Salale University

College of Natural Sciences
Department of Mathematics

MSc Thesis

Fitted Numerical Scheme for Solving Singularly Perturbed Parabolic Delay Differential Equation Involving Small Delay

By Nigusu Alemu Tamene

June 2023
Fitche ,Ethiopia

Salale University

College of Natural Sciences
Department of Mathematics

MSc Thesis

Fitted Numerical Scheme for Solving Singularly Perturbed Parabolic Delay Differential Equation Involving Small Delay

A Thesis Submitted to Department of Mathematics for the Partial Requirement of the Degree of Masters of Science in Mathematics (Numerical Analysis)

By Nigusu Alemu Tamene

Advisor:Imiru Takele(PhD,Assistant Professor)

Co-advisor:Genanew Gofe (PhD,Associate Professor)

June 2023
Fitche , Ethiopia

Salale University

College of Natural Sciences

Approval Sheet For Submitting Thesis

A Thesis on: Fitted Numerical Scheme for Solving Singularly Perturbed Parabolic Delay Differential Equation Involving Small Delay

Submitted by:

______________________	______________	__________
Name of Candidate	Signature	Date

Approval by:

1.______________________	______________	__________
Name of Advisor	Signature	Date
2.______________________	______________	__________
Name of Co-Advisor	Signature	Date
3.______________________	______________	__________
Name of External Examiner	Signature	Date
4.______________________	______________	__________
Name of Internal Examiner	Signature	Date
5.______________________	______________	__________
Name of Chairman	Signature	Date
6.______________________	______________	__________
Name of Dean SGS	Signature	Date

Declaration

I declare that this thesis entitled "fitted numerical scheme for solving singularly perturbed parabolic delay differential equation involving small delay" is my original work and that all source materials used for this thesis have been properly acknowledged. This thesis has been submitted in partial fulfillment of the requirements for degree of Masters of Science in Numerical Analysis at Salale University. I earnestly declare that this thesis is not submitted to any institutions any where for the award of any academic degree,diploma or certificate.

Researcher

Name	Signature	Date

The work has been done under the supervision and approval of the advisor

1.

Name of Advisor	Signature	Date

2.

Name of Co-Advisor	Signature	Date

Acknowledgment

First of all I would like to thank the almighty God who helped me to do every activity in every time. Next to this,it is great pleasure and proud privilege to express my deepest and sincere gratitude to my advisor Dr.Imiru Takele (PhD,Assistant Professor) and my co-advisor Dr.Genanew Gofe (PhD,Associate Professor) for their persistent help,constructive suggestions,advise and limitless effort in encouraging me during my work,correcting and giving comments by devoting their time from the beginning of proposal development up to the end of thesis work.

I greatly appreciate them not only with their keen and unlimited constructive advises in every aspect of my thesis work,but also for their professional and personal ethic,that should be an icon for others.Thirdly I would like to thank my partners who shared us different ideas and supporting me by their own material is needed to complete this thesis.

Finally I would like to thank my families those tired with me until now and supporting me by financial as well as encouraging me in every day to day.

Abbreviations and Symbols

Abbreviations		
FDM	:	Finite Difference Method
FEM	:	Finite Element Method
DE	:	Differential Equation
PDE	:	Partial Differential Equation
ODE	:	Ordinary Differential Equation
SPDE	:	Singularly Perturbed Differential Equation
SPP	:	Singularly perturbed Problem
SPPDDE	:	Singularly Perturbed Parabolic Delay Differential Equation
SPDDE	:	Singularly Perturbed Delay Differential Equation
LTE	:	Local Truncation Error
GTE	:	Global Truncation Error
Symbols		
δ	:	Delay
ε	:	Singular Perturbation Parameter

Table of Contents

List of Tables

List of Figures

Abstract

This thesis is attempted to introduce on fitted numerical solution of singularly perturbed parabolic delay differential equation involving small delay.Due to the appearance of the multi-scale phenomena,it is not an easy task to solve this problem analytically. Consequently,a direct or analytical for fitted numerical scheme for singularly perturbed parabolic delay differential equation involving small delay is still lacking, so one has to rely on numerical method for this problem. The scheme comprises for this problem is an implicit Euler method to discretized time variable on uniform mesh and cubic spline in tension method on space variable.The solution of the problem exhibits a parabolic boundary layer in the neighborhood of x=0. The main purpose of this study is to develop and analyze the fitted numerical scheme for singularly perturbed parabolic delay differential equation involving small delay.

Chapter 1

Introduction

1.1 Background of the Study

Mathematical models are developed to help in understanding of physical phenomena. A model is simplified version of the process that approximate the behavior of its subject of phenomena. Due to the difficulties in ending the analytical solution of mathematical problems,leads to the development of numerical analysis. Numerical analysis is a branch of mathematics that provides tools and methods for solving mathematical/real-life problems in numerical form (Gautschi, 2011).

Numerical analysis is concerned with the mathematical derivation of numerical methods,designing an algorithm for numerical methods and implementation of these algorithm on computers. It is also focuses on analysis of the errors associated with numerical methods to solve the mathematical problems.

Many problem in science and engineering can be modeled by differential equations involving small parameters. These models often yield equation involving derivatives of one or more dependent variables concerning one or more independent variable and this equation are called a differential equations(DEs). Those differential equations in which the unknown function is a function of only one independent variable are called ordinary differential equation(ODEs). A differential equation involving partial derivatives of unknown functions concerning two or more independent variables are called partial differential equations(PDEs).

A parabolic partial differential equation (PPDEs) are a type of partial equations that are used to describe a wide variety of time dependent phenomena,including heat conduction,particle diffusion and pricing derivatives of investment instruments. Such type of phenomena are extremely difficult (even impossible) to solve their exact solution mathematically. In this circumstance approximate solutions and the development of numerical methods to obtain approximate solution becomes necessary. To that extent, several numerical methods such as FDM,FEM and spline approximations methods,among others have developed in the literature based on the nature and type of the differential equation to solve these problems.

A singular perturbation was introduced by Prandtl at the Third International Congress of mathematicians in 1904. Many practical problems such as mathematical layer theory or approximation of the solutions of various problems are described by differential equation involving large or small parameter. A singularly perturbed differential equation (SPDE) is a differential equation in which the highest order derivative is multiplied by small positive parameter.

The points of the domain where coefficients of the convection term in the singularly perturbed differential equation vanishes are known as the turning points. The solution of such type of differential equation exhibits boundary layers or interior layers depending upon the nature of convection and reaction term.If the turning point occurs at the boundary of the domain then the problem is called as a boundary turning problem, otherwise it is an interior turning point problem.

Singularly perturbed problems with boundary turning point attracted the attention of various researchers due to their importance in the modeling of many real life phenomenons in engineering and science. (Hanks, 1971) stated single boundary turning point problem arise in the modeling if heat and mass transport flow near an oceanic rise due to linear velocity distribution.

The multiple boundary turning point problems arise in the modeling of the thermal boundary layers in laminar flow Schlichting (1979). The mathematical modeling of the process of the convective transport of diffusing substance (heat,matter) when rate of the flow from one of the boundaries is proportional to the distance from the boundary results into singularly perturbed problem(SPP) with a boundary turning point.

1.2 Statement of the Problem

In the past and recent years studies much interest have been given to solve unsteady Singularly perturbed parabolic delay differential equations, due to their wide applicability in modeling of processes in various application fields. Finding the solution to these problems has a significant role to capture the behavior of the physical phenomena of the problems. Most of the existing published works deal with the numerical treatment of the classical singularly perturbed parabolic delay differential equations.

(Gupta et al., 2018), Higher order numerical approximation time-dependent singularly perturbed differential difference convection diffusion equations.
(Yüzbaşı and Sezer, 2013) used an exponential collection method for finding solution of second order singularly perturbed delay differential equations.

(File et al., 2013) presented a computational method to solve equation with negative shift whose solution has boundary layer. In this scheme author reduced the second order SPDDE to first order equation and then employed numerical integration and interpolations.
(Doğan et al., 2012) A parameter-uniform numerical method for time-dependent singularly perturbed differential-difference equations. (Ramesh and Kadalbajoo, 2008) Upwind and midpoint upwind difference methods for time-dependent differential difference equations with layer behavior.
Mbroh et al. (2021)proposed the Crank Nicolson finite difference to discretized the time and a fitted finite difference scheme to discretized space derivative based on the mid point downwind scheme for a singularly perturbed degenerate parabolic problem.

Singh et al. (2022) developed a numerical scheme to solve singularly perturbed convection diffusion type degenerate parabolic problems using the Crank Nicolson scheme to discretized temporal direction and quadratic spline collocation method to discretized in space direction for singularly perturbed degenerated parabolic problems.

Rai and Yadav (2021) considered a numerical method which consists of backward Euler scheme for time discretization on uniform mesh and a combination of midpoint upwind and central difference scheme for the space discretization on modified shishkin mesh. Further, they increased the order of convergence in time direction by Richardson extrapolation for singular perturbed delay parabolic convection diffusion problems with degenerate coefficient.

Ku Sahoo and Gupta (2021) discussed scheme consists of an implicit Euler method on uniform mesh in time and simple upwind scheme on piecewise uniform mesh in the space for singularly perturbed parabolic problem with a boundary turning point. Then, they applied the Richardson extrapolation on scheme in both time and space direction to improve the order of convergence.

Woldaregay and Duressa (2021) numerical treatment of singularly perturbed parabolic delay differential equation is solved. However, the methods suggested above for unsteady Singularly perturbed parabolic delay differential equations,classical numerical methods on a uniform mesh fail to approximate the singularly perturbed parabolic delay differential equations. The author require an unacceptably large number of mesh points to sustain the approximation because the mesh width depends on the perturbation parameter ε. This limitation of the conventional numerical methods has encouraged researchers to develop delay numerical techniques that perform well enough independent of the ε.

Recently,Gelu and Duressa (2022) proposed an implicit trapezoidal method for time discretization on uniform mesh and second order central difference scheme for space discretization on Shishkin mesh on singularly perturbed parabolic turning point problem with Robin boundary condition.

Nevertheless, from this thesis the solution methodologies to solve the singularly perturbed parabolic delay differential equations is at infant stage and it needs a lot of studies. This motivates the researchers to develop and analyze a parameter uniform numerical schemes for solving the singularly perturbed parabolic delay differential equations.

Therefore,the main purpose of this study is to construct implicit Euler method to discretize the time variable and cubic spline in tension to discretize the space variable to solve singularly perturbed parabolic problems with Dirichlet boundary condition.

Owing to this,the present study attempted to answer the following question:

. How do we describe fitted numerical scheme for solving singularly perturbed parabolic delay differential equation involving small delay?

. How to provide a layer resolving parameter uniform method with sufficient accuracy?

. To what extent the proposed method converges?

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this study is to formulate the implicit Euler and cubic spline in tension method for solving singularly perturbed parabolic delay differential equation involving small delay.

1.3.2 Specific Objectives

The specific objectives for singularly perturbed parabolic delay differential equations are:

1. To develop the method for solving the problem
2. To develop ϵ uniform numerical scheme for solving singularly perturbed parabolic delay differential equation
3. To establish the stability and convergence of the proposed numerical scheme.

1.4 Significance of the Study

The future behaviors of singularly perturbed parabolic delay differential equation problems are assumed to be described by their solutions. However, it is not easy to solve singularly perturbed parabolic delay differential equation due to the presence of a thin boundary layer in the solution.

The outcomes of this study may have the following importance:

⇒ To provide numerical method for solving singularly perturbed parabolic problems with involving small delay.

⇒ Use as reference materials for scholars who works on this area.

⇒ Helps the graduate students to acquire research skill and scientific procedures.

1.5 Delimitation of the Study

Even though singularly perturbed parabolic delay differential equations are vast topics,this thesis is delimited to focus on fitted numerical scheme for solving singularly perturbed parabolic delay differential equation involving small delay.

$$\frac{\partial u(x,t)}{\partial t} - \epsilon \frac{\partial^2 u(x,t)}{\partial x^2} + a(x)\frac{\partial u(x,t)}{\partial x}(x-\delta,t) + b(x)u(x-\delta,t) = f(x,t) \tag{1.5.1}$$

on the domain $D = \Omega_x \times \Omega_t = (0,1) \times (0,T]$ for some fixed number $T > 0$.
Subject to the following initial and interval boundary conditions.

$$\begin{cases} u(x,0) = u_0(x), x\epsilon\omega_x \\ u(0,t) = \phi(0,t), t\epsilon\omega_t, \\ u(1,t) = \psi(1,t), t\epsilon\omega_t. \end{cases} \tag{1.5.2}$$

Chapter 2

Review of Related Literatures

2.1 Singular Perturbation Theory

Singular perturbation problem was first introduced byPrandtl (1905) during his talk on fluid motion with small friction presented at the Third International Congress of Mathematicians in Heidelberg,in which he demonstrated that fluid flow past over the body can be divide in two regions,a boundary layer and outer region.However the term" Singular perturbation" was first used by Friedrichs and Wasow (1946) in a paper presented at a seminar on non-linear vibrations at New York University. The solution of singular perturbation problems typically contain layers.
Prandtl (1905) originally introduced the term boundary layer,but this term came into more general following the work of Wasow (1942). The study of many theoretical and applied problems in science and technology leads to boundary value problem for singularly perturbed differential equations that have a multi-scale character.

However, most of the problems cannot be completely solved by analytic techniques. Consequently,numerical Simulations has fundamental importance in gaining some useful insights on the solutions of the singularly perturbed differential equations.Singularly perturbed problems arise in the modeling of various modern complicated processes such as fluid flow at high Reynolds number,water quality problems in rivers networks,convective heat transport problem with large perfect number drift diffusion equation of semiconductor device modeling electromagnetic field problem in moving media, financial modeling of option pricing turbulence model,simulation of oil extraction from under-ground reservoirs,theory of plates and shells,atmospheric pollution, groundwater transport and chemical reactor theory.

In the modeling of these processes,characterized by dominant convection and/or intensive reactions,one can observe boundary and interior layers whose width depends on the perturbation parameters can be arbitrarily small. On the other hand, the dominant itself,where the problem in question is considered,can be extremely large, even unbounded,compared to the available computational resources(especially in multidimensional problems for systems of equations).
A complicated geometry of the domains, and or lack of sufficient smoothness (or compatibility) of the problem data may result in singular solutions in which different parts are their own specific scales. Standard numerical methods applied to such multi-scale problems gives unsatisfactorily large errors,which make these methods inapplicable for practical use. Thus,it is of considerable scientific interest to develop a solid mathematical theory and specific computational methods for singularly perturbed multi-scale problems and related problems arising from applications. perturbation theory is a subject which studies the effect of small parameter in the mathematical model problems in DEs.

In Mathematics,more precisely in perturbation theory, a SPP is a problem containing a small parameter that cannot be approximated by setting the parameter value to zero. During the last few years much progress has been made in the theory and in the computer implementation of the numerical treatment of singular perturbation problems. These problems depend on a small positive parameter in such a way that the solution of varies rapidly in some parts and varies slowly in some other parts. The main concern with singularly perturbation problems is the rapid growth or decay of the solution in one or more narrow "layer region(s) ".

The term singular perturbation appears to have been the first coined by Friedrichs and Wasow (1946). Waso continued to contribute to the area of asymptotic methods over many years and his book " Asymptotic expansion for ordinary differential equation " Wasow (1965),attracted much in the area of singular perturbed boundary value problems.

2.2 Singularly Perturbed Partial Differential Equation

A singularly perturbed parabolic delay differential equation is a partial differential equation(PDE) in which the highest order derivative is multiplied by a small positive parameter $\varepsilon(0 < \varepsilon \ll 1)$. Developing numerical schemes for singularly perturbed parabolic delay differential equation is very tough and the developed numerical schemes for this problem are rare.

According to Roos et al. (2008) the basic application of these equations are in the Navier Stoke's equations,in modeling and analysis of heat and mass transfer process when the thermal conductivity and diffusion coefficients are small and the rate of reaction is large.The presence of a small parameter in the given differential equation leads difficulty to obtain satisfactory numerical solutions.Thus,numerical treatment of the singularly perturbed parabolic initial-boundary value problem is problematic because of the presence of boundary layers in its solution.

Woldaregay and Duressa (2021) studied the following class of singularly perturbed parabolic delay problem on the domain $\Omega = \Omega_x \times \Omega_t = (0,1) \times (0,T]$.

$$\frac{\partial u(x,t)}{\partial t} - \epsilon \frac{\partial^2 u(x,t)}{\partial x^2} + a(x)\frac{\partial u}{\partial x}(x-\delta,t) + b(x)u(x-\delta,t) = f(x,t) \tag{2.2.1}$$

Subject to the following initial and boundary conditions.

$$\begin{cases} u(x,0) = u_0(x), x\epsilon\Omega_x \\ u(0,t) = \phi(0,t), t\epsilon\Omega_t, \\ u(1,t) = \psi(1,t), t\epsilon\Omega_t \end{cases} \tag{2.2.2}$$

Where $0 < \epsilon \ll 1$ is singular perturbation parameter and δ is delay satisfying $\delta < \epsilon$. The function $a(x), b(x), f(x,t), u_0(x)$,$\phi(x,t)$ and$\psi(1,t)$ are assumed to be sufficiently smooth and bounded with $b(x) \geq b^* > 0$, for some constantb^*.

The authors constructed a numerical method for a class of singularly perturbed parabolic delay problem with a boundary turning point consisting of non standard finite difference operator on fitted mesh to discretize space variable and θ-method for discretize the time variable for a class of singularly perturbed parabolic delay problem with a boundary turning point. They proved that the method converge uniformly with respect to perturbation parameter.

Rai and Yadav (2021) considered a numerical method which consists of backward Euler scheme for time discretization on uniform mesh and a combination of midpoint upwind and central difference scheme for the space discretization on modified shishkin mesh. Further, they increased the order of convergence in time direction by Richardson extrapolation for singular perturbed delay parabolic convection diffusion problems with degenerate coefficient.

(Kumar and Kadalbajoo, 2012) numerical treated of singularly perturbed delay differential equation of second order using B-spline collocation method. In this survey the author has selected piecewise uniform mesh known as shishkin mesh which is adequate to handle singularly perturbed problems.

Remarks: Spline based methods provide more accurate results for singularly perturbed parabolic delay differential equation with small delay, large delay and for the SPPDDE with delay as well as advance.

(Yüzbaşı and Sezer, 2013) used an exponential collection method for finding solution of second order singularly perturbed delay differential equations.

(File et al., 2013) presented a computational method to solve equation with negative shift whose solution has boundary layer. In this scheme author reduced the second order SPDDE to first order equation and then employed numerical integration and interpolations. Author claim that the available asymptotic expansion methods for solving singular perturbed problems are difficult to apply as it is not easy to find appropriate asymptotic expansion in the inner and outer regions and matching of the coefficients of the inner and outer solution expansions is also a process that need skills.

(Swamy et al., 2015) proposed computational method for singularly perturbed delay differential equation of second order with twin layers or oscillatory behavior. Layer or oscillation behavior of the delay differential equation discussed depending on sign of (a(x) + b(x)). The layer behavior of the solution diminish as the delay increase and the solution exhibit oscillation behavior.

In general, this survey indicates that the development of the solution methodologies to solve equation has received very little attention from the research. Developing a sounding computational methods for solving model problem in equations are still at the preliminarily stage and it needs a lot of investigations. This gap initiates the researchers to consider model problem in equation.

Therefore,the main purpose of this study is to construct implicit Euler method to discretize the time variable and cubic spline in tension to discretize the space variable to solve singularly perturbed parabolic problems with Dirichlet boundary condition.

Chapter 3

Research Design and Methodology

3.1 Study Area and Period

This study was conducted under mathematics department, college of natural science in Salale University from January 2023 to June 2023

3.2 Study Design

This study employed both documentary review design and experimental design.

3.3 Source of Informations

The relevant source of information for this study are books and published articles.

3.4 Mathematical Procedures

In order to achieve the stated objectives, the study was followed the following mathematical procedures:

1. Defining the problem.
2. Formulating numerical scheme for the problem.
3. Establishing the stability and convergence analysis for the formulated scheme.
4. Developing an algorithm and writing MATLAB code for the presented scheme.
5. Validating the scheme using numerical examples.

6. Presenting the results using appropriate presentation (using tables, graphs).
7. Discussing and providing conclusion.

Chapter 4

Description of Numerical Method and Analysis

In this chapter description of the problem, properties of continuous problem, formulation of the numerical method, stability and convergence analysis, numerical examples and discussion of the numerical results are presented.

4.1 Description of the Problem

In this study, we consider the following SPPDDE's of the form

$$\frac{\partial u(x,t)}{\partial t} - \epsilon\frac{\partial^2 u(x,t)}{\partial x^2} + a(x)\frac{\partial u}{\partial x}(x-\delta,t) + b(x)u(x-\delta,t) = f(x,t) \tag{4.1.1}$$

on the domain $D = \Omega_x \times \Omega_t = (0,1) \times (0,T]$ for some fixed number $T > 0$ with initial and interval boundary conditions.

$$\begin{cases} u(x,0) = u_0(x), x \in \Omega_x = (0,1), \\ u(0,t) = \phi(0,t), t \in \Omega_t = (0.T], \\ u(1,t) = \psi(1,t), t \in \Omega_t, \end{cases} \tag{4.1.2}$$

where $0 < \epsilon \ll 1$ is singular perturbation parameter and δ is delay satisfying $\delta < \epsilon$. The function $a(x), b(x), f(x,t), u_0(x)$,$\phi(x,t)$ and $\psi(1,t)$ are assumed to be sufficiently smooth and bounded with $b(x) \geq b^* > 0$, for some constant b^*.

Solution of (4.1.1) and (4.1.2) exhibits boundary layer Gupta et al. (2018) and position of the layer depends on the conditions:If $a(x) < 0$ left layer exists. If $a(x) > 0$ right layer exists. For the case of $\delta < \epsilon$, using Taylor's series approximation for the terms containing delay $u(x-\delta,t)$ and $u_x(x-\delta,t)$ is valid Tian (2002). Since we assumed $\delta < \epsilon$, we approximate (4.1.1) and (4.1.2) by

$$\frac{\partial u(x,t)}{\partial t} - c_\epsilon(x)\frac{\partial^2 u(x,t)}{\partial x^2} + p(x)\frac{\partial u(x,t)}{\partial x} + b(x)u(x,t) = f(x,t). \tag{4.1.3}$$

Subject to the initial and boundary conditions.

$$\begin{cases} u(x,0) = u_0(x), x \in \Omega_x = (0,1), \\ u(0,t) = \phi(0,t), t \in \Omega_t = (0.T], \\ u(1,t) = \psi(1,t), t \in \Omega_t, \end{cases} \tag{4.1.4}$$

where $c_\epsilon(x) = \epsilon - \frac{\delta^2}{2}b(x) + \delta a(x)$ and $p(x) = a(x) - \delta b(x)$. For small values of δ, (4.1.1) and (4.1.2) within (4.1.3) and (4.1.4) are asymptotically equivalent. We assume
$0 < c_\epsilon(x) \leq \epsilon^2 - \frac{\delta^2}{2}b^* + \delta a^* = c_\epsilon$ where b^* and a^* are the lower bound for $b(x)$ and $a(x)$ respectively. We assume also $p(x) \geq p^* > 0$, implies occurrence of a boundary layer near x=1.

4.2 Properties of Continuous Problem

Lemma 4.2.1. *(Continuous maximum principle.) Let z be a sufficiently smooth function defined on D which satisfies z(x,t)* $\geq 0, (x,t) \in \partial D$ *and Lz(x,t)* $\geq 0, (x,t) \in D$. *Then implies that z(x,t)* $\geq 0, \forall (x,t) \in \overline{D}$

Proof. Let (x^*, t^*) be such that $z(x^*, t^*) = \min_{(x,t)\in\overline{D}} z(x,t)$ and suppose that $z(x^*, t^*) < 0$. It is clear that $(x^*, t^*) \notin \partial D$. From the theory in extrema calculus, since $z(x^*, t^*) = \min_{(x,t)\in\overline{D}}$ which implies $z_x(x^*, t^*) = 0, z_t(x^*, t^*) = 0$,and $z_{xx}(x^*, t^*) \geq 0$ and implies that $Lz(x^*, t^*) < 0, (x,t) \in D$ which is contradiction to the assumption that made above

$$Lz(x^*, t^*) \geq 0, (x,t) \in D.$$

Hence, $z(x^*, t^*) \geq 0, \forall (x,t) \in \overline{D}$. □

Lemma 4.2.2 (Uniform Stability Estimate.). *Let u(x,t) be the solution of the problem in eq.*(4.1.3) *and eq.*(4.1.4).*Then, we obtain the bound*

$$|u(x,t)| \leq \zeta^{-1}\|f\| + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\}$$

.

Proof. We define two barrier functions $\vartheta^{\pm}(x,t)$ as

$$\vartheta^{\pm}(x,t) = \zeta^{-1}\|f\| + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(x,t).$$

At the initial value, we have

$$\vartheta^{\pm}(x,0) = \zeta^{-1}\|f\| + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(x,t) \geq 0.$$

On the boundaries, we have

$$\vartheta^{\pm}(0,t) = \zeta^{-1}\|f\| + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(0,t) \geq 0.$$

$$\vartheta^{\pm}(1,t) = \zeta^{-1}\|f\| + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(1,t) \geq 0.$$

and for differential operator L

$$\begin{aligned}\vartheta^{\pm}(x,t) &= \vartheta^{\pm}_t(x,t) - c_\epsilon \vartheta^{\pm}_{xx}(x,t) + p(x)\vartheta^{\pm}_x(x,t) + q(x)\vartheta^{\pm}(x,t)\\ &= (0 \pm u_t(x,t)) - c_\epsilon(0 \pm u_{xx}(x,t)) + p(x)(0 \pm u_x(x,t))\\ &+q(x)\left(\zeta^{-1}\|f\| + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm u(x,t)\right)\\ &= q(x)\zeta^{-1}\|f\| + \max\{|u_0(x)|, |\phi(0,t)|, |\psi(1,t)|\} \pm f(x,t)\end{aligned}$$

$\geq 0, since q(x) \geq \zeta > 0$ which implies $L\vartheta^{\pm}(x,t) \geq 0$.
Hence by using maximum principle, we obtain $\vartheta^{\pm}(x,t) \geq 0, \forall (x,t) \in \overline{D}$.

□

4.3 Formulation of the Numerical Method

We derive the numerical method employing the implicit Euler method and the cubic spline in tension method for the time and space variable derivative, respectively.
Consider the partition of the solution domain [0,1] × [0,T].
D^M_τ $=\{j\tau, 0 < j \leq M\}$ and $D^N_l = \{il, 0 < i \leq N\}$ with temporal and spatial mesh sizes $\tau = T/M$ and $l = 1/N$,respectively.Here M and N denote the number of nodal points in the temporal and spatial directions.

4.3.1 Time discretization

To get a uniform partition $\Omega^M_t = t_j = \triangle t$, j= 0,1,2,...,M of the domain $\Omega_t = [0,T]$ in the time direction,we divide [0,T] into M mesh subintervals with step length $\triangle t$ = T/M at (j +1)th time interval. Let $u^{j+1}(x)$ be the approximation of $u(x, t_{j+1})$ at $(j+1)^{th}$ time level.
Then we discretized the (4.1.1) by using implicit Euler method as
$\frac{u^{j+1}(x)-u^j(x)}{\triangle t}$ - $c_\epsilon(x)\frac{\partial^2 u^{j+1}(x)}{\partial x^2} + p(x)\frac{\partial u^{j+1}(x)}{\partial x} + b(x)u^{j+1}(x) = f^{j+1}(x)$
$-c_\epsilon(x)\frac{\partial^2 u^{j+1}(x)}{\partial x^2} + p(x)\frac{\partial u^{j+1}(x)}{\partial x} + (b(x) + \frac{1}{\triangle t})u^{j+1}(x) = f^{j+1}(x) + \frac{u^j(x)}{\triangle t})$.
Then we obtain the following equation:

$$-c_\epsilon(x)\frac{\partial^2 u^{j+1}(x)}{\partial x^2} + p(x)\frac{\partial u^{j+1}(x)}{\partial x} + q(x)u^{j+1}(x) = g^{j+1}(x) \quad (4.3.1)$$

Where $q(x) = b(x) + \frac{1}{\triangle t}$ and $g^{j+1}(x) = f^{j+1}(x) + \frac{u^j(x)}{\triangle t}$.

To establish the convergence, we represent the local truncation error which is denoted by LTE as e_{j+1} such that

$$e_{j+1} = Lu^{j+1}(x) - Z^i$$

,
where u^{j+1} is the approximate solution of(4.1.1) and$Z^i = g^{j+1}(x)$ also the global error is the sum of LTE at each time level i.e...,
$E_j = \sum_{k=1}^{j} e_k$
The following Lemmas estimate the bound for LTE and GTE

Lemma 4.3.1. *Having* $\left|\frac{\partial^k u(x,t)}{\partial t^k}\right| \leq C, \forall (x,t)\epsilon\overline{D}$*,k=0,1,2*
The local error estimate in the temporal direction is given by

$$\|e_{j+1}\|_\infty \leq C(\triangle t)^2, \textit{for some constant } C$$

.

Proof. Using the series expansion to $u\left(x, t_{j+1}\right)$,we obtain

$$u\left(x, t_{j+1}\right) = u\left(x, t_j\right) + \triangle\, t u_t\left(x, t_j\right) + o\left((\triangle t)^2\right)$$

By substitution we obtain

$$\frac{u\left(x, t_{j+1}\right) - u\left(x, t_j\right)}{\triangle t} = u_t(x, t_j) + o((\triangle t)^2)$$
$$= -\left(-c_\epsilon u(x, t_j)_{xx} + p(x)u(x, t_j)_x + q(x)u(x, t_j) - f(x, t_j)\right) + o((\triangle t)^2)$$

since error satisfies the differential equations. So the local error satisfies the semi-discrete operator

$$L^{\triangle t} e^{j+1} = o\left((\triangle t)^2\right), e^{j+1}(0) = 0 = e^{j+1}(1)$$

Hence, applying the maximum principle, we obtain

$$\left\|e^{j+1}\right\| \leq C\left(\triangle\right)^2$$

□

Next, we need to give the bound for the global error of the semi discretization. Let denote TE^{j+1} be the global error estimate up to the $(j+1)^{th}$ time step.

Lemma 4.3.2. *Let $E_j = u(x,t_j) - u(x,t_j)$ be the global error estimate in the time direction. Then the following bound holds*

$$\|E_j\|_\infty \leq C(\triangle t)$$

Proof. From Lemma 4.3.1 it follows that
$\|E_j\|_\infty = || \sum_{k=1}^{j} ||_\infty \leq ||e_1||_\infty + ||e_2||_\infty + ... + ||e_j||_\infty$
$\leq C(\triangle t)$ for some constant C. □

4.3.2 Space Discretization

In this section,we introduce cubic spline in tension method on shishkin mesh for the solution of eq.(4.1.1). Shishkin meshes are piecewise-uniform mesh which condense more mesh points in the boundary layer as $\epsilon \to 0$. Here we approximate the resulting eq.(4.1.1) by applying the cubic spline in tension method as described below.

A function $S^{j+1}(x,\tau) \in C^2[0,1]$ which interpolates $u^{j+1}(x)$ at the mesh points x_i,i= 0,1,2,...,N depends on a parameter $\tau > 0$ reduces to cubic spline in [0,1] as $\tau \to 0$ is called parametric cubic spline function. In $[x_i, x_{i+1}]$, the parametric cubic spline function $S^{j+1}(x,\tau) = S^{j+1}(x)$ satisfies the differential equation Pramod Chakravarthy et al. (2017);

$$\begin{aligned}\frac{\partial^2 S^{j+1}(x)}{\partial x^2} + \tau S^{j+1}(x) = &\left[\frac{\partial^2 S^{j+1}(x_i)}{\partial x^2} + \tau S^{j+1}(x_i)\right]\left(\frac{x_{i+1}-x}{l}\right) + \\ &\left[\frac{\partial^2 S^{j+1}(x_{i+1})}{\partial x^2} + \tau S^{j+1}(x_{i+1})\right]\left(\frac{x-x_i}{l}\right),\end{aligned} \tag{4.3.2}$$

where $S^{j+1}(x_i) = u_i^{j+1}$ and $\tau > 0$ is known to be cubic spline in tension.

Solving eq.(4.3.2) we obtain

$$S^{j+1}(x) = Ae^{\frac{\lambda}{l}x} + Be^{\frac{\lambda}{l}x} + \left[\frac{Z_i^{j+1} - \tau u_i^{j+1}}{\tau}\right]\left(\frac{x - x_{i+1}}{l}\right) + \left[Z_{i+1}^{j+1} - \tau u_{i+1}^{j+1}\right]\left(\frac{x_i - x}{l}\right). \tag{4.3.3}$$

The arbitrary constants A and B can be determined using interpolate conditions.
$S^{j+1}(x_{i+1}) = u_{i+1}^{j+1}$, $S^{j+1}(x_i) = u_i^{j+1}$
putting $\lambda = l\tau^{1/2}$ and $Z_k^{j+1} = \frac{\partial^2 S^{j+1}(x_k)}{\partial x^2}, k = i, i\pm 1$,we have

$$S^{j+1}(x) = \frac{l^2}{\lambda^2 sinh\lambda}\left[Z_{i+1}^{j+1} sinh\frac{\lambda(x-x_i)}{l} + Z_i^{j+1} sinh\frac{\lambda(x_{i+1}-x)}{l}\right]$$
$$-\frac{l^2}{\lambda^2}\left[\left(\frac{x-x_i}{l}\right)\left(Z_{i+1}^{j+1}-\frac{\lambda^2}{l}u_{i+1}^{j+1}\right)+\left(\frac{x_{i+1}-x}{l}\right)\left(Z_i^{j+1}-\frac{\lambda^2}{l}u_i^{j+1}\right)\right] \quad (4.3.4)$$

Differentiating eq.(4.3.4) and taking $x \to x_i$, we obtain

$$\frac{\partial S^{j+1}(x_i^+)}{\partial x} = \frac{u_{i+1}^{j+1}-u_i^{j+1}}{l} - \frac{l}{\lambda^2}\left[\left(1-\frac{\lambda}{sinhl}\right)Z_{i+1}^{j+1} - (1-\lambda coth\lambda)Z_i^{j+1}\right] \quad (4.3.5)$$

Proceeding similarly in the interval (x_{i-1}, x_i),we get

$$\frac{\partial S^{j+1}(x_i^-)}{\partial x} = \frac{u_{i+1}^{j+1}-u_i^{j+1}}{l} + \frac{l}{\lambda^2}\left[-\left(1-\lambda coth\lambda\right)Z_i^{j+1} + \left(1-\frac{\lambda}{sinh\lambda}\right)Z_{i+1}^{j+1}\right] \quad (4.3.6)$$

Equating eq.(4.3.5) and eq.(4.3.6) at x_i, we obtain

$$\frac{u_{i+1}^{j+1}-u_i^{j+1}}{l} + \frac{l}{\lambda^2}\left[-\left(1-\lambda coth\lambda\right)Z_i^{j+1} + \left(1-\frac{\lambda}{sinh\lambda}\right)Z_{i+1}^{j+1}\right]$$
$$= \frac{u_{i+1}^{j+1}-u_i^{j+1}}{l} - \frac{l}{\lambda^2}\left[\left(1-\frac{\lambda}{sinhl}\right)Z_{i+1}^{j+1} - (1-\lambda coth\lambda)Z_i^{j+1}\right] \quad (4.3.7)$$

Rearranging, we get the following tridiagonal system

$$\lambda_1 Z_{i+1}^{j+1} + 2\lambda_2 Z_i^{j+1} + \lambda_1 Z_{i-1}^{j+1} = \frac{u_{i+1}-2u_i+u_{i-1}}{l^2}, i = 1, 2, ..., N-1. \quad (4.3.8)$$

Where $\lambda_1 = \frac{1}{\lambda^2}\left(1-\frac{\lambda}{sinh\lambda}\right)$, $\lambda_2 = \frac{1}{\lambda^2}\left(\lambda coth\lambda - 1\right)$

For the choice of parameters $\lambda_1 + \lambda_2 = 1/2$ is consistent and suitable for solving the given differential equations.

Equation (4.3.1) can be written as, for k=$i, i \pm 1$

$$c_\epsilon(x)\frac{\partial^2 u^{j+1}(x_k)}{\partial x^2} = p(x)\frac{\partial u^{j+1}(x_k)}{\partial x} + q(x)u^{j+1}(x_k) - g^{j+1}(x_k), \quad (4.3.9)$$

where $\frac{\partial u^{j+1}(x_k)}{\partial x}$ is approximated by

$$\begin{cases} \frac{\partial u_{i-1}^{j+1}}{\partial x} = \frac{-u_{i+1}^{j+1}+u_i^{j+1}-3u_{i-1}^{j+1}}{2l}, \\ \frac{\partial u_i^{j+1}}{\partial x} = \frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2l}, \\ \frac{\partial u_{i+1}^{j+1}}{\partial x} = \frac{3u_{i+1}^{j+1}-4u_i^{j+1}+u_{i-1}^{j+1}}{2l}\} \end{cases} \quad (4.3.10)$$

Taking eq.(4.3.10) into eq.(4.3.9),we obtain

$$\begin{cases} c\epsilon(x_{i-1})\frac{\partial^2 u_{i-1}^{j+1}}{\partial x^2} = p(x_{i-1})\left(\frac{-u_{i+1}^{j+1}+4u_i^{j+1}-3u_{i-1}^{j+1}}{2l}\right) + q(x_{i-1})u_{i-1}^{j+1} - g_{i-1}^{j+1}, \\ \\ c\epsilon(x_i)\frac{\partial^2 u_i^{j+1}}{\partial x^2} = p(x_i)\left(\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2l}\right) + q(x_i)u_i^{j+1} - g_i^{j+1}, \\ \\ c\epsilon(x_{i+1})\frac{\partial^2 u_{i+1}^{j+1}}{\partial x^2} = p(x_{i+1})\left(\frac{3u_{i+1}^{j+1}-4u_i^{j+1}+u_{i-1}^{j+1}}{2l}\right) + q(x_{i+1})u_{i+1}^{j+1} - g_{i+1}^{j+1}, \end{cases} \tag{4.3.11}$$

putting eq.(4.3.11) into eq.(4.3.8), we obtain

$$\frac{\lambda_1}{c\epsilon}\left[p_{i-1}\left(\frac{-u_{i+1}^{j+1}+4u_i^{j+1}-3u_{i-1}^{j+1}}{2l}\right) + q_{i-1}u_{i-1}^{j+1} - g_{i-1}^{j+1}\right] + \frac{2\lambda_2}{c\epsilon}\left[p_i\left(\frac{u_{i+1}^{j+1}-U_{i-1}^{j+1}}{2l}\right) + q_i u_i^{j+1} - g_i^{j+1}\right]$$
$$+ \frac{\lambda_1}{c\epsilon}\left[p_{i+1}\left(\frac{3u_{i+1}^{j+1}-4u_i^{j+1}+u_{i-1}^{j+1}}{2l}\right) + q_{i+1}u_{i+1}^{j+1} - g_{i+1}^{j+1}\right] = \left(\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{l^2}\right)$$

multiplying both sides by $c\epsilon$,we get

$$c\epsilon\left(\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{l^2}\right) = \left(\frac{-3\lambda_1 p_{i-1}}{2l} + \lambda_1 q_{i-1} - \frac{\lambda_2 p_i}{l} + \frac{\lambda_1 p_{i+1}}{2l}\right)u_{i-1}^{j+1}$$
$$+ \left(\frac{2\lambda_1 p_{i-1}}{l} + 2\lambda_2 q_i - \frac{2\lambda_1 p_{i+1}}{l}\right)u_i^{j+1} + \left(\frac{-\lambda_1 p_{i-1}}{2l} + \frac{\lambda_2 q_i}{l} + \frac{3\lambda_1 p_{i+1}}{2l} + \lambda_1 q_{i+1}\right)u_{i+1}^{j+1}$$
$$- \lambda_1 g_{i-1}^{j+1} - 2\lambda_2 g_i^{j+1} + \lambda_1 g_{i+1}^{j+1} \tag{4.3.12}$$

To handle the effect of ϵ a constant fitting factor $\gamma(\rho)$ is multiplied on the term containing ϵ as

$$\gamma(\rho)\, c\epsilon\left(\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{l^2}\right) = \left(\frac{-3\lambda_1 p_{i-1}}{2l} + \lambda_1 q_{i-1} - \frac{\lambda_2 p_i}{l} + \frac{\lambda_1 p_{i+1}}{2l}\right)u_{i-1}^{j+1}$$
$$+ \left(\frac{2\lambda_1 p_{i-1}}{l} + 2\lambda_2 q_i - \frac{2\lambda_1 p_{i+1}}{l}\right)u_i^{j+1} + \left(\frac{-\lambda_1 p_{i-1}}{2l} + \frac{\lambda_2 q_i}{l} + \frac{3\lambda_1 p_{i+1}}{2l} + \lambda_1 q_{i+1}\right)u_{i+1}^{j+1}$$
$$- \lambda_1 g_{i-1}^{j+1} - 2\lambda_2 g_i^{j+1} + \lambda_1 g_{i+1}^{j+1}. \tag{4.3.13}$$

Multiplying eq.(4.3.13) by l and taking the limit as $l \to 0$,we get

$$\lim_{l\to 0}\gamma(\rho)\, c\epsilon\left(\frac{-u_{i+1}^{j+1}+4u_i^{j+1}-3u_{i-1}^{j+1}}{2l}\right) - \frac{p(0)}{2}\left(u_{i+1}^{j+1} - u_{i_1}^{j+1}\right) = 0. \tag{4.3.14}$$

For the problems of with the layer at the right end of the interval from the theory of singular perturbations,the solution of eq.(4.3.1) is of the form O'Malley (1991).

$$u^{j+1}(x) \approx u_0^{j+1}(x) + \frac{p(1)}{p(x)}\left(u^{j+1}(2) - u_0^{j+1}(x)\right) exp\left(-p(x)\frac{1-x}{c\epsilon}\right) + o(\epsilon). \tag{4.3.15}$$

where $u_0^{j+1}(x)$ is the solution of the reduced problem

$$p(x)\frac{\partial u_0^{j+1}(x)}{\partial x}+q(x)u_0^{j+1}(x)=g^{j+1}(x)$$

with

$$u^{j+1}(1)=u^{j+1}(2)$$

Taking Taylor's series expansion for p(x) about the point x=1 and restricting to their first term eq.(4.3.15) becomes

$$u^{j+1}(x) \quad \approx \quad u_0^{j+1}(x) \;+\; \left(u^{j+1}(2)-u_0^{j+1}(x)\right)exp\left(-p(1)\frac{1-x}{c\epsilon}\right) \;+\; o\left(\epsilon\right) \quad (4.3.16)$$

now from eq.(4.3.16),we have

where $\rho=\frac{1}{c\epsilon}$, plugging the above equations into eq.(4.3.14) gives the required fitting factor

$$\gamma\left(\rho\right)=\frac{p(0)\rho}{2}coth\left(\frac{p(1)\rho}{2}\right) \quad (4.3.17)$$

Finally,from eq.(4.3.13) and eq.(4.3.17),we get

$$E_\epsilon^{N,M}u_i^{j+1}=H_i^{j+1}, i=1,2,...,N-1 \quad (4.3.18)$$

where

$$\lim_{l\to 0}U_{(il)}^{j+i}\approx u_0^{j+1}(0)+\left(u^{j+1}(2)-u_0^{j+1}(1)\right)exp\left(-p(1)\left(\frac{1}{c\epsilon}-i\rho\right)\right)+o\left(\epsilon\right),$$
$$\lim_{l\to 0}u_{(i-1)l}^{j+i}\approx u_0^{j+1}(0)+\left(u^{j+1}(2)-u_0^{j+1}(1)\right)exp\left(-p(1)\left(\frac{1}{c\epsilon}-i\rho+\rho\right)\right)+o\left(\epsilon\right),$$
$$\lim_{l\to 0}U_{(i+1)l}^{j+i}\approx u_0^{j+1}(0)+\left(u^{j+1}(2)-u_0^{j+1}(1)\right)exp\left(-p(1)\left(\frac{1}{c\epsilon}-i\rho-\rho\right)\right)+o\left(\epsilon\right),$$

$$\begin{cases} E_\epsilon^{N,M}u_i^{j+1}=X_i^{-}+X_i^{c}u_i^{j+1}+X_i^{+}u_{i+1}^{j+1} \\ X_i^{-}=-\frac{\gamma(\rho)c\epsilon}{l^2}-\frac{3\lambda_1 p_{i-1}}{2h}-\frac{\lambda_2 p_i}{l}+\frac{\lambda_1 p_{i+1}}{2h}+\lambda_1 q_{i-1}, \\ X_i^{c}=-\frac{2\gamma(\rho)c\epsilon}{l^2}-\frac{2\lambda_1 p_{i-1}}{l}+2\lambda_2 q_i+\frac{2\lambda_1 p_{i+1}}{l}, \\ X_i^{+}=-\frac{\gamma(\rho)c\epsilon}{l^2}-\frac{\lambda_1 p_{i-1}}{2l}+\frac{\lambda_2 p_i}{l}+\frac{3\lambda_1 p_{i+1}}{2l}+\lambda_1 q_{i+1}, \\ H_i^{j+1}=\lambda_1 g_{i-1}^{j+1}+2\lambda_2 g_i^{j+1}+\lambda_1 g_{i+1}^{j+1} \end{cases}$$

For sufficiently small l,the above matrix is non singular and $|X_i^c| \geq |X_i^-| + |X_i^+|$ (i.e..,the matrix are diagonally). Hence,by Nichols (1989),the matrix is M-matrix and have an inverse.Therefore,the system of equations can be solved by matrix inverse.

4.4 Analysis of the Method

In this section,we demonstrate convergence analysis through error analysis.

4.4.1 Convergence Analysis

Lemma 4.4.1. *(Discrete maximum principle)*

Assume that the mesh function $Z^{j+1}(x_i)$ satisfies $Z^{j+1}(x_0) \geq 0$ and $Z^{j+1}(x_N) \geq 0$. If $L^{h,\triangle t} Z^{j+1}(x_i) \geq 0$ for $1 \leq i \leq N-1$, then $Z^{j+1}(x_i) \geq 0$

Proof. Let choose k such that $Z^{j+1}(x_k) = min_{xi} Z^{j+1}(x_i)$,$1 \leq i \leq N-1$. If $Z^{j+1}(x_k) \geq 0$, the proof completed. We can see that $Z^{j+1}(x_{k+1}) - Z^{j+1}(x_k) \geq 0$ and $Z^{j+1}(x_k) - Z^{j+1}(x_{k-1}) \leq 0$.Then we obtain $L^{h,\triangle t} Z^{j+1}(x_k) < 0$ which contradicts $L^{h,\triangle t} Z^{j+1}(x_k) < o$. Hence the assumption is wrong. We conclude that $Z^{j+1}(x_i) \geq 0, \forall i, 0 \leq i \leq N$. □

Lemma 4.4.2. *(Discrete Uniform Stability)*

The solution of u_i^{j+1} of eq.(4.3.18) at $(j+1)^{th}$ time level and $\Gamma = min_{0 \leq i \leq N} \{q_i\}$,where Γ is some positive constant is bounded as
$\|u_i^{j+1}\| \leq \frac{\|E_\epsilon^{N,M} u_i^{j+1}\|}{\Gamma} + max\left\{|u_0^{j+1}|, |u_N^{j+1}|\right\}$

Proof. Let define barrier functions $\left(\beta_i^{j+1}\right)^{\pm} = V \pm u_i^{j+1}$,
$whereV = \frac{\|E_\epsilon^{N,M} u_i^{j+1}\|}{\Gamma} + max\left\{|u_0^{j+1}|, |u_N^{j+1}|\right\}$ on the boundary points,we obtain
$\left(\beta_i^{j+1}\right)^{\pm} = u \pm u_0^{j+1} = \frac{\|E_\epsilon^{N,M} u_i^{j+1}\|}{\Gamma} + \max\left\{|u_0^{j+1}|, |u_N^{j+1}|\right\} \pm u_2^{j+1}(N) \geq 0.$ □

Now, on the discretized spatial domain Ω_l^N, we have

$$
\begin{aligned}
E_\epsilon^{N,M}\left(\beta_i^{j+1}\right)^{\pm} &= E_\epsilon^{N,M}\left(V \pm u_i^{j+1}\right)\\
&= \left(-\frac{\gamma(\rho)c\epsilon}{l^2} - \frac{3\lambda_1 p_{i-1}}{2l} - \frac{\lambda_2 p_i}{l} + \frac{\lambda_1 p_{i+1}}{2l} + \lambda_1 q_{i-1}\right)\left(V \pm u_{i-1}^{j+1}\right)\\
&\quad + \left(\frac{2\gamma(\rho)c\epsilon}{l^2} - \frac{2\lambda_1 p_{i-1}}{l} + 2\lambda_2 q_i + \frac{2\lambda_1 p_{i+1}}{l}\right)\left(V \pm U_i^{j+1}\right)\\
&\quad + \left(-\frac{\gamma(\rho)c\epsilon}{l^2} - \frac{\lambda_1 p_{i-1}}{2l} + \frac{\lambda_2 p_i}{l} + \frac{3\lambda_1 p_{i+1}}{2l} + \lambda_1 q_{i+1}\right)\left(V \pm u_{i+1}^{j+1}\right)
\end{aligned}
$$

.

$$
\begin{aligned}
&= \pm\left(-\frac{\gamma(\rho)c\epsilon}{l^2} - \frac{3\lambda_1 p_{i-1}}{2l} - \frac{\lambda_2 p_i}{l} + \frac{\lambda_1 p_{i+1}}{2l} + \lambda_1 q_{i-1}\right)\left(u_{i-1}^{j+1}\right)\\
&\quad \pm \left(\frac{2\gamma(\rho)c\epsilon}{l^2} - \frac{2\lambda_1 p_{i-1}}{l} + 2\lambda_2 q_i + \frac{2\lambda_1 p_{i+1}}{l}\right)\left(u_i^{j+1}\right)\\
&\quad \pm \left(-\frac{\gamma(\rho)c\epsilon}{l^2} - \frac{\lambda_1 p_{i-1}}{2l} + \frac{\lambda_2 p_i}{l} + \frac{3\lambda_1 p_{i+1}}{2l} + \lambda_1 q_{i+1}\right)\left(u_{i+1}^{j+1}\right)\\
&\qquad + \left(\lambda_1 q_{i-1} + 2\lambda_2 q_i + \lambda_1 q_{i+1}\right)V, ,
\end{aligned}
$$

$$
\begin{aligned}
&\pm\left(\lambda_1 g_{i-1}^{j+1} + 2\lambda_1 g_i^{j+1} + \lambda_1 g_{i+1}^{j+1}\right) + \left(\lambda_1 q_{i-1} + 2\lambda_2 q_i + \lambda_1 q_{i+1}\right)V\\
&\quad = \left(\lambda_1 q_{i-1} + 2\lambda_2 q_i + \lambda_1 q_{i+1}\right)\left(\frac{\left\|E_\epsilon^{N,M} u_i^{j+1}\right\|}{\Gamma}\right) + max\left\{\left|u_0^{j+1}\right|, \left|u_N^{j+1}\right|\right\}\\
&\qquad \mp\left(\lambda_1 g_{i-1}^{j+1} + 2\lambda_1 g_i^{j+1} + \lambda_1 g_{i+1}^{j+1}\right) \geq 0, since q(x_i) \geq \Gamma > 0
\end{aligned}
$$

on applying lemma (4.4.1),we obtain $\left(\beta_i^{j+1}\right)^{\pm} \geq 0$,for all $x_i \in \Omega_l^N$ .Hence,the desired bound is obtained.

Lemma 4.4.3. *The local truncation error in space discretization of the discrete problem eq.*(4.3.18) *is given as*

$$max_{ij}\left|u^{j+1}(x_i) - u_i^{j+1}\right| \leq cl^2$$

where c is a constant independent of ϵ and l

Proof. From the truncation error of eq (4.3.10),we have

$$
\begin{cases}
e\prime_{i-1} = \frac{\partial u^{j+1}(x_{i-1})}{\partial x} - \frac{\partial u_{i+1}^{j+1}}{\partial x} = \frac{l^2}{3}\partial^3 u^{j+1}(x_i) + \frac{l^3}{12}\frac{\partial^4 u^{j+1}(x_i)}{\partial x^4} + \frac{l^4}{30}\frac{\partial^5 u^{j+1}(p_i)}{\partial x^5}\\[2ex]
e\prime_i = \frac{\partial u^{j+1}(x_i)}{\partial x} - \frac{\partial u_i^{j+1}}{\partial x} = -\frac{l^2}{6}\frac{\partial^2 u^{j+1}(x_i)}{\partial x^3} - \frac{l^4}{120}\frac{\partial^5 u^{j+1}(p_i)}{\partial x^5}\\[2ex]
e\prime_{i+1} = \frac{\partial u^{j+1}(x_{i+1})}{\partial x} - \frac{\partial u_{i+1}^{j+1}}{\partial x} = \frac{l^3}{3}\frac{\partial^3 u^{j+1}(x_i)}{\partial x^3} + \frac{l^3}{12}\frac{\partial^4 u^{j+1}(x_i)}{\partial x^4} + \frac{l^3}{30}\frac{\partial^5 u^{j+1}(p_i)}{\partial x^5}
\end{cases}
\tag{4.4.1}
$$

where $x_{i-1} < \psi < x_{i+1}$.Substituting
$\Gamma c\epsilon Z_k^{j+1} = p_k \frac{\partial u_k^{j+1}}{\partial x} q_k u_k^{j+1} - g_k^{j+1}, k = i, i \pm 1$ into eq.(4.2.8) , we get

$$\gamma c\epsilon \left(u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}\right) = l^2\lambda_1 \left(p_{i-1}\frac{\partial u_{i-1}^{j+1}}{\partial x} + q_{i-1}u_{i-1}^{j+1} - g_{i-1}^{j+1}\right)$$
$$+ 2l^2\lambda_2 \left(p_i\frac{\partial u_i^{j+1}}{\partial x} + q_i u_i^{j+1} - g_i^{j+1}\right) + l^2\lambda_1 \left(p_{i+1}\frac{\partial u_{i+1}^{j+1}}{\partial x} + q_{i+1}u_{i+1}^{j+1} - g_{i+1}^{j+1}\right). \quad (4.4.2)$$

Considering the corresponding exact solution to eq.(4.4.2) ,we have

$$\gamma c\epsilon \left(u^{j+1}(x_{i-1}) - 2u^{j+1}(x_i) + u^{j+1}x(_{i+1})\right) = l^2\lambda_1 p(x_{i-1})\frac{\partial u^{j+1}(x_{i-1})}{\partial x}$$
$$+ l^2\lambda_1 \left(q(x_{i-1})u^{j+1}(x_{i-1}) - q^{j+1}(x_{i-1})\right) + 2l^2\lambda_2 \left(p(x_i)\frac{\partial u^{j+1}(x_i)}{\partial x} + q(x_i)u^{j+1}(x_i)\right)$$
$$- 2l^2\lambda_2 g^{j+1}(x_i) + l^2\lambda_1 \left(p(x_{i+1})\frac{\partial u^{j+1}(x_{i+1})}{\partial x} + q(x_{i+1})u^{j+1}(x_{i+1}) - g^{j+1}(x_{i+1})\right). \quad (4.4.3)$$

Subtracting eq.(4.4.2) from eq.(4.4.3) and denoting
$e_k = u^{j+1}(x_k) - u_k^{j+1}$ for k=i,$i \pm 1$ we arrive at

$$\left(\gamma c\epsilon - l^2\lambda_1 q_{i-1}\right) e_{i-1} + \left(-2\gamma c\epsilon - 2l^2\lambda_2 q_i\right) e_i + \left(\gamma c\epsilon - l^2\lambda_1 qi + 1\right) e_{i+}$$
$$= l^2 \left(\lambda_1 p_{i-1} e\prime_{i-1} + 2\lambda_2 p_i e\prime_i + \lambda_1 p_{i+1} e\prime_{i+1}\right). \quad (4.4.4)$$

Inserting eq.(4.4.1) into eq.(4.4.4),we obtain

$$\left(\gamma c\epsilon - l^2\lambda_1 q_{i-1}\right) e_{i-1} + \left(-2\gamma c\epsilon - 2l^2\lambda_2 q_i\right) e_i + \left(\gamma c\epsilon - l^2\lambda_1 qi + 1\right) e_{i+}$$
$$= \frac{l^4}{3}\left(\lambda_1 p_{i-1} - \lambda_2 p_i + \lambda_1 p_{i+1}\right)\frac{\partial^3 u^{j+1}(x_i)}{\partial x^3} + \frac{l^5}{12}\left(-\lambda_1 p_{i-1} + \lambda_1 p_{i+1}\right)\frac{\partial^4 u^{j+1}(x_i)}{\partial x^4}$$
$$+ \frac{l^6}{60}\left(2\lambda_1 p_{i-1} - \lambda_2 p_i + 2\lambda_1 p_{i+1}\right)\frac{\partial^5 U^{j+1}(p_i)}{\partial x^5} \quad (4.4.5)$$

Using the expression $p_{i-1} = p_i - lp_i' + \frac{l^2}{2!}p(2)p_i$ and
$p_{i+1} = p_i + lp_i' + \frac{l^2}{2!}p(2)p_i$ in eq.(4.4.5),we have

$$\left(\gamma c\epsilon - l^2\lambda_1 q_{i-1}\right) e_{i-1} + \left(-2\gamma c\epsilon - 2l^2\lambda_2 q_i\right) e_i + \left(\gamma c\epsilon - l^2\lambda_1 qi + 1\right) e_{i+} = T_i(l) \quad (4.4.6)$$

where $T_i(l) = \frac{l^4}{3}\left(2\lambda_1 - \lambda_2\right) p_i \frac{\partial^3 u^{j+1}(x_i)}{\partial x^3} + o(l^6)$
Therefore,$T_i(l) = o(l^4)$ for the choice of parameters $\lambda_1 + \lambda_2 = 1/2$
Equation (4.4.6) can be written as a matrix form;

$$\left(\Lambda - \eta\right) E = V \quad (4.4.7)$$

where $\Lambda = trid\left(-\gamma c\epsilon, 2\gamma c\epsilon, -\gamma c\epsilon\right)$, $\eta = trid\left(l^2\lambda_1 q_{i-1}, l^2\lambda_1 q_i, l^2\lambda_1 q_{i+1}\right)$,
E=$[e_1, e_2, ..., e_{N-1}]^T$ and $V = [-V_1(l), -V_2(l), ..., -V_{N-1}(l)]^T$

following Adivi Sri Venkata and Palli (2017) ,it cn show that

$$\|E\| \leq \frac{c}{l^2} \times o(l^4) = C(l^2) \tag{4.4.8}$$

where c is constant, independent of l and ϵ. $\square$

Theorem 4.1: Let u(x,t) be solution of problem (4.4.4) at each grid point(x_i, t_{j+1}) and $u^{j+1}i$ be its approximate solution obtained by the proposed scheme given in eq.(4.3.18).Then the error estimate for the fully discrete method is given by

$$\max_{ij} \left|u(x_i, t_{j+1}) - u_i^{j+1}\right| \leq c\left((\tau) + l^2\right)$$

Proof. From the triangular inequality,we have

$$\begin{aligned} \max_{ij} \left|u(x_i, t_{j+1}) - u_i^{j+1}\right| &= \max_{ij} \left|u(x_i, t_{j+1}) - u^{j+1}(x_i) + u^{j+1}(x_i) - u_i^{j+1}\right| \\ &\leq \max_{ij} \left|u(x_i, t_{j+1}) - u^{j+1}(x_i)\right| + \max_{ij} \left|u^{j+1}(x_i) - u_i\right| \end{aligned} \tag{4.4.9}$$

$\square$

Theorem.4.2.(Main Convergence Theorem)Let $u(x_i.t_j)$ be the solution of continuous problem and U_i^j be the solution of discrete problem, then the parameter uniform error estimate is given by

$\left|U_i^j - u(x_i, t_j)\right| \leq C\left(\triangle t^2 + N^{-2}\ln^2 N\right)$,

where C is a constant independent of ϵ and the mesh parameters N and $\triangle t$

4.5 Numerical Examples,Results and Discussions

In this section, we carry out numerical experiment in order to corroborate the applicability of the proposed method.Since the exact solutions for the given examples are unknown, we use the double mesh principle to calculate the absolute error.Two model examples have been presented to illustrate the efficiency of proposed method. In both cases, we performed the numerical experiments by taking $\lambda_1 = 1e{-}010$ and $\lambda_2 = 4.999999999e{-}01$.As the exact solutions of the considered examples are not known, we calculate the maximum point-wise error for each ϵ and δ. For each ϵ,we can determine the maximum point-wise errors and rate of convergence using the following formula defined as

$$E_{\epsilon,\delta}^{N,M} = \max_{0 \leq i \leq N; [0,T]} \left|u^{N,M}(x_i, t_j) - u^{2N,2M}(x_i, t_j)\right|$$

and

$$r_{\epsilon,\delta}^{N,M} = \log_2 \left(\frac{E_{\epsilon,\delta}^{N,M}}{E_{\epsilon,\delta}^{2N,2M}} \right),$$

where $u^{N,M}(x_i, t_j)$ denote the numerical solution obtained at (N,M) mesh points whereas $u^{N,M}(x_i, t_j)$ denote the numerical solution at (2N,2M) mesh points.The uniform error and uniform rate of convergence are completed by the following formulas.

$$E^{N,M} = \max \left| E_{\epsilon,\delta}^{N,M} \right|$$

and

$$r^{N,M} = \log_2 \left(\frac{E^{N,M}}{E^{2N,2M}} \right)$$

Example 4.5.1. $a(x) = 2 - x^2$,$b(x) = x^2 + 1 + \cos(\pi x)$ and $f(x) = 10t^2 \exp(-t)(1 - x)$ for $u_0(x) = 0$,$0 \le x \le 1$ and $\phi(x,t) = 0, x \in [-\delta, 0], \psi(1,t) = 0$ for final time T=1.

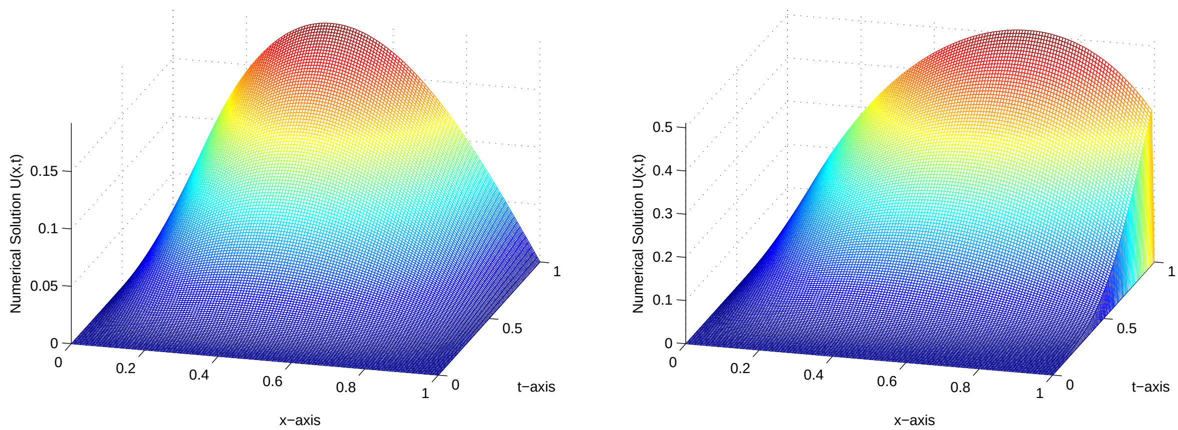
Example 4.5.2. $a(x) = 2 - x^2$, $b(x) = 3 - x$ and $f(x) = \exp(t) \sin(\pi x(1 - x))$ for $u_0(x) = 0$,$0 \le x \le 1$ and $\phi(x,t) = 0, x \in [-\delta, 0], \psi(1,t) = 0$ for final time T=1.

Table 4.1: Maximum absolute error of 4.5.1 for $M = N$

$\varepsilon \downarrow$	N=16	N=32	N=64	N=128	N=256	N=512
Present Method						
10^{-0}	1.3376e-04	5.8444e-05	2.7169e-05	1.3081e-05	6.4159e-06	3.1770e-06
10^{-2}	8.6393e-03	3.0718e-03	9.1610e-04	2.6734e-04	8.3377e-05	2.8965e-05
10^{-4}	1.0106e-02	5.3075e-03	2.7223e-03	1.3786e-03	6.9369e-04	3.4723e-04
10^{-6}	1.0106e-02	5.3074e-03	2.7223e-03	1.3785e-03	6.9369e-04	3.4795e-04
10^{-8}	1.0106e-02	5.3074e-03	2.7223e-03	1.3785e-03	6.9369e-04	3.4795e-04
10^{-10}	1.0106e-02	5.3074e-03	2.7223e-03	1.3785e-03	6.9369e-04	3.4795e-04
$E^{-N,M}$	1.0106e-02	5.3074e-03	2.7223e-03	1.3785e-03	6.9369e-04	3.4795e-04
$r^{N,M}$	0.9291	0.9632	0.9817	0.9907	0.9954	-
Results in Woldaregay and Duressa (2021) Before Richardson Extrapolation						
10^{-4}	1.4608e-02	8.1605e-03	4.3079e-03	2.2125e-03	-	-
10^{-6}	1.4608e-02	8.1600e-03	4.3077e-03	2.2124e-03	-	-
10^{-8}	1.4608e-02	8.1600e-03	4.3077e-03	2.2124e-03	-	-
10^{-10}	1.4608e02	8.1600e03	4.3077e03	2.2124e03	-	-
$E^{-N,M}$	1.4608e-02	8.1600e-03	4.3077e-03	2.2124e-03	-	-
$r^{N,M}$	0.8401	0.9217	0.9613	-	-	-

Table 4.2: Maximum absolute error of 4.5.2 for $M = N$

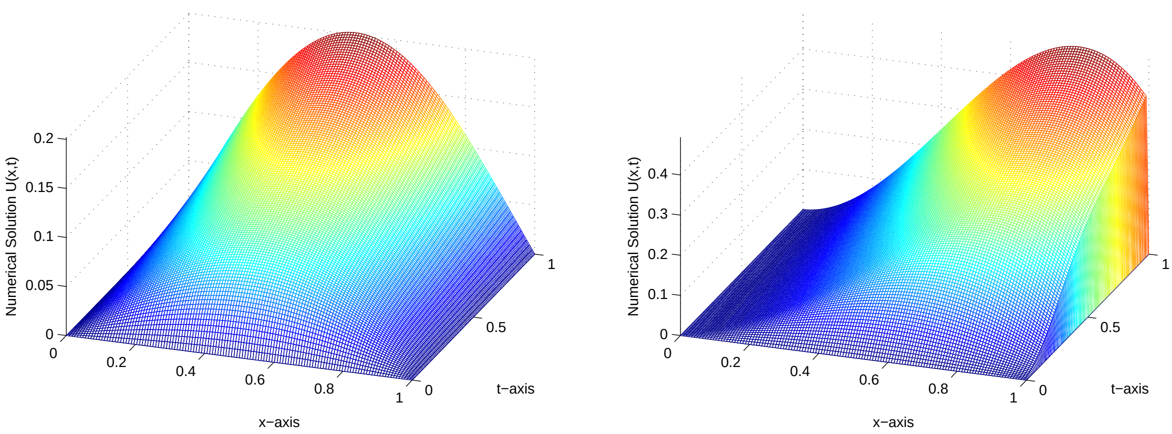
$\varepsilon \downarrow$	N=16	N=32	N=64	N=128	N=256	N=512
Present Method						
10^{-0}	2.1094e-03	1.1690e-03	6.2039e-04	3.1983e-04	1.6238e-04	8.1820e-05
10^{-2}	8.3012e-03	3.8273e-03	1.7262e-03	8.1359e-04	3.9597e-04	1.9558e-04
10^{-4}	9.0345e-03	6.2292e-03	3.5568e-03	1.9435e-03	1.0263e-03	5.2733e-04
10^{-6}	9.0345e-03	6.2291e-03	3.5567e-03	1.9435e-03	1.0264e-03	5.2908e-04
10^{-8}	9.0345e-03	6.2291e-03	3.5567e-03	1.9435e-03	1.0264e-03	5.2908e-04
10^{-10}	9.0345e-03	6.2291e-03	3.5567e-03	1.9435e-03	1.0264e-03	5.2908e-04
$E^{-N,M}$	9.0345e-03	6.2291e-03	3.5567e-03	1.9435e-03	1.0264e-03	5.2908e-04
$r^{N,M}$	0.5364	0.8084	0.9188	0.92106	0.9560	-
Results in Woldaregay and Duressa (2021)Before Richardson Extrapolation						
10^{-4}	9.2814e-03	6.5095e-03	3.8026e-03	2.0167e-03	-	-
10^{-6}	9.2806e-03	6.5094e-03	3.8028e-03	2.0170e-03	-	-
10^{-8}	9.2806e-03	6.5094e-03	3.8028e-03	2.0170e-03	-	-
10^{-10}	9.2806e-03	6.5094e-03	3.8028e-03	2.0170e-03	-	-
$E^{-N,M}$	9.2814e03	6.5095e03	3.8028e03	2.0170e03	-	-
$r^{N,M}$	0.5118	0.7755	0.9149	-	-	



(a) $\varepsilon = 10^0, N = M = 128$ (b) $\varepsilon = 10^{-10}, N = M = 128$

Figure 4.1: Numerical solution of Example 4.5.1 for $N = M = 128$

Solution of example (4.5.1) and (4.5.2) exhibits a right boundary layer. As one observes in figures above,as the perturbation parameter, ϵ goes small; the boundary layer formation becomes more visible. In Tables(4.1) and (4.2),the maximum absolute error, the uniform error and the uniform rate of convergence of the scheme is given for different values of ϵ and mesh numbers.The calculated $E_{\epsilon,\delta}^{N,M}$, $r_{\epsilon,\delta}^{N,M}$, $E^{N,M}$, $r^{N,M}$ for the test examples (4.5.1) and (4.5.2) with different values of N,M,ϵ and δ are presented in tables (4.1) and (4.2).From these tables,we can easily see the maximum absolute error decrease as the step sizes decrease



(a) $\varepsilon = 10^0, N = M = 128$ (b) $\varepsilon = 10^{-10}, N = M = 128$

Figure 4.2: Numerical solution of Example 4.5.2 for $N = M = 128$

for all values of ϵ, which reveals an ϵ-uniform convergence of the proposed algorithm.The numerical results show that the proposed method gives more accurate than methods in Woldaregay and Duressa (2021) from the figures,one can observe that as ϵ goes small strong boundary layer is created near x=1. As the size of δ increases,the thickness of layer increases. The numerical results computed for example (4.5.1) are displays in table (4.1). Also comparisons of the present method with the methods in the literatures are presented in table (4.1) for example (4.5.1).From this table, it can be observed that the proposed method is accurate than the others in terms of parameter uniform error estimates as well as the order of convergence.Table (4.2) displays numerical results for example (4.5.2) and comparison of ϵ - uniform maximum point wise errors and corresponding ϵ-uniform order of of convergence between present result and the method in the literature.

Figure (4.1a), provides the graph of numerical solution of example (4.5.1) for $\epsilon = 10^0$ and N=M=128. In Figure (4.1b), provides the graph of numerical solution of example (4.5.1) for $\epsilon = 10^{-10}$ and N=M=128. In Figure (4.2a), provides the graph of numerical solution of example (4.5.2) for $\epsilon = 10^0$ and N=M=128. In Figure (4.2b), provides the graph of numerical solution of example (4.5.2) for $\epsilon = 10^{-10}$ and N=M=128.

Chapter 5

Conclusion And Recommendation

5.1 Conclusion

In this thesis,we propose fitted numerical scheme for solving singularly perturbed parabolic delay differential equation involving small delay. The method comprises an implicit Euler method to discretize the time variable on uniform mesh and cubic spline in tension method in space variable. Some properties of discrete problems that ensured the stability of the method were presented and used to analyze the convergence.This analysis resulted in both the space and time variables.Stability of the scheme is investigated using construction of barrier function for the solution bound. Uniform convergence of the scheme is proved. Applicability of of the scheme is investigated by considering two test examples. Effects of the perturbation parameter on the solution is shown using figures and tables.The scheme is accurate,stable and uniformly convergent.

5.2 Recommendation

In this thesis,fitted numerical scheme was developed to solving singularly perturbed parabolic delay differential equation problem.In the future, the method used in this thesis can be extended to other types of time dependent singularly perturbed problems.

Bibliography

Adivi Sri Venkata, R. K. and Palli, M. M. K. (2017). A numerical approach for solving singularly perturbed convection delay problems via exponentially fitted spline method. *Calcolo*, 54:943–961.

Doğan, N., Ertürk, V. S., and Akın, Ö. (2012). Numerical treatment of singularly perturbed two-point boundary value problems by using differential transformation method. *Discrete Dynamics in Nature and Society*, 2012.

File, G., Reddy, Y., et al. (2013). Computational method for solving singularly perturbed delay differential equations with negative shift. *International Journal of Applied Science and Engineering*, 11(1):101–113.

Friedrichs, K. O. and Wasow, W. R. (1946). Singular perturbations of non-linear oscillations.

Gautschi, W. (2011). *Numerical analysis*. Springer Science & Business Media.

Gelu, F. W. and Duressa, G. F. (2022). Parameter-uniform numerical scheme for singularly perturbed parabolic convection–diffusion robin type problems with a boundary turning point. *Results in Applied Mathematics*, 15:100324.

Gupta, V., Kumar, M., and Kumar, S. (2018). Higher order numerical approximation for time dependent singularly perturbed differential-difference convection-diffusion equations. *Numerical Methods for Partial Differential Equations*, 34(1):357–380.

Hanks, T. C. (1971). Model relating heat-flow values near, and vertical velocities of mass transport beneath, oceanic rises. *Journal of Geophysical Research*, 76(2):537–544.

Ku Sahoo, S. and Gupta, V. (2021). Second-order parameter-uniform finite difference scheme for singularly perturbed parabolic problem with a boundary turning point. *Journal of Difference Equations and Applications*, 27(2):223–240.

Kumar, D. and Kadalbajoo, M. (2012). Numerical treatment of singularly perturbed delay differential equations using b-spline collocation method on shishkin mesh. *Jnaiam*, 7(3-4):73–90.

Mbroh, N. A., Noutchie, C. O., and Massoukou, R. Y. (2021). A second order fitted operator finite difference scheme for a singularly perturbed degenerate parabolic problem. *International Journal of Nonlinear Analysis and Applications*, 12(Special Issue):677–687.

Nichols, N. (1989). On the numerical integration of a class of singular perturbation problems. *Journal of optimization theory and applications*, 60:439–452.

O'Malley, R. E. (1991). *Singular perturbation methods for ordinary differential equations*, volume 89. Springer.

Pramod Chakravarthy, P., Dinesh Kumar, S., and Nageshwar Rao, R. (2017). Numerical solution of second order singularly perturbed delay differential equations via cubic spline in tension. *International Journal of Applied and Computational Mathematics*, 3:1703–1717.

Prandtl, L. (1905). Über flüssigkeits-bewegung bei kleiner reibung, verhandlungen, iii. *Int. Math. Kongresses, Tuebner, Leipzig*, pages 484–491.

Rai, P. and Yadav, S. (2021). Robust numerical schemes for singularly perturbed delay parabolic convection-diffusion problems with degenerate coefficient. *International Journal of Computer Mathematics*, 98(1):195–221.

Ramesh, V. and Kadalbajoo, M. K. (2008). Upwind and midpoint upwind difference methods for time-dependent differential difference equations with layer behavior. *Applied Mathematics and Computation*, 202(2):453–471.

Roos, H.-G., Stynes, M., and Tobiska, L. (2008). Robust numerical methods for singularly perturbed differential equations, volume 24 of springer series in computational mathematics.

Schlichting, H. (1979). Boundary layer theory, mcgraw-hill, new york, 1979. *FIGURE CAPTIONS solid curve displays the exact solution. The difference between the exact solution and the eighth QLM iteration for all t in the figure is less than*, pages 10–10.

Singh, S., Kumar, D., and Ramos, H. (2022). A uniformly convergent quadratic b-spline based scheme for singularly perturbed degenerate parabolic problems. *Mathematics and Computers in Simulation*, 195:88–106.

Swamy, D. K., Phaneendra, K., Babu, A. B., and Reddy, Y. (2015). Computational method for singularly perturbed delay differential equations with twin layers or oscillatory behaviour. *Ain Shams Engineering Journal*, 6(1):391–398.

Tian, H. (2002). The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag. *Journal of Mathematical Analysis and Applications*, 270(1):143–149.

Wasow, W. (1965). Asymptotic expansions for ode. *Interscience pub.*

Wasow, W. R. (1942). *ON BOUNDARY LAYER PROBLEMS IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATION*. New York University.

Woldaregay, M. M. and Duressa, G. F. (2021). Accurate numerical scheme for singularly perturbed parabolic delay differential equation. *BMC Research Notes*, 14:1–6.

Yüzbaşı, Ş. and Sezer, M. (2013). Exponential collocation method for solutions of singularly perturbed delay differential equations. In *Abstract and Applied Analysis*, volume 2013. Hindawi.