Kadalbajoo (2012) and Angasu et al. (2021).

Table 4.2: The computed maximum absolute errors ($E^N_{\varepsilon,\delta}$) and uniform maximum absolute errors (E^N) for Example 4.5.2

$\varepsilon \downarrow$	N=32	64	128	256	512	1024
Present Method						
2^{-0}	3.7924e-06	9.4846e-07	2.3708e-07	5.9270e-08	1.4816e-08	3.7055e-09
2^{-4}	1.4633e-04	3.5477e-05	8.8140e-06	2.2022e-06	5.5028e-07	1.3756e-07
2^{-8}	9.4991e-04	4.2221e-04	1.4702e-04	3.6363e-05	8.0495e-06	1.9507e-06
2^{-12}	1.0230e-03	5.3925e-04	2.7683e-04	1.3987e-04	6.5651e-05	2.5677e-05
2^{-16}	1.0232e-03	5.3933e-04	2.7686e-04	1.4027e-04	7.0600e-05	3.5417e-05
2^{-20}	1.0232e-03	5.3933e-04	2.7687e-04	1.4027e-04	7.0601e-05	3.5417e-05
E^N	1.0232e-03	5.3933e-04	2.7687e-04	1.4027e-04	7.0601e-05	3.5417e-05

Table 4.3: Maximum absolute errors for different values of delay parameter for $\varepsilon = 0.1$.

$\delta \downarrow$	N=32	64	128	256	512	1024
	Example 4.5.1					
$\delta = 0$	8.2428e-05	2.0572e-05	5.0818e-06	1.2666e-06	3.1642e-07	7.9090e-08
$\delta = 0.1\varepsilon$	7.2720e-05	1.7610e-05	4.3909e-06	1.0955e-06	2.7386e-07	6.8455e-08
$\delta = 0.2\varepsilon$	6.3093e-05	1.5415e-05	3.8316e-06	9.5744e-07	2.3927e-07	5.9819e-08
$\delta = 0.3\varepsilon$	5.4502e-05	1.3573e-05	3.3783e-06	8.4466e-07	2.1111e-07	5.2773e-08
$\delta = 0.4\varepsilon$	4.8615e-05	1.2050e-05	3.0071e-06	7.5138e-07	1.8783e-07	4.6954e-08
	Example 4.5.2					
$\delta = 0$	1.2888e-04	3.1150e-05	7.7212e-06	1.9262e-06	4.8128e-07	1.2031e-07
$\delta = 0.1\varepsilon$	1.1331e-04	2.7785e-05	6.9344e-06	1.7311e-06	4.3264e-07	1.0816e-07
$\delta = 0.2\varepsilon$	1.0109e-04	2.5127e-05	6.2559e-06	1.5632e-06	3.9071e-07	9.7677e-08
$\delta = 0.3\varepsilon$	9.2411e-05	2.2816e-05	5.6862e-06	1.4212e-06	3.5524e-07	8.8808e-08
$\delta = 0.4\varepsilon$	8.4184e-05	2.0837e-05	5.2061e-06	1.3007e-06	3.2516e-07	8.1288e-08

To demonstrate the effect of delay on the right boundary layer of the solution of (4.5.1), graphs for different values of delay parameter ε, mesh size h and perturbation parameter ε are plotted in Figures 4.1 and 4.1. From Figure 4.1 one can see that, the perturbation parameter goes small the graphs of the solution forms strong boundary layer.

Table 4.4: Comparison of the computed numerical rate of convergence for Example 4.5.1 and Example4.5.2.

$\varepsilon \downarrow$	N=32	64	128	256	512
Example 4.5.1					
Present Method					
2^{-0}	1.9970	1.9995	1.9999	1.9996	2.0010
2^{-4}	1.9881	1.9970	1.9993	1.9998	2.0000
2^{-8}	1.3758	1.7638	1.9321	1.9824	1.9956
Results in (Kadalbajoo and Ramesh, 2007) Upwind					
2^{-4}	0.7710	0.7795	0.8028	0.8213	0.8410
2^{-8}	0.7211	0.7808	0.8424	0.8753	0.8867
Results in (Kumar and Kadalbajoo, 2012) B-spline					
2^{-4}	2.0396	2.0097	2.0024	2.0006	2.0002
2^{-8}	1.5532	1.5755	1.5766	1.5996	1.5945
Example 4.5.2					
2^{-0}	1.9995	2.0002	2.0000	2.0001	1.9994
2^{-4}	2.0443	2.0090	2.0009	2.0007	2.0001
2^{-8}	1.1698	1.5219	2.0155	2.1755	2.0449

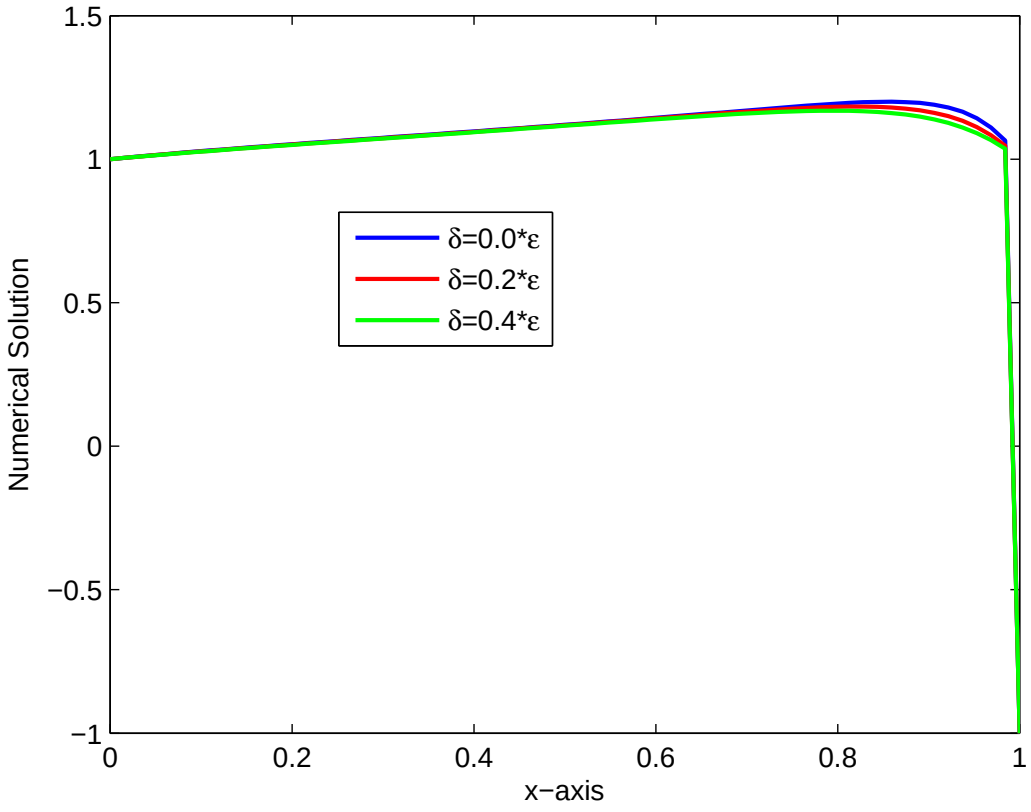

Figure 4.1: Effect of delay on the solution profile for $\varepsilon = 0.1$ in Example 4.5.1.

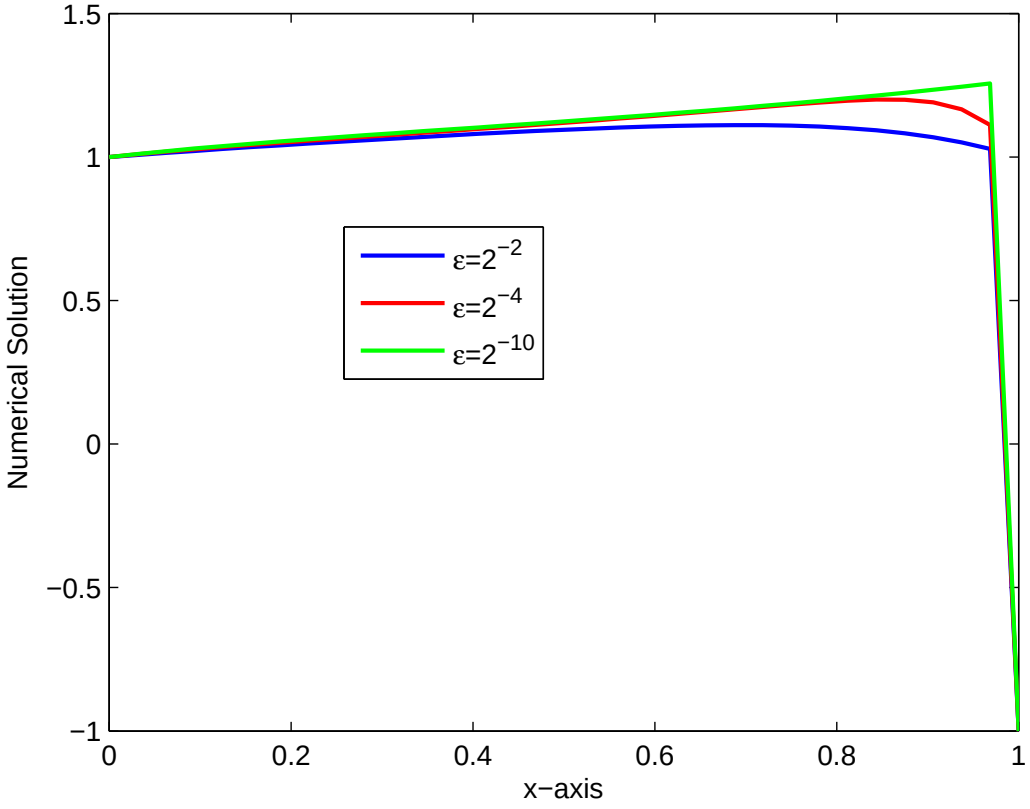

Figure 4.2: Effect of ε on the solution showing boundary layer formation in Example 4.5.1.

Chapter 5

Conclusion and Recommendation

5.1 Conclusion

In this thesis, we have developed, analyzed, and implemented a parameter-uniform convergent fitted computational method for solving singularly perturbed ordinary differential equations with small delays which has right boundary layer. To develop this method, we used a Taylor series expansion for the approximation of the terms with delay and the non-polynomial cubic spline method for the approximation of the resulting equation. It is shown that the developed method converges uniformly with respect to the perturbation parameter ε which is second-order accurate. Two test examples are presented to illustrate the applicability and efficiency of the proposed scheme. It is observed that the proposed computational method gives more accurate numerical results than the results in (Kumar and Kadalbajoo, 2012; Kadalbajoo and Ramesh, 2007) and (Angasu et al., 2021). Thus, from the results presented, we have seen that the proposed method is an efficient parameter-uniform convergent way to approximate the solution of the singularly perturbed differential equation having small delays.

5.2 Recommendation

In this work, a parameter-uniform convergent fitted computational method for solving singularly perturbed ordinary differential equations with small delays is presented. However, one can extend this method for higher order singularly perturbed BVPs, discontinuous coefficients problems, exhibiting interior layer singularly perturbed BVPs, and others.

References

Adilaxmi, M., Bhargavi, D., and Reddy, Y. (2019). An initial value technique using exponentially fitted non standard finite difference method for singularly perturbed differential-difference equations. *Applications and Applied Mathematics: An International Journal (AAM)*, 14(1):16.

Amiraliyev, G. M. and Cimen, E. (2010). Numerical method for a singularly perturbed convection–diffusion problem with delay. *Applied Mathematics and Computation*, 216(8):2351–2359.

Angasu, M. A., Duressa, G. F., and Woldaregay, M. M. (2021). Exponentially fitted numerical scheme for singularly perturbed differential equations involving small delays. *Journal of applied mathematics & informatics*, 39(3_4):419–435.

Bender, C. M. and Orszag, S. A. (2013). *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory*. Springer Science & Business Media.

Derstine, M., Gibbs, H., Hopf, F., and Kaplan, D. (1982). Bifurcation gap in a hybrid optically bistable system. *Physical Review A*, 26(6):3720.

File, G. and Edosa, F. (2017). Higher order compact finite difference method for singularly perturbed one dimensional reaction diffusion problems. *Journal of the Nigerian Mathematical Society*, 36(3):491–502.

File, G., Reddy, Y., et al. (2013). Computational method for solving singularly perturbed delay differential equations with negative shift. *International Journal of Applied Science and Engineering*, 11(1):101–113.

Gadissa, G. and File, G. (2019). Fitted fourth order scheme for singularly perturbed delay convection-diffusion equations. *Ethiopian Journal of Education and Sciences*, 14(2):102–118.

Glizer, V. (2003). Asymptotic analysis and solution of a finite-horizon h control problem for singularly-perturbed linear systems with small state delay. *Journal of Optimization Theory and Applications*, 117:295–325.

Johnson, C. (1983). Uniform numerical methods for problems with initial and boundary layers (ep doolan, jjh miller and wha schilders).

Kadalbajoo, M. K. and Ramesh, V. (2007). Numerical methods on shishkin mesh for singularly perturbed delay differential equations with a grid adaptation strategy. *Applied Mathematics and Computation*, 188(2):1816–1831.

Kadalbajoo, M. K. and Reddy, Y. (1989). Asymptotic and numerical analysis of singular perturbation problems: a survey. *Applied Mathematics and computation*, 30(3):223–259.

Kellogg, R. B. and Tsan, A. (1978). Analysis of some difference approximations for a singular perturbation problem without turning points. *Mathematics of computation*, 32(144):1025–1039.

Kevorkian, J. K. and Cole, J. D. (2012). *Multiple scale and singular perturbation methods*, volume 114. Springer Science & Business Media.

Kumar, D. and Kadalbajoo, M. (2012). Numerical treatment of singularly perturbed delay differential equations using b-spline collocation method on shishkin mesh. *Jnaiam*, 7(3-4):73–90.

Lange, C. G. and Miura, R. M. (1994). Singular perturbation analysis of boundary value problems for differential-difference equations. v. small shifts with layer behavior. *SIAM Journal on Applied Mathematics*, 54(1):249–272.

Longtin, A. and Milton, J. G. (1988). Complex oscillations in the human pupil light reflex with mixed and delayed feedback. *Mathematical Biosciences*, 90(1-2):183–199.

Mackey, M. C. (1997). Mathematical models of hematopoietic cell replication and control. *Case Studies in Mathematical Modeling–Ecology, Physiology and Cell Biology. New Jersey, Prentice-Hall*, pages 151–182.

Mackey, M. C. and Glass, L. (1977). Oscillation and chaos in physiological control systems. *Science*, 197(4300):287–289.

Miller, J. J., O'riordan, E., and Shishkin, G. I. (1996). *Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions*. World scientific.

Nayfeh, A. H. (2011). *Introduction to perturbation techniques*. John Wiley & Sons.

Nicaise, S. and Xenophontos, C. (2013). Robust approximation of singularly perturbed delay differential equations by the hp finite element method. *Computational Methods in Applied Mathematics*, 13(1):21–37.

O'malley, R. E. et al. (1991). *Singular perturbation methods for ordinary differential equations*, volume 89. Springer.

Phaneendra, K. and Lalu, M. (2019). Numerical solution of singularly perturbed delay differential equations using gaussion quadrature method. In *Journal of Physics: Conference Series*, volume 1344, page 012013. IOP Publishing.

Prandtl, L. (1905). Über flüssigkeits% bewegung bei kleiner reibung, verhand# lungen, iii. *Int. Math*, 13:740–766.

Robert Jr, E. et al. (2012). *Singular perturbation methods for ordinary differential equations*, volume 89. Springer Science & Business Media.

Roos, H.-G. (2008). *Robust numerical methods for singularly perturbed differential equations*. Springer.

Sekar, E. and Tamilselvan, A. (2019). Singularly perturbed delay differential equations of convection–diffusion type with integral boundary condition. *Journal of Applied Mathematics and Computing*, 59:701–722.

Selvi, P. A. and Ramanujam, N. (2016). An iterative numerical method for singularly perturbed reaction–diffusion equations with negative shift. *Journal of Computational and Applied Mathematics*, 296:10–23.

Shah, N. A., Agarwal, P., Chung, J. D., El-Zahar, E. R., and Hamed, Y. S. (2020). Analysis of optical solitons for nonlinear schrödinger equation with detuning term by iterative transform method. *Symmetry*, 12(11):1850.

Shishkin, G. I. and Shishkina, L. P. (2008). *Difference methods for singular perturbation problems*. Chapman and Hall/CRC.

Subburayan, V. and Ramanujam, N. (2013). An initial value technique for singularly perturbed reaction-diffusion problems with a negative shift. *Novi Sad J. Math*, 43(2):67–80.

Tian, H. (2002). The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag. *Journal of Mathematical Analysis and Applications*, 270(1):143–149.

Varga, R. S. (1962). Iterative analysis. *New Jersey*, 322.

Vasil'Eva, A. B., Butuzov, V. F., and Kalachev, L. V. (1995). *The boundary function method for singular perturbation problems*. SIAM.

Wasow, W. R. (1942). *On Boundary Layer Problems in the Theory of Ordinary Differential Equation*. New York University.

Woldaregay, M. M., Debela, H. G., and Duressa, G. F. (2022). Uniformly convergent fitted operator method for singularly perturbed delay differential equations. *Computational Methods for Differential Equations*, 10(2):502–518.

Woldaregay, M. M. and Duressa, G. F. (2019). Parameter uniform numerical method for singularly perturbed parabolic differential difference equations. *Journal of the Nigerian Mathematical Society*, 38(2):223–245.

Woldaregay, M. M. and Duressa, G. F. (2021). Robust numerical scheme for solving singularly perturbed differential equations involving small delays. *Appl. Math. E-Notes*, 21:622–633.

Woldaregay, M. M. and Duressa, G. F. (2022). Fitted numerical scheme for singularly perturbed differential equations having two small delays. *Caspian Journal of Mathematical Sciences (CJMS)*, 11(1):98–114.