



**SCHOOL OF GRADUATE STUDIES
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF MATHEMATICS**

PROGRAMME: MSc

**MSc THESIS ON
THE ANALYTICAL SOLUTION OF SOME FRACTIONAL ORDINARY
DIFFERENTIAL EQUATIONS BY ADOMIAN DECOMPOSITION METHOD**

BY: ASFEW ENDALU KENAH

ADVISOR: BOKA KUMSA (PhD, ASSOC. PROF.)

**MAY, 2026 G. C
NEKEMTE, ETHIOPIA**



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As a member of examining board of the final MSc thesis open defense, we certify that we have read and evaluated thesis by Asfew Endalu Kenaha under **“The Analytical Solution Of Some Fractional Ordinary Differential Equations By Adomian Decomposition Method”** and recommend that the thesis be accepted as fulfilling the thesis requirement for degree of master of science in differential equation.

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I hereby certify that all the correction and recommendation suggested by the board of examiners are incorporated into the final thesis entitled **“The Analytical Solution of Some Fractional Ordinary Differential Equations by Adomian Decomposition Method”** by Asfew Endalu.

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DECLARATION

This is to certify that this thesis is entitled “**The Analytical Solution of Some Fractional Ordinary Differential Equations by Adomian Decomposition Method**” is accepted in partial fulfillment of the requirements for the award of MSc. degree in differential equation by the school of graduate studies, Wallaga University through the college of natural and computational sciences, done by Asfew Endalu is genuine work carried out by him under my guidance. The matter embodied in this thesis work has not been submitted earlier for award of any degree. The assistance and help received during the course of this investigation have been dually acknowledged. Therefore, I recommend that it can be accepted as fulfilling the research thesis requirements.

Boka Kumsa (PhD, Assoc. prof.)

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Signature

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DEDICATION

This thesis is dedicated to all my parents and my entire supporters for their encouragement of moral and providing me any support accompanied with love for the success of my work.

BIOGRAPHICAL SKETCH

The author Asfew Endalu Kenaha was born in Haru district, West Wollega zone, Oromia Regional State. He attended his elementary school at Allebenta primary and elementary school from 2003-2010 G.C. He was then promoted to Guyi secondary and preparatory school to attend his next educational level from 2012-2014 G.C. Then he joined Mettu University in 2015 G.C and graduated with BSc. degree by applied mathematics in July 2017 G.C. Then, he has been employed by west Wallaga zonal educational office being a teacher in July 2020 G.C. Then, he started a master of degree and joined to follow MSc. program in field of differential equation. Now he prepared his thesis on “The Analytical Solution of Some Fractional Ordinary Differential Equations by Adomian Decomposition Method”.

STATEMENT OF THE AUTHOR

First, I declare that this thesis is my real work and all sources of materials used for the thesis have been accurately acknowledged. This thesis has been submitted in partial fulfillment of the requirements for the MSc degree in the field of differential at Wallaga University and reserved at the university library to be made available for borrowers under the rule of the library. I seriously declare that this thesis is not submitted to any other institution anywhere for the award of any academic degree, diploma, or certificate. Brief quotations from this thesis are allowed without special permission, provided that accurate acknowledgment of the source is made. Requests for permission for extended quotation from or reproduction of this research in whole or in part may be granted by the head of the major department or the dean of the school of graduate studies when his other judgment the pro permission must be obtained from the author.

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Place: Wallaga University

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LIST OF ABBREVIATIONS AND ACRONYMS

ADM.....	Adomian Decomposition Method
FIE.....	Fractional Integral Equation
FIVPs.....	Fractional Order Initial Value Problems
FODEs.....	Fractional Ordinary Differential Equations
HPM.....	Homotopy Perturbation Method
MATLAB.....	Matrix Laboratory
ODEs.....	Ordinary Differential Equations
VIM.....	Variational Iteration Method

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ABSTRACT

This paper contains the development and application of Adomian Decomposition Method (ADM) to solve fractional-order differential equations (FODEs). Classical differential equations are well-studied, but many natural and engineering processes are better modeled by fractional derivatives, which capture memory and hereditary effects. Because of their capacity to simulate physical phenomena with memory and hereditary characteristics, fractional ordinary differential equations (FODEs), which involve derivatives of non-integers(fractional) order, have attracted a lot of interest. It is a usable system with fractal capacitance, anomalous diffusion, and viscoelastic materials. However, obtaining analytical solutions for fractional differential equations remains a major challenge due to their nonlocal nature and the complexity of fractional operators.

To address this problem, this study employs the Adomian Decomposition Method (ADM), a semi-analytical iterative technique that decomposes nonlinear terms into rapidly convergent series. The method is applied to fractional differential equations expressed in the Caputo sense, with solutions obtained for both linear and nonlinear cases. The decomposition generates successive terms without requiring linearization or perturbation, thereby simplifying the solution process.

The results demonstrate that ADM provides accurate approximations that converge quickly to the exact solution. Moreover, the method captures the influence of different fractional orders ($0 < \beta \leq 1$) on the solution behavior, showing how fractional dynamics interpolate between classical integer-order solutions and memory-dependent systems.

Simulations were carried out on test problems, and the approximate ADM solutions were compared with exact solutions where available. Tables and plots of errors confirm the reliability and efficiency of the approach. The findings indicate that ADM is a powerful and flexible tool for solving fractional differential equations and can be extended to more complex models in applied mathematics, physics, and engineering.

Keyword: Adomian decomposition method, ordinary differential equation, fractional ordinary differential equations, Analytical solutions of FODEs by ADM

CHAPTER ONE

1 INTRODUCTION

1.1 Background of the study

The study of differential equations is as old as the development of the calculus by Newton and Leibniz in the 17th century. Originally, motivation was provided by important questions about change and motion on earth and in the heavens. This holds the truth that states differential equations are mathematical equations that contain a function with one or more independent variables and connect that function with its derivatives.

Differential equations form the backbone of mathematical modeling in science and engineering. By relating a function with its derivatives, they enable us to describe how quantities change over time or space. Ordinary differential equations (ODEs) involving derivatives with respect to a single independent variable are central in modeling processes in physics, biology, engineering, and economics. Examples include Newton's second law in mechanics, population growth models in biology, and charge-discharge behavior in electrical circuits.

The significant role of ODEs is reflected in how they capture dynamic behavior across contexts: for instance, in physics, ODEs describe motion (e.g., mass-spring systems), heat conduction, and electrical current dynamics. In biology, they model phenomena like logistic population growth, neuron firing rates, and enzyme kinetics. In engineering, ODEs guide design and control of systems such as control circuits, signal processing, and thermal management.

However, not all phenomena fit neatly within the realm of integer-order derivatives. Many natural and engineered systems exhibit memory or history-dependent behavior that classical ODEs cannot fully capture.

To address limitations in modeling real-world systems with memory effects, fractional calculus extends differentiation and integration to non-integer (fractional) orders. Fractional differential equations (FDEs) have seen growing use, because they provide a more accurate framework for processes with hereditary properties. For example,

Viscoelastic materials; (e.g., polymers, biological tissues) exhibit behavior dependent not only on current stress or strain but also on their deformation history effectively captured with fractional derivatives (Podlubny, 1999).

Anomalous diffusion (in complex media); where particle spread deviates from classical Brownian motion can be described by fractional derivatives representing “long memory” in transport (Metzler & Klafter, 2000).

In control systems; fractional-order controllers can offer superior performance over integer-order counterparts in managing systems with long-memory dynamics (Monje *et al.*, 2010).

The FODE generalizes the classical ODE by replacing integer-order derivatives with fractional counterparts (e.g., Caputo, & Riemann Liouville derivatives). Despite their modeling power, analytic solutions to FODEs are far more difficult to obtain than those for classical ODEs; closure in closed form exists only in limited cases.

Several analytical and numerical techniques have been developed to tackle Fractional Ordinary Differential Equations. Those methods are;

Laplace Transform Methods: For linear FODEs with constant coefficients and well-posed initial conditions, the Laplace transform method can yield analytical solutions in terms of Mittag-Leffler functions (Kilbas, 2006). Though elegant, the approach becomes cumbersome for variable coefficients or nonlinear forms.

Power Series and Frobenius Methods: These extend classical series expansions to include fractional powers. However, convergence may be slow and application limited to specific structures (Diethelm & Ford, 2010).

Variational Iteration Method (VIM): A semi-analytical technique generating iterations that converge to the solution by correcting a functional, often used in nonlinear FODEs (He, 1998). It relies on constructing correction functional that increasingly approximate the solution.

Homotopy Perturbation Method (HPM): Combining homotopy techniques with perturbation series to gradually deform a difficult problem into a simpler one, then extract a series-form

solution (He, 1998). Like VIM, HPM offers manageable iterations but may require small parameters or initial guesses tuned to specific problems.

ADM, pioneered by George Adomian in the late 1980s and early 1990s, has emerged as a robust and flexible semi-analytical method. It decomposes the solution into an infinite series, with nonlinear terms expanded into Adomian polynomials often leading to rapidly converging series without linearization or small-parameter assumptions.

As it was rose above, fractional ordinary differential equation can be solved by different methods including ADM. Compared to VIM and HPM, ADM stands out for:

Systematic handling of nonlinearity via Adomian polynomials, avoiding perturbation assumptions, thus applicable when the solution or parameters are not “small”,

Fast convergence, in many practical problems, with just a few series terms giving accurate approximations (Wazwaz, 2015). Those raised one shows that ADM is available and effective in finding analytic solution of fractional ordinary differential equation.

These strengths make ADM particularly attractive for FODEs where nonlinearity and memory effects both play roles. Nonetheless, its success hinges on demonstrating rigorous convergence, error estimation, and applicability across varying fractional orders and equation classes.

ADM has seen widespread application not only in integer-order ODEs but increasingly in FODEs, especially with Caputo derivatives (Momani & Odibat, 2006). Applications include time-fractional diffusion, chaotic systems, and fluid dynamics. For example, consider the following fractional differential equation.

$$D^\beta u(x) + Nu(x) + Ru(x) = g(x)$$

In this case D^β is fractional differential operator β , and applying its inverse integral operator $J^\beta = D^{-\beta}$ to above equation we obtain;

$$u(x) = J^\beta(g(x) - Ru(x) - Nu(x))$$

Here ADM decomposes u and nonlinear term as sum of infinite series as follow.

$$u_0 = c + J^\beta(g(x))$$

$$u_{n+1} = -J^\beta(Ru_n + A_n)$$

This determine that the ADM decomposes the solution and nonlinear terms into infinite series form to handle the analytical solution without requiring linearization, discretization and perturbation which illustrates the effectiveness and robustness of the method.

Despite its effectiveness, ADM is underexplored for linear and nonlinear FODEs across different fractional orders means especially not applied for orders $0 < \beta \leq 1$ broadly as that of integer orders, the performance of comparing approximate and exact solution widely and convergence rate were the gap identified by this thesis.

1.2 Statement of the problem

In current year, fractional order differential equations have emerged as powerful application in many fields like physics, Biology, Chemistry, and for modeling complex physical and engineering system that exhibit memory, hereditary properties or anomalous diffusion phenomena that cannot be accurately captured by classical integer order models.

By their nature, solving of such FODEs is difficult. Because of they exist in hereditary property. As a result of this difficulty in finding the solution, most of the numerical techniques often require discretization or linearization when applied to FODEs. This motivates the semi-analytical methods that offer recursive structures and avoid linearization, or discretization. One such promising technique is the ADM which provides an effective means of obtaining approximate solution to linear and nonlinear FODEs without the need for discretization and linearization. This method considers initial condition, first term of the series solution, and decomposing nonlinear functions in terms of special polynomial called Adomians polynomials and finding the successive terms of the series solution by recursive relation using Adomians polynomials. Thus central problems addressed were, while many analytical and numerical techniques exist for solving classical order, few applied both analytical structure and computational efficiency that ADM offers for FODEs. Limited research quantifies its accuracy by comparing it to exact solution across varying β and the convergence case studies as the order decreases. Therefore, this study was done to solve these problems.

1.3 Thesis questions

This section aims to address the following basic questions.

1. How can ADM be systematically applied to obtain approximate analytical solutions for a given fractional differential equation?
2. How accurate is the ADM solution compared to the exact solution for different fractional orders, and how does the fractional order affect convergence?
3. Does the ADM converge faster or slower for fractional orders less than one, what is the trend in the relative error as the order decreases and Can this method effectively extended to justify FODEs by modeling?

1.4 Objective of the study

1.4.1 The General objective

The general objective of the study was to find the analytical solution of some FODEs by ADM.

1.4.2 The Specific objectives

To achieve this aim, the study is guided by the following specific objectives and their corresponding justifications:

1. To apply the ADM to a selected fractional ordinary differential equation involving linear and nonlinear terms
2. To compute and compare the approximate ADM solutions with the exact analytical solutions for different fractional orders
3. To analyze the convergence behavior and relative error of the ADM for fractional-order cases and evaluating the effectiveness, accuracy and applicability of the ADM by simulation.

1.5 Significance of the study

The notion of FODEs has been widely accepted and used in a wide range of phenomena like in Viscoelastic materials (these material exhibit stress-strain relationships that depend on past deformations. Unlike ideal elastic, viscoelastic material remembers how they stretched,

Anomalous diffusion (in complex media particles do not diffuse in standard way-their movement is influenced by past position and collision, and in Biological system (Neuronal signals or gene expressions depend on past stimuli-not just the current signal. Since those applications are of FODEs, this study was significant for variety of reasons. First of all it applies the ADM on selected FODEs. This was strongly necessary to know how ADM solves FODEs. Additionally the study was outlining the comparison between approximate and exact solutions. So that it understood their relative error.

1.6 Delimitation of the study

The thesis was conducted under the stream of analytical solution of some FODEs by ADM. Thus it would delimit to analyze and discuss by considering some FODEs by ADM.

1.7 Limitations of the thesis

During conduct of the study, various limitations were there. The first and foremost one is lack of sufficient internet serves in the area, lack of necessarily reference materials, due to the security case an absence of exchanging idea with senior mathematics students are some of the limitations of this thesis.

1.8 Organization of thesis

This thesis contains 5 Chapters and it is organized as; chapter one provides introduction and available mathematical preliminaries of concepts of the analytical solution of FODEs such as basic definition, fundamental properties, unique properties of Gamma function etc. which would be used throughout this study. Chapter two provides a review of related literature, Chapter three reveals the methodology and mathematical procedure used to conduct the thesis on the given title. Chapter four is the discussion and results. Finally, Chapter five presents the brief summery and conclusion of the study.

1.9 Mathematical preliminary

1.9.1 Fundamental concepts

In this section we give a brief introduction and a background information summary of ordinary differential equations, since fractional ordinary differential equations is the consequences of Ordinary/ differential equation whenever integer order n extend to be fraction.

Definition 1.2.1 (Ordinary differential equation)

An Ordinary differential equation (ODE) is a differential equation that contains a function of one independent variable and its derivatives. A well-defined ODE must have the following features: It can be written in the form:

$$F(x, y, y', y'', \dots, y^n) = 0$$

Where the mathematical expression on the right hand side contains 1) variable x , 2) function y of x , and 3) some derivatives of y with respect to x ;

Definition 1.2.2 (Origin of fractional calculus)

Fractional calculus owes its origin to a question of whether the meaning of a derivative to an integer order n could be extended to still be valid when n not an integer is. This question was first raised by L'Hopital on September 30th, 1695.

Definition 1.2.3 (Fractional ordinary differential equation)

If an ordinary differential equation of integer order n elongated to non-integer order n , then it is called a fractional ordinary differential equation (FODE) (Ray & Bera, 2005).

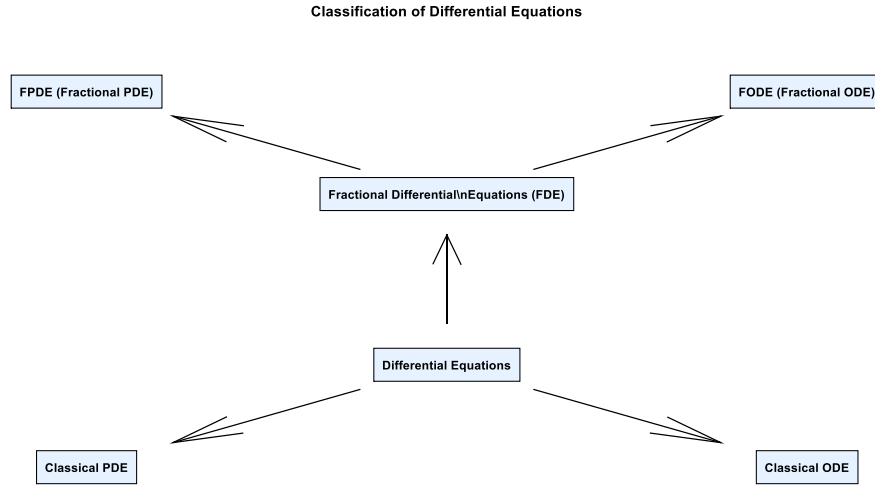


Figure 1: Class of differential equation

1.9.2 Useful mathematical functions

The two mathematical functions that inherently tied to fractional calculus and commonly be encountered are Gamma function and Mittag-Leffler Function.

1.9.2.1 Gamma Function

The Gamma function $\Gamma(x)$ is defined by integral as;

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad x \in \mathcal{R}^+$$

1.9.2.1.1 Unique properties of Gamma function

$$\left\{ \begin{array}{l} 1. \Gamma(x+1) = x\Gamma(x), x \in \mathcal{R}^+ \\ 2. \Gamma(x) = (x-1)! \\ 3. \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \end{array} \right. \quad (1)$$

1.10 Mittag-Leffler function

The Mittag-Leffler function is the direct generalization of the exponential function e^x and it plays a major role in fractional calculus. The one and two parameter representation of the Mittag-Leffler function can be defined in terms of power series as

$$(i) \ E_{\alpha}(x) = \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \alpha > 0 \quad (2)$$

$$(ii) \ E_{\alpha, \beta}(x) = \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta > 0 \quad (3)$$

CHAPTER TWO

2 REVIEW OF RELATED LITERATURE

In this section, we determined the capacities of fractional ordinary differential equations (FODEs) that represent memory-dependent and non-local dynamics in fields like biology, physics, and control systems which has spurred significant interest in their study. Without complete linearization, discretization or perturbation, the Adomian Decomposition Method (ADM) provides a semi-analytical solution by decomposing nonlinear operators using Adomian polynomials, allowing convergence to precise or approximate solutions. The issue that this study can solve is highlighted by this literature review, which critically looks at recent ADM implementations and their uses in solving FODEs.

Fractional calculus is a branch of mathematics that investigates the properties dealing with arbitrary order integral and differential operators (Hilfer, 2000).

In 1695, Leibniz and L'Hopital have discussed $1/2$ order derivative, which is regarded as the birth of fractional differential equation (Mainardi, 2022). For a long time, fractional calculus does not attract enough attention. It is only considered and studied by many mathematicians. However, in last few decades, fractional calculus has been studied more and more in applied sciences and engineering (Srivastava, 2020).

In the last few years, researchers introduced various problems involving fractional derivative and integral operators of arbitrary order (Srivastava, 2020). Some recent contributions to the theory of fractional differential (difference) equations and its applications can be varying (Mohammed *et al.*, 2020).

The study of fractional differential equations (FDEs) has gained significant attention in recent decades due to their ability to model real-world phenomena more accurately than classical integer-order differential equations. Applications of FDEs extend to diverse areas such as engineering, physics, chemistry, biology, control theory, and viscoelasticity, where memory and hereditary effects are important (Podlubny, 1999); (Kilbas., 2006). The application of ADM for the solution of this linear and nonlinear fractional differential equations has also been established by Shawagfeh, Saha Ray and Bera (Dehgan & Tatari, 2006; Mathai & Haubold, 2017).

Various analytical and numerical methods have been proposed to solve FDEs. Early approaches included the Laplace transform method, variational iteration method (He, 1999), and homotopy perturbation method (Jafari & Seifi, 2009). While these techniques provide valuable insights, they often require linearization, discretization, or perturbation, which can restrict their applicability to highly nonlinear systems.

ADM decomposes the solution into a rapidly convergent series without linearization or perturbation, while nonlinear terms are represented using Adomian polynomials. This advantage makes the method attractive for nonlinear problems in both integer-order and fractional-order systems.

Several researchers have extended ADM to fractional calculus. (Momani & Odibat, 2006) applied ADM to time-fractional diffusion equations with nonlocal initial conditions, obtaining series solutions efficiently. Again (Momani & Odibat, 2007) applied ADM to linear and nonlinear FDEs, demonstrating its reliability in generating approximate solutions. (Awadalla & Yameni, 2018) were used modeling exponential growth and exponential decay real phenomena by ψ -Caputo fractional derivative, which offers more convergence, & accuracy. (El-Sayed *et al.*, 2004) and (El-Mesiry *et al.*, 2005) studied numerical solution for mult-term fractional (arbitrary) orders differential equations while (Ray & Bera, 2005) investigated its application to nonlinear oscillators of fractional order. (Shawagfeh, 2002) extended the ADM to the solution of nonlinear FDEs in the Caputo sense. The study emphasized that ADM provides a systematic procedure for constructing approximate analytical solutions in the form of rapidly convergent series, without requiring perturbation techniques or linearization of the nonlinear operator. An essential aspect of ADM research is the comparison with other methods, such as truncated fractional power series, finite difference methods, or numerical solvers. (Ray & Bera, 2005) exemplifies the approach by comparing ADM solutions with a truncated fractional power series for a nonlinear FDE involving both integer and half-order derivatives.

More recently, (Dehghan & Tatari, 2006) analyzed ADM's efficiency in solving fractional-order models in viscoelasticity and fluid dynamics. These studies collectively highlight ADM's versatility in handling nonlinear FDEs across various domains.

While ADM frequently demonstrates rapid convergence, many studies focus on classical orders, and partial fractional differential equations. Means few focuses on fractional ordinary fractional differentials without clear term by term application of ADM, accuracy, and explicit error bounds remain scarce. For instance (Bulut *et al.*, 2013) and (Shawagfeh, 2002).

Generally, despite these developments, challenges remain concerning the applying ADM for FODEs broadly, justification of the solutions obtained to exact, and rigorous analysis of convergence and error estimates of ADM when applied to linear and nonlinear fractional order. The present work by this thesis addresses this gap by applying ADM on selected FODEs, offers numerical examples that compare ADM solutions with exact and numerical results, further validating the method's accuracy and stability, and additionally the paper presents proving the convergence and error estimation of ADM solutions under Lipschitz conditions, establishing convergence criteria, and explaining error bounds.

CHAPTER THREE

3 RESEARCH METHODOLOGY

This chapter addresses the methodology and mathematical procedure of the study, which plays a crucial role in the derivation of our main results. So the process of iteration is as follows.

3.1 The Study Area and Period

The study would be conducted in Wallaga University at the college of Natural and computational science under the department of mathematics From December 2024 G.C to August 2025 G.C. conceptually the study have been focused on the analytic solution of some FODEs by ADM.

3.2 Study Design

The study would be employ the documentary review design on analytical solution of some FODEs by ADM

3.3 Sources of data

First of all for this study, some literature and data that related with the FODEs theory has been collected from different sources. The sources of information for this study have been taken from different books, literature, published and unpublished articles and internet.

3.4 The Study procedures

After important materials and data for the study has been collected, the documentary review were organized through all necessary steps. The research process is analytical and theoretical based on deriving approximate solution to linear and nonlinear FODEs using ADM. The study involves;

1. Formulating the study problem by applying inverse Riemann Liouville operator.
2. Decomposing the solution and nonlinear terms by ADM into infinite series.
3. Determining the solution of selected FODEs.
4. Providing the illustrative examples with their simulation Matrix Laboratory (MATLAB) graph for applicability of the result.
5. Finally, it covered by conclusion and future work.

CHAPTER FOUR

4 THE MAIN RESULT AND DISCUSSION

4.1 Fractional calculus and ADM

In this section, we give some basic concepts of integration and differentiation, which are the consequences of fractional differentiation, properties of the fractional calculus theory which are used further in (Podlubny, 1999; Diethelm & Luchko, 2004; Rida, 2010) and ADM with the formulation of the problem in the Caputo sense were discussed.

4.2 Differentiation and integration

Differentiation and integration in calculus are often seen as two sides of the same coin. Derivative of a function represents the rate of change of the function with respect to an independent variable. Simply it tells us how fast something is changing at a given point, whereas integration represents the summation or accumulation of small quantities over an interval. Thus it finds the total effects of changes. Therefore derivative and integration are inverses of each other. More precisely let D^n be a derivative of order n , then D^{-n} is an integral operator of that order n derivative. In ordinary calculus the possible number for the order of a derivative and integral operator belongs to the integer number while, in fractional calculus the order of derivative and an integral operator has more flexibility. In fact when the order of this derivative or integral calculus extends to non-integer order then it is fractional calculus. In this fractional calculus the main two concepts considered were Riemann Liouville and Caputo fractional integrals and derivatives which are basic expert for fractional differential equation.

4.3 Fractional differential equations

4.3.1 Fractional ordinary differential equation.

Fractional differential equation is a type of differential equation that involves derivative of fractional order. Fractional differential equation extends ordinary differential equations by allowing the derivatives of non-integer order, enabling more accurate modeling of real world process with memory effects. Thus fractional ordinary differential equation which is circled by red box in the diagram below is a subclass of this type equation. Its classes are given by the following diagram.

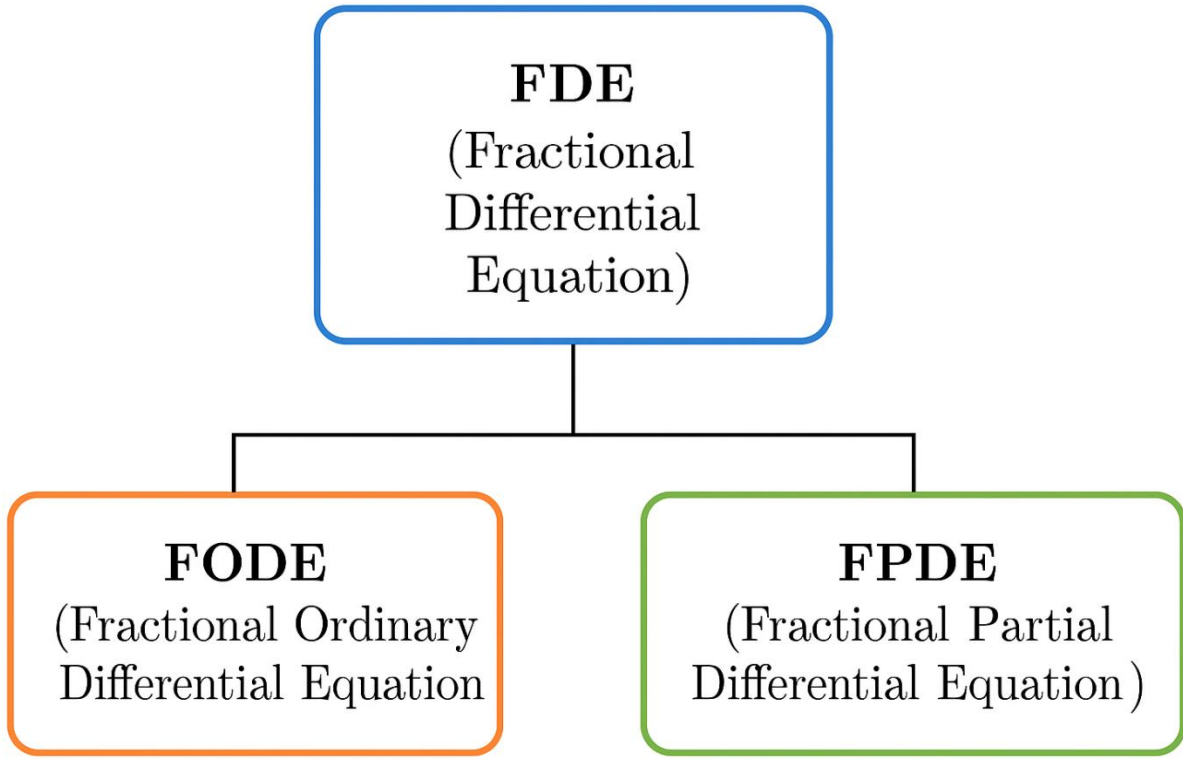


Figure 2: Classes of fractional differential equation

4.4 Fundamental Properties of Fractional Calculus

4.4.1 Fundamental Facts of the Fractional Calculus.

Firstly, we mention some of the fundamental properties of the fractional calculus. Fractional derivatives (and integrals as well) definitions may differ, but the most widely used definitions are those of Abel-Riemann (A-R) and Caputo. Following the nomenclature in a derivative of fractional order in the A-R sense is defined by (Mittal & Nigam, 2008).

$$D^\beta f(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \frac{d}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, & n-1 < \beta \leq n \\ \frac{d^n}{dt^n} f(t), & \beta = n \end{cases}$$

Where $n \in \mathbb{Z}^+$, and D^β is fractional derivative operator of order β . Thus

$$D^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau, \quad 0 < \beta \leq 1 \quad (4)$$

On the other hand according to the A-R, integral of fractional order is defined by implementing the integration of order $J^\beta = D^{-\beta}$ in the following manner.

$$J^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} f(\tau) d\tau; \quad t > 0, \beta > 0 \quad (5)$$

Another main definition of fractional derivative is that of Caputo (Podlubny, 1999). And it is defined by

$${}_c D^\beta f(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}} d\tau; n-1 < \beta \leq n \\ \frac{d^n}{dt^n}; \beta = n \end{cases} \quad (6)$$

The fundamental feature of the Caputo fractional derivative is that presented by (Katatbeh & Belgacem, 2011).

$$J^\beta [{}_c D^\beta f(t)] = f(t) - \sum_{k=0}^{\infty} f^{(k)}(0^+) \frac{t^k}{k!} \quad (7)$$

Over all when we come to some of the fundamental properties of a fractional integration and differentiation this has been introduced to the literature by (Podlubny, 1999). Among these we mention;

$$1. J^\beta [t^n] = \frac{\Gamma(1+n)}{\Gamma(1+n+\beta)} t^{n+\beta} \quad (8)$$

$$2. D^\beta [t^n] = \frac{\Gamma(1+n)}{\Gamma(1+n-\beta)} t^{n-\beta} \quad (9)$$

$$\text{Note: The always properties of the Kernel integral of } \int_0^t (t - \tau)^{\beta-1} d\tau = \frac{t^\beta}{\beta} \quad (10)$$

4.5 Adomian decomposition method

In this section, the idea of the Adomian decomposition method (Daftardar-Gejji & Jafari, 2007; Daftardar-Gejji, & Bhalekar, 2008) was illustrated. Let us consider the nonlinear differential equation.

4.6 General description of ADM over FDE

The general FDE was considered as follows.

$$D_t^\beta u(x, t) + Nu(x, t) + Ru(x, t) = g(x, t), u(x, 0) = f(x) \quad (11)$$

In this case, D_t^β is a FDE operator with order β & its inverse $D_t^{-\beta}=J^\beta$ is a fractional integral operator where N is non-linear, R was remainder linear term & g was a given function. Solving for $D_t^\beta u$, equation (1) can be written as,

$$D_t^\beta u(x, t) = g(x, t) - Nu(x, t) - Ru(x, t) \quad (12)$$

Now, applying J^β to (2) using (Vance, 2014).

$$J^\beta(D_t^\beta u(x, t) = g(x, t) - Nu(x, t) - Ru(x, t))$$

Thus it was obtained that;

$$u(x, t) - u(x, 0) = J^\beta(g(x, t) - Nu(x, t) - Ru(x, t)) \quad (13)$$

ADM assumed that solution that u can be decomposed into infinite series (12) and nonlinear term Nu can be decomposed into infinite series so substituting both of infinite series to (14)

$$u(x, t) = u(x, 0) + J^\beta(g(x, t)) - \sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} R_n \quad (14)$$

4.6.1 Recursive Relation

The recursive relation is constructed from the first n th terms to compute successive terms in the solution series. For instance based on (14), the recursive relation of the solution (14) was as follows;

$$u_0 = f(x) + J^\beta g(x, t) \quad (15)$$

$$u_{n+1} = -J^\beta(A_n + Ru_n) \quad (16)$$

Note that the above explanation is the general description for ADM of FDE. However the problem statement is specifically described below.

4.7 Problem formulations

In this section the Caputo sense are preferred to use because the initial conditions are expressed in terms of integer order derivatives of the unknown functions like $y(0), y'(0), y''(0), \dots$ which are the same as classical ODEs so that they are physically meaningful, unlike in A-R the initial conditions involve fractional derivatives of the function. Now performing subsequently by applying the fractional differentiation of order β , let us discuss its formulation as next.

We consider the following nonlinear FDE of Caputo sense,

$$D_t^\beta y(t) = f(t, y(t)), 0 < \beta \leq 1, t > 0 \quad (17)$$

Subject to the initial condition

$$y(0) = 0 \quad (18)$$

Where D_t^β denotes the Caputo fractional derivatives of order β .

Applying the Riemann Liouville fractional operator J^β of order β to both side of (17) and using the property of Caputo derivatives with initial condition (18) and (5) we obtain equivalent fractional integral equation (FIE);

$$y(t) = y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y(s)) ds \quad (19)$$

We assume that 1. $f(t, y)$ is bounded on the domain $[0, T] \times \mathbb{R}$

$$\text{i.e. } |f(t, y)| \leq M, \text{ for some constant } M < \infty$$

2. $f(t, y)$ is Lipschitz continues in y .

$$\text{i.e. } |f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \text{ for all } y_1, y_2 \in \mathbb{R} \text{ with Lipschitz constant } L > 0$$

The above (19) is equivalent to fractional integral equation (FIE) representation. ADM decomposes the solution and non-linear term as a series form using Adomian polynomials. Thus;

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \text{ and } f(t, y(t)) = \sum_{n=0}^{\infty} A_n \quad (20)$$

Where the Adomian polynomial A_n is defined as;

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N \left(\sum_{k=0}^{\infty} \lambda^k y_k(t) \right) \Big|_{\lambda=0}$$

For all $n = 0, 1, 2, 3, \dots$

If A_n is described, then obtaining;

$$A_0 = \frac{1}{0!} \frac{d^0}{d\lambda^0} [N(\sum_{k=0}^0 \lambda^k u_0)]_{\lambda=0} = N(u_0),$$

$$A_1 = \frac{1}{1!} \frac{d^1}{d\lambda^1} [N(\sum_{k=0}^1 \lambda^k u_k)]_{\lambda=0} = u_1 N'(u_0),$$

$$A_2 = \frac{1}{2!} \frac{d^2}{d\lambda^2} [N(\sum_{k=0}^2 \lambda^k u_k)]_{\lambda=0} = \frac{u_1^2}{2!} N''(u_0), \text{ and so on then,}$$

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N(\sum_{k=0}^{\infty} \lambda^k y_k(t))|_{\lambda=0} \quad (21)$$

Substituting (20), (21) into (19) and assuming the solution in series form we conclude;

$$y(t) = \sum_{n=0}^{\infty} y_n(t) + \sum_{n=0}^{\infty} A_n,$$

Then we obtain the recursive relations

$$y_0(t) = y(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y_0(s)) ds \quad (22)$$

$$y_{n+1}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A_n(s) ds \text{ For } n \geq 0 \quad (23)$$

Finally from above (22) and (23) it was obtained, the ADM solution that can be expressed as an infinite series of (24) below.

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \quad (24)$$

Theorem 1

The series solution (16) of the problem (12)-(13) by using ADM converges, if $|y_0| < \infty$ and

$$0 < M < 1; M = \frac{LT^\beta}{\Gamma(\beta+1)} \text{ with satisfying the above two assumptions of (19).}$$

Proof

Consider the Banach space $(C[J], \|\cdot\|)$ then the space of all continuous function on J is the norm $\|y(t)\| = \max_{t \in J} |y(t)|$. Define the sequence $\{u_n\}$ such that $u_n = \sum_{k=0}^n y_k(t)$ the sequence of partial sums from the series solution $\sum_{k=0}^{\infty} y_k(t)$ since;

$$f(y) = f(\sum_{k=0}^{\infty} y_k(t)) = \sum_{k=0}^{\infty} A_k(y_0, y_1, \dots, y_k)$$

$$f(y_0) = f(u_0) = A_0$$

$$f(y_0 + y_1) = f(u_1) = A_0 + A_1$$

$$f(y_0 + y_1 + y_2) = f(u_2) = A_0 + A_1 + A_3$$

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$$f(u_n) = \sum_{k=0}^{\infty} A_k(y_0, y_1, y_3, \dots, y_k)$$

For $n \geq m$, suppose u_n and u_m be a partial sums. We need to prove that $\{u_n\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned} \|u_n - u_m\| &= \max_{t \in J} |u_n - u_m| = \max |\sum_{k=m}^{n-1} y_k(t)| \\ &= \max_{t \in J} \left| \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} \sum_{k=m}^{n-1} A_k d\tau \right| \\ &= \max_{t \in J} \left| \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} [f(u_{n-1}) - f(u_{m-1})] d\tau \right| \\ &\leq \frac{L}{\Gamma(\beta)} \max_{t \in J} \int_0^t (t - \tau)^{\beta-1} |u_{n-1} - u_{m-1}| d\tau \end{aligned}$$

Using the (5) and property 1 of (1) we get,

$$\begin{aligned} &\leq \frac{LT^\beta}{\Gamma(\beta+1)} \|u_{n-1} - u_{m-1}\| \\ &\leq M \|u_{n-1} - u_{m-1}\| \text{ For } M = \frac{LT^\beta}{\Gamma(\beta+1)} \end{aligned}$$

By considering $n = m+1$,

$$\begin{aligned} \|u_{m+1} - u_m\| &\leq M \|u_m - u_{m-1}\| \leq M^2 \|u_{m-1} - u_{m-2}\| \leq M^3 \|u_{m-2} - u_{m-3}\| \leq \dots \\ &\leq M^n \|u_2 - u_1\|. \end{aligned}$$

By applying the inequality of a triangle,

$$\begin{aligned} \|u_n - u_m\| &\leq \|u_{m+1} - u_m\| + \|u_{m+2} - u_{m+1}\| + \|u_{m+3} - u_{m+2}\| + \|u_{m+4} - u_{m+3}\| + \dots \\ &\quad + \|u_{n+1} - u_n\| \\ &\leq [M^n + M^{n+1} + M^{n+2} + \dots + M^{k-1}] \|u_2 - u_1\|. \\ &\leq M^n [1 + M + M^2 + \dots] \|u_2 - u_1\|. \end{aligned}$$

Using the geometric ratio we have,

$$\leq M^n \left(\frac{1}{1-M} \right) \|y_0\|, \text{ with in } \|u_2 - u_1\| = \|y_0\|$$

However $|y_0| \leq \infty$ and as $n \rightarrow \infty$ then $\|u_n - u_m\| \rightarrow 0$. Hence $\{u_n\}$ is a Cauchy sequence in the Banach space. Therefore the series $\sum_{n=0}^{\infty} y_n(t)$ converges and then the theorem is proved.

4.8 Results

4.8.1 Application of ADM

Supportive Examples

In this class several examples are presented to demonstrate the efficiency and effectiveness of the ADM in solving FODEs that were subjected to the initial conditions.

Example 1 A The nonhomogeneous FODE presented by (Bulut *et al.*, 2013) below was considered.

$${}^C D^\beta u(t) + u(t) = t^2 + \frac{\Gamma(2)}{\Gamma(2.5)} t^{3/2},$$

$$\text{With the initial condition } u(t) = 0 \quad (25)$$

Before we apply the Riemann Liouville integral operator on it, Let $f(t) = t^2 + \frac{\Gamma(2)}{\Gamma(2.5)} t^{3/2}$

Now, applying the Riemann Liouville fractional integral J^β on both side for the Caputo derivative and $0 < \beta \leq 1$, we have;

$$J^\beta ({}^C D^\beta u(t) + u(t) = f(t)); J^\beta ({}^C D^\beta u(t)) = u(t) - u(0) \text{ and } u(t) = 0$$

$$\text{This gives } u(t) + J^\beta u(t) = J^\beta f(t)$$

$$\text{Thus } u(t) = -J^\beta u(t) + J^\beta f(t)$$

ADM decomposes the solution as $u(t) = \sum_{n=0}^{\infty} u_n(t)$ and it uses the recursive relation as;

$$u_0 = J^\beta f(t), \quad u_{n+1}(t) = -J^\beta u_n(t), \quad (n \geq 0)$$

$$\text{Using (25), compute } u_0 = J^\beta f(t) = \left(J^\beta t^2 + \frac{\Gamma(2)}{\Gamma(2.5)} J^\beta t^{\frac{3}{2}} = \frac{2}{\Gamma(3+\beta)} t^{2+\beta} + \frac{\Gamma(2)}{\Gamma(2.5)} \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}+\beta)} t^{\frac{3}{2}+\beta} \right)$$

But the $\Gamma(\frac{5}{2}) = \Gamma(2.5)$, so that the second coefficient was simplified to 1.

$$\text{Thus } u_0(t) = \frac{2}{\Gamma(3+\beta)} t^{2+\beta} + \frac{1}{\Gamma(\frac{5}{2}+\beta)} t^{\frac{3}{2}+\beta}$$

Applying the recurrence in the same way yielded the simple pattern,

$$u_0 = (-1)^n \left[\frac{2}{\Gamma(3 + (n+1)\beta)} t^{2+(n+1)\beta} + \frac{1}{\Gamma(5/2 + (n+1)\beta)} t^{3/2+(n+1)\beta} \right], n \geq 0$$

Therefore, the ADM (series) solution is $u(t) = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{2t^{2+k\beta}}{\Gamma(3+k\beta)} + \frac{t^{3/2+k\beta}}{\Gamma(5/2+k\beta)} \right)$

So, the first four terms are $u_0(t) = \frac{2}{\Gamma(3+\beta)} t^{2+\beta} + \frac{1}{\Gamma(5/2+\beta)} t^{3/2+\beta}$,

$$u_1(t) = - \left(\frac{2}{\Gamma(3+2\beta)} t^{2+2\beta} + \frac{1}{\Gamma(5/2+2\beta)} t^{3/2+2\beta} \right),$$

$$u_2(t) = \frac{2}{3+3\beta} t^{2+3\beta} + \frac{1}{\Gamma(5/2+3\beta)} t^{3/2+3\beta},$$

$$u_3(t) = - \left(\frac{2}{\Gamma(3+4\beta)} t^{2+4\beta} + \frac{1}{\Gamma(5/2+4\beta)} t^{3/2+4\beta} \right),$$

So ADM solution of (25) was $u(t) = \sum_{n=0}^{\infty} u_n(t)$ for $n \geq 0$

Therefore $u(t) = u_0 + u_1 + u_2 + \dots$

$$u(t) = \frac{2}{\Gamma(3+\beta)} t^{2+\beta} + \frac{1}{\Gamma(5/2+\beta)} t^{3/2+\beta} - \left(\frac{2}{\Gamma(3+2\beta)} t^{2+2\beta} + \frac{1}{\Gamma(5/2+2\beta)} t^{3/2+2\beta} \right) + \frac{2}{3+3\beta} t^{2+3\beta} + \frac{1}{\Gamma(5/2+3\beta)} t^{3/2+3\beta} - \left(\frac{2}{\Gamma(3+4\beta)} t^{2+4\beta} + \frac{1}{\Gamma(5/2+4\beta)} t^{3/2+4\beta} \right) + \dots$$

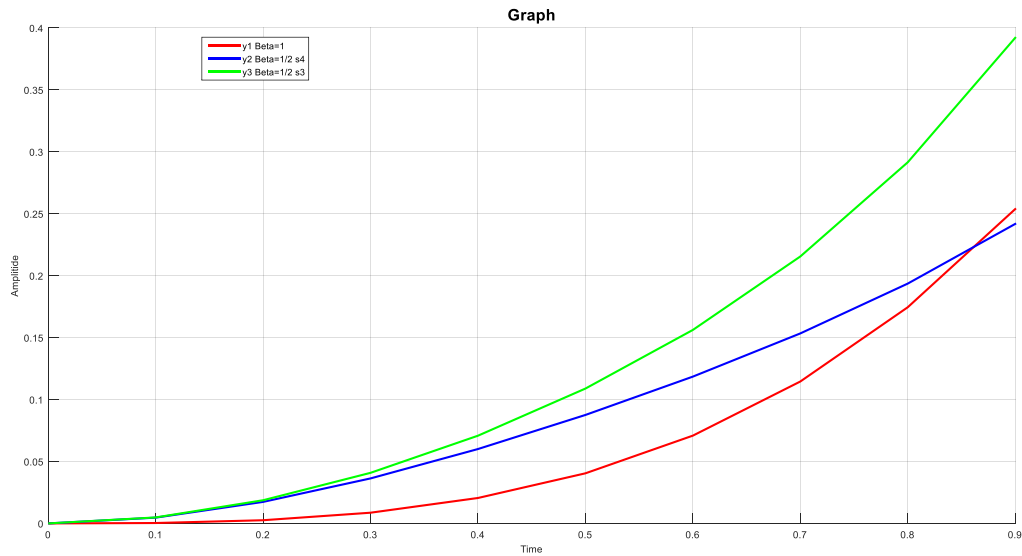


Figure 3: Graph of ADM of (25) when $\beta=1$ and $\frac{1}{2}$ for partial sum s_2, s_3, s_4

Example 1B

Consider the FODE ${}^C D_x^{\frac{1}{2}} y(x) = x^2$ subjected to the initial condition $y(0) = 0$

For the Caputo fractional derivative of order $0 < \beta < 1$, the standard left inverse is the Reimann Liouville fractional integral, $J^{1/2}$. Therefore the first step is applying $J^{1/2}$ to both side of this equation as follow,

$$J^{1/2} \left[{}^C D_x^{\frac{1}{2}} y(x) = x^2 \right]$$

Thus applying $J^{1/2}$ to both sides gave the equivalent integral equation:

$$y(x) - y(0) = J^{\frac{1}{2}}(x^2),$$

$$= y(x) = y(0) + J^{\frac{1}{2}}(x^2)$$

Using ADM steps, we can decompose $y(x) = \sum_{n=0}^{\infty} y_n(x)$ where $y_0(x) = 0$

To obtain the recurrence relation, the right hand side did not depend on y and therefore there were no non-linear Adomian polynomial needed. Thus,

$$y_1(x) = J^{\frac{1}{2}}(x^2),$$

$$y_{n+1}(x) = 0 \text{ for all } n \geq 1$$

So the ADM series truncated after the first nontrivial term.

$$\text{Therefore } y(x) = y_0(x) + y_1(x) = y(0) + J^{1/2}(x^2)$$

To evaluate the fractional integral, applying (9) and using $\Gamma(7/2) = \frac{15}{8}\sqrt{\pi}$, we obtain

$$\begin{aligned} J^{\frac{1}{2}}(x^2) &= y(0) + 2 * \frac{8}{15\sqrt{\pi}} x^{5/2} \\ &= y(0) + \frac{16}{15\sqrt{\pi}} x^{5/2} \end{aligned}$$

If $y(0) = 0$, is imposed, then the solution becomes,

$$y(x) = \frac{16}{15\sqrt{\pi}} x^{5/2}, \text{ and its verified by graph as follows}$$

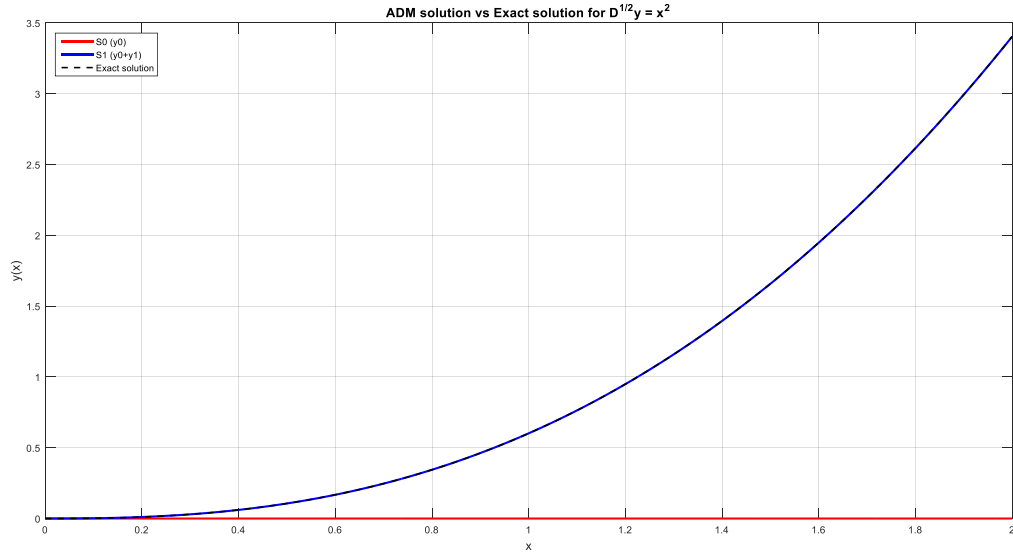


Figure 4: Graph of ADM solution of example 1B

Example 2

Consider the following FODE with the given initial condition to solve by ADM:

$$D^\beta y(t) = y^2 + 1 \quad (26)$$

$$m - 1 < \beta < m, 0 < t < 1, y^{(k)}(0) = 0, k = 0, 1, 3, \dots$$

To invert the given fractional differential equation (26), we use the fractional integral operator.

Thus applying Reimann Liouville Integral operator J^β , the obtained result was:

$$J^\beta(D^\beta y(t)) = J^\beta(y^2 + 1)$$

$$y(t) = J^\beta(y^2 + 1) \quad (27)$$

By assuming a decomposition method of y and the non-linearity $N(y)$ using (10) and (11), the Adomian polynomial of A_0 is;

$$A_0 = \frac{1}{0!} \frac{d^n}{d\lambda^n} N \left(\sum_{i=0}^k \lambda^i y_i \right) \quad \text{at } \lambda = 0$$

$$A_0 = N(y_0) \quad \text{Since } N(y) = y^2, N(y_0) = y_0^2$$

So, $A_0 = y_0^2$,

Accordingly the first three terms of Adomian polynomials were ;

$$A_1 = 2y_0y_1, \quad A_2 = 2y_0y_1 + y_1^2, \quad A_3 = 1/3[4y_1y_2 + y_0 + 2y_1^2], \dots$$

$$\text{The recursive of ADM of (26) is } \begin{cases} y_0 = J^\beta(1) = \frac{t^\beta}{\Gamma(\beta+1)} \\ y_{n+1} = J^\beta(A_n) \end{cases} \quad (28)$$

From (28), it is obvious that initial condition $y_0 = \frac{t^\beta}{\Gamma(\beta+1)}$ and also the first few terms are computed by letting $n \geq 0$ as follow,

$$y_1 = J^\beta(A_0)$$

$$\text{Since } A_0 = y_0^2, y_1 = J^\beta(A_0) = J^\beta\left(\left(\frac{t^\beta}{\Gamma(\beta+1)}\right)^2\right) = J^\beta\left(\frac{t^{2\beta}}{(\Gamma(\beta+1))^2}\right)$$

For $n=0$;

Applying property (9),

$$y_1 = J^\beta\left(\frac{t^{2\beta}}{(\Gamma(\beta+1))^2}\right) = \frac{1}{(\Gamma(\beta+1))^2} J^\beta(t^{2\beta}) = \frac{1}{(\Gamma(\beta+1))^2} \cdot \frac{\Gamma(2\beta+1)}{\Gamma(2\beta+\beta+1)} t^{2\beta+\beta} = \frac{1}{(\Gamma(\beta+1))^2} \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)} t^{3\beta}$$

$$\text{Thus } y_1 = \frac{\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2 \Gamma(3\beta+1)} t^{3\beta}$$

For $n=1$;

$$= y_2 = J^\beta(A_1) ; A_1 = 2y_0y_1$$

$$y_2 = 2J^\beta\left(\frac{t^\beta}{\Gamma(\beta+1)} \cdot \frac{\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2 \Gamma(3\beta+1)} t^{3\beta}\right)$$

$$y_2 = \frac{2\Gamma(2\beta+1)}{(\Gamma(\beta+1))^3 \Gamma(3\beta+1)} J^\beta(t^{4\beta})$$

$$y_2 = \frac{2\Gamma(2\beta+1)}{(\Gamma(\beta+1))^3 \Gamma(3\beta+1)} \cdot \frac{\Gamma(4\beta+1)}{\Gamma(5\beta+1)} t^{5\beta}$$

$$y_2 = \frac{2\Gamma(2\beta+1)\Gamma(4\beta+1)}{(\Gamma(\beta+1))^3\Gamma(3\beta+1)\Gamma(5\beta+1)} t^{5\beta}$$

For n=2;

$$y_3 = J^\beta(A_2) ; A_2 = 2y_0y_2 + y_1^2$$

$$A_2 = \left(\frac{4\Gamma(2\beta+1)\Gamma(4\beta+1)}{(\Gamma(\beta+1))^4\Gamma(3\beta+1)\Gamma(5\beta+1)} + \frac{(\Gamma(2\beta+1))^2}{(\Gamma(\beta+1))^4(\Gamma(3\beta+1))^2} \right) t^{6\beta} \quad \text{And}$$

$$\text{where, } y_1^2 = \left(\frac{(\Gamma(2\beta+1))^2}{(\Gamma(\beta+1))^2\Gamma(3\beta+1)} \right) t^{6\beta},$$

$$\text{And } 2y_0y_2 = \frac{4\Gamma(2\beta+1)\Gamma(4\beta+1)}{(\Gamma(\beta+1))^4\Gamma(3\beta+1)\Gamma(5\beta+1)} t^{6\beta}$$

$$y_3 = \left[\frac{4\Gamma(2\beta+1)\Gamma(4\beta+1)}{(\Gamma(\beta+1))^4\Gamma(3\beta+1)\Gamma(5\beta+1)} + \frac{(\Gamma(2\beta+1))^2}{(\Gamma(\beta+1))^4(\Gamma(3\beta+1))^2} \right] J^\beta(t^{6\beta}),$$

$$y_3 = \left[\frac{4\Gamma(2\beta+1)\Gamma(4\beta+1)}{(\Gamma(\beta+1))^4\Gamma(3\beta+1)\Gamma(5\beta+1)} + \frac{(\Gamma(2\beta+1))^2}{(\Gamma(\beta+1))^4(\Gamma(3\beta+1))^2} \right] \cdot \frac{\Gamma(6\beta+1)}{\Gamma(7\beta+1)} t^{7\beta}$$

Thus the recursive relation of the first four terms of the ADM solution is given by;

$$y(t) = y_0 + y_1 + y_2 + y_3$$

$$y(t) = \frac{t^\beta}{\Gamma(\beta+1)} + \frac{\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2\Gamma(3\beta+1)} t^{3\beta} + \frac{2\Gamma(2\beta+1)\Gamma(4\beta+1)}{(\Gamma(\beta+1))^3\Gamma(3\beta+1)\Gamma(5\beta+1)} t^{5\beta} +$$

$$\left[\frac{4\Gamma(2\beta+1)\Gamma(4\beta+1)}{(\Gamma(\beta+1))^4\Gamma(3\beta+1)\Gamma(5\beta+1)} + \frac{(\Gamma(2\beta+1))^2}{(\Gamma(\beta+1))^4(\Gamma(3\beta+1))^2} \right] \cdot \frac{\Gamma(6\beta+1)}{\Gamma(7\beta+1)} t^{7\beta} + \quad (29)$$

4.8.2 Case consideration

The ADM solution of (26) was generally solved above and it was given in (29). But that solution is not verified what would be the performance of the method as β vary in the given interval or as $\beta \in (0,1]$. Hence in order to justify it, here are considered cases, case 1 and case 2 as shown in below section.

I) Case 1

Table 1: The first 2-terms value of (29) for $0 \leq \mu < 1$

t	tan(t)	$\beta=1$	Error1	$\beta=1/2$	Error2	$\beta=1/3$	Error3
0	0	0	0	0	0	0	0
0.1000	0.1003	0.1003	0.0630	0.0569	0.0630	0.0373	0.0630
0.2000	0.2027	0.2027	0.1278	0.1167	0.1278	0.0749	0.1278
0.3000	0.3093	0.3090	0.1958	0.1822	0.1958	0.1135	0.1958
0.4000	0.4228	0.4213	0.2670	0.2563	0.2670	0.1558	0.2670
0.5000	0.5463	0.5417	0.3398	0.3420	0.3398	0.2065	0.3398
0.6000	0.6841	0.6720	0.4109	0.4421	0.4109	0.2732	0.4109
0.7000	0.8423	0.8143	0.4745	0.5594	0.4745	0.3678	0.4745
0.8000	1.0296	0.9707	0.5234	0.6968	0.5234	0.5062	0.5234
0.9000	1.2602	1.1430	0.5502	0.8573	0.5502	0.7100	0.5502

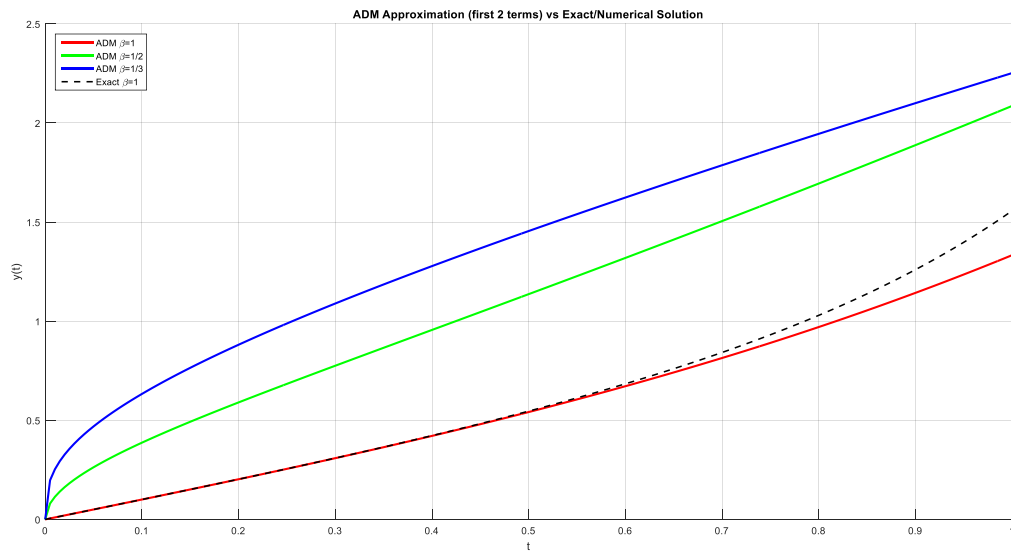


Figure 5: Graph of solution of (26) for first two terms

II) Case 2

Table 2: The first 3-terms value of (26) for $0 \leq \mu < 1$

t	tan(t)	$\beta=1$	Error1	$\beta=1/2$	Error2	$\beta=1/3$	Error3
0	0	0	0	0	0	0	0
0.1000	0.1003	0.1003	-0.0503	0.0569	-0.0503	0.1506	-0.0503
0.2000	0.2027	0.2027	-0.0988	0.1167	-0.0988	0.3015	-0.0988
0.3000	0.3093	0.3093	-0.1441	0.1824	-0.1441	0.4534	-0.1441
0.4000	0.4228	0.4227	-0.1862	0.2571	-0.1862	0.6090	-0.1862
0.5000	0.5463	0.5458	-0.2267	0.3443	-0.2267	0.7730	-0.2267
0.6000	0.6841	0.6824	-0.2689	0.4478	-0.2689	0.9530	-0.2689
0.7000	0.8423	0.8367	-0.3186	0.5718	-0.3186	1.1609	-0.3186
0.8000	1.0296	1.0144	-0.3830	0.7209	-0.3830	1.4126	-0.3830
0.9000	1.2602	1.2217	-0.4695	0.9007	-0.4695	1.7297	-0.4695

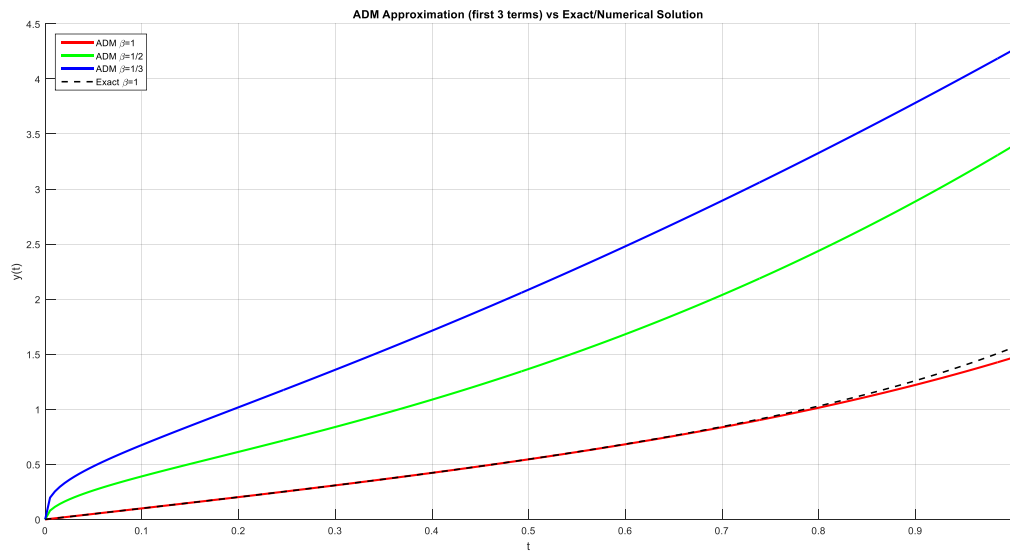


Figure 6: Graph of solution of (29) for the first three terms

Example 3

Consider nonlinear fractional ordinary differential equation below with the given initial condition,

$${}^C D_t^\beta u(t) = -u(t) + t, \quad 0 < \beta \leq 1; u(0) = 0, t \in (0, T] \quad (30)$$

Let $f(t, u) = -u(t) + t$ then we know that (30) above is a Caputo fractional equation. Applying Riemann Liouville fractional integral J^β on both sides of (30) we obtain,

$$u(t) = u(0) + J^\beta [-u(t) + t] = J^\beta [-u(t) + t] \quad (31)$$

By ADM the solution $u(t)$ can be decomposed as $u(t) = \sum_{n=0}^{\infty} u_n(t)$

$$\text{Thus } u_0(t) = 0, u_{n+1} = J^\beta [u_n(t) + t\delta_{k,0}] = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds \quad (32)$$

In order to compute the first few terms we use the above (32) as follows term wise.

$$u_0(t) = 0,$$

$$u_1(t) = J^\beta [t] = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ds,$$

$$\text{Using (9) } u_1(t) = \frac{\Gamma(2)}{\Gamma(\beta+2)} t^{\beta+1} = \frac{t^{\beta+1}}{\Gamma(\beta+2)}$$

$$u_2(t) = J^\beta (-u_1(t)) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \frac{s^{\beta+1}}{\Gamma(\beta+2)} ds = -\frac{\Gamma(\beta+2)}{\Gamma(2\beta+3)\Gamma(\beta+2)} t^{2\beta+2} = -\frac{t^{2\beta+2}}{\Gamma(2\beta+3)}$$

$$u_3(t) = J^\beta (-u_2(t)),$$

$$= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \frac{s^{2\beta+2}}{\Gamma(2\beta+3)} ds = \frac{\Gamma(2\beta+3)}{\Gamma(3\beta+4)\Gamma(2\beta+3)} t^{3\beta+3} = \frac{t^{3\beta+3}}{\Gamma(3\beta+4)}$$

So the ADM series solution is

$$\begin{aligned} u(t) &= u_0 + u_1 + u_2 + u_3 + \dots, \\ &= \frac{t^{\beta+1}}{\Gamma(\beta+2)} - \frac{t^{2\beta+2}}{\Gamma(2\beta+3)} + \frac{t^{3\beta+3}}{\Gamma(3\beta+4)} - \dots \end{aligned} \quad (33)$$

Note: The above (33) is the ADM solution of (30) and by theorem 1 we can check that (30) is converge using theorem 1 convergence condition. 1) Using $|f(t, u) - f(t, v)| = |u - v|$, the

Lipschitz constant in u is $L=1$, 2) for fractional integral bound is explained as $0 < M < 1$;

$M = \frac{LT^\beta}{\Gamma(\beta+1)} < 1$ thus $T < (\Gamma(\beta + 1))^{\frac{1}{\beta}}$. Under that condition the ADM iteration is contractive and the series converges.

3) Error bound: Because the series terms alternate in sign and (for fixed t) in magnitude eventually decrease, a practical error bound for the truncated sum $S_N = \sum_{k=1}^N a_k(t)$ is given by

$|u(t) - S_N(t)| \leq |a_{n+1}(t)| = \frac{t^{N+1(\mu+1)}}{\Gamma((N+1)(\mu+1)+1)}$ This is a conservative and useful estimate for choosing N to achieve a target tolerance.

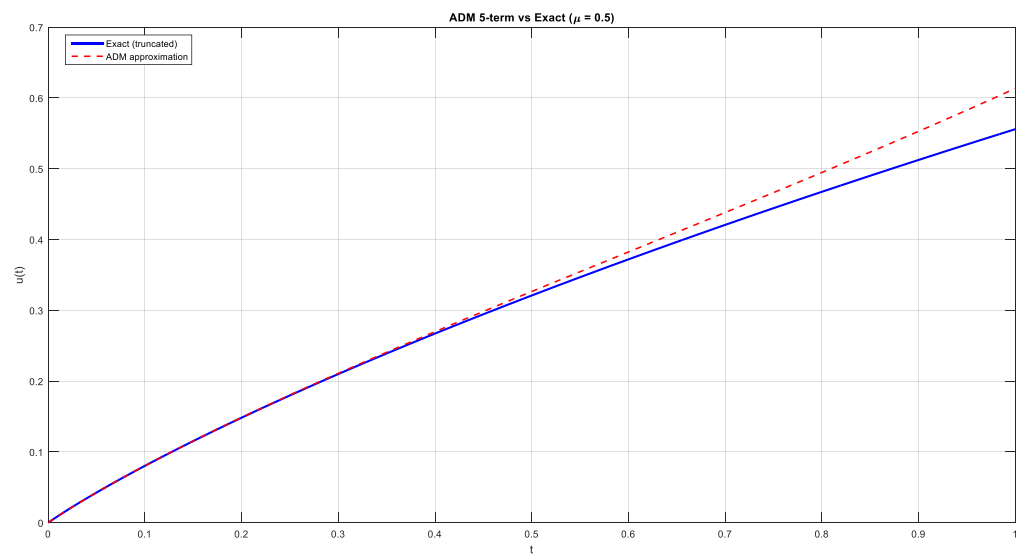
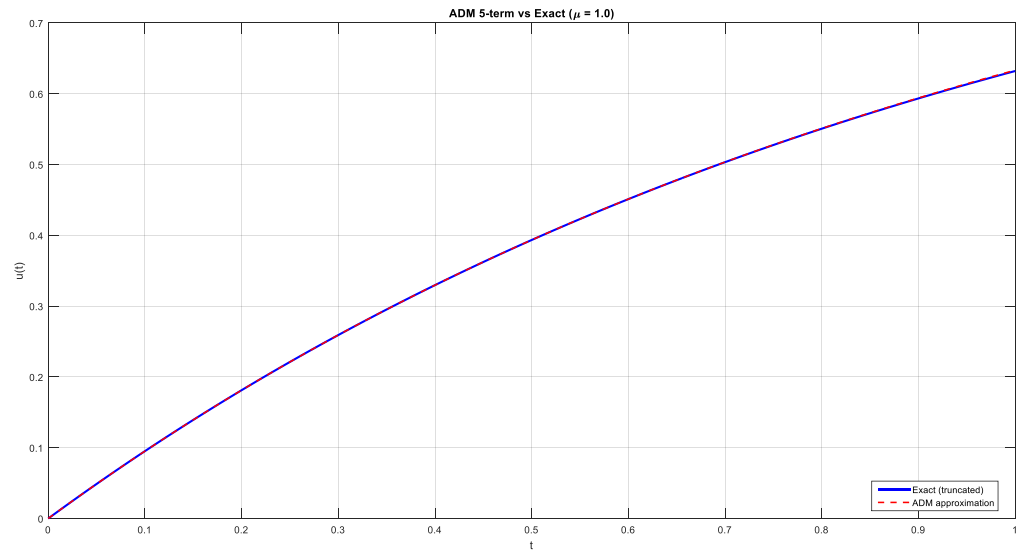
4) Using ratio: using Gamma function, Using Gamma asymptotics one finds the rough large k behavior of successive-term ratios is

$\left| \frac{a_{k+1}}{a_k} \right| \approx \frac{t^{\mu+1}}{((\mu+1)k)^{\mu+1}}$ Implies for a fixed t, the ratio tends to 0 as $k \rightarrow \infty$, confirming convergence.

The solution (33) above has been declared by simulation as follows with its error by graph when the order $\beta=1$, and $1/2$.

Table 3: The first 5-terms value of (33) for $0 \leq \mu < 1$ with its error bounds

Mu	t	Exact	ADM to 5 terms	Error percent
1	0.2	0.18127	0.18127	0.000004767
1	0.5	0.39347	0.39349	0.0051448
1	0.8	0.55067	0.551	0.059255
1	1	0.63212	0.63333	0.19186
0.5	0.2	0.14841	0.14867	0.17002
0.5	0.5	0.32104	0.32668	1.7563
0.5	0.8	0.4675	0.49482	5.8439
0.5	1	0.55596	0.61351	10.351



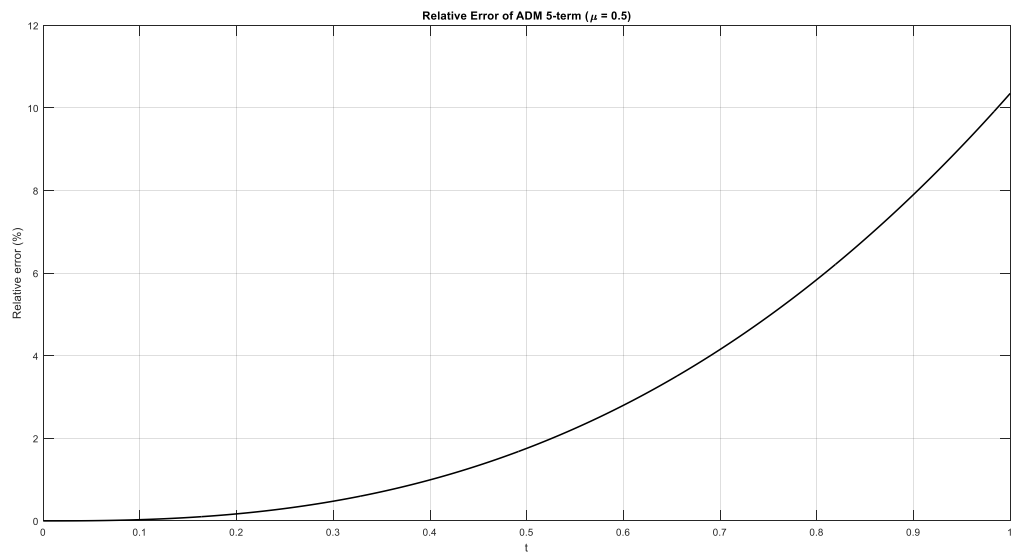
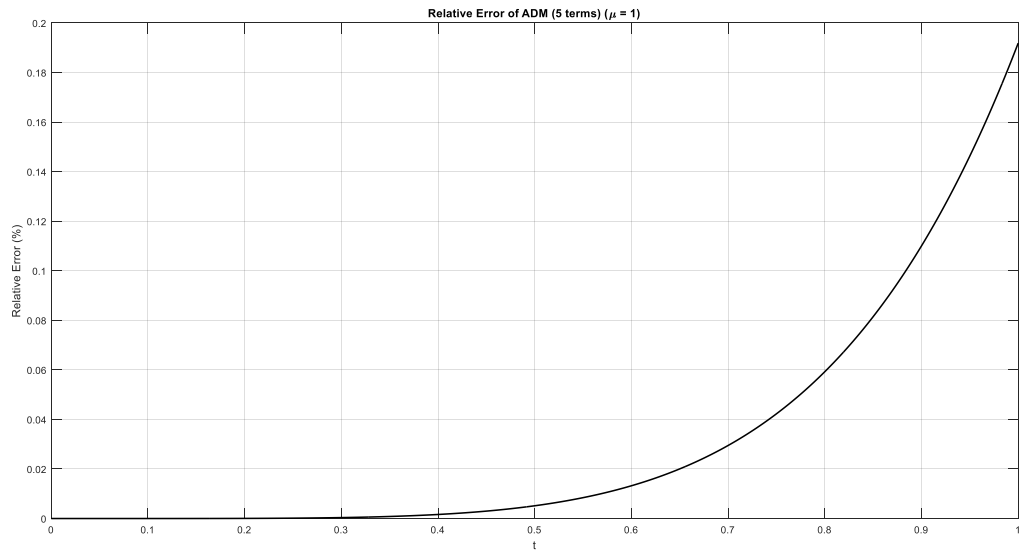


Figure 7: ADM Graph of the solution (33) versus exact with its error at first 5- terms

4.9 Discussion

To demonstrate the accuracy and efficiency of the analytic-numerical technique presented, four examples are considered and from the computational simulation results and graphs obtained, we observed as follows:

The proposed example 1A and 1B demonstrates that ADM is robust, rigorous and effective in solving linear and nonlinear fractional ordinary differential equations without linearization discretization and perturbation. Example 1A was presented by (Bulut *et al.*, 2013).but he solved by sumudu transform. And figure 3 and figure 4 show the accuracy of the solution.

The next example 2 was solved by Shawagfeh (2002) but each term was not explained and it was not justified by simulation. But in this paper it was presented by Table 1, 2 & figure 5, 6 also. From Tables 1 and 2 together with Figures 5 and 6, it is evident that ADM provides good approximations to the exact solution $y(t)=\tan(t)$ of equation (29). In Table 1, where the solution is truncated to the first two terms, the approximation is very close to the exact solution for small values of t . However, as t increases, the error becomes more significant, especially for fractional orders $\beta=1/2$ and $\beta=1/3$. This observation is confirmed in Figure 5, where the approximate solutions begin to deviate from the exact curve as t grows larger.

Table 2 extends the solution to the first three terms, leading to a marked improvement in accuracy. The errors are smaller compared to those in Table 1, and the approximations remain close to the exact solution over a wider range of t . Figure 6 clearly illustrates this improvement: the ADM curves with three terms follow the exact solution more closely, showing that the method converges rapidly when more terms are included in the decomposition.

Another important observation is that the accuracy of the ADM solution depends on the order β . For $\beta=1$, the approximation aligns very closely with the exact solution, as expected. When β decreases to $1/2$ and $1/3$, the deviation from the exact solution increases, highlighting the sensitivity of the method to fractional orders. Nevertheless, even in these cases, the inclusion of additional terms improves convergence.

Overall, the results demonstrate that ADM is an effective technique for approximating fractional differential equations. The method provides reliable results with only a few terms, and both tables and figures show that accuracy improves systematically as more terms are considered.

The nonlinear fractional ordinary differential equation (30) was solved using the ADM, and the resulting approximate solution was expressed in the series form (33). The series solution has an alternating structure; with successive terms involving increasing powers of t^μ scaled by Gamma functions. This form ensures convergence under the condition $t^\mu < \Gamma(\mu + 1)$, as established by the convergence theorem.

The ADM approximation demonstrates good accuracy even when truncated to a few terms. As shown in Figure 7, the approximation is highly reliable for small values of t , where the truncated series closely follows the exact solution. For $\mu = 1$, corresponding to the classical ODE, the method converges rapidly and only a few terms are sufficient to achieve high accuracy across the interval. However, for fractional orders such as $\mu = 1/2$, the convergence is slower, and deviations from the exact solution become more noticeable as t increases. This difference arises because smaller fractional orders reduce the rate of decay of successive terms, requiring more terms in the series to achieve the same accuracy.

The error curves obtained for the ADM approximation against the exact solution confirm the theoretical result of the convergence theorem. For the classical case ($\mu = 1$), the approximation error is very small (less than 0.2% for $t \leq 1$), which demonstrates the rapid convergence of the ADM series towards the exact exponential solution. In contrast, for the fractional case ($\mu = 0.5$), the error is slightly larger, reaching around 10% at ($t = 1$). This difference is expected since fractional derivatives introduce memory effects and slower decay, which naturally reduce the rate of convergence. Nevertheless, the relative error remains bounded and decreases for smaller values of (t), illustrating that the ADM partial sums form a Cauchy sequence as established in the convergence proof. The graphical results therefore provide strong numerical evidence that the ADM solution converges in both the integer-order and fractional-order cases, consistent with the theorem that guarantees convergence in the Banach space for ($0 < \mu < 1$).

CHAPTER FIVE

5 CONCLUSION AND FUTURE WORK

5.1 Conclusion

This thesis explored the application of ADM to solve FODEs with given initial conditions. The primary objective is to derive approximate analytical solutions using ADM and then it is to compare this with exact solutions to assess accuracy and applicative performance.

Fractional differential equations, characterized by derivatives of non-integer order, have proven valuable in modeling complex physical systems with memory and hereditary effects. Next to their usefulness, solving such equations especially when nonlinearity is presented remains a significant challenge using classical methods. In this work the Caputo fractional derivative is employed due to its natural incorporation of initial conditions commonly encountered in physical applications.

The ADM is chosen for its ability to handle both linear and nonlinear differential equations without requiring linearization or discretization. Solutions are expressed in terms of rapidly converging series, where nonlinear components were handled through especially constructed Adomian polynomials.

Several representations of FODEs were considered and for each case; the ADM was applied to FODEs to derive approximate solutions, the resulting series solution was compared to the known exact analytical solution and the error of this solution was conducted to quantify the accuracy of ADM across different values of β . The comparison was carried out for values $t=0.1, 0.2, \dots, 0.9$. The result demonstrated that the ADM approximation closely matched the exact solution within this time interval.

This comparison confirms that ADM is a powerful and reliable method for solving linear and non-linear FODEs. Its ability to approximate solutions with negligible error in cases such as orders like 1, $1/2$, and $1/3$ builds the performance of ADM accuracy for other fractional orders where exact solutions may not exist in closed form.

5.2. Future Work

ADM provides highly accurate approximations to the exact solutions with typically few errors after just a few terms in the decomposition series. The method remains effective and stable across different values of β , including fractional orders less than 1. The nearby behavior was consistent and the accuracy improved with the inclusion of additional terms in the ADM series.

This work demonstrates that the ADM is a powerful and flexible tool for solving FODEs. Its ability to produce analytical like solution without linearization and discretization makes it particularly active for both linear and nonlinear FODEs. Therefore one can extend the approach for higher accuracy over wider interval includes more terms in the ADM series.

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