



WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

**Numerical Treatment of Various Classes of
Two-Parameter Singularly Perturbed Parabolic
Problems**

PhD Dissertation

Shegaye Lema Cheru

May 27, 2026
Nekemte, Ethiopia



WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

**Numerical Treatment of Various Classes of
Two-Parameter Singularly Perturbed Parabolic
Problems**

A Dissertation Submitted to School of Graduate Studies in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics (Numerical Analysis)

Shegaye Lema Cheru

Advisor: Gemechis File Duressa (Professor)

Co-Advisor: Tariku Birabasa Mokennen (PhD)

May 27, 2026
Nekemte, Ethiopia



WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES
P.O.Box: 395, Nekemte, Ethiopia

APPROVAL SHEET FOR SUBMITTING FINAL DISSERTATION

As members of the Board of Examining of the **Final PhD Dissertation** open defense, we certify that we have read and evaluated the dissertation prepared by **Shegaye Lema Cheru** under the title “**Numerical Treatment of Various Classes of Two-Parameter Singularly Perturbed Parabolic Problems**” and recommend that the dissertation be accepted as fulfilling the dissertation requirement for the **PhD Degree in Mathematics (Numerical Analysis)**.

Name of Chairman	Signature	Date
------------------	-----------	------

Name of Internal Examiner	Signature	Date
---------------------------	-----------	------

Name of External Examine	Signature	Date
--------------------------	-----------	------

Final Approval and Acceptance :
Dissertation Approved by

Department, PGC	Signature	Date
-----------------	-----------	------

Name of Dean of College	Signature	Date
-------------------------	-----------	------

Certification of the Final Dissertation

I hereby certify that all the correction and recommendation suggested by the board of examiners are incorporated into the final dissertation entitled “**Numerical Treatment of Various Classes of Two-Parameter Singularly Perturbed Parabolic Problems**” by **Shegaye Lema Cheru**.

Name of Dean of SGS	Signature	Date
---------------------	-----------	------

Statement of the Author

I, Mr. **Shegaye Lema Cheru** hereby declare and affirm that the dissertation entitled “**Numerical Treatment of Various Classes of Two-Parameter Singularly Perturbed Parabolic Problems**” is my own work conducted under the supervision of **Gemechis File (Professor)** and **Tariku Birabasa Mekonnen (PhD)**. I affirm that this research has been conducted in accordance with established academic and ethical standards. The work presented in this dissertation is based on my own investigations, including the development, analysis, and implementation of numerical methods for singularly perturbed problems. All scholarly matter that is included in the dissertation has been given recognition through citation. I have adequately cited and referenced all the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and I have not misrepresented, fabricated, or falsified any idea/data/fact/source in my submission. This dissertation is submitted in partial fulfillment of the requirements for the degree of **Doctor of Philosophy (PhD) in Numerical Analysis** at Wollega University. I further declare that this dissertation has not been submitted to any other institution anywhere for the award of any academic degree, diploma or certificate. I understand that any violation of the above will be cause for disciplinary action by the University and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Name: **Shegaye Lema Cheru**

Signature: _____

Date: May 27, 2026

Department: **Mathematics**

College: **Natural and Computational Sciences**

Abbreviations and Symbols

Abbreviations

BVP	: Boundary value problem
FOM	: Fitted operator method
FDM	: Finite difference method
FDS	: Finite difference scheme
FMM	: Fitted mesh method
FEM	: Finite element method
IBVP	: Initial-boundary value problem
LTE	: Local truncation error
NS-FDM	: Non-standard finite difference method
DE	: Differential equation
DDE	: Differential difference equation
ODE	: Ordinary differential equation
PDE	: Partial differential equation
SPP	: Singularly perturbed problem
SPPP	: Singularly perturbed parabolic problem
SPDE	: Singularly perturbed differential equation
SPDDE	: Singularly perturbed differential difference equation
SPPDDE	: Singularly perturbed parabolic delay differential equation
SPODE	: Singularly perturbed ordinary differential equation
SPPDE	: Singularly perturbed partial differential equation
TP-SPPP	: Two-parameter singularly perturbed parabolic problem

Symbols

N	: Number of mesh points in spatial domain
M	: Number of mesh points in temporal direction
T	: Maximum time value
C	: Any generic constant independent of the perturbation parameters and step sizes
$C^0(\mathfrak{Q})$	: Continuous functions on the domain $(\mathfrak{Q})$
$C^1(\mathfrak{Q})$	: Functions with continuous first derivatives on the domain $(\mathfrak{Q})$
$C^2(\mathfrak{Q})$	: Functions with continuous first and second derivatives on the domain $(\mathfrak{Q})$
$C^{(k)}(\mathfrak{Q})$	: k times continuously differentiable function on the domain $(\mathfrak{Q})$
h	: Mesh size in spatial direction
k	: Mesh size in time direction
x, x_i	: Continuous and discrete spatial variables
t, t_j	: Continuous and discrete temporal variables
$\ \cdot\ , \ \cdot\ _\infty$	: Standard supremum norm
W	: The approximate solution to the analytical solution w
$w(x, t)$	: Solution of continuous problem
W_i, W_i^j	: Numerical solution
$\mathbb{L}_{\varepsilon, \mu}$	: Differential operator
$\mathbb{L}_{\varepsilon, \mu}^M$	: Semi-discrete difference operator
$\mathbb{L}_{\varepsilon, \mu}^{N, M}$	: Discrete difference operator
ε	: Perturbation parameter multiplying the diffusion term
μ	: Perturbation parameter multiplying the convection term
φ	: Discontinuity point
$O(\cdot)$	: Order symbol
τ	: Delay parameter
R_e	: Reynolds Number
P_ε	: Perturbation Problem
P_o	: Reduced Problem
$1D, 2D$	: One and two dimensional
$E_{\varepsilon, u}^{N, M}$	: Maximum Pointwise error
$\overline{R}_{\varepsilon, u}^{N, M}$	: Rate of convergence
$\mathfrak{J}^2$	: Denominator function which replaces h^2
$\delta_1, \delta_2, \delta_3, \delta_4$	: Transition parameters
f, ϖ	: Source functions
$\alpha, \alpha_1, \alpha_2$	: The minimum value of the convection coefficient on the domain for the corresponding problems
β	: The smallest positive value of the reaction coefficient on the given domain

Biographical Sketch

Mr. Shegaye Lema Cheru was born in 1991 G.C. in Meliyou Hamuse, Werra Jarso Woreda, North Shoa Zone, Oromia Regional State. He completed his primary education at Meliyou Hamuse and continued his secondary studies at Tuulu Milki Secondary School. He then attended Gohatsion Preparatory School, successfully completing his preparatory education. Following this, he joined Jimma University, where he earned a Bachelor's degree in Applied Mathematics on July 3, 2014 G.C. After graduation, he served as a mathematics teacher at Adama Secondary School, gaining valuable experience in teaching and academic mentoring. He later joined Samara University as an assistant lecturer, further contributing to higher education.

Motivated by a strong commitment to advanced academic research, he completed a Master of Science degree in Computational Mathematics at Hawassa University, graduating on August 18, 2020 G.C. During his master's studies, he conducted research entitled **"An Eco-Epidemiological Mathematical Model of Prey-Predator Species with Optimal Control."** This work led to the publication of two peer-reviewed research articles at the master's level. Since October 2022, he has been working as a Mathematics lecturer at Samara University, where he continues to engage in teaching and research. In November 2022, he enrolled in the School of Graduate Studies at Wallaga University to pursue his Ph.D. in Numerical Analysis, reflecting his ongoing commitment to advancing knowledge in his field.

Acknowledgment

First and foremost, I express my deepest gratitude to the Almighty God for His boundless grace and guidance throughout the course of this research. Secondly, I would like to extend my sincere appreciation to my supervisor, **Professor Gemechis File Duressa**, for his exceptional guidance, invaluable advice, and constant encouragement. His profound knowledge, constructive insights, and patient mentorship have been instrumental in shaping my understanding of the subject and in improving my research and scientific writing skills. My heartfelt thanks also go to my co-supervisor, **Dr. Tariku Birabasa Mekonnen**, for his insightful comments, critical suggestions, and unwavering support. His guidance has been vital in overcoming challenges and refining the quality of this dissertation. I am deeply grateful to the Department of Mathematics, Wallaga University, for granting me the opportunity to pursue this program and for the kind support of its academic staff, whose collaboration and friendship made my stay both productive and memorable. My sincere thanks also go to the Department of Mathematics, Jimma University, for their valuable cooperation and for facilitating part of my coursework. I owe special thanks to my parents, Lema Cheru and Asmare Ababu, whose unconditional love, sacrifices, and support, both spiritually and materially, have been the foundation of all my achievements. I am deeply grateful to my brother, Bulo Lema, for his generous financial support, which played a vital role in helping me complete my studies. I want to thank Abdeta Guluma, Lencha Tamiru, Feyisa Edosa and Dawit Kachine for their moral support during the study period. Finally, I extend my gratitude to all friends, colleagues, and individuals who have supported me, directly or indirectly, in completing this work. To anyone whose name does not appear here, please accept my heartfelt thanks and appreciation.

May 27, 2026

Shegaye Lema Cheru

Dedication

Dedicated

to

all my beloved family members.

Table of Contents

Statement of the Author	iii
Abbreviations and Symbols	iv
Biographical Sketch	vi
Acknowledgment	vii
Dedication	viii
List of Tables	xiii
List of Figures	xvi
List of Publications	xix
Abstract	xx
Chapter 1 General Introduction	1
1.1 Background of the study	1
1.1.1 Methods for solving Singular Perturbation Problems	5
1.1.2 Models Illustrating Singular Perturbations in PDEs	8
1.1.3 Overview of Two-Parameter Singular Perturbations	11
1.1.4 Basic Definitions and Terminologies	13
1.2 Statement of the Problem	16
1.3 Research Objectives	18
1.3.1 General Objective	18
1.3.2 Specific Objectives	18
1.4 Significance of the Study	19
1.5 Scope of the Study	19
1.6 Mathematical Procedure of the Study	20
1.7 Organization of the Dissertation	21

Chapter 2	Review of Related Literature	23
2.1	Singularly Perturbed Problems	23
2.2	Singularly Perturbed Delay Problems	24
2.3	Existing Studies on Two-Parameter Singularly Perturbed Problems	27
2.4	Two-Parameter Singular Perturbations with Discontinuous Data	31
Chapter 3	An Exponentially Fitted Gaussian Quadrature Method for Two-Parameter Singularly Perturbed Parabolic Problems	35
3.1	Introduction	36
3.2	Definition of the Problem	37
3.3	Properties of the Continuous Solution	37
3.4	Formulation of the Numerical Scheme	42
3.4.1	Temporal Semi-discretization	42
3.4.2	Spatial Semi-discretization	46
3.5	Convergence Analysis	52
3.6	Numerical Examples, Results, and Discussion	56
3.7	Conclusion	60
Chapter 4	Exponentially Fitted Simpson's One-Third Rule for Two- Parameter Singularly Perturbed Parabolic Convection-Diffusion Prob- lems	62
4.1	Introduction	63
4.2	Definition of the Problem	64
4.3	Properties of the Continuous Solution	65
4.4	Formulation of the Numerical Scheme	66
4.4.1	Temporal Semi-discretization	66
4.4.2	Spatial Semi-discretization	68
4.4.3	Derivation of the Fitting Factor	70
4.5	Convergence Analysis	72
4.6	Numerical Examples, Results and Discussion	78
4.7	Conclusion	85

Chapter 5	An Exponentially Fitted Simpson's One-Third Numerical Scheme for Two-Parameter Singularly Perturbed Parabolic Problems with Large Time Delay	86
5.1	Introduction	87
5.2	Definition of the Problem	88
5.3	Properties of the Continuous Solution	89
5.4	Formulation of the Numerical Scheme	92
5.4.1	Temporal Semi-discretization	92
5.4.2	Spatial Semi-discretization	96
5.4.3	Derivation of the Fitting Factor	98
5.5	Convergence Analysis	101
5.6	Numerical Examples, Results and Discussion	106
5.7	Conclusion	114
Chapter 6	Fitted Non-Polynomial Cubic Spline Method for Two-Parameter Singularly Perturbed Parabolic Problems with Discontinuous Coefficients	115
6.1	Introduction	116
6.2	Definition of the Problem	117
6.3	Properties of the Continuous Solution	118
6.4	Formulation of the Numerical Scheme	124
6.4.1	Temporal Semi-discretization	124
6.4.2	Spatial Semi-discretization and Non-Polynomial Cubic Spline Approach	125
6.4.3	Derivation of the Fitting Factor	129
6.5	Convergence Analysis	131
6.6	Numerical Examples, Results and Discussion	136
6.7	Conclusion	145
Chapter 7	Uniformly Convergent Numerical Methods for Two-Parameter Singularly Perturbed Parabolic Problems with Non-Smooth Data	146
7.1	Introduction	147
7.2	Definition of the Problem	148
7.3	Formulation of the Numerical Scheme	148

7.3.1	Temporal Semi-discretization	148
7.3.2	Determination of the Denominator Function	150
7.3.3	Non-Standard Finite Difference Methods	152
7.4	Convergence Analysis	156
7.5	Numerical Examples, Results and Discussion	162
7.6	Conclusion	172
Chapter 8	Mesh-Fitted Cubic Spline Method for Two-Parameter Singularly Perturbed Parabolic Problems with Discontinuous Convection and Source Terms	173
8.1	Introduction	174
8.2	Definition of the Problem	175
8.3	Formulation of the Numerical Scheme	175
8.3.1	Temporal Semi-discretization	176
8.3.2	Spatial Semi-discretization	177
8.3.3	Cubic Spline in Tension Method	179
8.4	Convergence Analysis	183
8.5	Numerical Examples, Results and Discussion	187
8.6	Conclusion	195
Chapter 9	General Conclusions and Recommendations	196
9.1	General Conclusions	196
9.2	Recommendations for the Future Work	198
	References	199

List of Tables

Table 3.1	Computed values of $E_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$, $\bar{R}_{\varepsilon,\mu}^{N,M}$ and $\bar{R}^{N,M}$ for Example 3.1 at $\mu = 10^{-6}$	57
Table 3.2	Computed maximum pointwise error and rate of convergence for Example 3.1 at $\mu = 10^{-6}$	58
Table 3.3	Comparison of $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}^{N,M}$ for Example 3.1.	58
Table 3.4	Computed maximum pointwise error and rate of convergence for Example 3.1 at $\varepsilon = 10^{-6}$ and varying values of μ	59
Table 3.5	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}^{N,M}$ for Example 3.2 at $\mu = 10^{-7}$. . .	60
Table 3.6	Comparison of $E_{\varepsilon,\mu}^{N,M}$ and $E^{N,M}$ for Example 3.2.	61
Table 3.7	Computed values of $E_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$, $\bar{R}^{N,M}$ for Example 3.2 at $\varepsilon = 10^{-9}$	61
Table 4.1	Maximum pointwise error and rate of convergence for Example 4.1 for $\mu = 10^{-6}$ and different values of ε	79
Table 4.2	Comparison of maximum pointwise error and rate of convergence for Example 4.1 with $\mu = 10^{-6}$ and various ε values. .	80
Table 4.3	Maximum pointwise error and rate of convergence for Example 4.1 for $\varepsilon = 10^{-6}$ and different values of μ	81
Table 4.4	Comparison of maximum pointwise error and rate of convergence for Example 4.2 with $\mu = 10^{-6}$ and various ε values. .	82
Table 4.5	Maximum pointwise error for Example 4.1 with $M = N = 64$	82
Table 4.6	Maximum pointwise error for Example 4.2 with $M = N = 32$	83
Table 4.7	Maximum pointwise error and convergence rate for Example 4.2 with different values of μ and $\varepsilon = 10^{-7}$	83
Table 4.8	Comparison of the maximum pointwise errors for Example 4.3 corresponding to different values of ε	84

Table 5.1	Computed $E_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 5.1 at $\mu = 10^{-9}$ and comparison with Sumit et al. (2020); Negero (2023).	107
Table 5.2	Computed maximum pointwise error and rate of convergence for Example 5.1 at $\mu = 10^{-4}$ and comparison with Govindarao et al. (2019) for $N = M$	108
Table 5.3	Maximum pointwise error, convergence rate and CPU time(in seconds) for Example 5.1 with $\varepsilon = 10^{-4}$ and $N = M$	109
Table 5.4	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 5.1 at $\varepsilon = 2^{-10}$ and comparison with Singh et al. (2022) for varying values of μ	110
Table 5.5	Computed $E_{\varepsilon,\mu}^{N,M}$, $\bar{R}_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$, $\bar{R}^{N,M}$ and CPU time (in seconds) for Example 5.2 at $\mu = 10^{-9}$ and comparison with Sumit et al. (2020); Negero (2023).	111
Table 5.6	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 5.2 at $\mu = 10^{-9}$ after Extrapolation.	111
Table 5.7	Maximum pointwise error, convergence rate and CPU time(in seconds) for Example 5.3 with $\varepsilon = 2^{-10}$ and varying μ values.	112
Table 5.8	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 5.3 at $\varepsilon = 10^{-10}$ after Extrapolation.	112
Table 6.1	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ when $\mu = 2^{-4}$ for Example 6.1.	137
Table 6.2	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for $\varepsilon = 2^{-12}$ for Example 6.2.	138
Table 6.3	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ with $\varepsilon = 2^{-8}$ for Example 6.3.	139
Table 6.4	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for $\mu = 2^{-10}$ and various values of ε for Example 6.4.	140
Table 6.5	Comparison of $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 6.1-6.4.	141
Table 7.1	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ when $\mu = 2^{-8}$ for Example 7.1.	163
Table 7.2	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 7.1 for $\mu = 2^{-12}$ and $\varphi = 0.5$	164
Table 7.3	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 7.2 for $\varepsilon = 10^{-8}$ and $\varphi = 0.5$	165
Table 7.4	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 7.2 for $\mu = 2^{-12}$	166

Table 7.5	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ for Example 7.3 for $\varepsilon = 10^{-6}$ and $\varphi = 0.5$	167
Table 7.6	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ when $\mu = 2^{-10}$ for Example 7.3. . .	168
Table 7.7	Computed $E_{\varepsilon,\mu}^{N,M}$ for Example 7.1 by taking $N = M = 128$. . .	168
Table 7.8	Computed $E_{\varepsilon,\mu}^{N,M}$ for Example 7.2 by taking $N = M = 128$. . .	169
Table 8.1	Computed $E_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ when $\mu = 2^{-4}$ for Example 8.1.	189
Table 8.2	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ when $\varepsilon = 2^{-8}$ for Example 8.1. . .	189
Table 8.3	Comparison of $E_{N,M}^{\varepsilon,\mu}$ for Example 8.1 with $\mu = 2^{-4}$	190
Table 8.4	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ for different ε, μ and φ for Example 8.1.	191
Table 8.5	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ when $\varepsilon = 2^{-8}$ for Example 8.2. . .	192
Table 8.6	Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ for different ε, μ and φ for Example 8.2.	192

List of Figures

Figure 3.1	The numerical solution profile for Example 3.1.	57
Figure 3.2	The numerical solution profile for Example 3.2.	57
Figure 3.3	Log-log graph illustrating the peak absolute error corresponding to Example 3.1.	58
Figure 3.4	The Log-Log plot of the maximum absolute error for Example 3.2.	59
Figure 4.1	The numerical solution profile for Example 4.1.	79
Figure 4.2	The behavior of spatial vs numerical solution graph for Example 4.1.	79
Figure 4.3	The numerical solution profile for Example 4.2.	80
Figure 4.4	The behavior of spatial vs numerical solution graph for Example 4.2.	81
Figure 4.5	The Log-Log plot for Example 4.1.	81
Figure 4.6	The Log-Log plot for Example 4.2.	83
Figure 4.7	The computed solution pattern for Example 4.3 is illustrated for the case where $N = M = 128$ and $\varepsilon = 10^{-4}$	84
Figure 5.1	The numerical solution profile for Example 5.1.	108
Figure 5.2	Surface plots for numerical Error for Example 5.1.	109
Figure 5.3	The numerical solution profile for Example 5.2.	110
Figure 5.4	The numerical solution profile for Example 5.3.	113
Figure 5.5	The Log-log plot for Example 5.1.	113
Figure 5.6	The Log-log plot for Example 5.2.	113
Figure 5.7	The Log-log plot for Example 5.3.	114
Figure 6.1	Domain $\mathfrak{Q} = [0, 1] \times (0, T]$	118

Figure 6.2	The numerical solution profile for Example 6.1.	138
Figure 6.3	The numerical solution profile for Example 6.2.	139
Figure 6.4	The numerical solution profile for Example 6.3.	140
Figure 6.5	The numerical solution profile for Example 6.4.	141
Figure 6.6	Numerical solution for Example 6.4 with $\varepsilon = 2^{-16}$ and $\mu = 2^{-16}$	142
Figure 6.7	Line plots for Example 6.1.	142
Figure 6.8	Line plots for Example 6.2 at different time-levels.	142
Figure 6.9	Line plots for Example 6.3 at different time-levels.	143
Figure 6.10	Line plots for Example 6.4 at different time-levels.	143
Figure 6.11	Line plots for Example 6.4 at different time-levels.	143
Figure 6.12	The Log-log plot of maximum absolute error.	144
Figure 6.13	Numerical error corresponding to Examples 6.2 and 6.4, respectively.	144
Figure 6.14	Numerical error corresponding to Examples 6.2 and 6.4, respectively.	144
Figure 7.1	Solution profiles for Example 7.1.	164
Figure 7.2	Solution plots for Example 7.1.	165
Figure 7.3	Numerical solutions for Example 7.1 with different source terms.	166
Figure 7.4	Log-log plots for Example 7.1 and Example 7.2.	167
Figure 7.5	Numerical solutions for Example 7.2 with $N = M = 128$	169
Figure 7.6	Line plot for Example 7.2 with $N = M = 128$	169
Figure 7.7	Numerical solution for Example 7.2 with different source terms.	170
Figure 7.8	Log-log plot for Example 7.3.	170
Figure 7.9	Solution profiles for Example 7.3 for $N = M = 128$	170
Figure 7.10	For Example 7.3, the numerical solution with $\varepsilon = 2^{-10}$, $\mu = 2^{-12}$	171
Figure 8.1	The computed solution profiles for Example 8.1 using $N = M = 128$	188
Figure 8.2	Numerical line plots illustrating Example 8.1 for several time increments.	190
Figure 8.3	Computed solution profiles for Example 8.1 with $N = 256$ and $M = 160$, illustrating the influence of varying φ values.	191

Figure 8.4	The computed solution profiles for Example 8.2 using $N = M = 128$	193
Figure 8.5	Numerical line plots illustrating Example 8.2 for several time increments.	193
Figure 8.6	Computed solution profiles for Example 8.2 with $N = 256$ and $M = 160$, illustrating the influence of varying φ values. . . .	193
Figure 8.7	Computed solution profiles for Example 8.2 with $N = M = 256$, $\varepsilon = 2^{-12}$, $\mu = 2^{-8}$, considering multiple continuous source terms.	194
Figure 8.8	Log-log plot of Maximum pointwise errors.	194

List of Publications

List of Papers Published

1. Cheru, S. L., Duressa, G. F., & Mekonnen, T. B. (2024). Gaussian quadrature method with exponential fitting factor for two-parameter singularly perturbed parabolic problem. BMC Research Notes, 17(1), 304.
2. Cheru, S. L., Duressa, G. F., & Mekonnen, T. B. (2024). Numerical integration method for two-parameter singularly perturbed time delay parabolic problem. Frontiers in Applied Mathematics and Statistics, 10, 1414899.
3. Cheru, S. L., Duressa, G. F., & Mekonnen, T. B. (2025). Robust numerical approach for singularly perturbed parabolic problems with discontinuous convection coefficients and source terms. Computational Methods for DEs.
4. Cheru, S. L., Duressa, G. F., & Mekonnen, T. B. (2025). Robust numerical methods for two-parameter singularly perturbed parabolic problems with non-smooth data. Scientific African, e03064.
5. Cheru, S. L., Duressa, G. F., & Mekonnen, T. B. (2026). Fitted-mesh cubic spline methods for layer-dominated singularly perturbed parabolic problems with non-smooth data. Iranian Journal of Numerical Analysis and Optimization, 10.22067/ijnao.2026.96617.1784 (**Articles in Press**).

List of Papers Under Review

1. Cheru, S. L., Duressa, G. F., & Mekonnen, T. B. Fitted Numerical Integration for Two-Parameter Singularly Perturbed Parabolic Problems.TJM

Abstract

This dissertation investigates two-parameter singularly perturbed parabolic problems, including models with and without time delay, and involving both continuous and discontinuous coefficients. The interaction of small perturbation parameters in the diffusion and convection terms induces complex multiscale layer structures, characterized by dual boundary layers of unequal thickness and sharp interior layers arising from temporal delay and non-smooth data. These features pose significant challenges for both analytical and numerical treatment. Since such problems frequently arise in scientific and engineering applications, the development of reliable numerical methods is essential for understanding the physical phenomena. To overcome the limitations of classical approaches, which often fail to capture the layer behavior accurately, parameter-uniform numerical schemes based on fitted-operator and fitted-mesh techniques are developed. The fitted-operator approach constructs discrete schemes on uniform meshes, while fitted-mesh strategy employs layer adapted non-uniform meshes. The proposed methods are rigorously analyzed for stability, consistency, and uniform convergence. Numerical experiments validate the theoretical results, demonstrating high accuracy, improved convergence behavior, and enhanced robustness compared to existing methods. The numerical results, presented through tables and surface plots clearly illustrate the influence of perturbation parameters, delay, and discontinuities on the solution structure.

Key words: Singularly perturbed problem; Two parameter; Parameter-uniform; Non-smooth data; Boundary and interior layers, Time delay.

Chapter 1

General Introduction

This chapter presents a coherent overview of the fundamental ideas underlying both regular and singular perturbation problems (SPPs), forming the analytical basis for the subsequent chapters. It highlights the theoretical principles and rationale behind fitted numerical techniques developed to tackle SPPs effectively. The discussion also references representative physical models where such problems naturally occur, underscoring their scientific and practical relevance. In addition, the chapter defines the research background, formulates the central problem, and specifies the objectives and scope of the study. It further emphasizes the motivation for undertaking this work, outlines its anticipated contributions, and concludes with a brief description of the dissertation's overall structure.

1.1 Background of the study

Many scientific and engineering phenomena are modeled by parameter-dependent differential equations (DEs) whose solutions vary with the parameter size. Such cases are called perturbation problems (P_ε), classified as either regular or singular. Consider a boundary value problem (BVP) P_ε with small parameter ε ($0 < \varepsilon \ll 1$) and solution $w(x, \varepsilon)$. If $w(x, \varepsilon)$ converges uniformly to $w(x, 0)$ as $\varepsilon \rightarrow 0$, the problem is regular. Otherwise, it is a SPP. In SPPs, the parameter ε multiplies the highest derivative, and setting $\varepsilon = 0$ reduces the equation's order, yielding the reduced problem P_0 . This order reduction distinguishes SPPs from regular perturbations.

The birth of SPPs was initially introduced by [Prandtl \(1905\)](#), a German engineer,

during his presentation on fluid motion with small friction at the third international congress of mathematics in Heidelberg. In his innovative work on the subject of boundary layer theory, he explained about how a quantity as small as the viscosity of common fluids such as water and air could play a crucial role in determining their flow. In his work, Prandtl proved that the flow about a body can be treated by dividing it into two regions: inside a very thin layer (which he called the boundary layer) in proximity to the body where the frictional effects are prominent, and the remaining as outside region. The boundary layer theory became the foundation stone for modern fluid dynamics. The term “singular perturbation” was first introduced by [Friedrichs & Wasow \(1946\)](#) in their work. Though Prandtl coined the name “boundary layer”, it got much greater generality in the substantial work of [Wasow \(1942\)](#).

A fundamental objective in scientific inquiry is to understand how causes influence outcomes, particularly in situations where a seemingly minor change produces a disproportionately large effect. This phenomenon is rigorously examined within perturbation theory, a framework that has evolved significantly since its early development ([Van Dyke, 1994](#)). Singular perturbation theory, in particular, focuses on explaining how small parameters can induce drastic and often non-uniform changes in the solution structure. Unlike regular perturbation problems, solutions of SPPs typically exhibit sharp gradients or layer-type behavior as $\varepsilon \rightarrow 0$, leading to substantial analytical and numerical difficulties ([Kadalbajoo & Gupta, 2010](#)). Consequently, the study of SPPs remains an active and important area of research, driven by both theoretical challenges and practical applications.

Singularly perturbed problems arise from DEs that contain a small perturbation parameter, for which the qualitative behavior of the solution in the limiting case differs significantly from that corresponding to finite parameter values. In many scientific and engineering applications, such problems naturally lead to boundary value formulations involving singularly perturbed differential equations (SPDEs), often exhibiting multiscale features and rapid variations in localized regions. Due to the inherent complexity of these problems, exact analytical solutions are rarely attainable. Consequently, reliable and efficient numerical methods play a crucial role in capturing the essential behavior and underlying dynamics of their solutions.

Singularly perturbed problems form a cornerstone in the mathematical modeling of complex physical and engineering phenomena. They appear across a broad spectrum of applications, including high-Reynolds number fluid flows (Hirsch, 1990), water quality modeling in river networks (Baumert et al., 2005), semiconductor drift-diffusion equations (Kevorkian & Cole, 2013), electromagnetic field interactions in moving media (Polak et al., 1987), oil reservoir simulations (Wolfshtein et al., 1975), and the theory of plates and shells (Huan-wen, 1984). Their significance also extends to geophysical fluid dynamics, large-scale atmospheric and oceanic circulation (Pudykiewicz, 1989), chemical kinetics, and optimal control (O'malley et al., 1991). Moreover, SPPs arise in aerodynamics, lubrication theory, and numerous branches of mathematics, physics, and biological modeling (Nayfeh, 2000; Verhulst, 2006; Miller et al., 1996).

A defining hallmark of SPPs is the presence of a small perturbation parameter that multiplies the highest-order derivatives in the governing DEs, producing rapid variations or sharp boundary and interior layers in the solution. A classical illustration is the Navier-Stokes equation in fluid dynamics:

$$\frac{\partial}{\partial x} (w^2 + p) + \frac{\partial}{\partial y} (wv) = \frac{1}{Re} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right), \quad (1.1)$$

subject to appropriate initial and boundary conditions, where w and v denote velocity components, p represents pressure and Re is Reynolds number (Hirsch, 1990). This dimensionless parameter, proportional to characteristic length and velocity scales while inversely proportional to kinematic viscosity, governs the flow regime. For large values of $Re \gg 1$, Eq. (1.1) reduces to a singularly perturbed form (Das & Natesan, 2013b). Further canonical examples and their analysis can be found in the comprehensive works of Miller (2006); Morton (1996), and Roos (2008).

Differential-difference equations (DDEs) that simultaneously incorporate delay or advance terms together with the influence of a singular perturbation parameter are classified as singularly perturbed differential-difference equations (SPDDEs) (Gupta et al., 2018). Within this framework, the terms positive shift and negative shift are often employed in the literature to denote advance and delay parameters, respectively. In general, SPDEs describe relationships between an unknown function and its derivatives evaluated at the same instant of time. However, in the case of singularly

perturbed delay differential equations, the system's present state is determined not only by its current conditions but also by its past history. This distinction in temporal dependence leads to different solution behaviors: while non-delay SPPs typically exhibit boundary layers, the inclusion of delay may additionally generate interior layers due to the effect of the shifted argument ([Lange & Miura, 1982](#)).

Singularly perturbed differential–difference equations arise in a wide range of models describing complex phenomena in science and engineering. These equations are particularly relevant in areas such as biosciences, control theory, and disease dynamics, where delayed responses and multiscale effects are inherent. Representative applications include models of neuronal variability ([Stein, 1967](#)), the human pupil light reflex ([Longtin & Milton, 1988](#)), optimal control and variational problems ([Glizer, 2000](#)), blood cell production processes ([Mahaffy et al., 1998](#)), population dynamics and epidemiology ([Kuang, 1993](#)), chemical kinetics ([Epstein, 1992](#)), immune system responses ([Marchuk, 1997](#)), and feedback control mechanisms ([Bellen & Zennaro, 2013](#)).

A defining characteristic of SPDDE is the emergence of sharp boundary and interior layers, whose thickness is governed by small perturbation parameters and may become exceedingly thin. This behavior reflects the intrinsic multiscale structure of the underlying processes, where convection and reaction effects often dominate diffusion. As a result, analytical treatment of such problems is generally difficult, if not infeasible, due to their inherent complexity. Consequently, numerical approximation becomes an essential tool for their study. However, classical numerical methods, when applied directly to multiscale SPDDEs, often fail to resolve the layer behavior accurately and may produce large errors ([Kadalbajoo & Gupta, 2010](#)). This limitation highlights the need for the development of specialized numerical frameworks that are capable of capturing the essential features of the solution without loss of accuracy. Therefore, a central objective in this field is to construct robust numerical techniques whose performance remains uniformly reliable, regardless of the presence of small perturbation parameters or localized singularities.

Over the past few decades, substantial progress has been achieved in both the qualitative and quantitative analysis of singularly perturbed ordinary differential equations (SPODEs) and various classes of singularly perturbed partial differential equations

(SPPDEs). Detailed surveys of these developments can be found in [Kadalbajoo & Reddy \(1989\)](#); [Kumar et al. \(2007\)](#); [Kadalbajoo & Gupta \(2010\)](#); [Roos \(2012\)](#); [Duressa et al. \(2023\)](#). Nevertheless, the construction of accurate computational methods for SPPDEs continues to be a challenging and active area of research. The primary difficulty lies in the presence of multiple small perturbation parameters, which induce complex multiscale solution structures characterized by sharp boundary and interior layers features that are not adequately captured by standard numerical approaches.

Motivated by these challenges, this dissertation is devoted to the development and rigorous analysis of parameter–uniform numerical methods for various classes of parabolic SPPs involving two small parameters. The main goal is to design robust computational schemes whose accuracy and stability are independent of the perturbation parameters, thereby ensuring their effectiveness for practical applications where such multiscale models frequently arise.

1.1.1 Methods for solving Singular Perturbation Problems

Singularly perturbed problems arise frequently in applied sciences and engineering, making the determination of their solutions both important and challenging. In most practical situations, exact analytical solutions are either extremely difficult or impossible to obtain due to the presence of small perturbation parameters and multiscale features. This limitation naturally motivates the development of accurate approximation techniques. The choice of an appropriate method depends on several factors, including the structure of the governing equations, the nature of the perturbation parameters, and available computational resources. Equally important is the requirement that any approximation preserves the essential physical properties of the underlying model ([Munyakazi, 2009](#)).

Broadly speaking, two principal approaches are employed in the study of SPPs: numerical methods and asymptotic methods. Numerical approaches aim to compute accurate approximations for specific problems and are generally applicable to a wide range of models through algorithmic procedures. In contrast, asymptotic methods focus on describing the qualitative behavior of solutions, particularly in the limiting case as the perturbation parameter tends to zero. Although asymptotic techniques

are typically restricted to simpler settings and require deeper analytical insight, they provide valuable information about the structure of the solution. In practice, these two approaches are complementary, as asymptotic analysis often guides the construction of efficient and robust numerical schemes.

A key difficulty in SPPs arises from the presence of small perturbation parameters, for example ε with $0 < \varepsilon \ll 1$, which induce multiscale solution behavior. Typically, the solution remains smooth over most of the domain but exhibits sharp variations within narrow boundary or interior layers. To analyze such behavior, the method of matched asymptotic expansions is widely used. In this framework, an outer solution describes the behavior away from layers, while an inner solution constructed using a suitable stretched variable captures the rapid variation inside the layer regions. These approximations are then matched to obtain a uniformly valid composite solution. A crucial step in this process is identifying the location and thickness of the layers, often guided by the principle of least degeneracy. This analytical framework not only explains the structure of SPP solutions but also provides insight for designing reliable numerical methods (Verhulst, 2006; Van Dyke, 1964; O’malley et al., 1991; Bush, 2018; Bender & Orszag, 2013; Miller, 2006).

Historically, early research on SPPs focused primarily on asymptotic analysis, emphasizing the limiting behavior of solutions as perturbation parameters vanish. However, from the mid–1960s onward, increasing attention was directed toward the development of numerical methods capable of resolving sharp layer phenomena. This shift led to substantial progress and the emergence of a rich body of literature addressing both theoretical and computational aspects of SPPs. Several influential monographs, including Bender & Orszag (1999, 2013); Miller et al. (1996); Shishkin & Shishkina (2008); Doolan et al. (1980); Roos (2008); Nayfeh & Balachandran (2008); O’malley et al. (1991); Kevorkian & Cole (2013); Holmes (2012); Eckhaus (2011); Eugene & Shishkin (2012), provide comprehensive treatments of these developments.

Despite these advances, the numerical treatment of SPPs remains challenging. Standard methods such as finite difference, finite element, collocation, and spectral methods often perform poorly on uniform meshes when applied directly to SPPs (Roos, 2008; Eugene & Shishkin, 2012). The main difficulty stems from the presence of steep

gradients in the solution, which require extremely fine meshes for accurate resolution. In fact, classical schemes may fail to reduce the maximum pointwise error unless the mesh size is comparable to the perturbation parameter (Farrell et al., 2000). For instance, when $\varepsilon = 10^{-20}$, an impractically large number of mesh points would be required, leading to excessive computational cost. This limitation motivates the development of ε -uniform numerical methods, whose convergence and accuracy remain independent of the perturbation parameter and depend only on the mesh size.

Over the past decades, the construction of ε -uniform numerical methods has evolved along two main directions: fitted operator methods (FOMs) and fitted mesh methods (FMMs). FOMs are based on modifying classical finite difference operators so that they incorporate the essential features of the SPP. In particular, the discrete operator is designed such that it accurately represents the exponential behavior associated with layer solutions. Typically implemented on uniform meshes, this approach was pioneered by Allen & Southwell (1955) in the context of viscous flow problems and has since been widely developed.

Fitted operator methods are effective for certain classes of problems, particularly those exhibiting regular boundary layers. However, it has been shown by Shishkin (1989) that uniform convergence cannot always be achieved using FOMs on uniform meshes, especially for problems with parabolic layers where characteristic directions align with the boundary. This limitation was further confirmed by Miller et al. (2004), highlighting the inherent difficulty of capturing such layer behavior with operator based modifications alone.

Detailed treatments of FOMs can be found in Doolan et al. (1980) and Morton (1996). These methods are attractive due to their simplicity of implementation and the fact that they do not require prior knowledge of layer location or thickness (Mickens, 1993; Patidar, 2008). Additionally, they are often computationally efficient and can achieve high accuracy in suitable settings (Das & Natesan, 2013a, 2015).

An alternative and widely used approach is the fitted mesh method (FMM), which combines standard discretization schemes with specially designed layer-adapted meshes. These meshes are refined in regions where sharp gradients occur and remain coarse elsewhere, allowing efficient resolution of layer structures. When sufficient information

about the layer behavior is available, FMMs can achieve parameter–uniform accuracy. They are particularly advantageous for handling higher–dimensional and more complex problems (Das & Natesan, 2013b), although constructing appropriate meshes can be challenging in practice.

Fitted meshes are commonly classified as graded meshes or piecewise–uniform meshes. Among these, the Bakhvalov mesh and the Shishkin mesh are the most widely used. The Bakhvalov mesh, introduced by Bakhvalov (1969), provides a smooth grading tailored to boundary layers, while the Shishkin mesh employs a simpler piecewise–uniform structure with fine resolution in layer regions and coarse resolution elsewhere (Shishkin, 1989). Due to its simplicity and effectiveness, the Shishkin mesh has become particularly popular in practical computations (Vulanović, 2001). Further details on FMMs can be found in Doolan et al. (1980); Miller et al. (1996); Roos (2008); Shishkin & Shishkina (2008); Eugene & Shishkin (2012).

In general, the performance of any ε –uniform numerical method is assessed based on three fundamental criteria: consistency, stability, and convergence. These properties ensure that the numerical solution accurately approximates the exact solution and remains robust as the perturbation parameter approaches zero (Neill & Hashemi, 2018). Whether based on fitted operators or layer-adapted meshes, a thorough understanding of these principles is essential for the reliable numerical treatment of SPPs arising in practical applications.

1.1.2 Models Illustrating Singular Perturbations in PDEs

Partial differential equations (PDEs) with singular perturbations describe a number of real-world phenomena. Here, we provide a list of PDE models that describe these circumstances as follows:

1. A time delayed diffusion system (control system) (Wang, 1963):

$$\frac{\partial w(x, t)}{\partial t} - \varepsilon \frac{\partial^2 w(x, t)}{\partial x^2} - \varsigma \zeta [w(x, t - \tau)] \left(\frac{\partial w(x, t)}{\partial x} \right) - f[w(x, t - \tau)] + w(x, t) = 0,$$

defined on a one-dimensional (1D) spatial domain $x \in (0, 1)$, where ς is the material strip velocity depending on spatial average of the time-delayed tem-

perature distribution $w(x, t - \tau)$, and f represents a distributed temperature source function depending on $w(x, t - \tau)$. Such equations describe systems whose dynamics rely not only on present states but also on past values, reflecting memory and latency effects. They naturally arise in biology (maturation, gestation), epidemiology (incubation periods), and engineering (feedback with processing delays). Because conventional PDEs cannot capture these after-effects, delayed parabolic models are particularly relevant in applications such as oil recovery (Ewing, 1983), chemical processes (Adomian & Rach, 1983), mechanical dynamics (Baranowski, 2010), and control theory (Van Harten & Schumacher, 1978).

2. Hutchinson's equation

$$\frac{\partial w(x, t)}{\partial t} - \varepsilon \frac{\partial^2 w(x, t)}{\partial x^2} - \lambda(1 - w(x, t - \tau)) = 0,$$

which arises as a rough model for the evolution of a population in mathematical ecology with density (Kumar et al., 2018).

3. Slow diffusion with heat production:

$$\begin{aligned} \frac{\partial T}{\partial t} - \varepsilon \frac{\partial^2 T}{\partial x^2} + \psi(x)[T - s(x)] - \varpi(x) &= 0, \\ T(0, t) = t_0, \quad T(x, 0) &= \varphi(x), \end{aligned}$$

where $T(x, t)$ is the temperature of the bar. The temperature of the neighborhood of bar is $s(x)$, $\psi(x)$ is the exchange coefficients of heat and $\psi(x) \geq q$ with $q > 0$ so the bar is nowhere isolated (Verhulst, 2006). Although this model does not include explicit delay, it shares structural similarities with delayed systems through strong source-diffusion interactions and the formation of layers.

4. Some financial data can be modeled by means of Black-Scholes equation:

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC &= 0, (s, t) \in \mathbb{R}^+, x \in (0, T], \\ C(S, T) &= \max(S - E, 0), S \in \mathbb{R}, \\ C(0, t) &= 0, \quad C(S, t) \rightarrow S, \quad \text{for } S \rightarrow \infty, t \in (0, T]. \end{aligned}$$

Here, $C(S, t)$ is the price of a European call option, where S and t are current

values of the underlying asset and time, respectively. The parameters δ, E, T , and r correspond to volatility, strike price, maturity time, and risk-free interest rate. The formulation exhibits several singular features, including an unbounded spatial domain and a piecewise smooth initial condition (Li et al., 2017).

5. The convection-diffusion problem

$$D \frac{\partial^2 w}{\partial x^2} - v \frac{\partial w}{\partial x} = \frac{\partial w}{\partial t}, \quad (1.2)$$

where D is dispersion, v is velocity, and w is concentration. Eq.(1.2) is referred to as the convection-diffusion equation with diffusion D . The concentration $w(x, t)$ is a function of space and time. Thus, we assume that D and v are real, scalar constants. From Eq.(1.2), the dispersion term is $D \frac{\partial^2 w}{\partial x^2}$ and the convection term is $v \frac{\partial w}{\partial x}$. Depending on the relative magnitudes of these contributions, the equation changes its character: when diffusion dominates, Eq.(1.2) reduces to the parabolic heat equation, while convection dominance yields a first-order hyperbolic form. Owing to this dual nature, Eq.(1.2) serves as a prototype for understanding a wide range of physical phenomena, particularly in the modeling of fluid flows where both parabolic and hyperbolic behaviors naturally emerge.

6. The reaction-diffusion problem

$$w_t = D \Delta w + f(w), \quad (1.3)$$

where $w = w(x, t)$. Eq.(1.3) consists of two main components: the diffusion term $D \Delta w$, which models the spread of the quantity w through space with diffusion coefficient D , and a reaction term $f(w)$, a smooth function that represents the local changes or interactions within the system. In essence, $D \Delta w$ governs the spatial dispersion, while $f(w)$ accounts for the reaction dynamics.

For further discussion on a class of singularly perturbed models interested readers referred to (Stein, 1967; Tuckwell, 1989) and references therein.

1.1.3 Overview of Two-Parameter Singular Perturbations

Up to this point, attention has been primarily directed toward SPPs involving a single small parameter. A broader and more intricate class of problems arises when two small parameters are present, commonly referred to as two-parameter singularly perturbed problems (TP-SPPs). In such cases, not only the highest-order derivative terms but also certain lower-order terms are multiplied by an additional small parameter, typically denoted by μ , thereby introducing further complexity into the model.

These problems frequently emerge in a wide range of scientific and engineering applications, including heat and mass transfer processes, elasticity theory, plate and shell analysis, reservoir simulation, chemical reactor dynamics, and boundary layer phenomena influenced by suction or injection. The study of convection–diffusion–reaction models with two perturbation parameters is particularly important, as it provides deeper insight into the intricate balance and interaction between diffusion and convection mechanisms.

A good examples of problems with two perturbation parameters is a 1D electromagnetic field for a moving media which is

$$-\frac{1}{\mu} \frac{\partial^2 w(x, t)}{\partial x^2} = -\sigma \left(v \frac{\partial w(x, t)}{\partial x} + \frac{\partial w(x, t)}{\partial t} \right),$$

where $w(x, t)$, μ , σ and v are the potential, permeability, conductivity and velocity of the moving media respectively (Hahn et al., 1987), and a 1D flow through unsaturated porous media given by

$$\frac{\partial \vartheta}{\partial t} - \frac{\partial}{\partial z} \left(\frac{\rho_r}{\rho} K \frac{\partial \psi}{\partial \vartheta} \frac{\partial \vartheta}{\partial z} \right) - \frac{\rho_r}{\rho} g \frac{\partial K}{\partial z} = 0,$$

where ρ_r is the bulk density of media, ρ is the fluid density, θ is moisture content on a dry weight basis, K is the coefficient of aqueous conductivity, ψ is the capillarity pressure potential, and g is a gravity constant. Through assumptions and conditions given in Bhathawala & Verma (1975), the equation is simplified to the form:

$$\varepsilon F''(\zeta) + 2\mu F'(\zeta) - F(\zeta) = 0;$$

where ε and μ are small perturbation parameters.

Problems of essential importance involving two perturbation parameters are typically formulated as follows:

$$\begin{aligned} (\varepsilon w'' + \mu a w' - b w = f)(x), \\ w(0) = \varphi_0, \quad w(1) = \varphi_1. \end{aligned} \tag{1.4}$$

The Eq.(1.4) represents a 1D steady-state ODE.

$$\begin{aligned} (\varepsilon w_{xx} + \mu a w_x - b w - w_t = f)(x, t), \\ w(x, 0) = \varphi(x), w(0, t) = \varphi_0(t), w(1, t) = \varphi_1(t). \end{aligned} \tag{1.5}$$

The equation in Eq.(1.5) represents a 1D time dependent parabolic PDE.

$$\begin{aligned} (\varepsilon \Delta w + \mu a \nabla w - b w = f)(x, y), \\ w(x, 0) = \varphi_l(x), \quad w(x, 1) = \varphi_r(x), w(0, y) = \varphi_0(y), w(1, y) = \varphi_1(y). \end{aligned} \tag{1.6}$$

The problem in Eq.(1.6) is a 2D elliptic PDE (Teofanov & Roos, 2007). It is evident that the BVPs and/or IBVPs in Eq.(1.4)-(1.6) exclude mixed derivatives and are restricted to simple sub-domains of at most 2Ds. The elliptic operator involved is the Laplacian, a choice made to reduce complexity in constructing uniformly convergent schemes and to preserve simplicity when extending to higher-dimensional geometries.

The class of problems represented by Eq. (1.4) has been extensively explored in the existing literature, whereas comparatively less attention has been given to the formulation in Eq. (1.5). Due to time limitations, the present study focuses on the formulation, development, and rigorous analysis of parameter-uniform numerical techniques for TP-SPPPs that are further complicated by time delays, discontinuous convection coefficients, and non-smooth source terms. These complexities are not merely of theoretical interest, they arise naturally in practical models describing transport and reaction processes in heterogeneous media, where abrupt physical or chemical transitions and delayed system responses are common. However, the inclusion of such effects leads to significant analytical challenges, including reduced smoothness of the exact solution, coupling of multiple scales, and the formation of unconventional

boundary and interior layer structures, all of which demand specially designed numerical approaches to ensure uniform accuracy and stability.

In the problem considered here, it needs an investigation as both perturbation parameters ε and μ tend to zero, roughly speaking the order of the DEs drops to zeroth-order which is completely different from the original. The behavior of solutions to these problems varies for: case(i) $\varepsilon/\mu^2 \rightarrow 0$, case(ii) $\varepsilon = \mu^2$, and case(iii) μ^2/ε goes to zero. The first case is, perhaps, the most interesting one as the second-order DEs degenerate to their zeroth-order. In case (ii), The DEs degenerate to first-order, that is, $\varepsilon = 0$, and in case(iii) the DEs resemble almost when $\mu = 0$ (reaction diffusion).

The study combines theoretical analysis with computational implementation to validate the proposed methods. The objectives include establishing rigorous error bounds that are independent of the perturbation parameters, designing stable and efficient algorithms, and testing them on benchmark problems. In doing so, this dissertation contributes to the broader goal of advancing reliable numerical tools for solving complex parabolic problems that arise in scientific and industrial applications.

1.1.4 Basic Definitions and Terminologies

This section is devoted to presenting some fundamental concepts, notations and conventions that will serve as foundation for the subsequent analysis in this dissertation.

Definition 1.1. (*Truncation error*). *A truncation error arises when an exact mathematical operation is replaced by an approximate one, typically due to representing an infinite process through a finite number of steps.*

In this work, we focused on errors associated with finite difference formulas and discretizations of DEs. These errors appear in two primary forms: local and global.

- The local truncation error quantifies the deviation between the DE and its discrete analogue at a grid point, serving as a measure of how accurately the finite difference scheme captures the original equation. Increasing the order of approximation, often by extending the stencil reduces this error and enhances accuracy (LeVeque, 1998).

- The global truncation error represents the cumulative effect of local errors, commonly evaluated in various norms such as the maximum, energy, or square norm.

In this dissertation, the maximum norm is adopted throughout, as it provides a sharper indication of boundary-layer behavior (Farrell et al., 2000), and is particularly effective for parabolic layer problems (Miller et al., 1996). As a standard goal, the analysis seeks to determine the smallest possible upper bound for the overall error.

Definition 1.2. (*Stability*). In a numerical system, stability describes how numerical errors that result from truncation, rounding, or initial data behave during computation. In order to prevent tiny disturbances from growing uncontrollably and distorting the solution, a system is deemed stable if such errors remain confined or decrease over subsequent iterations. On the other hand, the method is considered unstable if these errors increase indefinitely (Neill & Hashemi, 2018).

Definition 1.3. An M-matrix is a square matrix distinguished by non-negative diagonal entries and non-positive off-diagonal entries. Formally, a matrix $A = [a_{ij}]$ is an M-matrix if:

- $a_{ii} \geq 0$ for all diagonal elements,
- $a_{ij} \leq 0$ for all off-diagonal elements $i \neq j$.

M-matrices are also referred to as monotone matrices and possess structural properties that make them valuable in numerical analysis and applied mathematics.

Definition 1.4. A finite difference scheme is said to be monotone when its associated system matrix A possesses the monotonicity property; that is, either A or $-A$ qualifies as a monotone matrix, ensuring non-negativity preservation and stability of the discrete solution.

Definition 1.5. Finite difference methods (FDMs) approximate derivatives by replacing them with difference quotients based on nearby function values. Common types are forward, backward and central differences, and such approximations are derived using Taylor's series expansion for any mesh function w_i at a mesh points x_i with

mesh size $h = x_{i+1} - x_i$. Their standard notations are as follows:

$$\begin{aligned} D^+ w_i &= \frac{w_{i+1} - w_i}{h} + O(h), \\ D^- w_i &= \frac{w_i - w_{i-1}}{h} + O(h), \\ D^0 w_i &= \frac{w_{i+1} - w_{i-1}}{2h} + O(h^2), \\ \delta^2 w_i &= D^+ D^- w_i = \frac{w_{i+1} - w_i + w_{i-1}}{h^2} + O(h^2). \end{aligned}$$

Definition 1.6. (*Consistency*). Consistency refers to the extent to which a numerical scheme accurately represents the underlying DE. A scheme is said to be consistent with the original problem if, when the time step $\Delta t \rightarrow 0$ and the spatial mesh size $\Delta x \rightarrow 0$, the difference equation reduces to the corresponding DE. In other words, the local truncation error, the discrepancy between the exact differential operator and its discrete approximation must vanish as the grid is refined. Thus, consistency ensures that the discrete model faithfully approximates the continuous equation in the limit of infinitesimally small discretization parameters.

Definition 1.7. (*Convergence*). A numerical scheme is termed convergent if the solution obtained from discretized problem approaches the exact solution of the underlying DE as the mesh parameters, such as the spatial step and time step, tend to zero. Equivalently, convergence requires that the global error between the numerical approximation and the exact solution vanishes in the limit of infinitely fine discretization.

Theorem 1.1. A consistent and stable FDM is convergent. But the converse may not be true ([LeVeque, 1998](#); [Morton, 1996](#); [Peiró & Sherwin, 2005](#)).

Definition 1.8 (Rate of Convergence). The rate (or order) of convergence characterizes how efficiently a numerical method reduces its error as the mesh size decreases. In simple terms, if the step length is reduced by a certain factor, the rate of convergence indicates the corresponding reduction in the numerical error. It is determined by

$$R = \frac{\log(E^{h_1}) - \log(E^{h_2})}{\log(h_1) - \log(h_2)},$$

where E^{h_1} and E^{h_2} represent the maximum absolute errors for two uniform mesh sizes h_1 and h_2 , respectively. The computed rate should be consistent with the theoretical order of accuracy expected for the proposed numerical scheme ([O'Malley, 1974](#)).

Definition 1.9. (*Miller et al., 1996*) Let P_ε denote a family of SPPs depending on a small perturbation parameter $\varepsilon \in (0, 1]$. Denote by w_ε the exact solution and by $W^{N,M}$ its numerical approximation constructed on a mesh $D^{N,M}$, where N and M correspond to the temporal and spatial discretization parameters, respectively. A numerical scheme is said to be ε -uniformly convergent if there exist constants $C > 0$, $p > 0$, and $q > 0$, along with an integer N_0 , all independent of N, M , and ε , such that for all $N \geq N_0$ and $M \geq M_0$,

$$\sup_{0 < \varepsilon \leq 1} |W^{N,M} - w_\varepsilon|_{D^{N,M}} \leq C (N^{-p} + M^{-q}).$$

Here, p and q denote the ε -uniform orders of convergence in time and space, respectively, while C represents the ε -uniform error bound.

1.2 Statement of the Problem

Singularly perturbed parabolic problems frequently arise in diverse areas of science and engineering, including fluid dynamics, chemical kinetics, control theory, and biological systems, where small perturbation parameters introduce pronounced multi-scale behavior (*Miller et al., 2004; Kevorkian & Cole, 2013*). Such problems typically exhibit rapid solution variations confined to narrow regions, known as boundary or interior layers, whose accurate numerical resolution remains a long-standing challenge. The situation becomes even more intricate when multiple small parameters, time delays, discontinuous coefficients, or irregular source terms coexist within the model.

Two-parameter singularly perturbed boundary value problems are encountered in numerous real-world phenomena, such as reaction-diffusion processes, viscoelastic materials, thermal memory models, and population dynamics (*Miller et al., 1996; Kumar et al., 2018; Duressa, 2021; Zahra & El Mhlawy, 2013*). Time-delay effects introduce dependence on past states, reflecting memory-driven dynamics, while discontinuities in convection coefficients or source terms often correspond to abrupt physical transitions within the medium. These complexities pose significant difficulties for conventional numerical techniques, which often fail to maintain stability and accuracy across sharp layer regions, motivating the development of uniformly convergent schemes.

The behavior of TP-SPPPs is strongly influenced by the interplay between the small perturbation parameters ε and μ . Specifically, when $\mu = 1$, the system effectively reduces to a convection-diffusion equation, where boundary layers of thickness $O(\varepsilon)$ form near $x = 0$ and $x = 1$, with their location and intensity determined by the convection coefficients. In contrast, if $\mu = 0$, the system exhibits boundary layers of width $O(\sqrt{\varepsilon})$ at both endpoints, reflecting a different scaling of perturbation effects.

The qualitative characteristics of the solution, whether predominantly dissipative or dispersive are dictated by the ratio μ^2/ε . Dissipative effects dominate when $\varepsilon/\mu^2 \rightarrow 0$ as $\mu \rightarrow 0$, leading to solutions where diffusion smooths out sharp gradients. Conversely, dispersive effects become significant if $\mu^2/\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, resulting in oscillatory or wave-like behavior within the layers (O'malley, 1967). Understanding this parameter-dependent behavior is essential for designing robust numerical schemes that can accurately capture both boundary layer structures and the multiscale dynamics inherent in TP-SPPPs.

To tackle these multiscale complexities, a range of analytical and computational methods have been proposed. Asymptotic approaches provide valuable insights into layer structures but often fail to yield explicit solutions for realistic problems. Numerical techniques, such as finite difference, finite element, and spline-based methods have therefore become essential tools. However, classical schemes typically require extremely fine meshes to resolve steep gradients, resulting in excessive computational cost. To overcome this, advanced techniques including nonstandard finite differences (Patidar, 2008; Munyakazi, 2015), adaptive meshing, higher-order discretizations, and exponential fitting methods (Das & Natesan, 2012, 2013a; Kadalbajoo & Jha, 2012; Linß, 2009) have been introduced. More recently, parameter-uniform schemes have been extended to handle non-smooth data and discontinuous coefficients (Chandru et al., 2017, 2018, 2019; Roy & Jha, 2023; Singh et al., 2023).

Despite these contributions, existing research predominantly focuses on single-parameter formulations or assumes smoothness in coefficients and source terms. Comprehensive numerical investigations addressing TP-SPPPs with time delays, discontinuous coefficients, and irregular source functions remain limited. This gap underscores the pressing need for unified, parameter-uniform frameworks capable of maintaining ac-

curacy and stability across such heterogeneous conditions.

Therefore, the present study is devoted to the development and analysis of uniform numerical schemes for various classes of TP-SPPPs. The investigation encompasses both delayed and non-delayed cases, incorporating discontinuities in convection coefficients and source terms. The objective is to construct and rigorously analyze reliable schemes that ensure uniform convergence, while accurately capturing the influence of delays, discontinuities, and multiscale effects on the overall solution behavior.

1.3 Research Objectives

This dissertation is motivated by the need to develop, analyze, enhance, and optimize uniform convergent numerical methods for solving TP-SPPPs involving convection, reaction, and diffusion processes.

1.3.1 General Objective

The general objective of this study is to develop, analyze, and implement uniformly convergent numerical schemes for various classes of two-parameter singularly perturbed parabolic problems.

1.3.2 Specific Objectives

The specific objectives of this study are to:

- formulate fitted numerical schemes for solving a class of TP-SPPPs.
- establish the stability properties of the constructed numerical methods.
- establish parameter-uniform convergence of the proposed numerical schemes.
- investigate the influence of perturbation parameters and non-smooth data on the solution behavior, particularly in the boundary layer regions.

1.4 Significance of the Study

The problem under investigation possesses wide-ranging applications across science and engineering. Research in this area has long been central to advancing both theoretical mathematics and practical applications, and it continues to provide essential tools for tackling complex real-world problems (Nohel et al., 1997). Although this dissertation is primarily theoretical, its contributions extend beyond the realm of abstract analysis. The numerical methods developed here not only enhance accuracy, efficiency, and simplicity but also offer robust frameworks that can be adapted to applied research. These techniques can serve as valuable resources for scientists and engineers working on simulations, control systems, population dynamics, heat transfer, and other physical phenomena, thereby bridging the gap between mathematical theory and practical problem-solving.

1.5 Scope of the Study

This dissertation is delimited to a specific class of TP-SPPPs and concentrates on their formulation and numerical investigation across the following problem categories.

Governing Problem I

Consider a TP-SPPPs on the domain $\mathbb{G} = \Omega \times \Lambda$ as our first governing model as:

$$\left\{ \begin{array}{l} \mathbb{L}_{\varepsilon, \mu} w \equiv \left(\frac{\partial w}{\partial t} - \varepsilon \frac{\partial^2 w}{\partial x^2} - \mu a \frac{\partial w}{\partial x} + bw = \varpi \right) (x, t), \quad (x, t) \in \mathbb{G}, \\ w|_{\Lambda_b} = \zeta_b(x), \quad x \in \overline{\Omega}, \\ w|_{\Lambda_l} = \zeta_l(t), \quad w|_{\Lambda_r} = \zeta_r(t), \quad t \in \overline{\Lambda}, \end{array} \right. \quad (1.7)$$

where $0 < \varepsilon \ll 1$ and $0 \leq \mu \leq 1$ are the perturbation parameters.

Governing Problem II

The second model problem is considered as:

$$\left\{ \begin{array}{l} \mathbb{L}_{\varepsilon, \mu} w \equiv \left(\varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a \frac{\partial w}{\partial x} - bw - \frac{\partial w}{\partial t} \right) (x, t) = -cw(x, t - \gamma) + \varpi(x, t), \\ w|_{\Lambda_b} = \Upsilon_b(x, t), \\ w|_{\Lambda_l} = \Upsilon_l(t), \quad w|_{\Lambda_r} = \Upsilon_r(t), \end{array} \right. \quad (1.8)$$

where $0 < \varepsilon \ll 1$ and $0 \leq \mu \leq 1$ are the perturbation parameters.

Governing Problem III

As a third model problem, we consider a TP-SPPPs characterized by discontinuities in both the source term and the convection coefficient

$$\left\{ \begin{array}{l} \mathbb{L}_{\varepsilon, \mu} w \equiv \left(\varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a \frac{\partial w}{\partial x} - bw - c \frac{\partial w}{\partial t} = \varpi \right) (x, t), (x, t) \in \mathfrak{D}^- \cup \mathfrak{D}^+, \\ w(0, t) = \mathcal{B}(t) \quad \text{in } \mathfrak{D}_l, \quad w(1, t) = \mathcal{R}(t) \quad \text{in } \mathfrak{D}_r, \\ w(x, 0) = \mathcal{I}(x) \quad \text{in } \mathfrak{D}_b, \\ |[a](\varphi)| \leq C, \quad |[\varpi](\varphi)| \leq C, \\ a(x, t) \leq -\alpha_1 < 0, (x, t) \in \mathfrak{D}^-, \\ a(x, t) \geq \alpha_2 > 0, (x, t) \in \mathfrak{D}^+. \end{array} \right. \quad (1.9)$$

Here, $0 < \varepsilon \ll 1, 0 \leq \mu \leq 1$ are perturbation parameters and φ is a discontinuity point.

1.6 Mathematical Procedure of the Study

In this dissertation, to achieve the stated objective, we have followed the following mathematical procedures:

1. Defining the problems.
2. Examination of the analytical properties of the continuous solution.
3. Formulating numerical schemes for the problem under study.
4. Establishing the stability and convergence of the developed numerical schemes.
5. Developing an algorithm and writing MATLAB code for the presented schemes.
6. Validation of accuracy through carefully chosen numerical experiments.
7. Presentation of results in the form of tables and graphical illustrations.
8. Comparative analysis with existing methods reported in the literature.
9. Drawing conclusions based on the performance of the proposed schemes.

1.7 Organization of the Dissertation

The intent and content of this dissertation consists of nine chapters and it is organized as follows. Chapter 1 contains the background of the study and definitions of numerical analysis which are used in successive chapters of this dissertation. It also introduces the background of related work done and its progress in the field of SPPs. The theoretical framework of the study is presented in Chapter 2 and is achieved by surveying pertinent literature to the present work. The relevant literature is organized in publication year chronological order to show the advancement of methods used to solve SPPs clearly. In our review of the literature, we examined the methods used and identified gaps.

Chapters 3-8 constitute the core component of this dissertation, encompassing the principal methodologies, analytical formulations, and computational frameworks employed in the study. In Chapter 3, a parameter-uniform numerical approach is developed through a two-stage discretization strategy. The temporal derivative is approximated using the Crank-Nicolson method, ensuring second-order accuracy and temporal stability, while the spatial variable is discretized via the two-point Gaussian quadrature scheme, which enhances the precision of spatial integration and effectively captures the layer behavior inherent in the SPP formulated in Eq. (1.7).

Chapter 4 introduces a fitted operator-based numerical algorithm for Eq. (1.7) that further refines the uniform convergence characteristics of the solution. The method integrates the Crank-Nicolson time discretization with a Simpson's one-third rule for spatial integration, allowing improved accuracy and stability in the presence of steep boundary and interior gradients. This hybrid framework not only strengthens the numerical robustness but also provides a reliable foundation for analyzing complex SPPs with multiple small parameters.

Chapter 5 focuses on the numerical solution of Eq. (1.8) using parameter-uniform framework that effectively captures the influence of multiple small perturbation parameters and delay effects. The Crank-Nicolson scheme ensures second-order temporal accuracy, while a fitted spatial integration technique resolves the sharp gradients caused by singular perturbations. The formulation employs an adaptive operator-

fitting mechanism that balances the perturbation parameters within the spatial operator, achieving uniform convergence throughout the domain. The incorporated delay term, which adds dynamic complexity, is efficiently treated through the combined temporal-spatial discretization, preserving both stability and accuracy.

Chapters 6 and 7 focus on the problem formulated in Eq. (1.9), which represents a class of TP-SPPPs involving discontinuous data. These discontinuities, combined with the presence of multiple small perturbation parameters, give rise to complex layer structures that necessitate specialized numerical strategies for accurate resolution. In Chapter 6, a numerical scheme is developed by employing the implicit Euler method for temporal discretization, ensuring stability and reliability over fine temporal grids. The spatial variable is discretized using a non-polynomial spline technique constructed on a piecewise uniform mesh, which effectively captures sharp gradients and discontinuities across the domain while maintaining computational efficiency. Chapter 7 extends the study by developing a parameter-uniformly convergent scheme for the same problem. The implicit Euler method is used for temporal discretization, while a nonstandard finite difference technique ensures spatial accuracy with respect to the perturbation parameters. To enhance precision, Richardson extrapolation is employed, improving the convergence order and refining the numerical solution.

Chapter 8 focuses on the numerical treatment of Eq. (1.9), employing the implicit Euler method for time discretization and a cubic spline in tension for spatial approximation. The influence of perturbation parameters is effectively managed using a Shishkin mesh, ensuring uniform accuracy across layers. Finally, Chapter 9 summarizes the key findings of the proposed schemes, and outlines recommendations and future research directions for advancing singular perturbation analysis.

Chapter 2

Review of Related Literature

Singularly perturbed problems play a vital role in modeling physical, chemical, and biological processes characterized by sharp gradients or layer phenomena. Since Prandtl's pioneering work, research in singular perturbation theory has expanded enormously, making it difficult for any single study to capture the entire scope. The complexity increases further when small parameters, delay terms, or discontinuities are introduced, demanding specialized numerical methods for accurate solutions. This chapter reviews the key contributions in the literature, with emphasis on numerical techniques developed for tackling such challenging tasks.

2.1 Singularly Perturbed Problems

A considerable amount of research has been devoted to the development of uniformly convergent numerical techniques for SPDEs. In addressing such problems, extensive literature is available in the form of books and monographs that emphasize either asymptotic analysis or numerical methodologies (see, e.g., [Van Dyke \(1975\)](#); [O'malley et al. \(1991\)](#); [Miller et al. \(1996\)](#); [Farrell et al. \(2000\)](#); [Holmes \(2007\)](#); [Nayfeh & Balachandran \(2008\)](#); [Roos \(2008\)](#); [Shishkin & Shishkina \(2008\)](#); [Linß \(2009\)](#)). In addition to these foundational works, several comprehensive surveys have also been published, specifically reviewing the progress made in this area ([Kadalbajoo & Reddy, 1989](#); [Kadalbajoo & Patidar, 2002, 2003](#); [Kumar et al., 2007](#); [Kadalbajoo & Gupta, 2010](#); [Roos, 2012](#); [Duressa et al., 2023](#)).

Furthermore, there have been several published studies on recently developed uni-

formly convergent numerical methods for SPPs. For example, [Munyakazi & Patidar \(2014\)](#) formulated a fitted operator FDM for SPPs of the form:

$$\begin{aligned} Lw &:= (\varepsilon w'' + aw' - bw = f)(x), \quad x \in \Omega = (-1, 1), \\ w(-1) &= A, \quad w(1) = B, \end{aligned}$$

where A and B are given real constants and $\varepsilon \in (0, 1]$, and the coefficients $a(x)$, $b(x)$, and $f(x)$ are sufficiently smooth function in Ω . The suggested method is robust with respect to ε and is of order almost one. However, the authors improve the accuracy of the methods by employing Richardson extrapolation.

[Aga et al. \(2021\)](#) considered the SPP reaction-diffusion IBVP as:

$$\begin{aligned} \left(\varepsilon \frac{\partial^2 w}{\partial x^2} - bw - \frac{\partial w}{\partial t} \right) (x, t) &= f(x, t), \quad x \in \Omega = (0, 1) \times (0, 1), \\ w(x, 0) &= s(x), \quad \forall x \in [0, 1], \\ w(0, t) &= q_0(t), \quad w(1, t) = q_1(t), \quad \forall t \in [0, T], \end{aligned}$$

where ε is perturbation parameter and assume that the coefficient function is sufficiently smooth. The authors proposed a FOMs to solve problems whose solution exhibits twin boundary layers. The Richardson extrapolation technique was used to accelerate the rate of convergence.

2.2 Singularly Perturbed Delay Problems

In many practical scenarios, it is commonly assumed that a system evolves according to the principle of causality; meaning its future behavior depends solely on its current state, independent of past states. However, upon closer examination, this assumption often serves only as a first-order approximation. In reality, many systems are influenced by their historical states, requiring more accurate models that incorporate memory effects. Such systems are typically described by DEs with delay(or lag) terms, reflecting the dependence on past conditions.

Singularly perturbed problems with time delays are mathematical problems that combine DEs with a singular perturbation parameter, and a time delay parameter. These problems incorporate time delays, which introduce complexities by including memory

effects and feedback dynamics in the system. The presence of time delay can lead to oscillations, or the emergence of complex dynamics (Zak et al., 2003). The literature contains various numerical methods that have been developed for addressing these types of problems. These methods aim to accurately capture the dynamics of systems with time delay, enabling the analysis and prediction of their behavior with improved precision and reliability. For example, Ansari et al. (2007) investigated a SPPPs with time delays on $D = (0, 1) \times (0, T]$ of the form:

$$\begin{aligned} \mathbb{L}w_\varepsilon \equiv \left(\frac{\partial w_\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 w_\varepsilon}{\partial x^2} + aw_\varepsilon \right) (x, t) &= -bw_\varepsilon(x, t - \tau) + f(x, t), \quad (x, t) \in D, \\ w_\varepsilon(x, t) &= \phi_l(t), \quad w_\varepsilon(x, t) = \phi_r(t), \quad w_\varepsilon(x, t) = \phi_b(x, t), \end{aligned} \quad (2.1)$$

where $0 < \varepsilon \leq 1$ and $\tau > 0$ is a delay term and the coefficients are sufficiently smooth and bounded functions. The authors proposed a standard finite difference scheme (FDS), employing a spatially centered and temporally implicit discretization. The method proved highly effective in resolving boundary layer behavior and exhibited strong robustness with respect to ε . Moreover, the numerical solutions obtained showed uniform convergence in the maximum norm to the exact solution over the entire range of the perturbation parameter $\varepsilon \in (0, 1]$. It was rigorously demonstrated that the errors were bounded in the maximum norm by $C(N_x^{-2} \ln^{-2} N_x + N_t^{-1})$.

Bashier & Patidar (2011), the authors studied a SPPPs with time delay of the form given in Eq. (2.1). They proposed a fitted operator FDM to obtain its solution. The developed scheme was shown to be unconditionally stable and convergent with an accuracy of order $O(k + h^2)$. This performance was notably better than the method presented in Ansari et al. (2007). In the same year, the authors extended their work by developing a second-order fitted operator FDM specifically for SPPPs with time delay of convection-diffusion type. This method was constructed by replacing the classical differential operator with a fitted operator derived from the Crank-Nicolson discretization scheme. The proposed method was rigorously analyzed and found to be unconditionally stable and convergent with an improved accuracy of order $O(k^2 + h^2)$.

Gowrisankar & Natesan (2017), suggested ε -uniformly convergent numerical scheme comprises the backward-Euler method in the temporal direction with a constant step size and the resulting system of DEs is approximated by the up-wind FDM on a

piecewise uniform adaptive mesh for SPPPs with time delay on the domain $D = \Omega \times (0, T] = (0, 1) \times (0, T]$ and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \mathbb{L}_\varepsilon \right) w(x, t) = -b(x, t)w(x, t - \tau) + f(x, t), & (x, t) \in D, \\ w(x, t) = \varpi_b(x, t), & (x, t) \in \Gamma_b, \\ w(0, t) = \varpi_l(t), & \text{on } \Gamma_l = \{(0, t) : 0 \leq t \leq T\}, \\ w(1, t) = \varpi_r(t), & \text{on } \Gamma_r = \{(1, t) : 0 \leq t \leq T\}, \end{cases} \quad (2.2)$$

where $\mathbb{L}_\varepsilon w(x, t) = -\varepsilon \frac{\partial^2 w(x, t)}{\partial x^2} + c(x) \frac{\partial w}{\partial x} + a(x, t)w(x, t)$, $0 < \varepsilon \ll 1$ and $\tau > 0$. The functions $a(x, t), b(x, t), c(x)$ in $\overline{D}$ and $\varpi_b(x, t), \varpi_l(t), \varpi_r(t)$ in Γ are sufficiently smooth and bounded functions. The terminal time satisfy the condition $T = k\tau, k \in \mathbb{Z}^+$. The authors carried out the stability analysis of the method and obtained ε -uniform convergence of first-order both in time and space.

Several studies [Das & Natesan \(2015\)](#); [Salama & Al-Amery \(2017\)](#); [Das & Natesan \(2018\)](#); [Kumar & Kumari \(2019, 2020\)](#); [Podila & Kumar \(2020\)](#); [Govindarao & Mohapatra \(2020\)](#); [Kumar \(2021\)](#) have addressed the same problem (2.2) involving both small and large delay parameters in the time variable, proposing different numerical approaches such as the fitted FDS, compact finite difference technique, extended B-spline method, and hybrid schemes incorporating fitting strategies.

[Singh et al. \(2018\)](#) designed a domain decomposition method for solving singularly perturbed parabolic delay differential equation (SPPDDEs) by discretizing the problem using the backward Euler scheme in the time and FDS on Shishkin-mesh in spatial direction. The researchers considered the problem as:

$$\begin{aligned} \mathbb{L}w(x, t) &= -b(x, t)w(x, t - \tau) + f(x, t), & (x, t) \in D = (0, 1) \times (0, T], \\ w(0, t) &= \phi_l(t), \quad w(1, t) = \phi_r(t), \quad w(x, t) = \phi_b(x, t), \end{aligned}$$

where $\mathbb{L}w(x, t) := w_t - \varepsilon w_{xx} + aw$. Here, $0 < \varepsilon \ll 1, \tau > 0$ and $b \leq 0, a + b \geq 0$ on $\overline{D}$. Hence, the proposed scheme demonstrates uniform convergence with nearly second-order accuracy in space and first-order accuracy in time.

[Woldaregay et al. \(2023\)](#) considered a SPPDDE on $D = \Omega_x \times \Omega_t = (0, 1) \times (0, T]$ with

boundary ∂D , where $\overline{D} = \overline{\Omega}_x \times \overline{\Omega}_t = [0, 1] \times [0, T]$, which is given by:

$$\begin{aligned} w_t - \varepsilon w_{xx} + a(x)w_x + b(x, t)w(x, t) &= -c(x, t)w(x, t - \tau) + f(x, t), (x, t) \in D, \\ w(x, t) &= \phi_b(x, t), (x, t) \in [0, 1] \times [-\tau, 0], \\ w(0, t) &= \phi_l(t), t \in \Omega_t, w(1, t) = \phi_r(t), t \in \Omega_t, \end{aligned}$$

where $\varepsilon \in (0, 1]$ is a perturbation parameter and $\tau > 0$ is the delay parameter. To numerically solve a problem, the authors proposed a fitted exact difference method. This approach combines the Crank-Nicolson scheme for time discretization with a fitted mesh exact FDM for spatial discretization.

2.3 Existing Studies on Two-Parameter Singularly Perturbed Problems

Two-parameter singularly perturbed parabolic problems with and without time delay find numerous applications across various fields. For instance, in engineering, they are employed in modeling phenomena such as drift-diffusion in semiconductors (Polak et al., 1987), and fluid dynamics (Schlichting & Gersten, 2016; Govindarao et al., 2019).

To numerically solve these problems, researchers have devised uniformly convergent numerical methods. Numerous studies have contributed to the advancement of these techniques, focusing on ensuring their accuracy, stability, and reliability. These methods provide an effective and precise means for computing solutions to TP-SPPPs, both with and without time delay. Some of such studies were found in O'malley (1967); Vulcanović (2001); Roos & Uzelac (2003); Patidar (2008); Kadalbajoo & Yadaw (2012); Zahra & El Mhlawy (2013); Munyakazi (2015); Chandru et al. (2017, 2018); Mekonnen & Duressa (2020); Govindarao et al. (2019); Sumit et al. (2020); Bullo et al. (2021); Roy & Jha (2023); Negero et al. (2023).

Patidar (2008) consider the SPDE of the form:

$$\begin{aligned} \mathbb{L}_{\varepsilon, \mu} w &\equiv (\varepsilon w'' + \mu a w' - b w = f)(x), x \in \Omega = (0, 1), \\ w(0) &= \alpha_0, w(1) = \alpha_1, \end{aligned}$$

where $0 < \varepsilon$ and $\mu \leq 1$ are two small parameters. In his study, the author developed FOMs tailored for a class of SPDEs involving two small parameters associated with the derivatives. These techniques, classified as nonstandard FDMs, were employed to compute accurate numerical solutions to the problems.

[Kadalbajoo & Yadaw \(2012\)](#) investigated a time-dependent singularly perturbed reaction-diffusion BVP on $D = \Omega \times (0, T]$ with $\Omega = (0, 1)$ as:

$$\varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a(x) \frac{\partial w}{\partial x} - b(x)w(x) - \frac{\partial w}{\partial t} = f(x, t), (x, t) \in D,$$

$$w(x, 0) = w_0(x), w(0, t) = 0 = w(1, t) = 0, t \in [0, T],$$

with $0 < \varepsilon \ll 1$ and $0 < \mu \ll 1$, such that $\frac{\mu^2}{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$. The approach employs Rothe's method, an implicit finite difference scheme in time on a uniform mesh combined with a finite element method in space using a Shishkin-type piecewise uniform mesh. It has been demonstrated that the method is unconditionally stable and achieves an accuracy of order $O(N^{-2}(\ln N)^2 + \Delta t)$.

[Zahra & El Mhlawy \(2013\)](#) investigated a class of two-parameter singularly perturbed semi-linear BVPs of the form:

$$\mathbb{L}w(x) \equiv -\varepsilon_d w'' + \varepsilon_c p(x)w' + f(x, w) = 0, x \in (0, 1),$$

subject to $w(0) = \mu_1$ and $w(1) = \mu_2$, where $0 < \varepsilon_d \ll 1$ and $0 < \varepsilon_c \ll 1$. The authors employed an exponential spline method on a Shishkin mesh. The proposed method exhibits uniform convergence with respect to both perturbation parameters.

[Munyakazi \(2015\)](#) studied a class of time-dependent TP-SPPPs which respectively influence the diffusion and the convection terms. The problem is given by:

$$\mathbb{L}_{\varepsilon, \mu} w \equiv \varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a(x) \frac{\partial w}{\partial x} - b(x)w(x) - \frac{\partial w}{\partial t} = f(x, t), (x, t) \in (0, 1),$$

$$w(x, 0) = w_0(x), \quad x \in \overline{\Omega}, w(0, t) = w(1, t) = 0, \quad t \in [0, T],$$

where $0 < \varepsilon, \mu \leq 1$. To approximate the solution, the authors applied the backward Euler method for time discretization and nonstandard finite difference approach introduced by Mickens, for spatial discretization.

Mekonnen & Duressa (2020) investigated a 1D parabolic convection-diffusion problems characterized by the presence of two small perturbation parameters associated with the convection and diffusion terms. The governing equation is:

$$\begin{aligned}\mathbb{L}_{\varepsilon,\mu}w &\equiv \frac{\partial w}{\partial t} - \varepsilon \frac{\partial^2 w}{\partial x^2} - \mu a(x,t) \frac{\partial w}{\partial x} + b(x,t)w = f(x,t), (x,t) \in D, \\ w(x,0) &= w_0(x), x \in \overline{\Omega}, w(0,t) = 0 = w(1,t), t \in [0,T],\end{aligned}\tag{2.3}$$

where ε and μ are small positive parameters ($0 < \varepsilon \ll 1, 0 < \mu \ll 1$). To solve the problem numerically, the authors employed the Crank-Nicolson method for temporal discretization on a uniform mesh and applied a nonstandard FDS for spatial discretization. The results demonstrated that the scheme is more accurate than several existing methods reported in the literature.

Govindarao et al. (2019) examined a class of two-parameter singularly perturbed parabolic initial-boundary value problems (IBVPs) with time delay. The problem is defined on the domain $\Omega = (0,1) \times (0,T]$ and is governed by the following equation:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbb{L}_{\varepsilon,\mu}\right)w(x,t) &= -c(x,t)w(x,t-\tau) + f(x,t), x \in \Omega = (0,1) \times (0,T], \\ w(x,t) &= w_0(x,t), (x,t) \in \Gamma_b = [0,1] \times [-\tau,0], \\ w(0,t) &= \phi_l(t), \quad \text{on } \Gamma_l = \{(0,t) | 0 \leq t \leq T\}, \\ w(1,t) &= \phi_r(t), \quad \text{on } \Gamma_r = \{(1,t) | 0 \leq t \leq T\},\end{aligned}\tag{2.4}$$

where the differential operator is defined by

$$\mathbb{L}_{\varepsilon,\mu} := -\varepsilon w_{xx}(x,t) - \mu a(x)w_x(x,t) + b(x,t)w(x,t), G = \Omega \times (0,T].$$

They employed the implicit Euler method for time discretization and an upwind FDS on Shishkin meshes (standard and Bakhvalov-Shishkin) for space. The scheme was shown to be uniformly convergent and first-order accurate, independent of ε and μ .

In Gupta et al. (2019), the authors proposed a numerical approach for solving a time-dependent TP-SPPPs described by Eq. (2.3). The method applies the implicit Euler scheme for time discretization on a uniform mesh and a hybrid monotone difference operator on a piecewise-uniform Shishkin mesh for space. This hybrid operator

combines central-upwind and midpoint-upwind schemes for the outer and inner layer regions, respectively, achieving nearly second-order spatial accuracy. By incorporating Richardson extrapolation, the time discretization was enhanced to second-order accuracy, resulting in a uniformly convergent, higher-order method effective for problems with pronounced boundary and interior layers.

In [Sumit et al. \(2020\)](#), the authors studied a 1D singularly perturbed IBVP with delay, involving two small parameters, as given in Eq. (2.4). They proposed a hybrid monotone finite difference scheme on a rectangular mesh, combining the implicit Euler method for time discretization with a spatial hybrid operator formed from central, upwind, and midpoint differences. This formulation ensures stability and reliability under delay and perturbation effects, achieving nearly second-order spatial and first-order temporal convergence uniformly in both small parameters.

[Bullo et al. \(2019\)](#) formulated a higher-order uniform scheme by combining implicit Euler time discretization with a fitted central difference in space for Eq.(2.4). Using Richardson extrapolation, its accuracy was enhanced from second to fourth-order. Subsequently, [Bullo et al. \(2021\)](#) introduced a robust FDM for Eq.(2.4), employing Mickens type discretization with a fitted operator on a uniform mesh. Although initially first-order accurate, applying Richardson extrapolation improved accuracy.

[Negero et al. \(2023\)](#) introduced fitted operator FDMs for solving 1D,TP-SPPPs that involve a delay in the time variable. The problem is defined on the domain $D = \Omega_x \times (0, T]$, where $\Omega_x = (0, 1)$:

$$\mathbb{L}_{\varepsilon, \mu} w(x, t) \equiv w_t - \varepsilon w_{xx} - \mu a(x, t) w_x + b(x, t) w = g(x, t), (x, t) \in D, \quad (2.5)$$

subject to the initial and boundary conditions:

$$\begin{aligned} w(x, t) &= \Psi_b(x, t), (x, t) \in \Gamma_b = [0, 1] \times [-\tau, 0], \\ w(0, t) &= \Psi_l(t), \Gamma_l, w(1, t) = \Psi_r(t), \Gamma_r = \{(1, t) : 0 \leq t \leq T\}, \end{aligned} \quad (2.6)$$

where the source term is defined as $g(x, t) = -c(x, t)w(x, t - \tau) + f(x, t)$, with parameters $0 < \varepsilon \leq 1, 0 \leq \mu \leq 1$ and $\tau > 0$ denoting the time delay. To numerically approximate the solution, the Crank-Nicolson scheme was employed for time discretization,

while a non-standard fitted operator method was applied for spatial discretization.

[Ayele et al. \(2023\)](#) developed a numerical framework for TP-SPPPs with time delays, as formulated in (2.5-2.6). The interplay of the small perturbation parameters generates twin boundary layers near the spatial domain, with their intensity dictated by the parameter magnitudes. The scheme employs a θ -method for time discretization on a uniform grid, while the spatial domain is treated using a cubic spline formulation enhanced by an appropriate fitting factor. The authors also establish the scheme's stability and derive its corresponding error bounds.

[Daba et al. \(2024\)](#) developed a fitted collocation scheme based on third-degree B -splines for solving TP-SPPPs incorporating a time delay, as given in equations (2.5-2.6). The method integrates the Crank-Nicolson technique for temporal discretization with a cubic B -spline approach for spatial approximation. The numerical experiments conducted demonstrate that the proposed method achieves higher accuracy compared to several existing schemes in the literature. Furthermore, a detailed parameter-uniform convergence analysis confirms that the method is uniformly convergent with respect to ε and μ , thereby establishing its (ε, μ) -uniform convergence.

2.4 Two-Parameter Singular Perturbations with Discontinuous Data

Since TP-SPPPs are relevant to heat transport, chemical processes, fluid movement, and biological systems, they are extensively researched. Sharp border or inner layers are created by the presence of several tiny parameters, and complicated and realistic models are further enhanced by the addition of discontinuous convection coefficients and source terms, which are typical in heterogeneous or piecewise media.

[Roy & Jha \(2023\)](#) proposed method tailored for two-parameter singularly perturbed BVPs involving non-smooth data. The study focused on equations featuring discon-

tinuities in both the source and convection parameters, expressed as:

$$\begin{aligned}\mathbb{L}w(x) &\equiv (\varepsilon w'' + \mu aw' - bw = f)(x), x \in (\Omega^- \cup \Omega^+), \\ a(x) &\leq -\alpha_1 < 0, \quad \text{for } x \in \Omega^-, a(x) \geq \alpha_2 > 0, \quad \text{for } x \in \Omega^+, \\ |[a](d)| &\leq C, \quad |[f](d)| \leq C,\end{aligned}$$

where $0 < \varepsilon, \mu \ll 1, \alpha_1, \alpha_2 \in \mathbb{R}$, and $\Omega^- = (0, d), \Omega^+ = (d, 1), d \in \Omega = (0, 1)$. The source term and the convection coefficient are discontinuous at the point d . Their approach utilizes Shishkin-Bakhvalov type mesh combined with an upwind FDS to accurately capture sharp boundary and interior layers. The developed numerical scheme demonstrates uniform convergence concerning both perturbation parameters and attains first-order precision, proving its robustness and reliability in tackling problems involving discontinuities and steep solution gradients.

Chandru et al. (2017) developed a numerical method for solving TP-SPPPs with non-smooth data. The study focuses on a class of IBVPs defined on the domain $\bar{\Gamma} = \bar{\Omega}_x \times \Omega_t$, incorporating both reaction-diffusion and convection-diffusion effects. The governing equation is given by:

$$\mathbb{L}_{\varepsilon, \mu} w(x, t) \equiv (\varepsilon w_{xx} + \mu aw_x - bw - cw_t = f)(x, t), (x, t) \in (\Gamma^- \cup \Gamma^+), \quad (2.7)$$

with the following boundary and initial constraints:

$$\begin{aligned}w(x, t) &= p(x, t), (x, t) \in \Gamma_c, \\ w(x, t) &= q(x, t), (x, t) \in \Gamma_l, \quad w(x, t) = r(x, t), (x, t) \in \Gamma_r\end{aligned} \quad (2.8)$$

where, ε and μ are singular perturbation parameters satisfying $0 < \varepsilon \ll 1$ and $0 \leq \mu \leq 1$. The convection coefficient and the source term are assumed to be smooth within the subdomains $(\Gamma^- \cup \Gamma^+)$, with $a(x, t)$ satisfying the sign conditions: $a(x, t) \leq -\alpha_1 < 0, (x, t) \in \Gamma^-$ and $a(x, t) \geq \alpha_2 > 0, (x, t) \in \Gamma^+$, where, $\alpha = \min(\alpha_1, \alpha_2)$, and α_1, α_2 are positive constants. Furthermore, the authors accounted for the discontinuities in $a(x, t)$ and $f(x, t)$ at an interior point (d, t) , with the jumps satisfying: $|[a](d, t)| \leq C, |[f](d, t)| \leq C$, where the jump of ω at (d, t) is defined by $[\omega](d, t) = \omega(d+, t) - \omega(d-, t)$. The presence of these discontinuities

introduces interior layers near $x = d$, in addition to boundary layers caused by the small perturbation parameters. The proposed numerical method ensuring accurate resolution despite the non-smooth nature of the data.

[Chandru et al. \(2018\)](#) examined a parabolic convection-diffusion-reaction problem in which the diffusion and convection terms are governed by two distinct small perturbation parameters, as defined in Eqs. (2.7-2.8). The study further addressed cases where the convection coefficient and source term exhibit jump discontinuities within the domain. These discontinuities, combined with the small parameters, lead to the emergence of boundary and interior layers, posing significant numerical challenges. To address this, the authors proposed a parameter-uniform scheme that achieves nearly first-order accuracy in both spatial and temporal directions. Their method employs a Shishkin-type piecewise uniform spatial mesh along with a uniform time mesh, efficiently resolving the layer behavior.

In a later study, [Chandru et al. \(2019\)](#) extended this approach to the same class of problems, utilizing a backward Euler scheme for time discretization with uniform time steps and an upwind FDM on a pre-defined piecewise uniform mesh refined near anticipated layer regions. Both theoretical analysis and numerical experiments verified that the scheme maintains first-order temporal accuracy and nearly first-order spatial accuracy, while preserving parameter-uniform convergence consistent with their previous findings.

[Kumar & Kumari \(2021\)](#) introduced a numerical technique for SPPs characterized by small diffusion and convection parameters (ε, μ) , where both the convection and source terms contain jump discontinuities within the domain, as formulated in Eqs. (2.7-2.8). The method applies an implicit discretization framework on a predefined S-mesh. Owing to the existence of twin boundary layers and an interior layer, the solution exhibits a distinct structure compared to cases with smooth data, necessitating a specialized analytical treatment. A detailed convergence analysis confirmed that the scheme maintains parameter-uniform convergence, delivering second-order accuracy in time and nearly first-order accuracy in space.

[Singh et al. \(2023\)](#) recently investigated numerical approaches for problem (2.7-2.8) and developed an efficient spline-based technique for TP-SPPs with discontinuous

convection and source terms. The method employs the Crank-Nicolson scheme for temporal discretization and trigonometric B -spline basis functions for spatial approximation. Initially, it achieved nearly first-order spatial and second-order temporal accuracy. To further improve precision, Richardson extrapolation was applied, successfully enhancing the spatial convergence from first to second order. The approach was validated through two numerical experiments, which captured the expected layer structures and confirmed the theoretical convergence rates, clearly showing the effectiveness of the Richardson extrapolation in improving spatial accuracy.

In summary, extensive efforts have focused on designing uniformly convergent numerical schemes for TP-SPPPs in both steady-state and transient forms, owing to the absence of closed form analytical solutions. Most studies employ uniform temporal discretization using classical schemes such as backward or forward Euler and Crank-Nicolson, while spatial discretization emphasizes accurate resolution of boundary and interior layers through operator or mesh-fitting strategies, particularly on Shishkin-type meshes. Further developments have extended these techniques to address discontinuities and degeneracy in convection coefficients and source terms. The reviewed works also reveal that factors such as the convection coefficient's sign, perturbation magnitudes, and delay effects significantly shape the layer behavior. Although notable progress has been achieved, existing methods remain open to improvement, leaving scope for continued refinement and innovation in this research area.

Despite significant progress, time-dependent TP-SPPPs remain far less explored than their steady-state counterparts. Even with existing approaches, the accuracy obtained is often limited, and only a few studies report higher-order convergence. Although recent works mark important strides toward robust numerical schemes, achieving uniform convergence in the simultaneous presence of two perturbation parameters, particularly under non-smooth or discontinuous data, continues to be a major challenge. This research seeks to address that gap by developing stable and accurate schemes that ensure parameter-uniform convergence, thereby enhancing the reliability and applicability of numerical solutions to a broader class of such complex problems.

Chapter 3

An Exponentially Fitted Gaussian Quadrature Method for Two-Parameter Singularly Perturbed Parabolic Problems

This chapter focuses on the numerical analysis of a time-dependent two-parameter singularly perturbed IBVP Eq. (3.1), adopting a uniform mesh strategy that differs from the approaches presented in [Ganesh Kumar & Phaneendra \(2020\)](#); [Mekonnen & Duressa \(2020\)](#). The proposed numerical method is constructed through a two-stage procedure involving semi-temporal discretization followed by spatial discretization. Initially, the time variable is discretized using the Crank-Nicolson scheme. Subsequently, the resulting semi-discrete problems are further discretized in space by applying the *two-point Gaussian quadrature rule*, coupled with finite difference approximation for the first derivatives. To handle sharp solution variations, the fitting factor is derived based on singular perturbation theory. Numerical experiments confirm that the developed scheme exhibits uniform convergence and delivers improved accuracy compared to several existing techniques in the literature.

3.1 Introduction

Two-parameter singularly perturbed parabolic problems arise in models where the interaction of two small parameters generates multiple boundary layers and intricate multi-scale behavior. Compared to single parameter SPPs, these problems have received far less numerical attention. Although [O'malley \(1967\)](#) initiated asymptotic studies in the mid 1960s, the development of uniformly convergent numerical methods progressed slowly for several decades. The combined influence of two perturbation parameters leads to sharper gradients and complex layer interactions, making the construction of reliable discretization techniques significantly more challenging. Notable advances were achieved with the works of [Vulanović \(2001\)](#); [O'Riordan et al. \(2003\)](#); [Roos & Uzelac \(2003\)](#); [Chandru et al. \(2017\)](#); [Andisso & Duressa \(2024\)](#), which established important methodological foundations for TP-SPPPs.

Recent studies have shown that the two-point Gaussian quadrature is effective for problems involving sharp layer structures. While conventional integration techniques deteriorate in the presence of steep gradients, the two-point Gaussian quadrature integrates polynomials of degree up to three with high precision using only two nodes, making it well suited for weighted residual formulations, finite element methods(FEMs), and fitted finite difference schemes. Its versatility has been demonstrated in [Phaneendra & Lalu \(2019a\)](#) for SPODEs, in [Phaneendra & Lalu \(2019b\)](#) for delay DEs, and in [Tefera et al. \(2022\)](#) for parabolic convection-diffusion problems.

Motivated by these developments, this chapter formulates a uniform numerical scheme for time-dependent TP-SPPPs. The method employs the Crank–Nicolson scheme for time discretization to ensure temporal stability and second-order accuracy, and a fitted two-point Gaussian quadrature with an exponential fitting factor for spatial discretization on a uniform mesh. The fitted quadrature captures the intrinsic layer behavior within the spatial operator, enabling accurate approximation of both smooth and rapidly varying solution components. A rigorous stability and convergence analysis confirms that the resulting scheme achieves uniform convergence with respect to both perturbation parameters.

The chapter is organized as follows: Section [3.2](#) presents the mathematical formulation

of Eq. (3.1), while Section 3.3 discusses the fundamental properties of its continuous solution. The proposed numerical scheme is developed in Section 3.4, with its convergence analysis detailed in Section 3.5. Section 3.6 provides numerical experiments that illustrate the methods accuracy and effectiveness. The chapter concludes in Section 3.7 with a summary of the key findings.

3.2 Definition of the Problem

Consider a 1D, time-dependent parabolic convection-diffusion problems involving two perturbation parameters. The problem is defined over the domain $\mathbb{G} = \mathcal{Q} \times \Lambda$, where $\mathcal{Q} = (0, 1)$ and $\Lambda = (0, T]$. The boundary of the domain, $\Gamma = \Gamma_b \cup \Gamma_l \cup \Gamma_r$, consists of three parts: the left boundary $\Gamma_l = \{(x, t) : x = 0, t \in \overline{\Lambda}\}$, the bottom boundary $\Gamma_b = \{(x, t) : t = 0, x \in \overline{\mathcal{Q}}\}$, and the right boundary $\Gamma_r = \{(x, t) : x = 1, t \in \overline{\Lambda}\}$. Within this domain, we study the following problem:

$$\begin{cases} \mathbb{L}_{\varepsilon, \mu} w \equiv \left(\frac{\partial w}{\partial t} - \varepsilon \frac{\partial^2 w}{\partial x^2} - \mu a \frac{\partial w}{\partial x} + bw = \varpi \right) (x, t), & (x, t) \in \mathbb{G}, \\ w|_{\Gamma_b} = \zeta_b(x), & x \in \overline{\mathcal{Q}}, w|_{\Gamma_l} = \zeta_l(t), & t \in \overline{\Lambda}, \\ w|_{\Gamma_r} = \zeta_r(t), & t \in \overline{\Lambda}, \end{cases} \quad (3.1)$$

where $0 < \varepsilon \ll 1$ and $0 \leq \mu \leq 1$ are perturbation parameters that control the strenght of diffusion and convection effects, respectively. To guarantee the existence and uniqueness of the solution of Eq. (3.1), we impose the following assumptions:

- All functions involved in Eq. (3.1) are assumed to be sufficiently smooth, bounded, and possess continuous up to second derivatives.
- The coefficients satisfy the bounds $0 < \alpha \leq a(x, t)$, $0 < \varrho \leq b(x, t)$, $\forall (x, t) \in \overline{\mathbb{G}}$.
- Compatibility conditions are fulfilled at all corner points of the domain.

3.3 Properties of the Continuous Solution

This section examines key analytical properties of Eq. (3.1) defined over the 1D spatial domain $\overline{\mathbb{G}}$. The differential operator $\mathbb{L}_{\varepsilon, \mu}$ in Eq. (3.1) satisfies the following maximum principle and stability estimates.

Lemma 3.1. (*Continuous Maximum Principle*). Let $\Xi(x, t) \in C^2(\mathbb{G}) \cap C^0(\overline{\mathbb{G}})$. If $\Xi(x, t) \geq 0, \forall (x, t) \in \Gamma$ and $\mathbb{L}_{\varepsilon, \mu} \Xi(x, t) \geq 0, \forall (x, t) \in \mathbb{G}$ then, $\Xi(x, t) \geq 0, \forall (x, t) \in \overline{\mathbb{G}}$.

Proof. We can prove this lemma by contradiction. Assume there exists an arbitrary point (x^*, t^*) in the plane $\mathbb{G} = (0, 1) \times (0, T)$ such that $\Xi(x^*, t^*) = \min_{(x, t) \in \overline{\mathbb{G}}} \Xi(x, t)$, and suppose $\Xi(x^*, t^*) < 0$. By the definition of (x^*, t^*) , it is clear that $(x^*, t^*) \notin \{(0, 1)\} \times \{(0, T)\}$, which implies $(x^*, t^*) \in \mathbb{G}$. Applying the first and second derivative tests for multivariable functions in calculus, we have $\Xi_x(x^*, t^*) = 0, \Xi_t(x^*, t^*) = 0$, and $\Xi_{xx}(x^*, t^*) \geq 0$. Using the differential operator $\mathbb{L}_{\varepsilon, \mu}$ on $\Xi(x^*, t^*)$, we obtain

$$\begin{aligned} \mathbb{L}_{\varepsilon, \mu} \Xi(x^*, t^*) &= \frac{\partial \Xi(x^*, t^*)}{\partial t} - \varepsilon \frac{\partial^2 \Xi(x^*, t^*)}{\partial x^2} - \mu a(x^*, t^*) \frac{\partial \Xi(x^*, t^*)}{\partial x} + b(x^*, t^*) \Xi(x^*, t^*), \\ &= \varepsilon \frac{\partial^2 \Xi(x^*, t^*)}{\partial x^2} + b(x^*, t^*) \Xi(x^*, t^*) < 0, \end{aligned}$$

which leads to contradiction as $\mathbb{L}_{\varepsilon, \mu} \Xi(x, t) \geq 0$. Hence, $\Xi(x, t) \geq 0, \forall (x, t) \in \overline{\mathbb{G}}$. $\square$

Lemma 3.2. (*Continuous Stability Estimate*). The solution $w(x, t)$ of Eq. (3.1) is bounded and satisfies the estimate below:

$$\|w\| \leq |\varrho^{-1}| \|\varpi\| + \max \{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\}, \forall (x, t) \in \overline{\mathbb{G}}, \quad (3.2)$$

where $\|\cdot\|$ is the L_∞ norm given by $\|w\| = \max_{(x, t) \in \overline{\mathbb{G}}} |w(x, t)|$ and $0 < \varrho \leq b(x, t)$.

Proof. Let $\Xi^\pm(x, t)$ be two barrier functions defined by

$$\Xi^\pm(x, t) = |\varrho^{-1}| \|\varpi\| + \max \{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \pm w(x, t).$$

The barrier functions $\Xi^\pm(x, t)$ on the initial value is

$$\Xi^\pm(x, 0) = |\varrho^{-1}| \|\varpi\| + \max \{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \pm w(x, 0) \geq 0,$$

The barrier functions $\Xi^\pm(x, t)$ at the two end points are

$$\Xi^\pm(0, t) = |\varrho^{-1}| \|\varpi\| + \max \{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \pm w(0, t) \geq 0,$$

$$\Xi^\pm(1, t) = |\varrho^{-1}| \|\varpi\| + \max \{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \pm w(1, t) \geq 0,$$

When we consider the differential operator $\mathbb{L}_{\varepsilon,\mu}$, on $\overline{\mathbb{G}}$, we have

$$\begin{aligned}
\mathbb{L}_{\varepsilon,\mu}\Xi^\pm(x,t) &= \frac{\partial\Xi^\pm(x,t)}{\partial t} - \varepsilon\frac{\partial^2\Xi^\pm(x,t)}{\partial x^2} - \mu a(x,t)\frac{\partial\Xi^\pm(x,t)}{\partial x} + b(x,t)\Xi^\pm(x,t), \\
&= \frac{\partial}{\partial t} \left[|\varrho^{-1}| \|\varpi\| + \max\{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \pm w(x,t) \right] - \\
&\quad \varepsilon \frac{\partial^2}{\partial x^2} \left[|\varrho^{-1}| \|\varpi\| + \{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \pm w(x,t) \right] - \\
&\quad \mu a(x,t) \frac{\partial}{\partial x} \left[|\varrho^{-1}| \|\varpi\| + \max\{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \pm w(x,t) \right] \\
&\quad + b(x,t) \left[|\varrho^{-1}| \|\varpi\| + \max\{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \pm w(x,t) \right], \\
&= b(x,t) \left(|\varrho^{-1}| \|\varpi\| + \max\{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \pm w(x,t) \right) \pm \mathbb{L}_{\varepsilon,\mu}w(x,t), \\
&= b(x,t) \left(|\varrho^{-1}| \|\varpi\| + \max\{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\} \right) \pm \varpi(x,t), \\
&= b(x,t) \left((\max\{|\zeta_b(x)|, |\zeta_l(t)|, |\zeta_r(t)|\}) + |\varrho^{-1}| \|\varpi\| \right) \pm \varpi(x,t).
\end{aligned}$$

Using the definition of the continuous operator $\mathbb{L}_{\varepsilon,\mu}$ and the equality $b(x,t) \geq \varrho > 0$ implies $b(x,t)\varrho^{-1} \geq 1$, and since $\|\varpi\| \geq \varpi(x,t)$, we have $\mathbb{L}_{\varepsilon,\mu}\Xi^\pm(x,t) \geq 0$, $\forall (x,t) \in \overline{\mathbb{G}}$. Hence, the maximum principle given by Lemma 3.1 confirms $\Xi^\pm(x,t) \geq 0$, $\forall (x,t) \in \overline{\mathbb{G}}$, which yields the desired result. $\square$

Lemma 3.1 implies the uniqueness of the solution. Indeed, the difference of any two solutions satisfies the associated homogeneous problem with zero boundary and initial conditions, and hence vanishes identically by the maximum principle. The existence of a solution follows from the well-established theory of linear parabolic problems. Furthermore, Lemma 3.2 guarantees that the solution is bounded, ensuring stability. Consequently, the problem is well-posed.

Lemma 3.3. *The derivative of the solution w of Eq. (3.1) satisfy the bounds for all nonnegative integers k, m , such that $1 \leq k + 2m \leq 3$.*

1. If $\mu^2 \leq C\varepsilon$ (diffusion-dominated or not strong convective), then

$$\begin{aligned}
\|\partial_x^k \partial_t^m w\|_{L_\infty} &\leq \frac{C}{(\sqrt{\varepsilon})^k} \\
&\max \left\{ \|w\|_{L_\infty}, \sum_{i+2j=0}^2 (\sqrt{\varepsilon})^i \|\partial_x^i \partial_t^j \varpi\|_{L_\infty}, \sum_{i=0}^4 \left\| \frac{d^i \zeta_b}{dx^i} \right\|_{\Gamma_b} + \left\| \frac{d^i \zeta}{dt^i} \right\|_{\Gamma_l \cup \Gamma_r} \right\}.
\end{aligned}$$

2. If $\mu^2 \geq C\varepsilon$ (convection-dominated), then

$$\begin{aligned} \|\partial_x^k \partial_t^m w\|_{L_\infty} &\leq C \left(\frac{\mu}{\varepsilon}\right)^k \left(\frac{\mu^2}{\varepsilon}\right)^m \times \\ \max &\left\{ \|w\|_{L_\infty}, \sum_{i+2j=0}^2 \left(\frac{\varepsilon}{\mu}\right)^i \left(\frac{\varepsilon}{\mu^2}\right)^{j+1} \|\partial_x^i \partial_t^j \varpi\|_{L_\infty}, \sum_{i=0}^4 \left\| \frac{d^i \zeta_b}{dx^i} \right\|_{\Gamma_b} + \left\| \frac{d^i \zeta}{dt^i} \right\|_{\Gamma_l \cup \Gamma_r} \right\}, \end{aligned}$$

where C depends only on the coefficients a, b and their derivatives.

Proof. A proof for these bounds can be established using a conventional approach (refer to [Miller et al. \(1998\)](#) for details). To begin, we apply an appropriate variable stretching to reformulate the original problem into a more manageable form. Once the problem is expressed in the new variables, we utilize the local estimates provided in [Ladyzhenskaia et al. \(1968\)](#) to derive precise bounds for the transformed solution. $\square$

Lemma 3.4. *Let ι, m be a non-negative integers such that $0 \leq \iota + 2m \leq 4$. Then, the solution $w(x, t)$ of Eq. (3.1) and its partial derivatives satisfy the bound*

$$\left\| \frac{\partial^{\iota+m} w}{\partial x^\iota \partial t^m} \right\| \leq \begin{cases} C \varepsilon^{-\frac{\iota}{2}}, & \text{if } \mu^2 \leq \frac{\varepsilon \lambda}{\alpha}, \\ C \left(\frac{\mu}{\varepsilon}\right)^\iota \left(\frac{\mu^2}{\varepsilon}\right)^m, & \text{if } \mu^2 \geq \frac{\varepsilon \lambda}{\alpha}, \end{cases} \quad (3.3)$$

where $\lambda \approx \min_{(x,t) \in \bar{\Delta}} \frac{b(x,t)}{a(x,t)}$, and C is a positive constant independent of ε and μ .

Proof. The characteristics of $w(x, t)$ are influenced by the relative magnitudes of the perturbation parameters ε and μ . Accordingly, we examine two distinct scenarios:

1. $\mu^2 \leq \frac{\varepsilon \lambda}{\alpha}$: In this scenario, the diffusion effects are predominant over the convective contributions. By applying classical maximum principle arguments for parabolic equations with weak diffusion ([Ladyzhenskaia et al., 1968](#)), one can establish that the spatial derivatives exhibit the following scaling behavior:

$$\left\| \frac{\partial^\iota w}{\partial x^\iota} \right\| \leq C \varepsilon^{-\iota/2}, \quad 0 \leq \iota \leq 2. \quad (3.4)$$

Estimates for the time derivatives can be obtained from the structure of Eq. (3.1)

$$\frac{\partial w}{\partial t} = \varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a \frac{\partial w}{\partial x} - bw + \varpi(x, t),$$

and since $\mu^2 \ll \varepsilon$, we obtain

$$\left\| \frac{\partial^m w}{\partial x^m} \right\| \leq C, \quad m = 0, 1, 2. \quad (3.5)$$

Applying mathematical induction to the mixed-derivatives and synthesizing the results from Eq. (3.4) and Eq. (3.5), we deduce that

$$\left\| \frac{\partial^{\iota+m} w}{\partial x^\iota \partial t^m} \right\| \leq C \mu^{-\iota/2}. \quad (3.6)$$

2. $\mu^2 \geq \frac{\varepsilon \lambda}{\alpha}$: In this case, the convection term becomes significant relative to diffusion. Here, we apply a scaling argument of the form $\xi = \frac{\mu x}{\varepsilon}$ and $\tau = \frac{\mu^2 t}{\varepsilon}$ to balance convection and diffusion. Under this transformation, Eq. (3.1) is rewritten in a form where the derivatives are order-one with respect to ξ and τ . Transforming back to the original variables gives:

$$\frac{\partial^\iota}{\partial x^\iota} = \left(\frac{\mu}{\varepsilon} \right)^\iota \frac{\partial^\iota}{\partial \xi^\iota}, \quad \frac{\partial^m}{\partial t^m} = \left(\frac{\mu^2}{\varepsilon} \right)^m \frac{\partial^m}{\partial \tau^m}.$$

It follows that the mixed-derivatives obey the relation:

$$\left\| \frac{\partial^{\iota+m} w}{\partial x^\iota \partial t^m} \right\| \leq C \left(\frac{\mu}{\varepsilon} \right)^\iota \left(\frac{\mu^2}{\varepsilon} \right)^m. \quad (3.7)$$

The constant C appearing in Eqs. (3.6) and (3.7) does not depend on ε or μ but is determined entirely by the smoothness and upper bounds of the coefficient function $a(x, t)$, $b(x, t)$, and $\varpi(x, t)$. Integrating the results from both cases, that is Eqs. (3.6) and (3.7), we establish the piecewise bound:

$$\left\| \frac{\partial^{\iota+m} w}{\partial x^\iota \partial t^m} \right\| \leq \begin{cases} C \varepsilon^{-\frac{\iota}{2}}, & \text{if } \mu^2 \leq \frac{\varepsilon \lambda}{\alpha}, \\ C \left(\frac{\mu}{\varepsilon} \right)^\iota \left(\frac{\mu^2}{\varepsilon} \right)^m, & \text{if } \mu^2 \geq \frac{\varepsilon \lambda}{\alpha}, \end{cases}$$

which proves the Lemma. □

3.4 Formulation of the Numerical Scheme

This section outlines a numerical scheme developed to solve Eq. (3.1). The approach employs the *Crank-Nicolson method* to discretize the time derivative and *two-point Gaussian quadrature* is applied over a uniformly spaced grid for spatial derivatives, enabling accurate and efficient numerical integration. Together, these techniques provide a stable and precise framework for approximating the solution of the given problem.

3.4.1 Temporal Semi-discretization

The continuous problem (3.1) is semi-discretized in time over $[0, T]$ by employing the Crank-Nicolson scheme on a uniform temporal grid. The interval is divided into M equal subintervals, leading to a constant time step size $k = T/M$. The corresponding time mesh is defined as $\Lambda_t^M = \{t_j = jk : j = 0, 1, \dots, M, t_0 = 0, t_M = T\}$, where M is the total number of mesh elements along time axis. To construct the Crank-Nicolson scheme, the Taylor expansion of $w(x, t)$ about midpoint $(x, t_{j+\frac{1}{2}})$ is used:

$$w^{j+1}(x) = w^{j+\frac{1}{2}}(x) + \frac{k}{2} \frac{\partial w^{j+\frac{1}{2}}(x)}{\partial t} + \frac{k^2}{8} \frac{\partial^2 w^{j+\frac{1}{2}}(x)}{\partial t^2} + \frac{k^3}{48} \frac{\partial^3 w^{j+\frac{1}{2}}(x)}{\partial t^3} + \dots, \quad (3.8)$$

$$w^j(x) = w^{j+\frac{1}{2}}(x) - \frac{k}{2} \frac{\partial w^{j+\frac{1}{2}}(x)}{\partial t} + \frac{k^2}{8} \frac{\partial^2 w^{j+\frac{1}{2}}(x)}{\partial t^2} - \frac{k^3}{48} \frac{\partial^3 w^{j+\frac{1}{2}}(x)}{\partial t^3} + \dots. \quad (3.9)$$

Subtracting Eq. (3.9) from Eq. (3.8) eliminates the midpoint term $w^{j+\frac{1}{2}}(x)$, yielding

$$\frac{\partial w^{j+\frac{1}{2}}(x)}{\partial t} = \frac{w^{j+1}(x) - w^j(x)}{k} + TE^{j+\frac{1}{2}}(x), \quad (3.10)$$

where $TE^{j+\frac{1}{2}}(x)$ denotes the local truncation error (LTE), expressed as

$$TE^{j+\frac{1}{2}}(x) = \frac{k^3}{24} \frac{\partial^3 w^{j+\frac{1}{2}}(x)}{\partial t^3} + \text{higher-order terms}, \quad (3.11)$$

which is of order $\mathcal{O}(k^3)$. The semi-discrete form of Eq. (3.1) with time discretized and space remaining continuous takes the following form:

$$\begin{aligned} & -\varepsilon (w_{xx}(x))^{j+1} - \mu a^{j+1}(x) (w_x(x))^{j+1} + \left(b^{j+1}(x) + \frac{2}{k}\right) w^{j+1}(x) = \\ & \varepsilon (w_{xx}(x))^j + \mu a^j(x) (w_x(x))^j - \left(b^j(x) - \frac{2}{k}\right) w^j(x) + \varpi^{j+1}(x) + \varpi^j(x). \end{aligned} \quad (3.12)$$

At each time level $(j + 1)$, the semi-discrete approximation $w(x, t_{j+1})$ satisfies the following BVP in the spatial variable x :

$$\left\{ \begin{array}{l} \mathbb{L}_{\varepsilon, \mu}^M W^{j+1}(x) \equiv -\varepsilon (W_{xx}(x))^{j+1} - \mu a^{j+1}(x) (W_x(x))^{j+1} + \\ \quad \left(b^{j+1}(x) + \frac{2}{k}\right) W^{j+1}(x) = \mathcal{H}^j(x), \\ W^{j+1}(0) = \zeta_l(t_{j+1}), \quad W^{j+1}(1) = \zeta_r(t_{j+1}), \quad 0 \leq j \leq M, \\ W^0(x) = \zeta_b(x), \quad x \in (0, 1), \end{array} \right. \quad (3.13)$$

where

$$\mathcal{H}^j(x) = \varepsilon (W_{xx}(x))^j + \mu a^j(x) (W_x(x))^j - \left(b^j(x) - \frac{2}{k}\right) W^j(x) + \varpi^{j+1}(x) + \varpi^j(x).$$

Here, $W(x, t_{j+1})$ denotes the semi-discrete approximation to the exact solution $w(x, t_{j+1})$ of Eq. (3.1). To verify the uniform convergence of the proposed numerical method, we first analyze the semi-discrete problem (3.13). This analysis includes deriving the consistency estimates and establishing the stability properties, which together form the basis for the subsequent convergence proofs. The main analytical results are summarized in the following lemmas.

Lemma 3.5. (*Semi-discrete Maximum Principle*). *At the $(j + 1)$ th time level, let $\Psi(x, t_{j+1}) \in C^2(\overline{\mathcal{Q}})$ be a smooth function such that $\Psi(0, t_{j+1}) \geq 0$ and $\Psi(1, t_{j+1}) \geq 0$. If $\mathbb{L}_{\varepsilon, \mu}^M \Psi(x, t_{j+1}) \geq 0, \forall x \in \mathbb{G}$, then $\Psi(x, t_{j+1}) \geq 0, \forall x \in \overline{\mathcal{Q}}$.*

Proof. Let $(\chi^*, t_{j+1}) \in \{(x, t_{j+1}) : x \in \overline{\mathcal{Q}}\}$ be a point where

$$\Psi(\chi^*, t_{j+1}) = \min_{x \in \overline{\mathcal{Q}}} \Psi(x, t_{j+1}).$$

Assume, for contradiction, that $\Psi(\chi^*, t_{j+1}) < 0$. Then $(\chi^*, t_{j+1}) \notin \{(0, t_{j+1}), (1, t_{j+1})\}$,

implying $\chi^* \in \mathcal{Q}$. At this interior point, the necessary conditions hold:

$$\Psi_x(\chi^*, t_{j+1}) = 0, \quad \Psi_{xx}(\chi^*, t_{j+1}) \geq 0.$$

Applying the operator $\mathbb{L}_{\varepsilon, \mu}^M$ to $\Psi(\chi^*, t_{j+1})$ gives

$$\begin{aligned} \mathbb{L}_{\varepsilon, \mu}^M \Psi(\chi^*, t_{j+1}) &= -\varepsilon \Psi_{xx}(\chi^*, t_{j+1}) - \mu a(\chi^*, t_{j+1}) \Psi_x(\chi^*, t_{j+1}) \\ &\quad + \left(b(\chi^*, t_{j+1}) + \frac{2}{k} \right) \Psi(\chi^*, t_{j+1}) < 0, \end{aligned}$$

which contradicts the assumption $\mathbb{L}_{\varepsilon, \mu}^M \Psi(x, t_{j+1}) \geq 0$. Hence, the assumption is false, and it follows that $\Psi(\chi^*, t_{j+1}) \geq 0$. Therefore, $\Psi(x, t_{j+1}) \geq 0$ for all $x \in \overline{\mathcal{Q}}$. $\square$

To analyze the uniform convergence of the semi-discrete scheme defined in Eq. (3.13), we first construct *local truncation error* (LTE), which is $e_{j+1} = w(x, t_{j+1}) - W(x, t_{j+1})$, where $w(x, t_{j+1})$ represents the exact solution and $W(x, t_{j+1})$ denotes the semi-discrete numerical approximation. The function $W(x, t_{j+1})$ satisfies the discrete BVP:

$$\begin{cases} -\varepsilon W_{xx}(x, t_{j+1}) - \mu a(x, t_{j+1}) W_x(x, t_{j+1}) + B(x) W(x, t_{j+1}) = \mathcal{H}(x, t_j), \\ W(0, t_{j+1}) = \zeta_l(t_{j+1}), \quad W(1, t_{j+1}) = \zeta_r(t_{j+1}), \quad 0 \leq j \leq M, \\ W^0(x) = \zeta_b(x), \quad x \in (0, 1), \end{cases} \quad (3.14)$$

where $B(x) = b(x, t_{j+1}) + \frac{2}{k}$, and $\mathcal{H}(x, t_j) = \varepsilon W_{xx}(x, t_j) + \mu a(x, t_j) W_x(x, t_j) - B(x) W(x, t_j) + \varpi(x, t_{j+1}) + \varpi(x, t_j)$. Based on the preceding lemmas, the consistency and stability of the Crank-Nicolson time discretization can be established, both of which determine the scheme's temporal convergence behavior.

Lemma 3.6. *If the solution $w(x, t_{j+1})$ of Eq. (3.1) and its time derivatives are uniformly bounded for $(x, t) \in [0, 1] \times [0, T]$, independent of ε, μ, N , and M , then the LTE of Eq. (3.14) satisfies*

$$\|e_{j+1}\|_{\infty} \leq Ck^3.$$

Proof. The proof directly follows from Eqs. (3.10) and (3.11). $\square$

Lemma 3.7. (*Global Error Estimate*). *The global error estimate $\mathcal{E}_j$ at the (j) th time*

level is bounded by

$$\|\mathcal{E}_j\|_\infty \leq C(k^2), \forall j \leq T/k.$$

Proof. By summing the local error bounds derived in Lemma 3.6 over all time levels up to the (j) th step, and applying the stability properties of the semi-discretization scheme, we obtain the following global error estimate at (j) th time level.

$$\begin{aligned} \|\mathcal{E}_j\|_\infty &= \left\| \sum_{i=1}^j e_i \right\|_\infty, \quad j \leq \frac{T}{k}, \\ &= \|e_1\|_\infty + \|e_2\|_\infty + \dots + \|e_{j-1}\|_\infty + \|e_j\|_\infty, \\ &\leq C_1(k^3) + C_2(k^3) + C_3(k^3) + \dots + C_j(k^3), \\ &\leq C^*(j)(k^3), \quad \text{by Lemma 3.6,} \\ &\leq C^*(jk)(k^2) \leq C^*T(k^2) \leq C(k^2), \quad \text{where } C = C^*T. \end{aligned}$$

Here, C denotes a positive constant independent of the perturbation parameters and the time step k . Hence, the temporal semi-discretization exhibits uniform second-order accuracy, remaining stable and precise for all parameter values. $\square$

Lemma 3.8. *Let $W^j(x)$ denote the semi-discrete solution of Eq. (3.14). For derivatives up to order q , determined by the smoothness of $W^j(x)$, the following bound holds:*

$$\left| \frac{d^\beta W^j(x)}{dx^\beta} \right| \leq C \left(1 + \vartheta_1^\beta e^{-p\vartheta_1 x} + \vartheta_2^\beta e^{-p\vartheta_2(1-x)} \right), \quad 0 \leq \beta \leq q, \quad (3.15)$$

where $C > 0$ is a constant and p is any real number satisfying $0 < p < 1$.

Proof. To prove the bound for the derivatives of $W^j(x)$, we use the smoothness properties of the semi-discrete solution and exponential decay estimates for the coefficients ϑ_1 and ϑ_2 . By differentiating Eq. (3.15) β times with respect to x , we obtain a system of equations involving the derivatives of $W^j(x)$ up to order β . Using the exponential decay estimates for the coefficients ϑ_1 and ϑ_2 , there exist positive constants r, s , and p ($0 < p < 1$) such that $\vartheta_1 \leq r e^{-p\vartheta_1 x}$, $\vartheta_2 \leq s e^{-p\vartheta_2(1-x)}$, for all $x \in [0, 1]$. These estimates indicate that ϑ_1 and ϑ_2 decay exponentially as x approaches the boundaries of the interval $[0, 1]$. The constants r and s control the rate of decay, while p determines the speed of exponential decay. We can estimate each term in the system of

equations. Specifically, we have:

$$\begin{aligned} \left| \frac{d^\beta W^j(x)}{dx^\beta} \right| &\leq C \left(|1| + \left| \vartheta_1^\beta e^{-p\vartheta_1 x} \right| + \left| \vartheta_2^\beta e^{-p\vartheta_2(1-x)} \right| \right), \\ &= C \left(1 + \vartheta_1^\beta e^{-p\vartheta_1 x} + \vartheta_2^\beta e^{-p\vartheta_2(1-x)} \right). \end{aligned}$$

Therefore, for each derivative order β with $0 \leq \beta \leq q$, derivative bound holds true. $\square$

To analyze the boundary layer in Eq. (3.13), consider the characteristic equation

$$\varepsilon \lambda^2(x) + \mu a^{j+1}(x) \lambda(x) - \beta^{j+1}(x) = 0, \quad (3.16)$$

The roots of Eq. (3.16) define two continuous functions $\lambda_{1,2}(x) : [0, 1] \rightarrow \mathbb{R}$, given by

$$\begin{aligned} \lambda_1(x) &= -\frac{\mu a^{j+1}(x)}{2\varepsilon} + \sqrt{\left(\frac{\mu a^{j+1}(x)}{2\varepsilon} \right)^2 + \frac{\beta^{j+1}(x)}{\varepsilon}} \geq 0, \\ \lambda_2(x) &= -\frac{\mu a^{j+1}(x)}{2\varepsilon} - \sqrt{\left(\frac{\mu a^{j+1}(x)}{2\varepsilon} \right)^2 + \frac{\beta^{j+1}(x)}{\varepsilon}} \leq 0. \end{aligned} \quad (3.17)$$

These roots characterize the exponential decay of the solution near the boundaries, corresponding to layers at $x = 0$ and $x = 1$ (see (Shivhare et al., 2021; Mekonnen & Duressa, 2020)). Define $\varrho_1 = -\max_{x \in [0,1]} \lambda_2(x)$, $\varrho_2 = \min_{x \in [0,1]} \lambda_1(x)$, then the following cases arise:

- (i) If $\frac{\mu^2}{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$, then $\varrho_1 = \varrho_2 \approx \sqrt{\frac{\beta^{j+1}(x)}{\varepsilon}}$, where $0 < \beta^* < \beta^{j+1}$.
- (ii) If $\frac{\varepsilon}{\mu^2} \rightarrow 0$ as $\mu \rightarrow 0$, then $\varrho_1 \approx \frac{\mu a^*}{\varepsilon}$ and $\varrho_2 \approx 0$, where $0 < a^* < a^{j+1}$.
- (iii) When $\varrho_2 \ll \varrho_1$, the solution exhibits a dominant boundary layer near both $x = 0$ and $x = 1$.

3.4.2 Spatial Semi-discretization

In this section, we develop a fully discrete scheme by applying the *two-point Gaussian quadrature* method in the spatial domain within the framework of fitted operator techniques. To effectively capture the boundary layer behavior, a fitting factor is employed, the derivation of which is presented in the following section. To reformulate

the second-order DE into a first-order delay form with a small deviating argument, we apply the Taylor expansion to $W_x^{j+1}(x + \varepsilon)$:

$$W_x^{j+1}(x + \varepsilon) \approx W_x^{j+1}(x) + \varepsilon W_{xx}^{j+1}(x) + O(\varepsilon^2). \quad (3.18)$$

From Eq. (3.18), we obtain

$$\varepsilon W_{xx}^{j+1}(x) = W_x^{j+1}(x + \varepsilon) - W_x^{j+1}(x) + O(\varepsilon^2). \quad (3.19)$$

Substituting Eq. (3.19) into Eq. (3.12) yields an asymptotically equivalent first-order neutral-type delay equation:

$$\begin{aligned} & -(W_x(x + \varepsilon))^{j+1} + (W_x(x))^{j+1} - \mu a^{j+1}(x) (W_x(x))^{j+1} + \\ & \left(b^{j+1}(x) + \frac{2}{k} \right) W^{j+1}(x) = (W_x(x + \varepsilon))^j - (W_x(x))^j + \\ & \mu a^j(x) (W_x(x))^j - \left(b^j(x) - \frac{2}{k} \right) W^j(x) + \varpi^{j+1}(x) + \varpi^j(x). \end{aligned} \quad (3.20)$$

Since ε is small, the neglected term $O(\varepsilon^2)$ is negligible, making the transformation from Eq. (3.12) to Eq. (3.20) asymptotically valid (see (Norkin et al., 1973)). Next, we discretize the spatial domain $[0, 1]$ as $\mathcal{Q}_x^N = \{x_i = ih \mid i = 0, 1, \dots, N; h = 1/N\}$. Integrating Eq. (3.20) over $[x_{i-1}, x_i]$, $i = 1, \dots, N + 1$, gives

$$\begin{aligned} & - \int_{x_{i-1}}^{x_i} (W_x(x + \varepsilon))^{j+1} dx + \int_{x_{i-1}}^{x_i} (W_x(x))^{j+1} dx - \mu \int_{x_{i-1}}^{x_i} a^{j+1}(x) (W_x(x))^{j+1} dx \\ & + \int_{x_{i-1}}^{x_i} \left(b^{j+1}(x) + \frac{2}{k} \right) W^{j+1}(x) dx = \int_{x_{i-1}}^{x_i} \left[(W_x(x + \varepsilon))^j - (W_x(x))^j + \right. \\ & \left. \mu a^j(x) (W_x(x))^j + \left(b^j(x) - \frac{2}{k} \right) W^j(x) + \varpi^{j+1}(x) + \varpi^j(x) \right] dx. \end{aligned} \quad (3.21)$$

Using the Gauss–Legendre quadrature, the integral in Eq. (3.21) can be approximated over the interval $[-1, 1]$. For any sufficiently smooth function $\mathcal{F}(x) \in C^4[-1, 1]$, the two-point Gaussian quadrature rule is given by (Sastry, 2012; Tefera et al., 2022)

$$\int_{-1}^1 \mathcal{F}(x) dx \approx \mathcal{F}\left(-\frac{1}{\sqrt{3}}\right) + \mathcal{F}\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{135} \mathcal{F}^{(4)}(\mathcal{Y}), \quad -1 < \mathcal{Y} < 1, \quad (3.22)$$

where the last term represents the truncation error. To extend this formula to an arbitrary subinterval $[x_{i-1}, x_i]$, we have to use the transformation: $x = \frac{x_i - x_{i-1}}{2} \mathcal{J} + \frac{x_i + x_{i-1}}{2} = \frac{x_i - x_{i-1}}{2} \mathcal{J} + \frac{x_i - x_{i-1} + x_{i-1} + x_{i-1}}{2} = x_{i-1} + \frac{h}{2} \left(1 - \frac{1}{\sqrt{3}}\right)$. Thus $x = x_{i-1} + \kappa$, where $\kappa = \frac{h}{2} \left(1 - \frac{1}{\sqrt{3}}\right)$. Similarly, $x = x_i - \kappa$. But from Gauss-quadrature formula, we have

$$\int_{x_{i-1}}^{x_i} \mathcal{F}(x) dx = \int_{-1}^1 \mathcal{F}(\mathcal{J}) d\mathcal{J} \approx \frac{h}{2} [\mathcal{F}(x_{i-1} + \kappa) + \mathcal{F}(x_i - \kappa)], \quad \kappa = \frac{h}{2} \left(1 - \frac{1}{\sqrt{3}}\right). \quad (3.23)$$

The Gauss-Legendre rule in Eq. (3.23) employs the zeros of the Legendre polynomial $P_2(x)$ as integration nodes, providing exact results for all polynomials of degree up to three. Unlike the trapezoidal rule, which fixes the evaluation points at the subinterval endpoints, the Gaussian rule selects interior nodes $\{x_{i-1} + \kappa, x_i - \kappa\}$ to achieve higher accuracy with the same number of evaluations. Hence, substituting Eq. (3.23) into Eq. (3.21) yields the corresponding discrete approximation.

$$\begin{aligned} & -W^{j+1}(x_i + \varepsilon) + W^{j+1}(x_{i-1} + \varepsilon) + W^{j+1}(x_i) - w^{j+1}(x_{i-1}) - \\ & \quad \mu [a^{j+1}(x_i)W^{j+1}(x_i) - a^{j+1}(x_{i-1})W^{j+1}(x_{i-1})] + \\ & \quad \frac{h\mu}{2} \left[a'(x_i - \kappa, t_{j+1}) W^{j+1}(x_i - \kappa) + a'(x_{i-1} + \kappa, t_{j+1}) W^{j+1}(x_{i-1} + \kappa) \right] + \\ & \quad \frac{h}{2} \left[\left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) W^{j+1}(x_i - \kappa) + \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right) W^{j+1}(x_{i-1} + \kappa) \right] \\ & \quad = W^j(x_i + \varepsilon) - W^j(x_{i-1} + \varepsilon) - W^j(x_i) + W^j(x_{i-1}) + \mu [a^j(x_i)W^j(x_i) - \\ & \quad a^j(x_{i-1})W^j(x_{i-1})] - \frac{h\mu}{2} \left[a'(x_i - \kappa, t_j) W^j(x_i - \kappa) + a'(x_{i-1} + \kappa, t_j) W^j(x_{i-1} + \kappa) \right] \\ & \quad - \frac{h}{2} \left[\left(b^j(x_i - \kappa) - \frac{2}{k} \right) W^j(x_i - \kappa) + \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right) W^j(x_{i-1} + \kappa) \right] + \\ & \quad \frac{h}{2} [\varpi^{j+1}(x_i - \kappa) + \varpi^{j+1}(x_{i-1} + \kappa) + \varpi^j(x_i - \kappa) + \varpi^j(x_{i-1} + \kappa)]. \end{aligned} \quad (3.24)$$

Because the delay is of order ε , it is natural to linearize the shifted arguments through a Taylor series expansion. Accordingly, the delayed terms $W^{j+1}(x_i + \varepsilon)$, $W^{j+1}(x_{i-1} + \varepsilon)$, $W^j(x_i + \varepsilon)$, $W^j(x_{i-1} + \varepsilon)$, as well as $W^{j+1}(x_i - \kappa)$, $W^{j+1}(x_{i-1} + \kappa)$, $W^j(x_i - \kappa)$, $W^j(x_{i-1} + \kappa)$, are expanded about x_i or x_{i-1} by retaining only the first-order terms. For example, expanding $W^{j+1}(x_i + \varepsilon)$ about x_i yields $W^{j+1}(x_i + \varepsilon) \approx W^{j+1}(x_i) + \varepsilon W_x^{j+1}(x_i)$. The spatial derivative is then approximated using a finite difference scheme as $W_x^{j+1}(x_i) \approx$

$\frac{W^{j+1}(x_{i+1}) - W^{j+1}(x_i)}{h}$. Substituting this approximation gives

$$W^{j+1}(x_i + \varepsilon) \approx \left(1 - \frac{\varepsilon}{h}\right) W^{j+1}(x_i) + \frac{\varepsilon}{h} W^{j+1}(x_{i+1}).$$

Following the same procedure for the remaining shifted terms and introducing the compact notation $W_i^{j+1} = W^{j+1}(x_i)$, the discrete system corresponding to Eq. (3.24) can be expressed in a simplified and computationally efficient form as

$$\left\{ \begin{aligned} & \left[-\frac{\varepsilon}{h} + \mu a_{i-1}^{j+1} + \frac{\mu\kappa}{4} a' (x_i - \kappa, t_{j+1}) + \frac{\mu}{4} (2h - 3\kappa) a' (x_{i-1} + \kappa, t_{j+1}) + \right. \\ & \quad \left. \frac{\kappa}{4} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) + \frac{1}{4} (2h - 3\kappa) \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right) \right] W_{i-1}^{j+1} + \\ & \left[\frac{2\varepsilon}{h} - \mu a_i^{j+1} + \frac{h\mu}{2} a' (x_i - \kappa, t_{j+1}) + \mu\kappa a' (x_{i-1} + \kappa, t_{j+1}) \right. \\ & \quad \left. + \frac{h}{2} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) + \kappa \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right) \right] W_i^{j+1} + \\ & \left[-\frac{\varepsilon}{h} - \frac{\mu\kappa}{4} a' (x_i - \kappa, t_{j+1}) - \frac{\mu\kappa}{4} a' (x_{i-1} + \kappa, t_{j+1}) - \frac{\kappa}{4} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) \right. \\ & \quad \left. - \frac{\kappa}{4} \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right) \right] W_{i+1}^{j+1} = \\ & \left[\frac{\varepsilon}{h} - \mu a_{i-1}^j - \frac{\mu\kappa}{4} a' (x_i - \kappa, t_j) - \frac{\mu}{4} (2h - 3\kappa) a' (x_{i-1} + \kappa, t_j) - \right. \\ & \quad \left. \frac{\kappa}{4} \left(b^j(x_i - \kappa) - \frac{2}{k} \right) - \frac{1}{4} (2h - 3\kappa) \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right) \right] W_{i-1}^j + \\ & \left[-\frac{2\varepsilon}{h} + \mu a_i^j - \frac{h\mu}{2} a' (x_i - \kappa, t_j) - \mu\kappa a' (x_{i-1} + \kappa, t_j) - \frac{h}{2} \left(b^j(x_i - \kappa) - \frac{2}{k} \right) \right. \\ & \quad \left. - \kappa \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right) \right] W_i^j + \\ & \left[\frac{\varepsilon}{h} + \frac{\mu\kappa}{4} a' (x_i - \kappa, t_j) + \frac{\mu\kappa}{4} a' (x_{i-1} + \kappa, t_j) + \frac{\kappa}{4} \left(b^j(x_i - \kappa) - \frac{2}{k} \right) + \right. \\ & \quad \left. \frac{\kappa}{4} \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right) \right] W_{i+1}^j + \\ & \quad \frac{h}{2} [\varpi^{j+1}(x_i - \kappa) + \varpi^{j+1}(x_{i-1} + \kappa) + \varpi^j(x_i - \kappa) + \varpi^j(x_{i-1} + \kappa)]. \end{aligned} \right. \quad (3.25)$$

To account for the influence of the perturbation parameter, an exponential fitting factor $\sigma(\rho)$ is incorporated into Eq. (3.25) by multiplying the term associated with ε

as follows:

$$\left\{ \begin{aligned} & \left[-\frac{\varepsilon\sigma(\rho)}{h} + \mu a_{i-1}^{j+1} + \frac{\mu\kappa}{4} a'(x_i - \kappa, t_{j+1}) + \frac{\mu}{4} (2h - 3\kappa) a'(x_{i-1} + \kappa, t_{j+1}) \right. \\ & \quad \left. + \frac{\kappa}{4} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) + \frac{1}{4} (2h - 3\kappa) \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right) \right] W_{i-1}^{j+1} + \\ & \left[\frac{2\varepsilon\sigma(\rho)}{h} - \mu a_i^{j+1} + \frac{h\mu}{2} a'(x_i - \kappa, t_{j+1}) + \mu\kappa a'(x_{i-1} + \kappa, t_{j+1}) + \right. \\ & \quad \left. \frac{h}{2} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) + \kappa \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right) \right] W_i^{j+1} + \\ & \left[-\frac{\varepsilon\sigma(\rho)}{h} - \frac{\mu\kappa}{4} a'(x_i - \kappa, t_{j+1}) - \frac{\mu\kappa}{4} a'(x_{i-1} + \kappa, t_{j+1}) - \right. \\ & \quad \left. \frac{\kappa}{4} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) - \frac{\kappa}{4} \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right) \right] W_{i+1}^{j+1} = \\ & \left[\frac{\varepsilon\sigma(\rho)}{h} - \mu a_{i-1}^j - \frac{\mu\kappa}{4} a'(x_i - \kappa, t_j) - \frac{\mu}{4} (2h - 3\kappa) a'(x_{i-1} + \kappa, t_j) - \right. \\ & \quad \left. \frac{\kappa}{4} \left(b^j(x_i - \kappa) - \frac{2}{k} \right) - \frac{1}{4} (2h - 3\kappa) \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right) \right] W_{i-1}^j + \\ & \left[-\frac{2\varepsilon\sigma(\rho)}{h} + \mu a_i^j - \frac{h\mu}{2} a'(x_i - \kappa, t_j) - \mu\kappa a'(x_{i-1} + \kappa, t_j) - \right. \\ & \quad \left. \frac{h}{2} \left(b^j(x_i - \kappa) - \frac{2}{k} \right) - \kappa \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right) \right] W_i^j + \\ & \left[\frac{\varepsilon\sigma(\rho)}{h} + \frac{\mu\kappa}{4} a'(x_i - \kappa, t_j) + \frac{\mu\kappa}{4} a'(x_{i-1} + \kappa, t_j) + \frac{\kappa}{4} \left(b^j(x_i - \kappa) - \frac{2}{k} \right) \right. \\ & \quad \left. + \frac{\kappa}{4} \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right) \right] W_{i+1}^j + \\ & \quad \frac{h}{2} [\varpi^{j+1}(x_i - \kappa) + \varpi^{j+1}(x_{i-1} + \kappa) + \varpi^j(x_i - \kappa) + \varpi^j(x_{i-1} + \kappa)]. \end{aligned} \right. \quad (3.26)$$

Following the techniques in [Kadalbajoo & Jha \(2012\)](#); [Duressa & Mekonnen \(2023\)](#), the parameter $\sigma(\rho)$ with $\rho = \frac{h}{\varepsilon}$ is given by:

$$\sigma(\rho) = \begin{cases} \frac{\rho\mu a^{j+1}(0)}{2} \coth\left(\frac{\rho\mu a^{j+1}(0)}{2}\right), & \text{for } x = 0, \\ \frac{\rho\mu a^{j+1}(1)}{2} \coth\left(\frac{\rho\mu a^{j+1}(1)}{2}\right), & \text{for } x = 1, \end{cases} \quad (3.27)$$

This fitting factor is not chosen arbitrarily; rather, it is systematically derived so that the discrete scheme accurately replicates the inherent exponential behavior of the exact solution, which is the key characteristic of SPPs.

Using this definition, Eq. (3.26) can be expressed equivalently as a three-term recur-

rence relation of the form:

$$\left\{ \begin{array}{l} \mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1} \equiv A_i^- W_{i-1}^{j+1} + A_i^c W_i^{j+1} + A_i^+ W_{i+1}^{j+1} = \\ \quad B_i^- W_{i-1}^j + B_i^c W_i^j + B_i^+ W_{i+1}^j + F_j^j, \\ W_i^0 = \zeta_b(x_i), i = 1, \dots, N-1, \\ W_0^{j+1} = \zeta_l(t_{j+1}), \quad W_{N+1}^{j+1} = \zeta_r(t_{j+1}), 0 \leq j \leq M-1. \end{array} \right. \quad (3.28)$$

for $i = 0, 1, \dots, N$ and $j = 0, 1, \dots, M$ where

$$\left\{ \begin{array}{l} A_i^- = -\frac{\sigma(\rho)}{\rho} + \mu a_{i-1}^{j+1} + \frac{\mu\kappa}{4} a'(x_i - \kappa, t_{j+1}) + \frac{\mu}{4} (2h - 3\kappa) a'(x_{i-1} + \kappa, t_{j+1}) \\ \quad + \frac{\kappa}{4} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) + \frac{1}{4} (2h - 3\kappa) \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right), \\ A_i^c = \frac{2\sigma(\rho)}{\rho} - \mu a_i^{j+1} + \frac{h\mu}{2} a'(x_i - \kappa, t_{j+1}) + \mu\kappa a'(x_{i-1} + \kappa, t_{j+1}) + \\ \quad \frac{h}{2} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) + \kappa \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right), \\ A_i^+ = -\frac{\sigma(\rho)}{\rho} - \frac{\mu\kappa}{4} a'(x_i - \kappa, t_{j+1}) - \frac{\mu\kappa}{4} a'(x_{i-1} + \kappa, t_{j+1}) - \\ \quad \frac{\kappa}{4} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) - \frac{\kappa}{4} \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right), \\ B_i^- = \frac{\sigma(\rho)}{\rho} - \mu a_{i-1}^j - \frac{\mu\kappa}{4} a'(x_i - \kappa, t_j) - \frac{\mu}{4} (2h - 3\kappa) a'(x_{i-1} + \kappa, t_j) - \\ \quad \frac{\kappa}{4} \left(b^j(x_i - \kappa) - \frac{2}{k} \right) - \frac{1}{4} (2h - 3\kappa) \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right), \\ B_i^c = -\frac{2\sigma(\rho)}{\rho} + \mu a_i^j - \frac{h\mu}{2} a'(x_i - \kappa, t_j) - \mu\kappa a'(x_{i-1} + \kappa, t_j) - \\ \quad \frac{h}{2} \left(b^j(x_i - \kappa) - \frac{2}{k} \right) - \kappa \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right), \\ B_i^+ = \frac{\sigma(\rho)}{\rho} + \frac{\mu\kappa}{4} a'(x_i - \kappa, t_j) + \frac{\mu\kappa}{4} a'(x_{i-1} + \kappa, t_j) + \frac{\kappa}{4} \left(b^j(x_i - \kappa) - \frac{2}{k} \right) \\ \quad + \frac{\kappa}{4} \left(b^j(x_{i-1} + \kappa) - \frac{2}{k} \right), \\ F_i^j = \frac{h}{2} [\varpi^{j+1}(x_i - \kappa) + \varpi^{j+1}(x_{i-1} + \kappa) + \varpi^j(x_i - \kappa) + \varpi^j(x_{i-1} + \kappa)]. \end{array} \right. \quad (3.29)$$

This scheme is constructed using the two-point Gauss quadrature rule, ensuring high accuracy and stability even in the presence of boundary layers. The resulting discrete system in Eq. (3.28) is efficiently solved using the matrix inversion technique. For sufficiently fine spatial discretizations, it follows from the tridiagonal structure of

Eqs. (3.28) and (3.29) that

$$|A_i^c| \geq |A_i^-| + |A_i^+|. \quad (3.30)$$

This inequality confirms that the coefficient matrix corresponding to $\mathbb{L}_{\varepsilon,\mu}^{N,M}$ is *irreducibly diagonally dominant* and hence *non-singular*. Consequently, the system matrix qualifies as an M -matrix, guaranteeing the existence and uniqueness of the discrete solution. With the imposed boundary conditions, the matrix inversion method provides a stable and reliable means to compute the numerical solution of Eq. (3.28).

3.5 Convergence Analysis

We proved that the exact solution and its derivatives are uniformly bounded. Then, we derived bounds for temporal and spatial discretization errors to verify stability and consistency. Combining these results yielded the total error estimate, confirming convergence of the proposed scheme.

Lemma 3.9. (*Discrete Maximum Principle*). Assume $\Xi^{j+1}(x)$ be a mesh function satisfying, $\Xi(0, t_{j+1}) \geq 0$, $\Xi(1, t_{j+1}) \geq 0$, and $\mathbb{L}_{\varepsilon,\mu}^{N,M} \Xi(x_i, t_{j+1}) \geq 0$ on $\Lambda_x^N \times \Lambda_t^M$. Then $\Xi(x_i, t_j) \geq 0$ at each point $\bar{\Lambda}_x^N \times \bar{\Lambda}_t^M$, where Λ_x^N and Λ_t^M are discretized domain.

Proof. Let $(x_\vartheta, t_{j+1}) \in \{(x_i, t_{j+1}) : i = 1(1)N\}$ such that

$$\Xi^{j+1}(x_\vartheta) = \min_{1 \leq i \leq N} \Xi^{j+1}(x_i),$$

and let $\Xi^{j+1}(x_\vartheta) < 0$. From this $(x_\vartheta, t_{j+1}) \notin \{(0, t_{j+1}), (1, t_{j+1})\}$, which implies that $(x_\vartheta, t_{j+1}) \in \{(x_i, t_{j+1}) : i = 1(1)N - 1\}$. Then, from Eq. (3.28), we have

$$\mathbb{L}_{\varepsilon,\mu}^{N,M} \Xi_\vartheta^{j+1} = A_i^- \Xi_{\vartheta-1}^{j+1} + A_i^c \Xi_\vartheta^{j+1} + A_i^+ \Xi_{\vartheta+1}^{j+1} \leq 0,$$

since $|A_i^-| > 0$, $|A_i^c| > 0$ and $|A_i^+| > 0$, which is a contradiction. As a result, $\Xi^{j+1}(x_\vartheta) \geq 0$. Therefore $\Xi^{j+1}(x_i) \geq 0, \forall (x_i, t_{j+1}) \in \{(x_i, t_{j+1}) : i = 1(1)N - 1\}$. $\square$

Lemma 3.9 establishes that the discrete operator $\mathbb{L}_{\varepsilon,\mu}^{N,M}$ satisfies the discrete maximum principle, thereby ensuring that its coefficient matrix is monotone. Furthermore, since the matrix is irreducibly diagonally dominant, it constitutes an M -matrix, which in turn guarantees the existence and uniqueness of the discrete solution.

Lemma 3.10. (*Uniform Stability Estimate for the Discrete Scheme*). The solution W_i^{j+1} of the discrete scheme in Eq. (3.28) on $\mathbb{L}_{\varepsilon,\mu}^{N,M}$ gratifies the following estimate:

$$\|W_i^{j+1}\| \leq \xi^{-1} \|\mathbb{L}_{\varepsilon,\mu}^{N,M} W_i^{j+1}\| + \max \{|\zeta_l(t_{j+1})|, |\zeta_r(t_{j+1})|, |\zeta_b(x_i)|\}, \quad (3.31)$$

for $i = 1(1)N - 1$ and $0 < \varrho \leq b(x, t)$.

Proof. Let $\Gamma = \xi^{-1} \|\mathbb{L}_{\varepsilon,\mu}^{N,M} W_i^{j+1}\| + \max \{|\zeta_l(t_{j+1})|, |\zeta_r(t_{j+1})|, |\zeta_b(x_i)|\}$ and describe the barrier functions as:

$$(\nabla^\pm)_i^{j+1} = \Gamma \pm W_i^{j+1}. \quad (3.32)$$

The discrete function $\nabla^\pm(x_i, t_{j+1})$ on the boundary and initial conditions are

$$\begin{aligned} \nabla^\pm(0, t_{j+1}) &= \Gamma + \max \{|\Upsilon_l(t_{j+1})|, |\zeta_r(t_{j+1})|, |\zeta_b(0)|\} \pm \zeta_l(t_{j+1}) \geq 0, \\ \nabla^\pm(1, t_{j+1}) &= \Gamma + \max \{|\zeta_l(t_{j+1})|, |\zeta_r(t_{j+1})|, |\zeta_b(1)|\} \pm \Upsilon_r(t_{j+1}) \geq 0, \\ \nabla^\pm(x_i, 0) &= \Gamma + \max \{|\zeta_l(0)|, |\zeta_r(0)|, |\zeta_b(x_i, 0)|\} \pm \zeta_b(x_i, 0) \geq 0. \end{aligned}$$

Additionally, on the discretized domain, it is given as

$$\begin{aligned} \mathbb{L}_{\varepsilon,\mu}^{N,M} (\nabla^\pm)_i^{j+1} &= \Gamma \pm W_i^{j+1} = A_i^- (\Gamma \pm W_{i-1}^{j+1}) + A_i^c (\Gamma \pm W_i^{j+1}) + A_i^+ (\Gamma \pm W_{i+1}^{j+1}), \\ &= (A_i^- + A_i^c + A_i^+) \Gamma \pm H_i^j \geq 0, \end{aligned}$$

where H_i^j is right hand side of Eq. (3.28). The proof is right away when Lemma 3.9 is applied and initial conditions $\zeta_b(x)$ are integrated into the scheme. $\square$

Lemma 3.11 (Error in Gaussian quadrature). The definite integral is given by

$$I_{\mathcal{F}} = \int_{x_{i-1}}^{x_i} \mathcal{F}(x) dx,$$

and Gaussian quadrature with change of intervals is:

$$G(\mathcal{F}) = \frac{h}{2} \sum_{i=0}^n \left[A_i \mathcal{F} \left(\frac{x_i - x_{i-1}}{2} o_i + \frac{x_i + x_{i-1}}{2} \right) \right],$$

where o_i is the Gauss nodes on $[-1, 1]$. The error of Gaussian quadrature rule is given

by (Norkin et al., 1973; Sastry, 2012):

$$E(\mathcal{F}) = I(\mathcal{F}) - G(\mathcal{F}) = \frac{h^6}{(7!)400} W_{xxxx}(\nu), \quad x_{i-1} < \nu < x_i.$$

To establish a parametric uniform convergence of the discrete scheme of Eq. (3.28), let $TE^{j+1}(h)$ represent a LTE of the proposed discrete scheme. The LTE is given as:

$$TE^{j+1}(h) = \frac{h^2}{2} \left[\frac{\mu\kappa}{2} a'(x_i - \kappa, t_{j+1}) + \mu \left(\frac{h - 2\kappa}{2} \right) a'(x_{i-1} + \kappa, t_{j+1}) \right] + O(h^3). \quad (3.33)$$

We can rewrite Eq. (3.28) as follows:

$$\left(-\frac{\sigma(\rho)}{\rho} + \omega_i \right) W_{i-1}^{j+1} + \left(2\frac{\sigma(\rho)}{\rho} + \mu_i \right) W_i^{j+1} + \left(-\frac{\sigma(\rho)}{\rho} + z_i \right) W_{i+1}^{j+1} + \mathcal{K}_i + TE_i^{j+1}, \quad (3.34)$$

where $\mathcal{K}_i$ is the right hand side of Eq. (3.28) and

$$\left\{ \begin{array}{l} \omega_i = -\mu a_{i-1}^{j+1} + \frac{\mu\kappa}{4} a'(x_i - \kappa, t_{j+1}) + \frac{\mu}{4} (2h - 3\kappa) a'(x_{i-1} + \kappa, t_{j+1}) + \\ \quad \frac{\kappa}{4} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) + \frac{1}{4} (2h - 3\kappa) \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right), \\ \mu_i = -\mu a_i^{j+1} + \frac{h\mu}{2} a'(x_i - \kappa, t_{j+1}) + \mu\kappa a'(x_{i-1} + \kappa, t_{j+1}) + \\ \quad \frac{h}{2} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) + \kappa \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right), \\ z_i = -\frac{\mu\kappa}{4} a'(x_i - \kappa, t_{j+1}) - \frac{\mu\kappa}{4} a'(x_{i-1} + \kappa, t_{j+1}) - \\ \quad \frac{\kappa}{4} \left(b^{j+1}(x_i - \kappa) + \frac{2}{k} \right) - \frac{\kappa}{4} \left(b^{j+1}(x_{i-1} + \kappa) + \frac{2}{k} \right), \end{array} \right. \quad (3.35)$$

By imposing the boundary conditions $W^{j+1}(0) = \zeta_l(t^j)$ and $W^{j+1}(1) = \zeta_r(t^j)$, we obtain system of equations for each time level t_{j+1} , in matrix form as:

$$[\mathcal{A} + \mathcal{B}] W + \mathcal{X} + TE^{j+1}(h) = 0, \quad \text{where} \quad (3.36)$$

$$\begin{aligned} \mathcal{A} &= \text{trid} \left(-\frac{\sigma(\rho)}{\rho}, 2\frac{\sigma(\rho)}{\rho}, -\frac{\sigma(\rho)}{\rho} \right), \quad \mathcal{B} = \text{trid}(\omega_{i-1}, \mu_i, z_{i+1}), \\ \mathcal{X} &= \left[\mathcal{K}_1 + \left(\frac{\sigma(\rho)}{\rho} + \omega_1 \right) \zeta_l^{j+1}(0), \mathcal{K}_2, \dots, \mathcal{K}_{N-1} + \left(\frac{\sigma(\rho)}{\rho} + \mathcal{K}_{N-1} \right) \zeta_r^{j+1}(1) \right], \\ W &= [W_1^{j+1}, W_2^{j+1}, \dots, W_{N-1}^{j+1}]^T, \quad TE_i^{j+1}(h) = [TE_1^{j+1}, \dots, TE_{N-1}^{j+1}]^T. \end{aligned}$$

Let $\bar{W} = [\bar{W}_1^{j+1}, \bar{W}_2^{j+1}, \dots, \bar{W}_{N-1}^{j+1}]^T \cong W$ satisfy the following:

$$[\mathcal{A} + \mathcal{B}] \bar{U} + \mathcal{X} = 0, \quad (3.37)$$

Subtracting Eq. (3.36) from Eq. (3.37), we have

$$[\mathcal{A} + \mathcal{B}] \Psi = TE^{j+1}(h), \quad (3.38)$$

where $\Psi = \bar{W} - W = \bar{W}_i^{j+1} - W_i^{j+1} = [e_1, e_2, \dots, e_{i-1}]^T$ is the discretization error. Since ε is small, for a given small h , the matrix $[\mathcal{A} + \mathcal{B}]$ is irreducible and monotone (Tefera et al., 2022). Hence, there exists $[\mathcal{A} + \mathcal{B}]^{-1}$ and $[\mathcal{A} + \mathcal{B}]^{-1} \geq 0$. By manipulating matrix properties and the principles of matrix multiplication and norms, including their inverses, we establish that $\|\Psi\| \leq O(h^2)$. This implies that the spatial semi-discretization of the proposed method exhibits quadratic convergence.

Theorem 3.1. *Let $w(x_i, t_{j+1})$ and W_i^{j+1} denote the continuous solution of Eq. (3.1) and the numerical solution obtained from the fully discrete scheme Eq. (3.28), respectively. By comparing these two solutions, we derive the global error estimate for the fully discrete scheme, which is given by*

$$\sup_{0 < \varepsilon \leq 1, 0 \leq \mu \leq 1} \left[\max_{0 \leq i \leq N, 0 \leq j \leq M} |w(x_i, t_{j+1}) - W_i^{j+1}| \right] \leq C (k^2 + h^2), \quad (3.39)$$

where C is a constant independent of ε, μ, h and k .

Proof. To establish the theorem, we begin by considering the left-hand side of Eq. (3.39).

We introduce the semi-discrete solution $W^{j+1}(x_i)$ and employ the triangular inequality to decompose the expression as follows:

$$\begin{aligned} \max_{0 \leq i \leq N, 0 \leq j \leq M} |w(x_i, t_{j+1}) - W_i^{j+1}| &= \max_{0 \leq i \leq N, 0 \leq j \leq M} |w(x_i, t_{j+1}) - W^{j+1}(x_i) + W^{j+1}(x_i) - W_i^{j+1}|, \\ &\leq \max_{0 \leq i \leq N, 0 \leq j \leq M} |w(x_i, t_{j+1}) - W^{j+1}(x_i)| + \\ &\quad \max_{0 \leq i \leq N, 0 \leq j \leq M} |W^{j+1}(x_i) - W_i^{j+1}|, \\ &\leq C (k^2) + C (h^2) \leq C (k^2 + h^2). \end{aligned}$$

Thus, the claim follows. $\square$

3.6 Numerical Examples, Results, and Discussion

In this section, the performance of the proposed numerical scheme is validated through two representative test examples. To support the theoretical analysis, numerical results are tabulated, showing the maximum pointwise errors and the corresponding convergence rates. These quantities are evaluated using the double-mesh technique, which estimates the maximum pointwise error $E_{\varepsilon,\mu}^{N,M}$ and the numerical rate of convergence $\bar{R}_{\varepsilon,\mu}^{N,M}$ as follows:

$$E_{\varepsilon,\mu}^{N,M} = \max_{0 \leq i \leq N, 0 \leq j \leq M} |W_{i,j}^{N,M} - W_{i,j}^{2N,2M}|, \quad \bar{R}_{\varepsilon,\mu}^{N,M} = \frac{\log(E_{\varepsilon,\mu}^{N,M}) - \log(E_{\varepsilon,\mu}^{2N,2M})}{\log(2)},$$

where $W_{i,j}^{N,M}$ and $W_{i,j}^{2N,2M}$ denote the numerical solutions on meshes of sizes $N \times M$ and $2N \times 2M$, respectively. To further confirm parameter-uniform convergence with respect to ε and μ , the uniform maximum error $E^{N,M}$ and the uniform convergence rate $\bar{R}^{N,M}$ are computed by

$$E^{N,M} = \max_{\varepsilon,\mu} E_{\varepsilon,\mu}^{N,M}, \quad \bar{R}^{N,M} = \frac{\log(E^{N,M}) - \log(E^{2N,2M})}{\log(2)}.$$

These metrics provide a comprehensive assessment of the uniform convergence of the proposed scheme across various mesh densities and perturbation parameters.

Example 3.1. *Shivhare et al. (2021); Duressa & Mekonnen (2023).* Consider a class of time-dependent TP-SPPPs of the form Eq. (3.1)

$$\begin{cases} \frac{\partial w}{\partial t} - \varepsilon \frac{\partial^2 w}{\partial x^2} - \mu(1 + \exp(x)) \frac{\partial w}{\partial x} + (1 + x^5)w = -10 \exp(t^2)x^2(1 - x^2), \\ w(x, 0) = 0, w(0, t) = 0 = w(1, t). \end{cases} \quad (3.40)$$

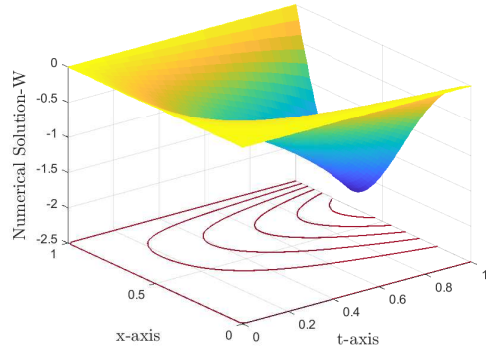
Example 3.2. *Ganesh Kumar & Phaneendra (2020); Das & Mehrmann (2016).* Consider a class of time-dependent TP-SPPPs of the form Eq. (3.1)

$$\begin{cases} \frac{\partial w}{\partial t} - \varepsilon \frac{\partial^2 w}{\partial x^2} - \mu(1 + x - x^2 + t^2) \frac{\partial w}{\partial x} + (1 + 5xt)w = (x^2 - x)(e^t - 1), \\ w(x, 0) = 0, w(0, t) = 0 = w(1, t). \end{cases} \quad (3.41)$$

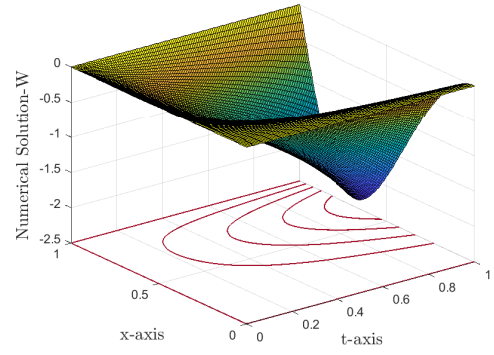
The computed maximum pointwise errors $E_{\varepsilon,\mu}^{N,M}$ and convergence rates $\bar{R}_{\varepsilon,\mu}^{N,M}$ for

Table 3.1: Computed values of $E_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$, $\bar{R}_{\varepsilon,\mu}^{N,M}$ and $\bar{R}^{N,M}$ for Example 3.1 at $\mu = 10^{-6}$.

ε ↓	$N = M = 16$	$N = M = 32$	$N = M = 64$	$N = M = 128$	$N = M = 256$	$N = M = 512$
10^{-4}	$1.6273e-02$ 1.9041	$4.3478e-03$ 1.9619	$1.1160e-03$ 1.9947	$2.8003e-04$ 2.0188	$6.9103e-05$ 2.0219	$1.7015e-05$ —
10^{-6}	$1.6372e-02$ 1.9009	$4.3839e-03$ 1.9527	$1.1325e-03$ 1.9774	$2.8759e-04$ 1.9901	$7.2394e-05$ 1.9957	$1.8152e-05$ —
10^{-8}	$1.6374e-02$ 1.9009	$4.3846e-03$ 1.9526	$1.1328e-03$ 1.9771	$2.8773e-04$ 1.9896	$7.2451e-05$ 1.9949	$1.8177e-05$ —
10^{-10}	$1.6374e-02$ 1.9009	$4.3846e-03$ 1.9526	$1.1328e-03$ 1.9770	$2.8776e-04$ 1.9892	$7.2480e-05$ 1.9940	$1.8195e-05$ —
10^{-12}	$1.6374e-02$ 1.9009	$4.3846e-03$ 1.9526	$1.1328e-03$ 1.9770	$2.8776e-04$ 1.9892	$7.2480e-05$ 1.9940	$1.8195e-05$ —
10^{-14}	$1.6374e-02$ 1.9009	$4.3846e-03$ 1.9526	$1.1328e-03$ 1.9770	$2.8776e-04$ 1.9892	$7.2480e-05$ 1.9940	$1.8195e-05$ —
$E_{\varepsilon,\mu}^{N,M}$	$1.6374e-02$	$4.3846e-03$	$1.1328e-03$	$2.8776e-04$	$7.2480e-05$	$1.8195e-05$
$\bar{R}^{N,M}$	1.9009	1.9526	1.9770	1.9892	1.9940	—

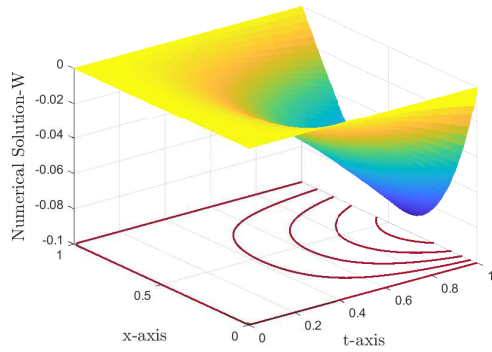


(a) $N = M = 64, \varepsilon = 10^{-2}, \mu = 10^{-8}$

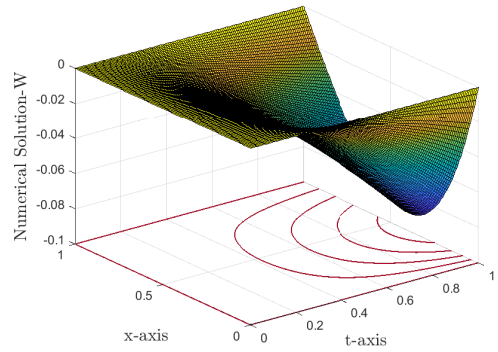


(b) $N = M = 128, \varepsilon = 10^{-12}, \mu = 10^{-6}$

Figure 3.1: The numerical solution profile for Example 3.1.



(a) $N = M = 64, \varepsilon = 10^{-4}, \mu = 10^{-2}$



(b) $N = M = 128, \varepsilon = 10^{-8}, \mu = 10^{-6}$

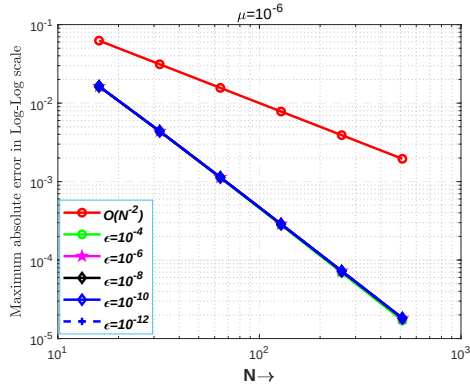
Figure 3.2: The numerical solution profile for Example 3.2.

Table 3.2: Computed maximum pointwise error and rate of convergence for Example 3.1 at $\mu = 10^{-6}$.

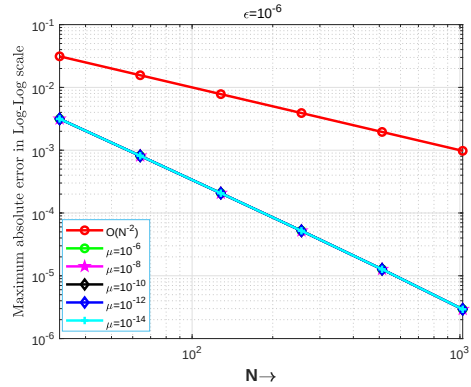
ε $\downarrow$	$N = 32$ $M = 10$	$N = 64$ $M = 20$	$N = 128$ $M = 40$	$N = 256$ $M = 80$	$N = 512$ $M = 160$	$N = 1024$ $M = 320$
10^{-6}	$3.1504e-03$ 1.9538	$8.1321e-04$ 1.9762	$2.0669e-04$ 1.9906	$5.2012e-05$ 2.0347	$1.2694e-05$ 2.1153	$2.9297e-06$ —
10^{-8}	$3.1502e-03$ 1.9542	$8.1293e-04$ 1.9772	$2.0647e-04$ 1.9884	$5.2034e-05$ 1.9936	$1.3066e-05$ 1.9953	$3.2772e-06$ —
10^{-10}	$3.1502e-03$ 1.9542	$8.1295e-04$ 1.9771	$2.0649e-04$ 1.9883	$5.2043e-05$ 1.9934	$1.3070e-05$ 1.9953	$3.2782e-06$ —
10^{-12}	$3.1502e-03$ 1.9542	$8.1295e-04$ 1.9771	$2.0649e-04$ 1.9883	$5.2043e-05$ 1.9934	$1.3070e-05$ 1.9953	$3.2782e-06$ —
10^{-14}	$3.1502e-03$ 1.9542	$8.1295e-04$ 1.9771	$2.0649e-04$ 1.9883	$5.2043e-05$ 1.9934	$1.3070e-05$ 1.9953	$3.2782e-06$ —
$E^{N,M}$	$3.1504e-03$	$8.1321e-04$	$2.0669e-04$	$5.2043e-05$	$1.3070e-05$	$3.2782e-06$
$\bar{R}^{N,M}$	1.9538	1.9762	1.9906	1.9934	1.9953	—

Table 3.3: Comparison of $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}^{N,M}$ for Example 3.1.

$\mu = 10^{-6}$	$N = 32$ $M = 10$	$N = 64$ $M = 20$	$N = 128$ $M = 40$	$N = 256$ $M = 80$	$N = 512$ $M = 160$
Proposed Method					
$E^{N,M}$	$3.1504e-03$	$8.1321e-04$	$2.0669e-04$	$5.2043e-05$	$1.3070e-05$
$\bar{R}^{N,M}$	1.9538	1.9762	1.9906	1.9934	—
Result in Duressa & Mekonnen (2023)(Exponentially Fitted Method)					
$E^{N,M}$	$3.9334e-03$	$1.1115e-03$	$2.9600e-04$	$7.6401e-05$	$1.9400e-05$
$\bar{R}^{N,M}$	1.8233	1.9088	1.9539	1.9775	—
Result in Shivhare et al. (2021)(Quadratic B-spline Collocation Method)					
$E^{N,M}$	$3.9420e-02$	$1.9392e-02$	$9.5791e-03$	$4.7594e-03$	$2.3718e-03$
$\bar{R}^{N,M}$	1.023	1.017	1.009	1.005	—



(a) Table 3.1

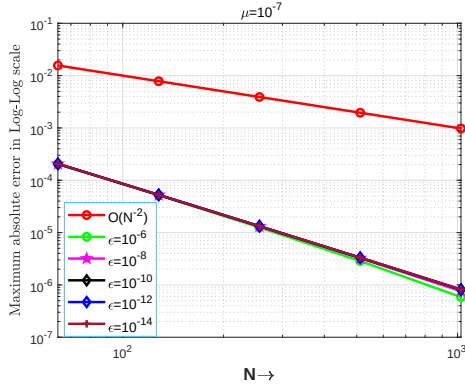


(b) Table 3.4

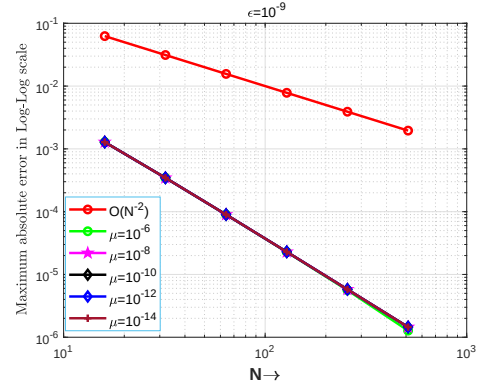
Figure 3.3: Log-log graph illustrating the peak absolute error corresponding to Example 3.1.

Table 3.4: Computed maximum pointwise error and rate of convergence for Example 3.1 at $\varepsilon = 10^{-6}$ and varying values of μ .

μ	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$\downarrow$	$M = 10$	$M = 20$	$M = 40$	$M = 80$	$M = 160$	$M = 320$
10^{-6}	$3.1504e-03$	$8.1321e-04$	$2.0669e-04$	$5.2012e-05$	$1.2694e-05$	$2.9297e-06$
	1.9538	1.9762	1.9906	2.0347	2.1153	—
10^{-8}	$3.1503e-03$	$8.1313e-04$	$2.0666e-04$	$5.1994e-05$	$1.2686e-05$	$2.9270e-06$
	1.9539	1.9762	1.9908	2.0351	2.1157	—
10^{-10}	$3.1503e-03$	$8.1313e-04$	$2.0666e-04$	$5.1994e-05$	$1.2686e-05$	$2.9269e-06$
	1.9539	1.9762	1.9908	2.0351	2.1158	—
10^{-12}	$3.1503e-03$	$8.1313e-04$	$2.0666e-04$	$5.1994e-05$	$1.2686e-05$	$2.9269e-06$
	1.9539	1.9762	1.9908	2.0351	2.1158	—
10^{-14}	$3.1503e-03$	$8.1313e-04$	$2.0666e-04$	$5.1994e-05$	$1.2686e-05$	$2.9269e-06$
	1.9539	1.9762	1.9908	2.0351	2.1158	—
$E^{N,M}$	$3.1504e-03$	$8.1321e-04$	$2.0669e-04$	$5.2012e-05$	$1.2694e-05$	$2.9297e-06$
$\bar{R}^{N,M}$	1.9538	1.9762	1.9906	2.0347	2.1153	—



(a) Table 3.5



(b) Table 3.7

Figure 3.4: The Log-Log plot of the maximum absolute error for Example 3.2.

Examples 3.1 and 3.2 are summarized in Tables 3.1–3.7 for various values of ε , μ , N , and M . Tables 3.3, 3.5, and 3.6 present a comparative analysis with results reported in Duressa & Mekonnen (2023); Das & Mehrmann (2016); Ganesh Kumar & Phaneendra (2020); Shivhare et al. (2021), where ε or μ are fixed or varied. The comparison clearly demonstrates that the proposed two-point Gaussian quadrature scheme delivers superior accuracy relative to existing methods. The numerical findings validate the theoretical predictions established in Lemma 3.7 and Theorem 3.1. As observed in Tables 3.1, 3.2, 3.4, 3.5, and 3.7, the error consistently decreases with mesh refinement and eventually stabilizes, confirming uniform convergence and second-order convergence. Figures 3.1 and 3.2 illustrate the numerical solution surfaces for Examples 3.1 and 3.2 under different parameter and mesh settings. The solutions exhibit

Table 3.5: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}^{N,M}$ for Example 3.2 at $\mu = 10^{-7}$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$\downarrow$	$M = 16$	$M = 32$	$M = 64$	$M = 128$	$M = 256$
Proposed Method					
10^{-6}	$2.0524e-04$	$5.1757e-05$	$1.2586e-05$	$2.8636e-06$	$5.8638e-07$
	1.9875	2.0399	2.1359	2.2879	—
10^{-7}	$2.0597e-04$	$5.2454e-05$	$1.3175e-05$	$3.2435e-06$	$7.6179e-07$
	1.9733	1.9932	2.0222	2.0901	—
10^{-8}	$2.0604e-04$	$5.2528e-05$	$1.3247e-05$	$3.3097e-06$	$8.1081e-07$
	1.9718	1.9874	2.0009	2.0293	—
10^{-9}	$2.0605e-04$	$5.2535e-05$	$1.3255e-05$	$3.3170e-06$	$8.1742e-07$
	1.9716	1.9867	1.9986	2.0207	—
10^{-10}	$2.0605e-04$	$5.2536e-05$	$1.3256e-05$	$3.3177e-06$	$8.1816e-07$
	1.9716	1.9867	1.9984	2.0197	—
10^{-12}	$2.0605e-04$	$5.2536e-05$	$1.3256e-05$	$3.3177e-06$	$8.1816e-07$
	1.9716	1.9867	1.9984	2.0197	—
$E^{N,M}$	$2.0605e-04$	$5.2536e-05$	$1.3256e-05$	$3.3177e-06$	$8.1816e-07$
$\bar{R}^{N,M}$	1.9716	1.9867	1.9984	2.0197	—
Result in Das & Mehrmann (2016)(Mesh-adaptive upwind scheme)					
10^{-6}	$9.6949e-4$	$4.9906e-4$	$2.5231e-4$	$1.2824e-4$	$6.446e-5$
	0.95802	0.98400	0.97638	0.99233	—
10^{-7}	$9.8712e-4$	$5.0049e-4$	$2.5485e-4$	$1.2853e-4$	$6.449e-5$
	0.97987	0.97368	0.98758	0.99493	—
10^{-8}	$9.51284e-4$	$5.0026e-4$	$2.5237e-4$	$1.2781e-4$	$6.442e-5$
	0.92720	0.98713	0.98156	0.98839	—
10^{-9}	$9.6744e-4$	$5.0012e-4$	$2.5461e-4$	$1.28036e-4$	$6.437e-5$
	0.95193	0.97394	0.99186	0.99188	—

distinct boundary layers on the left or right sides, depending on the magnitude of ε and μ , which is consistent with the expected physical behavior of SPPs. Furthermore, Figures 3.3 and 3.4 display the log-log plots of the maximum pointwise errors for Examples 3.1 and 3.2, while Figure 3.3a illustrates the same behavior for Example 3.1 using the data from Table 3.1. These plots clearly reveal that the proposed numerical method achieves uniform convergence as the perturbation parameters approach zero.

3.7 Conclusion

In this chapter, a TP-SPPPs was analyzed using the Crank–Nicolson scheme for time discretization and a *two-point Gaussian quadrature* with an exponential fitting factor for spatial approximation. The proposed technique provides an elegant and

Table 3.6: Comparison of $E_{\varepsilon,\mu}^{N,M}$ and $E^{N,M}$ for Example 3.2.

$\mu = 10^{-7}$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$
Present Method				
$E^{N,M}$	$2.0605e - 04$	$5.2536e - 05$	$1.3256e - 05$	$3.3177e - 06$
Result in Das & Mehrmann (2016)(Mesh-adaptive upwind scheme)				
$E^{N,M}$	$9.8712e - 04$	$5.0026e - 04$	$2.5485e - 04$	$1.2853e - 04$
Result in Ganesh Kumar & Phaneendra (2020)(Exponential fitting factor)				
$E^{N,M}$	$4.6780e - 04$	$1.192e - 04$	$3.0257e - 05$	$7.5982e - 06$

Table 3.7: Computed values of $E_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$, $\bar{R}^{N,M}$ for Example 3.2 at $\varepsilon = 10^{-9}$.

μ $\downarrow$	$(N, M) = 16$	$(N, M) = 32$	$(N, M) = 64$	$(N, M) = 128$	$(N, M) = 256$	$(N, M) = 512$
10^{-6}	$1.2818e - 03$ 1.8916	$3.4545e - 04$ 1.9457	$8.9673e - 05$ 1.9796	$2.2737e - 05$ 2.0194	$5.6084e - 06$ 2.1370	$1.2751e - 06$ —
10^{-8}	$1.2819e - 03$ 1.8911	$3.4561e - 04$ 1.9437	$8.9842e - 05$ 1.9713	$2.2911e - 05$ 1.9859	$5.7841e - 06$ 1.9942	$1.4519e - 06$ —
10^{-10}	$1.2819e - 03$ 1.8911	$3.4561e - 04$ 1.9437	$8.9844e - 05$ 1.9713	$2.2912e - 05$ 1.9855	$5.7858e - 06$ 1.9928	$1.4537e - 06$ —
10^{-12}	$1.2819e - 03$ 1.8911	$3.4561e - 04$ 1.9437	$8.9844e - 05$ 1.9713	$2.2912e - 05$ 1.9855	$5.7858e - 06$ 1.9928	$1.4537e - 06$ —
10^{-14}	$1.2819e - 03$ 1.8911	$3.4561e - 04$ 1.9437	$8.9844e - 05$ 1.9713	$2.2912e - 05$ 1.9855	$5.7858e - 06$ 1.9928	$1.4537e - 06$ —
$E^{N,M}$	$1.2819e - 03$	$3.4561e - 04$	$8.9844e - 05$	$2.2912e - 05$	$5.7858e - 06$	$1.4537e - 06$
$\bar{R}^{N,M}$	1.8911	1.9437	1.9713	1.9855	1.9928	—

competent means of addressing the difficulties posed by SPPs without relying on asymptotic expansions or fitted meshes. Owing to the dual perturbation parameters, boundary layers of varying thickness arise at both ends of the domain. The developed method attains second-order convergence in both time and space. Comparative studies further demonstrate that its accuracy matches or exceeds that of existing FDM (Das & Mehrmann, 2016; Ganesh Kumar & Phaneendra, 2020; Shivhare et al., 2021; Duressa & Mekonnen, 2023), confirming its stability and improved computational performance. The method ensures a clear formulation, uniform convergence, and strong stability across perturbation ranges. Its parameter-independent performance makes it particularly effective for convection-dominated flow problems with large Reynolds or Péclet numbers, offering a reliable and accurate alternative to traditional numerical schemes. The chapter concludes by noting that, although the developed scheme is uniformly convergent, further gains in spatial accuracy can be achieved through alternative fitted integration strategies. This motivates the development of an improved method presented in Chapter 4.

Chapter 4

Exponentially Fitted Simpson's One-Third Rule for Two-Parameter Singularly Perturbed Parabolic Convection-Diffusion Problems

Building on the numerical framework developed in Chapter 3, this chapter introduces an enhanced discretization strategy motivated by the need for improved spatial accuracy. While the previous chapter employed a two-stage approach using Crank–Nicolson time discretization and a two-point Gaussian quadrature rule in space, the resulting method is still limited in its ability to capture some fine scale solution features with higher-order precision, even though it is uniformly convergent. To address these limitations, this chapter advances the methodology by incorporating a FOM combined with Simpson's $1/3$ rule, offering a more accurate treatment of the spatial domain. The proposed scheme applies the Crank–Nicolson method on a uniform temporal mesh and employs the FOM Simpson hybrid in space, yielding improved approximation properties. A rigorous convergence analysis demonstrates the method's uniform convergence across varying perturbation parameters. Numerical experiments further validate its performance, demonstrating higher accuracy compared to existing methods and to the scheme developed in Chapter 3.

4.1 Introduction

A wide range of scientific and engineering phenomena give rise to TP-SPPPs, including water quality modeling in rivers, high Reynolds number Navier-Stokes flows (Hirsch, 1990), vibration or heat transfer processes such as turbulent channel flows (Kuerten et al., 2011), flow past circular cylinders (Ma & Duan, 2020), and high-speed gas dynamics (Van Dyke, 1975). In these systems, the highest-order spatial derivatives are multiplied by ε , producing boundary layers where the solution changes rapidly over narrow regions while remaining smooth elsewhere. This multiscale nature introduces major analytical and numerical challenges, as conventional methods demand extremely fine meshes to capture steep gradients, leading to high computational cost and possible instability. Therefore, parameter-uniform numerical techniques are essential for obtaining reliable approximations (Farrell et al., 2000; Eugene & Shishkin, 2012; Das & Natesan, 2018; Roos, 2012; Bullo et al., 2021; Duressa, 2021).

Several numerical strategies such as collocation, FDM, and FEMs have been proposed for TP-SPPPs (O'malley, 1967; Kadalbajoo & Yadaw, 2012; Das & Mehrmann, 2016; Zahra & El Mhlawy, 2013; Roos & Uzelac, 2003; Phaneendra & Mahesh, 2017; Patidar, 2008; Duressa & Mekonnen, 2023). Among them, integration-based approaches, particularly Simpson's one-third rule, offer a simple yet accurate framework for spatial discretization. This higher-order quadrature rule approximates integrals by fitting quadratic polynomials, achieving second-order accuracy and effectively capturing steep gradients compared with basic trapezoidal or midpoint rules.

When combined with FOMs, Simpson's rule enables the formulation of efficient FDS that preserve the key features of the exact solution without the need for excessive mesh refinement. Such schemes naturally handle variable coefficients and non-uniform grids, reduce discretization errors, and maintain uniform convergence making them well-suited for SPPs. To the best of our knowledge, no prior study has combined the Crank-Nicolson time discretization with Simpson's one-third rule enhanced by an exponential fitting factor for TP-SPPPs. In this chapter, we propose a uniform numerical scheme that integrates these techniques on a uniform spatial mesh. The method achieves high accuracy and parameter-uniform second-order convergence while remaining computationally efficient and easy to implement. It effectively re-

solves boundary layers, offering a reliable approach for a wide class of SPPPs.

The chapter is organized as follows: Section 4.2 formulates the problem, followed by an analysis of the continuous solution in Section 4.3. Section 4.4 outlines the proposed scheme, and Section 4.5 presents its convergence analysis. Numerical results are discussed in Section 4.6, while Section 4.7 concludes with key findings.

4.2 Definition of the Problem

The model problem introduced in Chapter 3 (Eq. (3.1)) is retained in this chapter, with only the essential assumptions and notations restated for clarity. The given data are assumed to be sufficiently smooth and to satisfy the required corner compatibility conditions at $(0, 0)$ and $(1, 0)$, ensuring well-posedness and supporting the subsequent analysis, as specified below.

$$\zeta_b(0) = \zeta_l(0), \quad \zeta_b(1) = \zeta_r(1), \quad \text{and} \quad (4.1)$$

$$\begin{cases} \frac{\partial \zeta_l(0)}{\partial t} - \varepsilon \frac{\partial^2 \zeta_b(0)}{\partial x^2} - \mu a(0, 0) \frac{\partial \zeta_b(0)}{\partial x} + b(0, 0) \varphi_b(0) = \varpi(0, 0), \\ \frac{\partial \zeta_r(0)}{\partial t} - \varepsilon \frac{\partial^2 \zeta_b(1)}{\partial x^2} - \mu a(1, 0) \frac{\partial \zeta_b(1)}{\partial x} + b(1, 0) \varphi_b(1) = \varpi(1, 0). \end{cases} \quad (4.2)$$

The compatibility conditions in Eqs. (4.1) and (4.2) guarantee the uniqueness of the solution. Moreover, the corner compatibility ensures the existence of constants Q_1 and Q_2 such that $\forall (x, t) \in \overline{\mathbb{G}}$ (Bobisud, 1968),

$$|w(x, t) - \zeta_b(x)| \leq Q_1 t, \quad |w(x, t)| \leq Q_2(1 - x).$$

Hence, if $w(x, t)$ is once differentiable in t , it follows from Kadalbajoo & Arora (2009); Kadalbajoo & Jha (2012) that $|w_t(x, t)| \leq C$. The problems defined by Eq. (3.1) represents TP-SPPPs, whose solutions exhibit sharp boundary layers at both ends of the spatial domain. Even with smooth data, rapid variations arise within thin transition regions whose thickness depends on the interaction between ε and μ . When $\mu^2/\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, layers of width $O(\sqrt{\varepsilon})$ appear near $x = 0$ and $x = 1$; conversely, if $\varepsilon/\mu^2 \rightarrow 0$ as $\mu \rightarrow 0$, the layer at $x = 0$ scales as $O(\mu)$ and at $x = 1$ as $O(\varepsilon/\mu)$ (Bullo et al., 2019). This general formulation encompasses classical convection-diffusion and

reaction-diffusion models (Kadalbajoo & Yadaw, 2012). For instance, setting $\mu = 1$ yields a convection-diffusion equation with a single boundary layer of thickness $O(\varepsilon)$ determined by the convection direction (Aga et al., 2019), whereas $\mu = 0$ reduces it to a reaction-diffusion type exhibiting twin layers of width $O(\sqrt{\varepsilon})$ at both boundaries (Ganesh Kumar & Phaneendra, 2020; Daba et al., 2024).

4.3 Properties of the Continuous Solution

This section analyzes the properties of exact solution to Eq. (3.1). To ensure well-posedness, the existence of a solution is established through the following lemmas.

Lemma 4.1. *Let $a, b, f \in C^l(\overline{\mathbb{G}})$, $\zeta_b \in C^{2+l}(\overline{\mathcal{Q}})$ and $\zeta_l, \zeta_r \in C^{1+l/2}(\overline{\Lambda})$ for some $0 < l < 1$. Assume the initial and boundary data satisfy Eqs. (4.1)- (4.2) and the time/space derivatives match at $(x, t) = (0, 0)$ and $(1, 0)$. Then the IBVP Eq. (3.1) has a unique classical solution $w(x, t) \in C^{2+l, 1+l/2}(\overline{\mathbb{G}})$.*

Proof. Because $0 < \varepsilon \ll 1$ but strictly positive, the principal part of the operator in x is $-\varepsilon \partial_x^2$. Hence the PDE is uniformly parabolic on $\overline{\mathbb{G}}$ (the parabolicity constant is $\varepsilon > 0$). The lower-order coefficients $\mu a(x, t)$ and $b(x, t)$ are continuous (indeed Hölder), so the problem fits the framework of linear, second-order, uniformly parabolic equations with Hölder coefficients (Ladyzhenskaia et al., 1968). To show its uniqueness, let w_1, w_2 be two solutions in the class $C^{2,1}(\overline{\mathbb{G}})$. Set $v = w_1 - w_2$. Then v satisfies:

$$\partial_t v - \varepsilon v_{xx} - \mu a(x, t) v_x + b(x, t) v = 0, \quad \text{in } \mathbb{G},$$

with homogeneous initial and boundary data $v(\cdot, 0) = 0, v(0, t) = v(1, t) = 0$. Multiply the PDE by v and integrate in $x \in (0, 1)$:

$$\int_0^1 v \partial_t v dx - \varepsilon \int_0^1 v v_{xx} dx - \mu \int_0^1 a v v_x dx + \int_0^1 b v^2 dx = 0.$$

Integrate by parts the diffusion term and estimate the convection term:

$$\frac{1}{2} \frac{d}{dt} \|v(\cdot, t)\|_{L_2}^2 + \varepsilon \|v_x(\cdot, t)\|_{L_2}^2 - \frac{\mu}{2} \int_0^1 a_x v^2 dx + \int_0^1 b v^2 dx = 0,$$

where we used $v(0, t) = v(1, t) = 0$. Since a_x, b are bounded on $\overline{\mathbb{G}}$, there exists

$C \geq 0$ (depending on $\|a_x\|_\infty, \|b\|_\infty, \mu$) such that

$$\frac{d}{dt} \|v(\cdot, t)\|_{L_2}^2 \leq C \|v(\cdot, t)\|_{L_2}^2.$$

By Gröwall's inequality and the initial condition $\|v(\cdot, 0)\|_{L_2} = 0$ and $\|v(\cdot, t)\|_{L_2} = 0$ for all t , hence $v = w_1 - w_2 = 0$ implies $w_1 = w_2$. Thus the solution is unique. $\square$

4.4 Formulation of the Numerical Scheme

The parabolic problem is semi-discretized in time using Rothe's method with a Crank–Nicolson scheme, and the resulting ODEs are discretized in space by Simpson's one-third rule on a uniform mesh.

4.4.1 Temporal Semi-discretization

This section introduces a Crank–Nicolson based numerical method for solving Eq. (3.1). The time domain $[0, T]$ is uniformly partitioned into M subintervals of size $k = T/M$, forming the mesh $\Lambda_t^M = \{t_j = k(j-1), j = 1, 2, \dots, M+1\}$. Through temporal semi-discretization of Eq. (3.1), the problem is reduced to a system of ODEs that forms the basis for the fully discrete scheme.

$$\begin{aligned} & \frac{\varepsilon}{2} [w_{xx}^j + w_{xx}^{j-1}] + \frac{\mu}{2} [a^j(x)w_x^j + a^{j-1}(x)w_x^{j-1}] - \frac{1}{2} [b^j(x)w^j(x) + b^{j-1}(x)w^{j-1}(x)] \\ &= \frac{w^j(x) - w^{j-1}(x)}{k} - \frac{1}{2} [\varpi^j(x) + \varpi^{j-1}(x)], \\ \Rightarrow & \frac{\varepsilon}{2} [w_{xx}^j + w_{xx}^{j-1}] + \frac{\mu}{2} [a^j(x)w_x^j + (a'(x))^j w^j(x) + (a'(x))^{j-1} w^{j-1}(x) + a^{j-1}(x)w_x^{j-1}] \\ &= \frac{1}{2} [b^j(x)w^j(x) + b^{j-1}(x)w^{j-1}(x)] + \frac{w^j(x) - w^{j-1}(x)}{k} - \frac{1}{2} [\varpi^j(x) + \varpi^{j-1}(x)] \\ & \quad + \frac{\mu}{2} [(a'(x))^j w^j(x) + (a'(x))^{j-1} w^{j-1}(x)], \\ \Rightarrow & \frac{\varepsilon}{2} [w_{xx}^j + w_{xx}^{j-1}] + \frac{\mu}{2} [a^j(x)w^j(x) + a^{j-1}(x)w^{j-1}(x)]' = \\ & \frac{1}{2} [b^j(x)w^j(x) + b^{j-1}(x)w^{j-1}(x)] + \frac{w^j(x) - w^{j-1}(x)}{k} - \frac{1}{2} [\varpi^j(x) + \varpi^{j-1}(x)] \\ & \quad + \frac{\mu}{2} [(a'(x))^j w^j(x) + (a'(x))^{j-1} w^{j-1}(x)]. \end{aligned}$$

The foregoing expression can now be reformulated as follows:

$$\left\{ \begin{array}{l} \frac{\varepsilon}{2} [w_{xx}^j + w_{xx}^{j-1}] + \frac{\mu}{2} [a^j(x)w^j(x) + a^{j-1}(x)w^{j-1}(x)]' - \\ \frac{1}{2} [b^j(x)w^j(x) + b^{j-1}(x)w^{j-1}(x)] = \frac{w^j(x) - w^{j-1}(x)}{k} - \\ \frac{1}{2} [\varpi^j(x) + \varpi^{j-1}(x)] + \frac{\mu}{2} [(a'(x))^j w^j(x) + (a'(x))^{j-1} w^{j-1}(x)], \\ w^j(0) = \zeta_l(t_j), \quad w^j(1) = \zeta_r(t_j), \quad 1 \leq j \leq M+1, \\ w^0(x) = \zeta_b(x), \quad x \in (0, 1). \end{array} \right. \quad (4.3)$$

At the (j) th time level, the semi-discrete solution $w(x, t_j)$ satisfies the following spatial DE derived from Eq. (3.1).

$$\left\{ \begin{array}{l} \mathbb{L}_{\varepsilon, \mu}^M w(x, t_j) := \frac{\varepsilon}{2} (w_{xx}(x))^j + \frac{\mu}{2} a^j(x) (w_x(x))^j - \\ \left(\frac{1}{2} b^j(x) + \frac{1}{k} \right) w^j(x) = G^j(x), \\ w^j(0) = \zeta_l(t_j), \quad w^j(1) = \zeta_r(t_j), \quad 1 \leq j \leq M+1, \\ w^0(x) = \zeta_b(x), \quad x \in (0, 1), \end{array} \right. \quad (4.4)$$

where $G^j(x) = -\frac{\varepsilon}{2} (w_{xx}(x))^{j-1} - \frac{\mu}{2} a^{j-1}(x) (w_x(x))^{j-1} + \left(\frac{1}{2} b^{j-1}(x) + \frac{1}{k} \right) w^{j-1}(x) + \frac{1}{2} [\varpi^j(x) + \varpi^{j-1}(x)]$. The operator $\mathbb{L}_{\varepsilon, \mu}^M$ obeys discrete maximum principle, ensuring the stability of Eq. (4.4) (Zahra & El Mhlawy, 2013). The semi-discretization yields the LTE $e^j(x) = w^j - \bar{w}^j$, where $\bar{w}^j$ denotes the numerical solution of the BVP.

Lemma 4.2. (*Estimation of LTE*). Suppose that $\left| \frac{\partial^l w(x, t)}{\partial t^l} \right| \leq C, (x, t) \in \overline{\mathbb{G}}, 0 \leq l \leq 2$. The LTE estimate in the temporal direction is given by

$$\|e^j\|_{\infty} \leq C_1 k^3. \quad (4.5)$$

Proof. Using Taylor series expansion, we have

$$w(x, t_j) = w(x, t_{j+1/2}) + \frac{k}{2} w_t(x, t_{j+1/2}) + \frac{k^2}{8} w_{tt}(x, t_{j+1/2}) + \frac{k^3}{12} w_{ttt}(x, t_{j+1/2}) + O(k^4), \quad (4.6)$$

$$w(x, t_{j-1}) = w(x, t_{j+1/2}) - \frac{k}{2} w_t(x, t_{j+1/2}) + \frac{k^2}{8} w_{tt}(x, t_{j+1/2}) - \frac{k^3}{12} w_{ttt}(x, t_{j+1/2}) + O(k^4). \quad (4.7)$$

Upon performing the subtraction of Eq. (4.7) from Eq. (4.6), we derive:

$$\begin{aligned} \frac{w(x, t_j) - w(x, t_{j-1})}{k} &= w_t(x, t_{j+1/2}) + O(k^2), \\ &= \varepsilon w_{xx} \left(x, t_j + \frac{k}{2} \right) + \mu a \left(x, t_j + \frac{k}{2} \right) w_x \left(x, t_j + \frac{k}{2} \right) - \\ &\quad b \left(x, t_j + \frac{k}{2} \right) w \left(x, t_j + \frac{k}{2} \right) - G \left(x, t_j + \frac{k}{2} \right), \end{aligned}$$

where

$$\begin{aligned} w \left(x, t_j + \frac{k}{2} \right) &= \frac{w(x, t_j) + w(x, t_{j-1})}{2} + O(k^2), \\ G \left(x, t_j + \frac{k}{2} \right) &= \frac{G(x, t_j) + G(x, t_{j-1})}{2} + O(k^2), \\ a \left(x, t_j + \frac{k}{2} \right) &= \frac{a(x, t_j) + a(x, t_{j-1})}{2}, \quad b \left(x, t_j + \frac{k}{2} \right) = \frac{b(x, t_j) + b(x, t_{j-1})}{2}. \end{aligned}$$

The solution for the LTE is

$$\begin{cases} L_{\varepsilon, \mu}^M e^j = O(k^3), \\ e^j(0) = \zeta_l(t_j), \quad e^j(1) = \zeta_r(t_j). \end{cases}$$

Hence, by invoking the maximum principle, the intended result is established. $\square$

As shown in Lemma 4.2, the LTE governs the total temporal discretization error. The global error at the j th time level is stated below.

Lemma 4.3. (*Global Error Estimate*). *The global error estimate at j th time level is*

$$\|E^j\|_{\infty} \leq C_3 k^2. \quad (4.8)$$

Proof. A detailed proof can be found in Chapter 3, Lemma 3.7. As a result, the proposed time semi-discrete formulation achieves second-order accuracy. $\square$

4.4.2 Spatial Semi-discretization

The spatial domain $[0, 1]$ is divided into N elements of size $h = 1/N$, forming the mesh $\mathcal{Q}_x^N = \{x_i = ih, i = 0, 1, \dots, N\}$, where x_i are the mesh points. Integrating

Eq. (4.3) over each interval (x_{i-1}, x_{i+1}) for $i = 1, 2, \dots, N-1$ yields

$$\begin{aligned} & \frac{\varepsilon}{2} \int_{x_{i-1}}^{x_{i+1}} [w_{xx}^j + w_{xx}^{j-1}] dx + \frac{\mu}{2} \int_{x_{i-1}}^{x_{i+1}} [(a^j w^j + a^{j-1} w^{j-1})'] dx \\ & - \frac{1}{2} \int_{x_{i-1}}^{x_{i+1}} [b^j w^j + b^{j-1} w^{j-1}] dx = \frac{1}{k} \int_{x_{i-1}}^{x_{i+1}} [w^j - w^{j-1}] dx \\ & - \frac{1}{2} \int_{x_{i-1}}^{x_{i+1}} [\varpi^j + \varpi^{j-1}] dx + \frac{\mu}{2} \int_{x_{i-1}}^{x_{i+1}} [(a')^j w^j + (a')^{j-1} w^{j-1}] dx, \end{aligned} \quad (4.9)$$

Rewriting Eq. (4.9), we obtain

$$\begin{aligned} & \frac{\varepsilon}{2} [w_x^j(x_{i+1}) - w_x^j(x_{i-1}) + w_x^{j-1}(x_{i+1}) - w_x^{j-1}(x_{i-1})] \\ & + \frac{\mu}{2} [a^j(x_{i+1})w^j(x_{i+1}) - a^j(x_{i-1})w^j(x_{i-1}) \\ & + a^{j-1}(x_{i+1})w^{j-1}(x_{i+1}) - a^{j-1}(x_{i-1})w^{j-1}(x_{i-1})] \\ & = \int_{x_{i-1}}^{x_{i+1}} R^j(x) dx. \end{aligned} \quad (4.10)$$

where

$$\begin{aligned} R^j(x)dx = & \frac{1}{2} [b^j(x)w^j(x) + b^{j-1}(x)w^{j-1}(x)] + \frac{1}{k} [w^j(x) - w^{j-1}(x)] - \\ & \frac{1}{2} [\varpi^j(x) + \varpi^{j-1}(x)] + \frac{\mu}{2} [(a'(x))^j w^j(x) + (a'(x))^{j-1} w^{j-1}(x)]. \end{aligned}$$

Adopting $w^j(x_i) = W_i^j$ and employing Simpson's 1/3rd integration rule, we obtain:

$$\begin{aligned} & \frac{\varepsilon}{2} [(W_x)_i^j - (W_x)_{i-1}^j + (W_x)_{i+1}^{j-1} - (W_x)_{i-1}^{j-1}] + \\ & \frac{\mu}{2} [a_{i+1}^j W_{i+1}^j - a_{i-1}^j W_{i-1}^j + a_{i+1}^{j-1} W_{i+1}^{j-1} - a_{i-1}^{j-1} W_{i-1}^{j-1}] \\ & = \frac{h}{3} [R_{i+1}^j + 4R_i^j + R_{i-1}^j]. \end{aligned} \quad (4.11)$$

We then approximate the first derivative in Eq.(4.11) using appropriate non-symmetric finite difference formulas.

$$\begin{cases} (W_x)_i^j \approx \frac{W_{i+1}^j - W_{i-1}^j}{2h}, \\ (W_x)_{i+1}^j \approx \frac{3W_{i+1}^j - 4W_i^j + W_{i-1}^j}{2h}, \\ (W_x)_{i-1}^j \approx \frac{-W_{i+1}^j + 4W_i^j - 3W_{i-1}^j}{2h}, \end{cases} \quad (4.12)$$

By inserting the approximations given in Eq. (4.12) into Eq. (4.11), we derive

$$\begin{aligned} & \frac{\varepsilon}{h} [W_{i+1}^j - 2W_i^j + W_{i-1}^j + W_{i+1}^{j-1} - 2W_i^{j-1} + W_{i-1}^{j-1}] + \\ & \frac{\mu}{2} [a_{i+1}^j W_{i+1}^j - a_{i-1}^j W_{i-1}^j + a_{i+1}^{j-1} W_{i+1}^{j-1} - a_{i-1}^{j-1} W_{i-1}^{j-1}] \\ & = \frac{h}{3} [R_{i+1}^j + 4R_i^j + R_{i-1}^j]. \end{aligned} \quad (4.13)$$

Introducing the fitting parameter σ , designed to accommodate the boundary layer structure, allows Eq. (4.13) to be rewritten in the following form

$$\begin{aligned} & \frac{\varepsilon\sigma}{h} [W_{i+1}^j - 2W_i^j + W_{i-1}^j + W_{i+1}^{j-1} - 2W_i^{j-1} + W_{i-1}^{j-1}] + \\ & \frac{\mu}{2} [a_{i+1}^j W_{i+1}^j - a_{i-1}^j W_{i-1}^j + a_{i+1}^{j-1} W_{i+1}^{j-1} - a_{i-1}^{j-1} W_{i-1}^{j-1}] \\ & = \frac{h}{3} [R_{i+1}^j + 4R_i^j + R_{i-1}^j]. \end{aligned} \quad (4.14)$$

Let $\rho = \frac{h}{\varepsilon}$ and taking the limit of Eq. (4.14) as $h \rightarrow 0$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{\sigma}{\rho} [W_{i+1}^j - 2W_i^j + W_{i-1}^j + W_{i+1}^{j-1} - 2W_i^{j-1} + W_{i-1}^{j-1}] + \\ & \lim_{h \rightarrow 0} \frac{\mu}{2} [a_{i+1}^j W_{i+1}^j - a_{i-1}^j W_{i-1}^j + a_{i+1}^{j-1} W_{i+1}^{j-1} - a_{i-1}^{j-1} W_{i-1}^{j-1}] = 0. \end{aligned} \quad (4.15)$$

4.4.3 Derivation of the Fitting Factor

In the framework of singular perturbation theory (O'malley et al., 1991), the solution of Eq. (3.1) at the (j) th time level admits an asymptotic decomposition that captures its inherent multiscale nature, characterized by smooth components together with boundary layer corrections. Motivated by this structure and following the exponential fitting strategy in Duressa & Mekonnen (2023), the fitting parameter σ is constructed so that the discrete scheme accurately reproduces the underlying exponential behavior of the exact solution within the layer regions. Consequently, σ is defined as

$$\sigma = \begin{cases} \frac{\rho\mu a^j(0)}{2} \coth\left(\frac{\rho\mu a^j(0)}{2}\right), & \text{for } x = 0, \\ \frac{\rho\mu a^j(1)}{2} \coth\left(\frac{\rho\mu a^j(1)}{2}\right), & \text{for } x = 1, \end{cases} \quad (4.16)$$

where $\rho = h/\varepsilon$. This formulation ensures that the numerical method remains stable and uniformly accurate, effectively resolving the sharp boundary layers without loss of accuracy as $\varepsilon \rightarrow 0$. Substituting Eq. (4.16) into (4.14) and solving for R_i^j gives

$$\begin{aligned}
& \left[\frac{\sigma}{\rho} - \frac{\mu}{2} a_{i-1}^j - \frac{h}{6} b_{i-1}^j - \frac{h}{3k} - \frac{h}{6} \mu (a')_{i-1}^j \right] W_{i-1}^j + \\
& \left[-\frac{2\sigma}{\rho} - \frac{2h}{3} b_i^j - \frac{4h}{3k} - \frac{2h}{3} \mu (a')_i^j \right] W_i^j + \\
& \left[\frac{\sigma}{\rho} + \frac{\mu}{2} a_{i+1}^j - \frac{h}{6} b_{i+1}^j - \frac{h}{3k} - \frac{h}{6} \mu (a')_{i+1}^j \right] W_{i+1}^j = \\
& \left[-\frac{\sigma}{\rho} + \frac{\mu}{2} a_{i-1}^{j-1} + \frac{h}{6} b_{i-1}^{j-1} - \frac{h}{3k} + \frac{h}{6} \mu (a')_{i-1}^{j-1} \right] W_{i-1}^{j-1} + \\
& \left[\frac{2\sigma}{\rho} + \frac{2h}{3} b_i^{j-1} - \frac{4h}{3k} + \frac{2h}{3} \mu (a')_i^{j-1} \right] W_i^{j-1} + \\
& \left[-\frac{\sigma}{\rho} - \frac{\mu}{2} a_{i+1}^{j-1} + \frac{h}{6} b_{i+1}^{j-1} - \frac{h}{3k} + \frac{h}{6} \mu (a')_{i+1}^{j-1} \right] W_{i+1}^{j-1} - \\
& \frac{h}{6} [\varpi_{i+1}^j + \varpi_{i+1}^{j-1}] - \frac{2h}{3} [\varpi_i^j + \varpi_i^{j-1}] - \frac{h}{6} [\varpi_{i-1}^j + \varpi_{i-1}^{j-1}].
\end{aligned} \tag{4.17}$$

The scheme in Eq. (4.17) can be written as three term recurrence relation:

$$\begin{aligned}
\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^j &= A_i^- W_{i-1}^j + A_i^c W_i^j + A_i^+ W_{i+1}^j = B_i^- W_{i-1}^{j-1} + B_i^c W_i^{j-1} + B_i^+ W_{i+1}^{j-1} + F_i^j, \\
W_i^1 &= \zeta_b(x_i), \quad W_0^j = \zeta_l(t_j), \quad W_{N+1}^j = \zeta_r(t_j),
\end{aligned} \tag{4.18}$$

for $i = 0, 1, \dots, N$ and $j = 1, 2, \dots, M + 1$ where

$$\left\{ \begin{aligned}
A_i^- &= \frac{\sigma}{\rho} - \frac{\mu}{2} a_{i-1}^j - \frac{h}{6} b_{i-1}^j - \frac{h}{3k} - \frac{h}{6} \mu (a')_{i-1}^j, \\
A_i^c &= -\frac{2\sigma}{\rho} - \frac{2h}{3} b_i^j - \frac{4h}{3k} - \frac{2h}{3} \mu (a')_i^j, \\
A_i^+ &= \frac{\sigma}{\rho} + \frac{\mu}{2} a_{i+1}^j - \frac{h}{6} b_{i+1}^j - \frac{h}{3k} - \frac{h}{6} \mu (a')_{i+1}^j, \\
B_i^- &= -\frac{\sigma}{\rho} + \frac{\mu}{2} a_{i-1}^{j-1} + \frac{h}{6} b_{i-1}^{j-1} - \frac{h}{3k} + \frac{h}{6} \mu (a')_{i-1}^{j-1}, \\
B_i^c &= \frac{2\sigma}{\rho} + \frac{2h}{3} b_i^{j-1} - \frac{4h}{3k} + \frac{2h}{3} \mu (a')_i^{j-1}, \\
B_i^+ &= -\frac{\sigma}{\rho} - \frac{\mu}{2} a_{i+1}^{j-1} + \frac{h}{6} b_{i+1}^{j-1} - \frac{h}{3k} + \frac{h}{6} \mu (a')_{i+1}^{j-1}, \\
F_i^j &= -\frac{h}{6} [\varpi_{i+1}^j + \varpi_{i+1}^{j-1}] - \frac{2h}{3} [\varpi_i^j + \varpi_i^{j-1}] - \frac{h}{6} [\varpi_{i-1}^j + \varpi_{i-1}^{j-1}].
\end{aligned} \right. \tag{4.19}$$

The resulting linear system is solved using the matrix inversion technique, ensuring an efficient and straightforward computational approach. Examining the tri-diagonal structure of the system defined by Eqs. (4.18) and (4.19), it is evident that the coefficients satisfy the inequalities:

$$|A_i^c| \geq |A_i^-| + |A_i^+| \quad \text{and} \quad |B_i^c| \geq |B_i^-| + |B_i^+|. \quad (4.20)$$

These relations demonstrate that the coefficient matrix associated with $\mathbb{L}_{\varepsilon,\mu}^{N,M}$ is irreducibly diagonally dominant, which is a key property in guaranteeing stability and uniqueness of the solution. According to Nichols (1989), such matrices are classified as M -matrices, and therefore they possess a well-defined inverse. Consequently, the linear system can be reliably solved using the matrix inversion method, ensuring both numerical stability and accuracy. The diagonal dominance and M -matrix property not only confirm solvability but also imply that the numerical scheme preserves monotonicity, which is particularly important for SPPs to prevent spurious oscillations near boundary layers. This makes the FOM especially suitable for problems where sharp gradients or layers are present.

4.5 Convergence Analysis

We have shown that the continuous solution and its derivatives are bounded. Errors from temporal discretization are quantifiable, and spatial error analysis ensures the stability and consistency of the fully discrete scheme.

Lemma 4.4. (*Discrete Maximum Principle*). Assume $\Xi^j(x)$ be a mesh function satisfying, $\Xi(0, t_j) \geq 0$, $\Xi(1, t_j) \geq 0$, and $\mathbb{L}_{\varepsilon,\mu}^{N,M} \Xi(x_i, t_j) \geq 0$ on $\mathcal{Q}_x^N \times \Lambda_t^M$. Then $\Xi(x_i, t_j) \geq 0$ at each point $\overline{\mathcal{Q}}_x^N \times \overline{\Lambda}_t^M$.

Proof. A detailed proof can be found in Chapter 3, Lemma 3.9. This Lemma indicates that $\mathbb{L}_{\varepsilon,\mu}^{N,M}$ satisfies the discrete maximum principle, yielding a monotone and irreducibly diagonally dominant M -matrix. This ensures the existence of a unique discrete solution. The following lemma establishes its uniform stability. $\square$

Lemma 4.5. The solution W_i^j of Eq. (3.1) at the j th time level satisfies the estimate

given below and is bounded:

$$\|W_i^j\| \leq \varrho^{-1} \|\mathbb{L}_{\varepsilon,\mu}^{N,M} W_i^j\| + \max \{|\zeta_l(t_j)|, |\zeta_r(t_j)|, |\zeta_b(x_i)|\}, \quad (4.21)$$

for $i = 1, 2, \dots, N-1$ and $0 < \varrho \leq b(x, t)$.

Proof. A detailed proof can be found in Chapter 3, Lemma 3.10. $\square$

To demonstrate uniform convergence of the discrete scheme in Eq. (4.18), let $TE_x(h)$ denote the LTE at the j th time level for the proposed method:

$$TE_x(h) = A_i^- W_{i-1}^j + A_i^c W_i^j + A_i^+ W_{i+1}^j - (B_i^- W_{i-1}^{j-1} + B_i^c W_i^{j-1} + B_i^+ W_{i+1}^j + F) \quad (4.22)$$

To calculate the spatial truncation error, use Eq. (4.18) at $x_v, v = i, i \pm 1$ in Eq. (4.22).

$$\begin{aligned} TE_x(h) = & \left[A_i^- W_{i-1}^j - h (\varepsilon W_{xx}^j + \mu a^j(x) W_x^j - \beta^j(x) W^j)_{i-1} \right] + \\ & \left[A_i^c W_i^j - h (\varepsilon W_{xx}^j + \mu a^j(x) W_x^j - \beta^j(x) W^j)_i \right] + \\ & \left[A_i^+ W_{i+1}^j - h (\varepsilon W_{xx}^j + \mu a^j(x) W_x^j - \beta^j(x) W^j)_{i+1} \right] + \\ & \left[B_i^- W_{i-1}^{j-1} - h (\varepsilon W_{xx}^{j-1} + \mu a^{j-1}(x) W_x^{j-1} - \beta^{j-1}(x) W^{j-1})_{i-1} \right] + \\ & \left[B_i^c W_i^{j-1} - h (\varepsilon W_{xx}^{j-1} + \mu a^{j-1}(x) W_x^{j-1} - \beta^{j-1}(x) W^{j-1})_i \right] + \\ & \left[B_i^+ W_{i+1}^{j-1} - h (\varepsilon W_{xx}^{j-1} + \mu a^{j-1}(x) W_x^{j-1} - \beta^{j-1}(x) W^{j-1})_{i+1} \right] \end{aligned} \quad (4.23)$$

Replacing a Taylor series expansion of $W_{i\pm 1}^j, W_{i\pm 1}^{j-1}, (W_x)_{i\pm 1}^j, (W_x)_{i\pm 1}^{j-1}, (W_{xx})_{i\pm 1}^j$ and $(W_{xx})_{i\pm 1}^{j-1}$ into Eq. (4.23), we have

$$\begin{aligned} TE_x(h) = & [\varpi_0^j W_i^j + \varpi_0^{j-1} W_i^{j-1}] + [\varpi_1^j W_{x,i}^j + \varpi_1^{j-1} W_{x,i}^{j-1}] + \\ & [\varpi_2^j W_{xx,i}^j + \varpi_2^{j-1} W_{xx,i}^{j-1}] + [\varpi_3^j W_{xxx,i}^j + \varpi_3^{j-1} W_{xxx,i}^{j-1}], \end{aligned} \quad (4.24)$$

where

$$\begin{aligned} \varpi_0^j &= A_i^- + A_i^c + A_i^+ + h (\beta_{i-1}^j + \beta_i^j + \beta_{i+1}^j), \\ \varpi_0^{j-1} &= - (B_i^- + B_i^c + B_i^+) + h (\beta_{i-1}^{j-1} + \beta_i^{j-1} + \beta_{i+1}^{j-1}), \\ \varpi_1^j &= h (A_i^+ - A_i^-) - h\mu (a_{i-1}^j + a_i^j + a_{i+1}^j) - h^2 (\beta_{i-1}^j - \beta_{i+1}^j), \\ \varpi_1^{j-1} &= h (B_i^- - B_i^+) - h\mu (a_{i-1}^{j-1} + a_i^{j-1} + a_{i+1}^{j-1}) + h^2 (\beta_{i+1}^{j-1} - \beta_{i-1}^{j-1}), \end{aligned}$$

$$\begin{aligned}
\varpi_2^j &= \frac{h^2}{2} (A_i^- + A_i^+) - 3h\varepsilon + h^2\mu (a_{i-1}^j - a_{i+1}^j) + \frac{h^3}{2} (\beta_{i-1}^j + \beta_{i+1}^j), \\
\varpi_2^{j-1} &= -\frac{h^2}{2} (A_i^+ + A_i^-) - 3h\varepsilon + h^2\mu (a_{i-1}^{j-1} - a_{i+1}^{j-1}) + \frac{h^3}{2} (\beta_{i-1}^{j-1} + \beta_{i+1}^{j-1}), \\
\varpi_3^j &= \frac{h^3}{6} (A_i^+ - A_i^-) - \frac{h^3}{2}\mu (a_{i-1}^j + a_{i+1}^j) - \frac{h^4}{6} (\beta_{i-1}^j - \beta_{i+1}^j), \\
\varpi_3^{j-1} &= \frac{h^3}{6} (B_i^- - B_i^+) - \frac{h^3}{2}\mu (a_{i-1}^{j-1} + a_{i+1}^{j-1}) - \frac{h^4}{6} (\beta_{i+1}^{j-1} - \beta_{i-1}^{j-1}).
\end{aligned}$$

Substituting and simplifying the values of $A_i^-, A_i^c, A_i^+, B_i^-, B_i^c, B_i^+$ and restricting the expansion of coefficients to their first term gives $\varpi_0^j = \varpi_0^{j-1} = \varpi_1^j = \varpi_1^{j-1} = 0$ and

$$\begin{aligned}
\varpi_2^j &= \frac{h^2\sigma}{\rho} - 3h\varepsilon - \frac{h^3}{6} \left(11(a')_i^j + 5\beta_j^i \right) + O(h^3), \\
\varpi_2^{j-1} &= \frac{h^2\sigma}{\rho} - 3h\varepsilon - \frac{h^3}{6} \left(11(a')_i^{j-1} + 5\beta_{j-1}^i \right) + O(h^3).
\end{aligned}$$

Thus using the boundedness of coefficient functions, we get

$$|TE_x(h)| \leq \varpi_2^j W_{xx,i}^j + \varpi_2^{j-1} W_{xx,i}^{j-1} \leq h\varepsilon(\sigma - 3)W_{xx,i}^j, \quad \text{since } \rho = \frac{h}{\varepsilon}. \quad (4.25)$$

By utilizing the series representation of the hyperbolic cotangent function, the following inequality is established.

$$\coth(x) = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + O(x^5) \Rightarrow x \coth(x) = 1 + \frac{x^2}{3} + O(x^4),$$

where $x = \frac{\rho\mu a^j(x_i)}{2}$. Now $|\sigma - 3| = \left| 1 + \frac{\left(\frac{\rho\mu a^j(x_i)}{2} \right)^2}{3} - 3 \right| \leq C\rho^2\mu^2$. Then

$$|TE_x(h)| \leq h\varepsilon (C\rho^2\mu^2) W_{xx,i}^j = Ch^3 \frac{\mu^2}{\varepsilon} W_{xx,i}^j. \quad (4.26)$$

Thus, considering the ratio of μ^2/ε and applying Lemma 3.8, we have

$$|TE_x(h)| = Ch^3 \frac{\mu^2}{\varepsilon}. \quad (4.27)$$

The numerical scheme Eq. (4.18) can be written as

$$\left(\frac{\sigma}{\rho} + u_i \right) W_{i-1}^j + \left(-\frac{2\sigma}{\rho} + y_i \right) W_i^j + \left(\frac{\sigma}{\rho} + z_i \right) W_{i+1}^j + K_i + TE_i = 0, \quad (4.28)$$

where

$$\left\{ \begin{array}{l} u_i = -\frac{\mu}{2}a_{i-1}^j - \frac{h}{6}b_{i-1}^j - \frac{h}{3k} - \frac{h}{6}\mu(a')_{i-1}^j, \\ y_i = -\frac{h}{6}b_{i+1}^j - \frac{h}{3k} - \frac{h}{6}\mu(a')_{i+1}^j, \\ z_i = \frac{\mu}{2}a_{i+1}^j - \frac{h}{6}b_{i+1}^j - \frac{h}{3k} - \frac{h}{6}\mu(a')_{i+1}^j, \\ K_i = \begin{bmatrix} -\frac{\sigma}{\rho} + \frac{\mu}{2}a_{i-1}^{j-1} + \frac{h}{6}b_{i-1}^{j-1} - \frac{h}{3k} + \frac{h}{6}\mu(a')_{i-1}^{j-1} \\ \frac{2\sigma}{\rho} + \frac{2h}{3}b_i^{j-1} - \frac{4h}{3k} + \frac{2h}{3}\mu(a')_i^{j-1} \\ -\frac{\sigma}{\rho} - \frac{\mu}{2}a_{i+1}^{j-1} + \frac{h}{6}b_{i+1}^{j-1} - \frac{h}{3k} + \frac{h}{6}\mu(a')_{i+1}^{j-1} \end{bmatrix} W_{i-1}^{j-1} + \\ \begin{bmatrix} \frac{2\sigma}{\rho} + \frac{2h}{3}b_i^{j-1} - \frac{4h}{3k} + \frac{2h}{3}\mu(a')_i^{j-1} \\ -\frac{\sigma}{\rho} - \frac{\mu}{2}a_{i+1}^{j-1} + \frac{h}{6}b_{i+1}^{j-1} - \frac{h}{3k} + \frac{h}{6}\mu(a')_{i+1}^{j-1} \end{bmatrix} W_i^{j-1} + \\ \begin{bmatrix} -\frac{\sigma}{\rho} + \frac{\mu}{2}a_{i-1}^{j-1} + \frac{h}{6}b_{i-1}^{j-1} - \frac{h}{3k} + \frac{h}{6}\mu(a')_{i-1}^{j-1} \\ -\frac{\sigma}{\rho} - \frac{\mu}{2}a_{i+1}^{j-1} + \frac{h}{6}b_{i+1}^{j-1} - \frac{h}{3k} + \frac{h}{6}\mu(a')_{i+1}^{j-1} \end{bmatrix} W_{i+1}^{j-1} - \\ \frac{h}{6} [\varpi_{i+1}^j + \varpi_{i+1}^{j-1}] - \frac{2h}{3} [\varpi_i^j + \varpi_i^{j-1}] - \frac{h}{6} [\varpi_{i-1}^j + \varpi_{i-1}^{j-1}]. \end{array} \right. \quad (4.29)$$

By enforcing the boundary conditions $W^j(0) = \zeta_l(t^j)$ and $W^j(1) = \zeta_r(t^j)$ on Eq. (4.28), a system of algebraic equations corresponding to each time level $t_j = jk$, can be formulated in matrix representation as follows:

$$[E + Z]W + P + TE_x(h) = 0, \quad (4.30)$$

where

$$E = \left(\frac{\sigma}{\rho}, -\frac{2\sigma}{\rho}, \frac{\sigma}{\rho} \right) = \begin{bmatrix} -\frac{2\sigma}{\rho} & \frac{\sigma}{\rho} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{\sigma}{\rho} & -\frac{2\sigma}{\rho} & \frac{\sigma}{\rho} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{\sigma}{\rho} & -\frac{2\sigma}{\rho} & \frac{\sigma}{\rho} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{\sigma}{\rho} & -\frac{2\sigma}{\rho} & \frac{\sigma}{\rho} \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{\sigma}{\rho} & -\frac{2\sigma}{\rho} \end{bmatrix},$$

$$Z = (u_i, y_i, z_i) = \begin{bmatrix} y_1 & z_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ u_2 & y_2 & z_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & u_3 & y_3 & z_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & u_{N-2} & y_{N-2} & z_{N-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & u_{N-1} & y_{N-1} \end{bmatrix},$$

$P = \left[K_1 + \left(\frac{\sigma}{\rho} + u_1 \right) \zeta_l^j(0), K_2, K_3, \dots, K_{N-2}, K_{N-1} + \left(\frac{\sigma}{\rho} + Z_{N-1} \right) \zeta_r^j(1) \right]$,
 $W = [W_1^j, W_2^j, \dots, W_{N-1}^j]^T$, and the truncation error at the mesh points is $TE_i = [TE_1, \dots, TE_{N-1}]^T$. Let $\bar{W} = [\bar{W}_1^j, \dots, \bar{W}_{N-1}^j]^T \cong W$ satisfy the following equation

$$[E + Z] \bar{W} + P = 0 \quad (4.31)$$

Subtracting Eq. (4.30) from Eq. (4.31) yields:

$$[E + Z] \Xi = TE_x(h), \quad (4.32)$$

where $\Xi = \bar{W}_i^j - W_i^j = [e_1, e_2, \dots, e_{i-1}]^T$ is the discretization error. Let $|a_i| \leq \chi_1$, $|b_i| \leq \chi_2$ and $|a'_i| \leq \chi_3$, so that if $z_{i,j}$ is the (i, j) th element of the matrix Z , then we have

$$\begin{aligned} |Z_{i,i+1}| = |z_i| &\leq \frac{\mu}{2} \chi_1 - \frac{h}{6} (\chi_2 + \mu \chi_3) - \frac{h}{3k}, \quad i = 1, 2, \dots, N-2, \\ |Z_{i,i-1}| = |w_i| &\leq -\frac{\mu}{2} \chi_1 - \frac{h}{6} (\chi_2 + \mu \chi_3) - \frac{h}{3k}, \quad i = 2, \dots, N-1. \end{aligned}$$

Thus, as $h \rightarrow 0$ or h is sufficiently small,

$$\begin{cases} -\frac{\sigma}{\rho} + |Z_{i,i+1}| \leq \left(-\frac{\sigma}{\rho} + \frac{\mu}{2} \chi_1 \right) \neq 0, & i = 1, 2, \dots, N-2 \\ -\frac{\sigma}{\rho} + |Z_{i,i-1}| \leq \left(-\frac{\sigma}{\rho} - \frac{\mu}{2} \chi_1 \right) \neq 0, & i = 2, \dots, N-1. \end{cases} \quad (4.33)$$

Hence, the matrix $[E + Z]$ is irreducible (Daba & Duressa, 2022). For the j th time level of matrix $(E + Z)$, let S_i be the sum of elements of the i th row. Hence, we have

$$\begin{aligned} S_1 &= \sum_{j=1}^{N-1} z_{1,j} = -\frac{\sigma}{\rho} + \frac{\mu}{2} a_{i+1}^j - \frac{2h}{3} b_i^j - \frac{h}{6} b_{i+1}^j - \frac{2h}{3} \mu (a')_i^j - \frac{h}{6} (a')_{i+1}^j - \frac{5h}{3k}, \quad \text{for } i = 1, \\ S_{N-1} &= \sum_{j=1}^{N-1} z_{N-1,j} = -\frac{\sigma}{\rho} - \frac{\mu}{2} a_{i-1}^j - \frac{2h}{3} b_i^j - \frac{h}{6} b_{i-1}^j - \frac{2h}{3} \mu (a')_i^j - \frac{h}{6} (a')_{i-1}^j - \frac{5h}{3k}, \quad \text{for } i = N, \\ S_i &= \sum_{j=2}^{N-1} z_{i,j} = \frac{\mu}{2} (a_{i+1}^j - a_{i-1}^j) - \frac{h}{6} (b_{i-1}^j + 4a_i^j + b_{i+1}^j) - \\ &\quad \frac{h\mu}{6} ((a')_{i-1}^j + 4(a')_i^j + (a')_{i+1}^j) - \frac{2h}{k} + O(h^2), \quad \text{for } i = 2(1)N-2. \end{aligned} \quad (4.34)$$

For sufficiently small h , the matrix $(E + Z)$ is monotone (Daba & Duressa, 2022), so

$(E + Z)^{-1}$ exists and $(E + Z)^{-1} \geq 0$. Hence, from Eq. (4.32), we have

$$\Xi = (E + Z)^{-1} T E_x(h). \quad (4.35)$$

An alternative representation of Eq. (4.35) can be formulated as follows:

$$\|\Xi\| = \|(E + Z)^{-1}\| \|T E_x(h)\|. \quad (4.36)$$

Let $\bar{z}_{ki}$ denote the (k, i) th entry of the matrix $(E + Z)^{-1}$. Since $\bar{z}_{ki} \geq 0$, the corresponding matrix operator can be expressed as follows:

$$\sum_{i=1}^{N-1} \bar{z}_{ki} \times S_i = 1, \quad k = 1, 2, \dots, N-1. \quad (4.37)$$

Consequently, Eq. (4.37) is obtained.

$$\sum_{i=1}^{N-1} \bar{z}_{ki} \leq \frac{1}{\min_{1 \leq i \leq N-1} S_i} \leq \frac{1}{\Theta_i} \leq \frac{1}{|\Theta_i|}, \quad k = 1, 2, \dots, N-1, \quad (4.38)$$

where $\Theta_i = \min_{1 \leq i \leq N} \{S_0, S_i, S_{N-1}\}$. Thus, from Eqs. (4.28), (4.36) and (4.38), we obtain:

$$\Xi_i = \sum_{i=1}^{N-1} \bar{z}_{ki} T E_i(h) \leq \frac{O(h^2)}{|\Theta_i|}, \quad i = 1, 2, \dots, N-1. \quad (4.39)$$

For this reason, it follows that $\|\Xi\| \leq O(h^2)$. This clearly indicates that the semi-discretization procedure employed in the spatial domain exhibits a second-order (quadratic) rate of convergence, thereby confirming the accuracy and robustness of the proposed numerical scheme in capturing spatial variations.

Theorem 4.1. (*Uniform convergence*). Let $w(x_i, t_j)$ and W_i^j denote the solutions of the continuous problem (3.1) and the discrete problem (4.18), respectively. The error associated with the fully discrete scheme can then be estimated by the inequality

$$|w(x_i, t_j) - W_i^j| \leq C (k^2 + h^2), \quad (4.40)$$

which demonstrates that the numerical method attains second-order accuracy in both the temporal and spatial discretizations.

4.6 Numerical Examples, Results and Discussion

This section presents extensive numerical experiments, reporting maximum absolute errors and corresponding convergence rates for the proposed scheme. All computations, carried out in MATLAB, validate the theoretical findings and are demonstrated through three illustrative examples. The maximum pointwise error $E_{\varepsilon,\mu}^{N,M}$ and the convergence rate $\bar{R}_{\varepsilon,\mu}^{N,M}$ of the scheme are evaluated through the double-mesh technique described in Chapter 3, Section 3.6.

Example 4.1. Consider singularly perturbed IBVP in [Shivhare et al. \(2021\)](#)

$$\begin{cases} \frac{\partial w}{\partial t} - \varepsilon \frac{\partial^2 w}{\partial x^2} - \mu(1+x) \frac{\partial w}{\partial x} + w = -16x^2(1-x)^2, \\ w(x, 0) = 0, w(0, t) = 0 = w(1, t). \end{cases}$$

Example 4.2. Consider TP-SPPPs in [Duressa & Mekonnen \(2023\)](#)

$$\begin{cases} \frac{\partial w}{\partial t} - \varepsilon \frac{\partial^2 w}{\partial x^2} - \mu(1 + \exp(x)) \frac{\partial w}{\partial x} + (1 + x^5)w = -10 \exp(t^2)x^2(1 - x^2), \\ w(x, 0) = 0, w(0, t) = 0 = w(1, t). \end{cases}$$

Example 4.3. Consider the singularly perturbed parabolic IBVP in [Aga et al. \(2019\)](#)

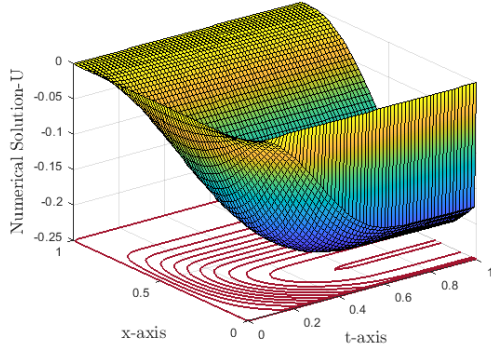
$$\begin{cases} \frac{\partial w}{\partial t} - \varepsilon \frac{\partial^2 w}{\partial x^2} + (1 + x^2 + \frac{1}{2} \sin(\pi x)) \frac{\partial w}{\partial x} + (1 + x^2 \frac{1}{2} \sin(\frac{\pi t}{2}))w = \\ \quad x^3(1-x)^3 t(1-t) \sin(\pi t), \\ w(x, 0) = 0, w(0, t) = 0 = w(1, t). \end{cases}$$

Example 4.3 exhibits a distinct structure compared to Examples 4.1 and 4.2. While the first two involve TP-SPPPs, Example 4.3 is governed by a single perturbation parameter, making its analytical and numerical behavior comparatively different. To further examine the effectiveness and adaptability of the proposed numerical integration technique, we introduce this additional example as a complementary test case.

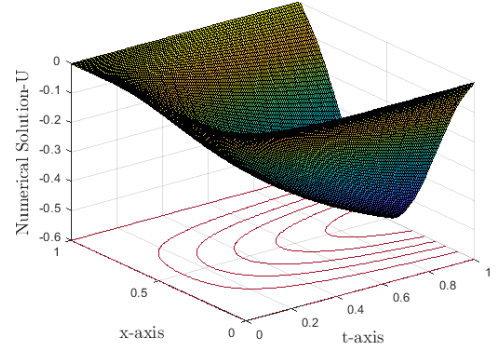
Using the derived formulas, numerical results for the test examples are presented through tables and plots. Table 4.1 lists the maximum absolute errors $E_{\varepsilon,\mu}^{N,M}$ and cor-

Table 4.1: Maximum pointwise error and rate of convergence for Example 4.1 for $\mu = 10^{-6}$ and different values of ε .

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 16$	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
10^{-8}	$8.9594e-05$	$2.2407e-05$	$5.6004e-06$	$1.3999e-06$	$3.4995e-07$	$8.7487e-08$
	1.9995	2.0003	2.0002	2.0001	2.0000	—
10^{-10}	$8.9592e-05$	$2.2399e-05$	$5.5889e-06$	$1.3911e-06$	$3.4476e-07$	$8.4720e-08$
	1.9009	2.0028	2.0063	2.0126	2.0000	—
10^{-12}	$8.9592e-05$	$2.2399e-05$	$5.5889e-06$	$1.3911e-06$	$3.4476e-07$	$8.4720e-08$
	2.0000	2.0028	2.0063	2.0126	2.0248	—
10^{-14}	$8.9592e-05$	$2.2399e-05$	$5.5889e-06$	$1.3911e-06$	$3.4476e-07$	$8.4720e-08$
	2.0000	2.0028	2.0063	2.0126	2.0248	—
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-22}	$8.9592e-05$	$2.2399e-05$	$5.5889e-06$	$1.3911e-06$	$3.4476e-07$	$8.4720e-08$
	2.0000	2.0028	2.0063	2.0126	2.0248	—

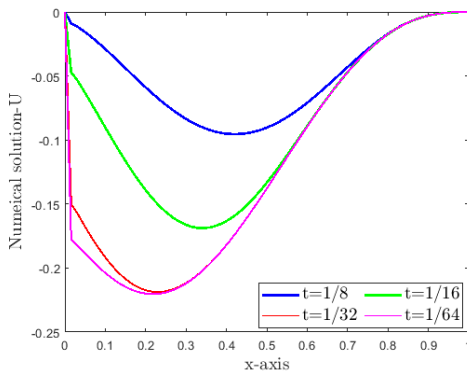


(a) $N = M = 64, \varepsilon_1 = 10^{-8}, \mu = 1$

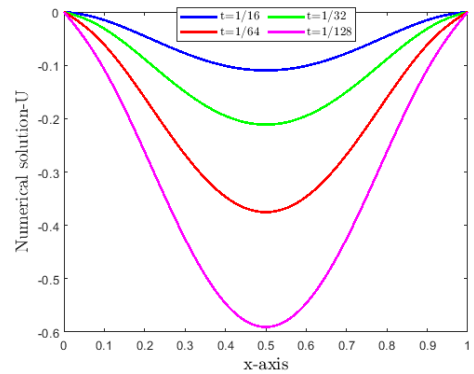


(b) $N = M = 128, \varepsilon_1 = 10^{-2}, \mu = 10^{-10}$

Figure 4.1: The numerical solution profile for Example 4.1.



(a) $N = M = 64, \varepsilon_1 = 10^{-8}, \mu = 1$



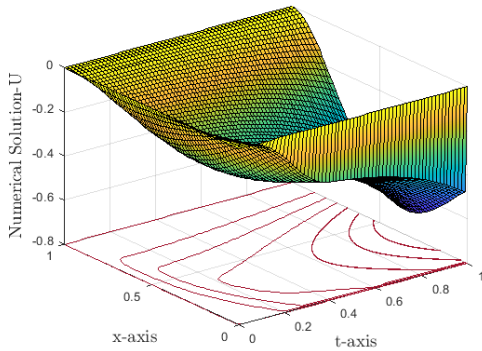
(b) $N = M = 128, \varepsilon_1 = 10^{-8}, \mu = 10^{-10}$

Figure 4.2: The behavior of spatial vs numerical solution graph for Example 4.1.

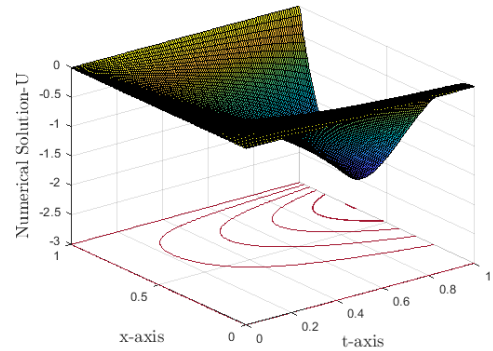
responding convergence rates $\overline{R}_{\varepsilon, \mu}^{N, M}$ for Example 4.1, obtained via the proposed fitted integration technique for improved accuracy and stability. As shown in Table 4.2, the results not only confirm the theoretical predictions but also outperform those in

Table 4.2: Comparison of maximum pointwise error and rate of convergence for Example 4.1 with $\mu = 10^{-6}$ and various ε values.

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 10$	$M = 20$	$M = 40$	$M = 80$	$M = 160$
Present Method					
10^{-2}	$2.2527e-04$	$5.6208e-05$	$1.4055e-05$	$3.5132e-06$	$8.7826e-07$
	2.0028	1.9997	2.0002	2.0001	
10^{-4}	$2.2889e-04$	$6.1601e-05$	$1.6622e-05$	$3.8331e-06$	$9.4099e-07$
	1.8936	1.8900	2.1165	2.0263	
10^{-6}	$2.2902e-04$	$5.7426e-05$	$1.4367e-05$	$3.5923e-06$	$8.9812e-07$
	1.9957	1.9990	1.9998	1.9999	
10^{-8}	$2.2898e-04$	$5.7415e-05$	$1.4364e-05$	$3.5915e-06$	$8.9791e-07$
	1.9957	1.9990	1.9998	2.0000	
10^{-10}	$2.2898e-04$	$5.7404e-05$	$1.4355e-05$	$3.5863e-06$	$8.9507e-07$
	1.9960	1.9996	2.0001	2.0024	
10^{-12}	$2.2898e-04$	$5.7404e-05$	$1.4355e-05$	$3.5863e-06$	$8.9507e-07$
	1.9960	1.9996	2.0001	2.0024	
Result from Shivhare et al. (2021) (by Quadratic B-spline Method)					
10^{-2}	$8.6053e-03$	$4.4951e-03$	$2.2857e-03$	$1.1438e-03$	$5.7093e-04$
	0.937	0.976	0.999	1.002	
10^{-4}	$8.6095e-03$	$4.4529e-03$	$2.2631e-03$	$1.1406e-03$	$5.7256e-04$
	0.951	0.976	0.988	0.994	
10^{-6}	$8.6014e-03$	$4.4513e-03$	$2.2628e-03$	$1.1406e-03$	$5.7356e-04$
	0.950	0.976	0.988	0.994	
10^{-8}	$8.6006e-03$	$4.4512e-03$	$2.2628e-03$	$1.1406e-03$	$5.7356e-04$
	0.950	0.976	0.988	0.994	
10^{-10}	$8.6006e-03$	$4.4512e-03$	$2.2628e-03$	$1.1406e-03$	$5.7356e-04$
	0.950	0.976	0.988	0.994	
10^{-12}	$8.6006e-03$	$4.4512e-03$	$2.2628e-03$	$1.1406e-03$	$5.7356e-04$
	0.950	0.976	0.988	0.994	



(a) $N = M = 64, \varepsilon_1 = 10^{-6}, \mu = 1$

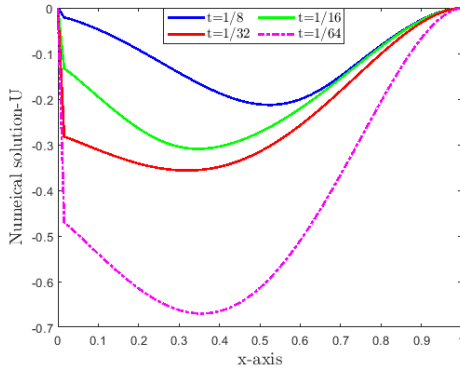


(b) $N = M = 128, \varepsilon_1 = 10^{-12}, \mu = 10^{-4}$

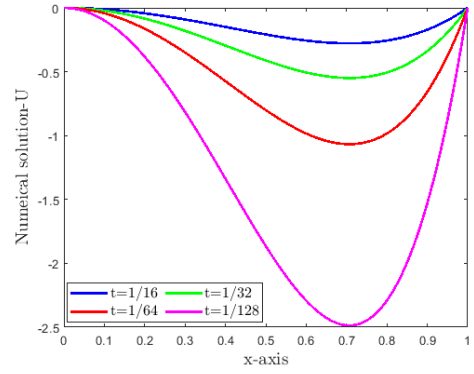
Figure 4.3: The numerical solution profile for Example 4.2.

Table 4.3: Maximum pointwise error and rate of convergence for Example 4.1 for $\varepsilon = 10^{-6}$ and different values of μ .

μ	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 10$	$M = 20$	$M = 40$	$M = 80$	$M = 160$
10^{-4}	$2.2561e-04$	$5.6241e-05$	$1.4034e-05$	$3.5065e-06$	$9.7711e-07$
	2.0041	2.0027	2.0008	1.8434	
10^{-6}	$2.2902e-04$	$5.7426e-05$	$1.4367e-05$	$3.5923e-06$	$8.9812e-07$
	1.9957	1.9990	1.9998	1.9999	
10^{-8}	$2.2902e-04$	$5.7426e-05$	$1.4367e-05$	$3.5923e-06$	$8.9812e-07$
	1.9957	1.9989	1.9998	1.9999	
10^{-10}	$2.2902e-04$	$5.7426e-05$	$1.4367e-05$	$3.5923e-06$	$8.9812e-07$
	1.9957	1.9989	1.9998	1.9999	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-22}	$2.2902e-04$	$5.7426e-05$	$1.4367e-05$	$3.5923e-06$	$8.9812e-07$
	1.9957	1.9989	1.9998	1.9999	

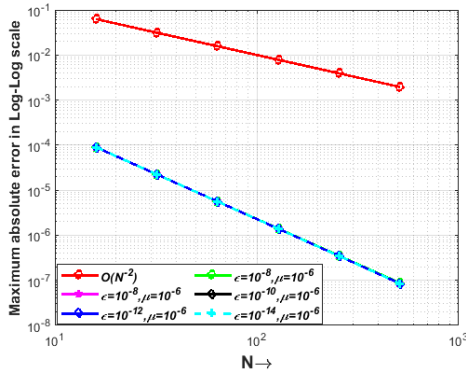


(a) $N = M = 64, \varepsilon_1 = 10^{-6}, \mu = 1$

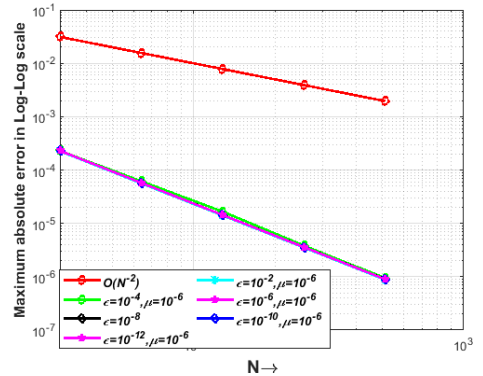


(b) $N = M = 128, \varepsilon_1 = 10^{-12}, \mu = 10^{-4}$

Figure 4.4: The behavior of spatial vs numerical solution graph for Example 4.2.



(a) Table 4.1



(b) Table 4.2

Figure 4.5: The Log-Log plot for Example 4.1.

Table 4.4: Comparison of maximum pointwise error and rate of convergence for Example 4.2 with $\mu = 10^{-6}$ and various ε values.

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 10$	$M = 20$	$M = 40$	$M = 80$	$M = 160$
Present Method					
10^{-2}	$3.9346e-03$	$1.1105e-03$	$2.9572e-04$	$7.6325e-05$	$1.9390e-05$
	1.8250	1.9089	1.9540	1.9768	
10^{-4}	$4.1211e-03$	$1.1714e-03$	$3.1261e-04$	$8.0790e-05$	$2.0536e-05$
	1.8148	1.9058	1.9521	1.9762	
10^{-6}	$4.1228e-03$	$1.1719e-03$	$3.1276e-04$	$8.0827e-05$	$2.0546e-05$
	1.8148	1.9057	1.9521	1.9760	
10^{-8}	$4.1226e-03$	$1.1718e-03$	$3.1272e-04$	$8.0817e-05$	$2.0543e-05$
	1.8148	1.9058	1.9521	1.9760	
10^{-10}	$4.1226e-03$	$1.1717e-03$	$3.1268e-04$	$8.0789e-05$	$2.0526e-05$
	1.8150	1.9058	1.9525	1.9767	
10^{-12}	$4.1226e-03$	$1.1717e-03$	$3.1268e-04$	$8.0789e-05$	$2.0526e-05$
	1.8150	1.9058	1.9525	1.9767	
Result in Shivhare et al. (2021) (By Quadratic B-spline Collocation Method)					
10^{-2}	$3.6825e-02$	$1.8188e-02$	$8.5040e-03$	$4.2227e-03$	$2.1179e-03$
	1.018	1.097	1.010	0.995	
10^{-4}	$3.9442e-02$	$1.9359e-02$	$9.5692e-03$	$4.7539e-03$	$2.3691e-03$
	1.027	1.016	1.009	1.005	
10^{-6}	$3.9402e-02$	$1.9391e-02$	$9.5773e-03$	$4.7594e-03$	$2.3717e-03$
	1.023	1.018	1.009	1.005	
10^{-8}	$3.9418e-02$	$1.9392e-02$	$9.5791e-03$	$4.7594e-03$	$2.3718e-03$
	1.023	1.017	1.009	1.005	
10^{-10}	$3.9420e-02$	$1.9392e-02$	$9.5791e-03$	$4.7594e-03$	$2.3718e-03$
	1.023	1.017	1.009	1.005	
10^{-12}	$3.9420e-02$	$1.9392e-02$	$9.5791e-03$	$4.7594e-03$	$2.3718e-03$
	1.023	1.017	1.009	1.005	

Table 4.5: Maximum pointwise error for Example 4.1 with $M = N = 64$.

$\mu \downarrow \varepsilon \rightarrow$	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}
10^{-6}	$5.6122e-06$	$5.6008e-06$	$5.5891e-06$	$5.5891e-06$	$5.5891e-06$
10^{-8}	$5.6121e-06$	$5.6136e-06$	$5.6135e-06$	$5.6133e-06$	$5.6133e-06$
10^{-10}	$5.6121e-06$	$5.6136e-06$	$5.6136e-06$	$5.6136e-06$	$5.6136e-06$
10^{-12}	$5.6121e-06$	$5.6136e-06$	$5.6136e-06$	$5.6136e-06$	$5.6136e-06$
10^{-14}	$5.6121e-06$	$5.6136e-06$	$5.6136e-06$	$5.6136e-06$	$5.6136e-06$

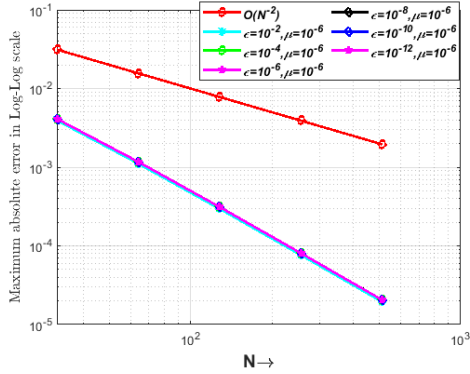
[Shivhare et al. \(2021\)](#), demonstrating the robustness and efficiency of the proposed scheme. Table 4.3 reports the errors and convergence rates for $\varepsilon = 10^{-6}$ under varying mesh sizes and decreasing μ , confirming the method's stability and uniform convergence even under strong perturbations. Furthermore, Tables 4.5 and 4.6 show that

Table 4.6: Maximum pointwise error for Example 4.2 with $M = N = 32$.

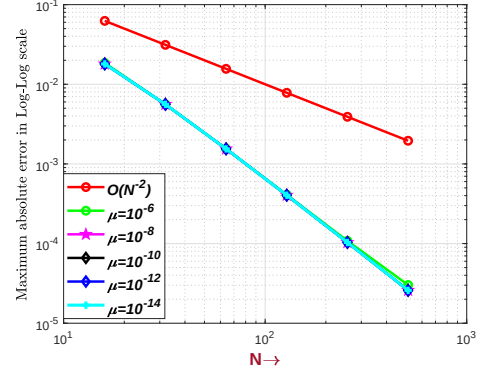
$\mu \downarrow \varepsilon \rightarrow$	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
10^{-4}	$9.6360e-04$	$7.9007e-04$	$7.9067e-04$	$7.9067e-04$	$7.9067e-04$
10^{-6}	$5.1809e-04$	$4.8024e-04$	$4.7994e-04$	$4.7993e-04$	$4.7993e-04$
10^{-8}	$5.1359e-04$	$4.8024e-04$	$4.8024e-04$	$4.8024e-04$	$4.8024e-04$
10^{-10}	$5.1355e-04$	$4.8024e-04$	$4.8024e-04$	$4.8024e-04$	$4.8024e-04$
10^{-12}	$5.1355e-04$	$4.8024e-04$	$4.8024e-04$	$4.8024e-04$	$4.8024e-04$

Table 4.7: Maximum pointwise error and convergence rate for Example 4.2 with different values of μ and $\varepsilon = 10^{-7}$.

$\mu \downarrow$	$(N, M) = 16$	$(N, M) = 32$	$(N, M) = 64$	$(N, M) = 128$	$(N, M) = 256$	$(N, M) = 512$
10^{-6}	$1.8528e-02$	$5.6001e-03$	$1.5493e-03$	$4.0953e-04$	$1.0811e-04$	$3.0213e-05$
	1.7262	1.8538	1.9196	1.9215	1.8393	—
10^{-8}	$1.7993e-02$	$5.5951e-03$	$1.5444e-03$	$4.0459e-04$	$1.0330e-04$	$2.5906e-05$
	1.6852	1.8571	1.9325	1.9696	1.9955	—
10^{-10}	$1.7993e-02$	$5.5951e-03$	$1.5443e-03$	$4.0454e-04$	$1.0325e-04$	$2.5862e-05$
	1.6852	1.8572	1.9326	1.9701	1.9972	—
10^{-12}	$1.7993e-02$	$5.5951e-03$	$1.5443e-03$	$4.0454e-04$	$1.0325e-04$	$2.5862e-05$
	1.6852	1.8572	1.9326	1.9701	1.9972	—
10^{-14}	$1.7993e-02$	$5.5951e-03$	$1.5443e-03$	$4.0454e-04$	$1.0325e-04$	$2.5862e-05$
	1.6852	1.8572	1.9326	1.9701	1.9972	—



(a) Table 4.4



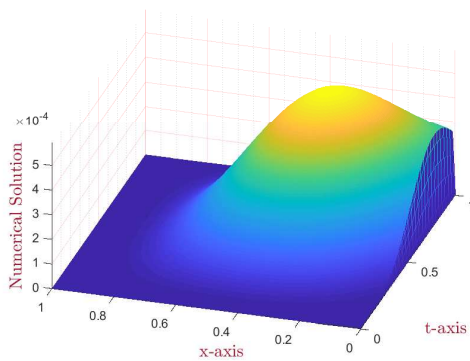
(b) Table 4.7

Figure 4.6: The Log-Log plot for Example 4.2.

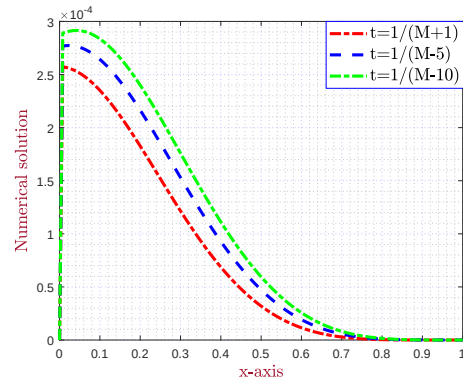
the maximum error remains stable as $(\varepsilon, \mu) \rightarrow 0$, verifying parameter-uniform convergence and mesh refinement consistency. The results summarized in Tables 4.1-4.7 closely align with theoretical estimates, confirming second-order accuracy. Comparative analysis in Tables 4.2 and 4.4 further emphasize the higher accuracy of the proposed approach over existing methods. Finally, Table 4.8 compares the scheme with those in [Aga et al. \(2019\)](#); [Mekonnen & Duressa \(2020\)](#) for Example 4.3, showing consistently smaller errors and superior overall performance.

Table 4.8: Comparison of the maximum pointwise errors for Example 4.3 corresponding to different values of ε .

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 16$	$M = 32$	$M = 64$	$M = 128$	$M = 256$
Present Method					
10^{-0}	$5.2925e-07$	$1.1702e-07$	$2.8905e-08$	$7.2239e-09$	$1.8057e-09$
10^{-1}	$2.4156e-06$	$6.0369e-07$	$1.5077e-07$	$3.7683e-08$	$9.4202e-09$
10^{-2}	$1.3617e-05$	$4.3614e-06$	$1.1815e-06$	$3.0182e-07$	$7.5863e-08$
10^{-3}	$1.6642e-05$	$8.4431e-06$	$4.2416e-06$	$1.9013e-06$	$6.5198e-07$
10^{-4}	$1.6642e-05$	$8.4435e-06$	$4.2763e-06$	$2.1531e-06$	$1.0806e-06$
10^{-5}	$1.6642e-05$	$8.4435e-06$	$4.2763e-06$	$2.1531e-06$	$1.0806e-06$
Result in Aga et al. (2019) (By Quadratic B-spline Collocation Method)					
10^{-0}	$8.7112e-05$	$1.8949e-05$	$4.1396e-06$	$1.0346e-06$	$2.5881e-07$
10^{-1}	$3.4987e-04$	$8.7838e-05$	$2.1985e-05$	$5.4998e-06$	$1.3751e-06$
10^{-2}	$1.7357e-03$	$6.7495e-04$	$1.9879e-04$	$5.2071e-05$	$1.3176e-05$
10^{-3}	$1.8971e-03$	$1.0473e-03$	$5.5534e-04$	$2.7537e-04$	$1.0935e-04$
10^{-4}	$1.8971e-03$	$1.0473e-03$	$5.5579e-04$	$2.8673e-04$	$1.4569e-04$
10^{-5}	$1.8971e-03$	$1.0473e-03$	$5.5579e-04$	$2.8673e-04$	$1.4569e-04$
Result in Mekonnen & Duressa (2020) (By Fitted B-spline Collocation Method)					
10^{-0}	$5.9148e-7$	$1.4804e-7$	$3.6913e-8$	$9.2256e-9$	$2.3063e-9$
10^{-1}	$3.7660e-6$	$9.4219e-7$	$2.3590e-7$	$5.8994e-8$	$1.4750e-8$
10^{-2}	$3.9402e-02$	$5.9951e-6$	$1.6882e-6$	$4.3637e-7$	$1.1003e-7$
10^{-3}	$1.8817e-5$	$9.1973e-6$	$4.5437e-6$	$2.1630e-6$	$8.3111e-7$
10^{-4}	$1.8817e-5$	$9.1973e-6$	$4.5465e-6$	$2.1630e-6$	$1.1272e-6$
10^{-5}	$1.8817e-5$	$9.1973e-6$	$4.5465e-6$	$2.1630e-6$	$1.1272e-6$



(a) Surface plot



(b) Line plot

Figure 4.7: The computed solution pattern for Example 4.3 is illustrated for the case where $N = M = 128$ and $\varepsilon = 10^{-4}$.

To illustrate the underlying physical behavior of the solutions, surface plots for Examples 4.1, 4.2, and 4.3 are depicted in Figures 4.1, 4.3, and 4.7a. These visualizations, generated for various values of ε, μ, N , and M , vividly capture the solution profiles

and provide deeper insight into the problem’s dynamic characteristics. Figures 4.2, 4.4, and 4.7b present line plots that vividly illustrate how the boundary layer behavior evolves as the perturbation parameters ε and μ vary. These plots clearly reveal the increasing steepness of the boundary layers with decreasing parameter values, highlighting the singularly perturbed nature of the problem. Furthermore, Figures 4.5 and 4.6 display the log-log plots of the maximum pointwise errors for Examples 4.1 and 4.2. These plots serve as a powerful tool for visually assessing the convergence characteristics of the proposed method. From the observed linear trends in the log-log graphs, it can be inferred that the scheme exhibits (ε, μ) –uniform convergence even as the perturbation parameters become exceedingly small. This behavior confirms the robustness and reliability of the method in capturing boundary layer effects without deterioration in accuracy for diminishing parameter values.

4.7 Conclusion

In this chapter, a convergent numerical scheme has been formulated for 1D TP-SPPPs. The method integrates Crank–Nicolson time discretization with a fitted Simpson’s one-third rule, effectively capturing sharp boundary layers induced by small perturbation parameters. A detailed convergence analysis, supported by the model assumptions Eq. (3.1), confirms that the scheme attains uniform convergence with respect to both mesh size and perturbation parameters. Extensive numerical tests that show second-order accuracy and validate the theoretical results include error tables, log-log plots, and graphical visualizations. Comparative results consistently produce less maximum pointwise errors than existing approaches, further demonstrating the improved precision and resilience of the suggested strategy.

Motivated by practical applications in which diffusion–reaction processes are influenced by time delays, the present work naturally extends to more complex models in the next chapter. Time delays introduce memory effects and modify the structure of boundary layers, requiring additional analytical care to preserve stability and parameter-uniform convergence. Consequently, Chapter 5 advances the methodology by adapting the fitted Crank–Nicolson–Simpson framework to TP-SPPPs with delay, thereby generalizing the numerical strategy developed thus far.

Chapter 5

An Exponentially Fitted Simpson's One-Third Numerical Scheme for Two-Parameter Singularly Perturbed Parabolic Problems with Large Time Delay

Building upon the uniformly convergent fitted schemes developed in Chapters 3 and 4, this chapter extends the numerical framework to a more challenging class of equations: TP-SPPPs with time delay. Delay effects arise naturally in models with after-reaction, transport lag, or memory-driven processes and they significantly complicate the analytical structure of boundary and interior layers. These additional difficulties necessitate a refinement of the previously developed techniques. To address this, the Crank–Nicolson scheme is employed for time discretization on a uniform mesh, ensuring temporal stability and accuracy, while spatial discretization utilizes a fitted Simpson's one-third rule to resolve layer structures. A parameter-uniform error analysis confirms second-order convergence in both space and time. Extensive numerical results demonstrate uniform convergence and robust performance of the proposed approach when applied to delayed TP-SPPPs without known exact solutions.

5.1 Introduction

Many phenomena in physics, biology, and chemistry such as population dynamics, epidemiology, circadian rhythms, and tumor growth are modeled by SPDDs. The inclusion of delay terms introduces memory effects that capture the influence of past states, making these models more realistic. However, the coexistence of delay terms, boundary layers, and multiscale behavior renders their analytical and numerical treatment highly challenging. Classical analytical methods often fail to handle these complexities, leaving numerical approximation as the most practical approach (Wu, 2012; Negero et al., 2023; Ayele et al., 2023; Singh et al., 2018, 2022).

Conventional numerical schemes lose accuracy/stability when perturbation parameters become small, resulting in oscillatory or divergent solutions. To address this, parameter-uniform numerical methods have been developed that maintain accuracy independent of perturbation size, efficiently resolving layers without requiring dense meshes. Notable studies include Ayele et al. (2023), Sumit et al. (2020), Negero (2023), Singh et al. (2022), and Mohye et al. (2023), who employed FDMs, spline-based, and nonstandard schemes to achieve uniform convergence for time-delay SPDDs.

Building on these advancements, this work introduces a reliable scheme for 1D time-dependent SPDDs with dual small parameters and time delay. The temporal variable is discretized using the Crank–Nicolson method on a uniform mesh, while spatial discretization employs a fitted numerical integration technique. Unlike standard FDMs, this approach naturally accommodates variable coefficients and captures boundary layers with higher precision by approximating integral averages instead of pointwise derivatives. The resulting formulation produces a compact and stable scheme with uniform convergence, preserving bounded error estimates as $(\epsilon, \mu) \rightarrow 0$ and providing a reliable framework for solving time-delay SPDDs.

The chapter is structured as follows: Section 5.2 formulates the problem, Section 5.4 develops the proposed numerical scheme and Section 5.5 provides its error analysis. Section 5.6 presents numerical experiments supporting the theory, while Section 5.7 concludes the chapter with key findings.

5.2 Definition of the Problem

Consider a 1D time-dependent SPPDDE with two small positive parameters, defined on the rectangular domain $\overline{\Delta} = \Delta \cup \partial\Delta$, where $\Delta = (0, 1) \times (0, T]$ and $\partial\Delta = \Lambda_b \cup \Lambda_l \cup \Lambda_r$, with $\Lambda_b = [0, 1] \times [-\tau, 0]$, $\Lambda_l = \{0\} \times (0, T]$, and $\Lambda_r = \{1\} \times (0, T]$. The governing problem is formulated as:

$$\left\{ \begin{array}{l} \mathbb{L}_{\varepsilon, \mu} w := (\varepsilon w_{xx} + \mu a w_x - b w - w_t)(x, t) = \\ \quad -c(x, t)w(x, t - \tau) + \varpi(x, t), (x, t) \in \Delta, \\ w|_{\Lambda_b} = \Upsilon_b(x, t), \\ w|_{\Lambda_l} = \Upsilon_l(t), \quad w|_{\Lambda_r} = \Upsilon_r(t), \end{array} \right. \quad (5.1)$$

where $0 < \varepsilon \ll 1$ and $0 \leq \mu \leq 1$ are perturbation parameters. The coefficients $(a, b, c, \varpi, \Upsilon_b)(x, t)$, $\Upsilon_l(t)$, $\Upsilon_r(t)$ are assumed to be sufficiently smooth and satisfy $0 < \alpha \leq a(x, t)$, $0 < \beta \leq b(x, t)$, $0 < \varsigma \leq c(x, t)$, $(x, t) \in \overline{\Delta}$. At corner points, $(0, 0)$, $(0, 1)$, $(-\tau, 0)$, and $(-\tau, 1)$ the compatibility conditions are

$$\begin{aligned} w(0, 0) &= \Upsilon_l(0), w(1, 0) = \Upsilon_r(0), w(0, -\tau) = \Upsilon_l(\tau), w(1, -\tau) = \Upsilon_r(-\tau), \\ \varepsilon (\Upsilon_b)_{xx}(0, 0) + \mu a(0, 0) (\Upsilon_b)_x(0, 0) - b(0, 0) (\Upsilon_b)(0, 0) - (\Upsilon_b)_t(0, 0) &= \\ &= -c(0, 0) (\Upsilon_b)(0, -\tau) + \varpi(0, 0), \\ \varepsilon (\Upsilon_b)_{xx}(1, 0) + \mu a(1, 0) (\Upsilon_b)_x(1, 0) - b(1, 0) (\Upsilon_b)(1, 0) - (\Upsilon_b)_t(1, 0) &= \\ &= -c(1, 0) (\Upsilon_b)(1, -\tau) + \varpi(1, 0), \end{aligned}$$

for $\Upsilon_b(x, t)$ (initial-boundary data) satisfies appropriate compatibility criteria at the two corners, $(0, 0)$ and $(1, 0)$. In order to establish the existence of continuous solution $w(x, t) \in C^{1+d/2, 2+d}(\overline{\Delta})$ with $d \in (0, 1)$, the assumptions of Hölder continuity are made on all the coefficients and IBVPs involved in Eq. (5.1). Having a time delay $\tau > 0$, in Eq. (5.1), the entire time domain $[-\tau, k\tau]$ for $k \in \mathbb{N}$, is divided into several subdomains. For the analysis purpose $k = 2$ is used. Because, in the region, $[0, 1] \times [0, \tau]$ the solution depends upon the solution in $[0, 1] \times [-\tau, 0]$, which is known and hence, the analysis is similar as we do for any SPPDEs having no time delay (Kumar, 2021). So, our analysis is done for $[0, 1] \times [\tau, 2\tau]$, where $w(x, t - \tau)$ is the solution in

$[0, 1] \times [0, \tau]$, which is known. Now, Eq. (5.1) can be rewritten as follows:

$$\left(\mathbb{L}_{\varepsilon, \mu} - \frac{\partial}{\partial t} \right) w(x, t) = f(x, t), \quad (x, t) \in \Delta, \quad (5.2)$$

where

$$\begin{aligned} \mathbb{L}_{\varepsilon, \mu} w(x, t) &= \left(\varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a \frac{\partial w}{\partial x} - bw \right) (x, t), \quad \text{and} \\ f(x, t) &= \begin{cases} \varpi(x, t) - c(x, t) \Upsilon_b(x, t - \tau), & \text{for } (x, t) \in (0, 1] \times (0, \tau], \\ \varpi(x, t) - c(x, t) w(x, t - \tau), & \text{for } (x, t) \in (0, 1) \times (\tau, T], \end{cases} \end{aligned} \quad (5.3)$$

with interval and boundary condition stated in Eq. (5.1).

5.3 Properties of the Continuous Solution

In this section, the analytical properties of Eq. (5.1) are explored. The analysis first establishes a minimum principle and then derives bounds on the solution's derivatives.

Lemma 5.1 (Minimum Principle). *Let $\Xi(x, t) \in C^2(\Delta) \cap C^0(\overline{\Delta})$ satisfy $\Xi(x, t) \geq 0$ for all $(x, t) \in \partial\Delta$ and $\mathbb{L}_{\varepsilon, \mu} \Xi(x, t) \leq 0$ in Δ . Then, $\Xi(x, t) \geq 0$ for all $(x, t) \in \overline{\Delta}$.*

Proof. Assume, for contradiction, that there exists a point $(\chi^*, v^*) \in \overline{\Delta}$ such that

$$\Xi(\chi^*, v^*) = \min_{(x, t) \in \overline{\Delta}} \Xi(x, t) < 0.$$

Since $\Xi(x, t) \geq 0$ on $\partial\Delta$, the minimum must occur at an interior point $(\chi^*, v^*) \in \Delta$.

By the first and second derivative test, we have

$$\Xi_x(\chi^*, v^*) = 0, \quad \Xi_t(\chi^*, v^*) = 0, \quad \text{and} \quad \Xi_{xx}(\chi^*, v^*) \geq 0.$$

Evaluating $\mathbb{L}_{\varepsilon, \mu} \Xi$ at this point gives

$$\begin{aligned} \mathbb{L}_{\varepsilon, \mu} \Xi(\chi^*, v^*) &= \varepsilon \Xi_{xx}(\chi^*, v^*) + \mu a(\chi^*, v^*) \underbrace{\Xi_x(\chi^*, v^*)}_{=0} - b(\chi^*, v^*) \Xi(\chi^*, v^*) - \underbrace{\Xi_t(\chi^*, v^*)}_{=0} \\ &= \varepsilon \Xi_{xx}(\chi^*, v^*) - b(\chi^*, v^*) \Xi(\chi^*, v^*) > 0, \end{aligned}$$

since $\varepsilon > 0$, $\Xi_{xx}(\chi^*, v^*) \geq 0$, and $\Xi(\chi^*, v^*) < 0$ while $b(x, t) > 0$. This contradicts

the assumption $\mathbb{L}_{\varepsilon,\mu}\Xi(x,t) \leq 0$ in Δ . Hence, our assumption is false, and $\Xi(x,t) \geq 0$ for all $(x,t) \in \overline{\Delta}$. $\square$

Lemma 5.2. *The solution w of Eq.(5.1) is uniformly bounded, that is, there exists a constant C such that*

$$\|w\| \leq C.$$

Proof. Consider the auxiliary function $z(x,t)$ defined by

$$z(x,t) = \begin{cases} w(x,t) - \Upsilon_b(x,0), & \text{for } 0 \leq t \leq T, \\ w(x,t) - \Upsilon_b(x,t), & \text{for } -\tau \leq t \leq 0, \end{cases}$$

where $\Upsilon_b(x,0)$ is a smooth extension of the boundary data. The function z satisfies the following differential equation in Δ_x :

$$\mathbb{L}_{\varepsilon,\mu}z - z_t = r(x,t)$$

where the residual term $r(x,t)$ is given by:

$$\begin{aligned} r(x,t) &= -c(x,t)w(x,t-\tau) + \varpi(x,t) - \left(\mathbb{L}_{\varepsilon,\mu} - \frac{\partial}{\partial t} \right) \Upsilon_b(x,0), \\ &= \varpi - \varepsilon \Upsilon_{xx}(x,t) + \mu a(x,t)\Upsilon_b(x,t) - b(x,t)\Upsilon(x,t) - c(x,t)\Upsilon(x,t-\tau). \end{aligned}$$

The corresponding boundary conditions for z are

$$z|_{\Lambda_b} = 0, z|_{\Lambda_l} = \Upsilon_l(t) - \Upsilon_b(0,0), z|_{\Lambda_r} = \Upsilon_r(t) - \Upsilon_b(1,0),$$

where Λ_b, Λ_l , and Λ_r denote the bottom, left, and right boundaries of the space-time domain, respectively. To establish the uniform bound, we introduce a suitable barrier function $v(x,t)$ defined by:

$$v(x,t) = \begin{cases} Ct, & \text{for } t \in [0, T], \\ 0, & \text{for } t \in [-\tau, 0]. \end{cases}$$

where $C > 0$ is a sufficiently large constant. This function satisfies:

$$\left(\mathbb{L}_{\varepsilon, \mu} - \frac{\partial}{\partial t} \right) v = -C(1 + bt),$$

together with homogeneous boundary conditions $v|_{\Lambda_b} = 0$ and $v|_{\Lambda_l} = 0$. By comparing $\left| \left(\mathbb{L}_{\varepsilon, \mu} - \frac{\partial}{\partial t} \right) z \right|$ with $\left| \left(\mathbb{L}_{\varepsilon, \mu} - \frac{\partial}{\partial t} \right) v \right|$, and using the fact that $|z(x, t)| \leq v(x, t)$ on the parabolic boundary $\partial\Delta$, the maximum principle implies that $|z(x, t)| \leq v(x, t)$ holds in the closure of $\overline{\Delta}$. Finally, by choosing $\Upsilon_b(x, 0) = 0$ (without loss of generality, since Υ_b is arbitrary and smooth), we obtain

$$|w(x, t)| = |z(x, t) + \Upsilon_b(x, 0)| \leq Ct, \quad \text{for } (x, t) \in \overline{\Delta}.$$

Since $t \in [0, T]$ it follows that $\|w\| \leq C$, where C is independent of the perturbation parameters. For comprehensive derivations and detailed reasoning, the reader is referred to the proof presented in (Kumar, 2021). $\square$

Lemma 5.3 (Uniform Stability Estimate). *Let $w(x, t)$ be the solution to the continuous problem (5.1), then it satisfies the bound*

$$\|w(x, t)\|_{\overline{\Delta}} \leq |\zeta^{-1}| \|\varpi(x, t)\| + \max \{ \|\Upsilon_b(x, t)\|, \|\Upsilon_l(t)\|, \|\Upsilon_r(t)\| \}, \forall (x, t) \in \overline{\Delta}, \quad (5.4)$$

where $0 < \zeta \leq (b(x, t) + c(x, t))$ and $\|\cdot\|_{\overline{\Delta}}$ is used to denote maximum norm given by $\|w(x, t)\|_{\overline{\Delta}} = \max_{(x, t) \in \overline{\Delta}} |w(x, t)|$.

Proof. The proof is provided in Chapter 3, Lemma 3.2. $\square$

Lemma 5.4. *For $k, m \in \mathbb{Z}^+$ satisfying $0 \leq k + 2m \leq 4$, the solution $w(x, t)$ and its derivatives of Eq. (5.1), satisfy the bound:*

$$\left\| \frac{\partial^{k+m} w}{\partial x^k \partial t^m} \right\| \leq \begin{cases} C\varepsilon^{-\frac{k}{2}} & \text{for } \mu^2 \leq \frac{\varepsilon \mathfrak{N}}{\alpha}, \\ C \left(\frac{\mu}{\varepsilon} \right)^k \left(\frac{\mu^2}{\varepsilon} \right)^m & \text{for } \mu^2 \geq \frac{\varepsilon \mathfrak{N}}{\alpha}, \end{cases}$$

where $\mathfrak{N} \approx \min_{(x, t) \in \overline{\Delta}} \frac{b(x, t)}{a(x, t)}$ and the constant C is independent of ε and μ and depends only on the bound derivatives of the coefficients and the source term.

Proof. A complete justification of this result is available in the work of Kumar (2021) as well as in Lemma 3.4 of Chapter 4. $\square$

5.4 Formulation of the Numerical Scheme

To construct an appropriate numerical scheme for Eq. (5.1), we adopt Rothe's method. In the first stage, the time variable is discretized using the Crank-Nicolson method. Subsequently, spatial discretization is carried out on a uniform mesh by applying a fitted numerical integration technique based on Simpson's one-third rule. The main distinction in this section lies in extending the methodology applied to TP-SPPPs in Chapter 4 to the more intricate case of TP-SPPPs with time delay.

5.4.1 Temporal Semi-discretization

The time domain $[0, T]$ is partitioned into a uniform mesh. A parameter τ is chosen such that $T = k\tau$, with $k > 1$. Two sets of temporal nodes are introduced.

$$\Lambda_t^M = \left\{ t_j = kj, j = 0(1)M, t_M = T, k = \frac{T}{M} \right\},$$

$$\Lambda_\tau^m = \left\{ t_j = jk, j = 0(1)m, t_m = \tau, k = \frac{\tau}{m} \right\}.$$

Here, M and m denote the number of mesh points in $[0, T]$ and $[-\tau, 0]$, respectively. To obtain a semi-discrete form of the problem, the Crank-Nicolson FDM is applied in the temporal direction. Derivation of the scheme for $w(x, t)$ at the intermediate time level $(x, t_{j+1/2})$ involves Taylor series expansions of w^{j+1} and w^j , leading to the Crank-Nicolson formulation through a systematic procedure outlined below:

$$w^{j+1}(x) = w^{j+1/2}(x) + \frac{k}{2} \frac{dw^{j+1/2}(x)}{dt} + \frac{k^2}{8} \frac{d^2w^{j+1/2}(x)}{dt^2} + \frac{k^3}{48} \frac{d^3w^{j+1/2}(x)}{dt^3} + \dots \quad (5.5)$$

$$w^j(x) = w^{j+1/2}(x) - \frac{k}{2} \frac{dw^{j+1/2}(x)}{dt} + \frac{k^2}{8} \frac{d^2w^{j+1/2}(x)}{dt^2} - \frac{k^3}{48} \frac{d^3w^{j+1/2}(x)}{dt^3} + \dots \quad (5.6)$$

Subtracting Eq. (5.5) from Eq. (5.6) removes the term $w^{j+1/2}(x)$, leading to:

$$\frac{dw^{j+1/2}(x)}{dt} = \frac{w^{j+1}(x) - w^j(x)}{k} + \mathfrak{T}^{j+1/2}(x), \quad (5.7)$$

where $\mathfrak{T}^{j+1/2}(x)$ represents the LTE of the scheme, defined as

$$\mathfrak{T}^{j+1/2}(x) = \frac{k^3}{24} \frac{d^3w^{j+1/2}(x)}{dt^3} + \text{Terms of higher degree} \quad (5.8)$$

which is order three. By incorporating the previously described discretization into problem (5.1), the semi-discretized form of the scheme can be written as

$$\begin{aligned}
& \frac{\varepsilon}{2} [w_{xx}^{j+1} + w_{xx}^j] + \frac{\mu}{2} [a^{j+1}(x)w_x^{j+1} + a^j(x)w_x^j] - \frac{1}{2} [b^{j+1}(x)w^{j+1}(x) + b^j(x)w^j(x)] \\
& = \frac{w^{j+1}(x) - w^j(x)}{k} + TE^{j+1/2}(x) + \frac{1}{2} [\varpi^{j+1}(x) + \varpi^j(x)] - \\
& \quad \frac{1}{2} [c^{j+1}(x)w^{j+1-m}(x) + c^j(x)w^{j-m}(x)], \\
\Rightarrow & \frac{\varepsilon}{2} [w_{xx}^{j+1} + w_{xx}^j] + \frac{\mu}{2} [a^{j+1}(x)w_x^{j+1} + (a'(x))^{j+1}w^{j+1}(x) + (a'(x))^jw^j(x) + a^j(x)w_x^j] = \\
& \quad \frac{1}{2} [b^{j+1}(x)w^{j+1}(x) + b^j(x)w^j(x)] + \frac{w^{j+1}(x) - w^j(x)}{k} + \frac{1}{2} [\varpi^{j+1}(x) + \varpi^j(x)] - \\
& \quad \frac{1}{2} [c^{j+1}(x)w^{j+1-m}(x) + c^j(x)w^{j-m}(x)] + \frac{\mu}{2} [(a'(x))^{j+1}w^{j+1}(x) + (a'(x))^jw^j(x)], \\
\Rightarrow & \frac{\varepsilon}{2} [w_{xx}^{j+1} + w_{xx}^j] + \frac{\mu}{2} [a^{j+1}(x)w^{j+1}(x) + a^j(x)w^j(x)]' = \\
& \quad \frac{1}{2} [b^{j+1}(x)w^{j+1}(x) + b^j(x)w^j(x)] + \frac{w^{j+1}(x) - w^j(x)}{k} + \frac{1}{2} [\varpi^{j+1}(x) + \varpi^j(x)] - \\
& \quad \frac{1}{2} [c^{j+1}(x)w^{j+1-m}(x) + c^j(x)w^{j-m}(x)] + \frac{\mu}{2} [(a'(x))^{j+1}w^{j+1}(x) + (a'(x))^jw^j(x)],
\end{aligned}$$

where $(')$ denotes derivative of the function. The foregoing expression is rewritten as:

$$\left\{ \begin{aligned} & \mathbb{L}_{\varepsilon, \mu}^M w^{j+1}(x) := \frac{\varepsilon}{2} [w_{xx}^{j+1} + w_{xx}^j] + \frac{\mu}{2} [a^{j+1}(x)w^{j+1}(x) + a^j(x)w^j(x)]' - \\ & \quad \frac{1}{2} [b^{j+1}(x)w^{j+1}(x) + b^j(x)w^j(x)] = \frac{w^{j+1}(x) - w^j(x)}{k} + \frac{1}{2} [\varpi^{j+1}(x) + \varpi^j(x)] - \\ & \quad \frac{1}{2} [c^{j+1}(x)w^{j+1-m}(x) + c^j(x)w^{j-m}(x)] + \frac{\mu}{2} [(a'(x))^{j+1}w^{j+1}(x) + (a'(x))^jw^j(x)], \\ & w^{j+1}(0) = \Upsilon_l(t_{j+1}), \quad w^{j+1}(1) = \Upsilon_r(t_{j+1}), \quad 0 \leq j \leq M, \\ & w^{j+1}(x) = \Upsilon_b(x, t_{j+1}), \quad (x, t_{j+1}) \in [0, 1] \times [-\tau, 0]. \end{aligned} \right. \quad (5.9)$$

By performing additional simplifications, Eq. (5.9) takes the form

$$\left\{ \begin{aligned} & \mathbb{L}_{\varepsilon, \mu}^M w^{j+1}(x) := \frac{\varepsilon}{2} (w_{xx}(x))^{j+1} + \frac{\mu}{2} [(a^{j+1}(x)w^{j+1}(x))' - (a'(x))^{j+1}w^{j+1}(x)] \\ & \quad - \left(\frac{1}{2}b^{j+1}(x) + \frac{1}{k} \right) w^{j+1}(x) = \mathfrak{G}^{j+1}(x), \\ & w^{j+1}(0) = \Upsilon_l(t_{j+1}), \quad w^{j+1}(1) = \Upsilon_r(t_{j+1}), \quad 0 \leq j \leq M, \\ & w^{j+1}(x) = \Upsilon_b(x, t_{j+1}), \quad (x, t_{j+1}) \in [0, 1] \times [-\tau, 0], \end{aligned} \right. \quad (5.10)$$

where

$$\begin{aligned}\mathfrak{G}^{j+1}(x) = & -\frac{\varepsilon}{2}(w_{xx}(x))^j + \frac{\mu}{2} \left[(a'(x))^j w^j(x) - (a^j(x) w^j(x))' \right] + \left(\frac{1}{2} b^j(x) - \frac{1}{k} \right) w^j(x) \\ & + \frac{1}{2} [\varpi^{j+1}(x) + \varpi^j(x)] - \frac{1}{2} [c^{j+1}(x) w^{j+1-m}(x) + c^j(x) w^{j-m}(x)].\end{aligned}$$

Applying the initial condition, the right-hand side of Eq. (5.10) is evaluated as

$$\mathfrak{G}^{j+1}(x) = \begin{cases} -\frac{\varepsilon}{2}(w_{xx}(x))^j + \frac{\mu}{2} \left[(a'(x))^j w^j(x) - (a^j(x) w^j(x))' \right] + \\ \left(\frac{1}{2} b^j(x) - \frac{1}{k} \right) w^j(x) - \frac{1}{2} [c^{j+1}(x) \Upsilon_b(x, t_{j+1-m}) + c^j(x) \Upsilon_b(x, t_{j-m})] \\ + \frac{1}{2} [\varpi^{j+1}(x) + \varpi^j(x)], j = 0(1)m, \\ -\frac{\varepsilon}{2}(w_{xx}(x))^j + \frac{\mu}{2} \left[(a'(x))^j w^j(x) - (a^j(x) w^j(x))' \right] + \\ \left(\frac{1}{2} b^j(x) - \frac{1}{k} \right) w^j(x) - \frac{1}{2} [c^{j+1}(x) w(x, t_{j+1-m}) + c^j(x) w(x, t_{j-m})] \\ + \frac{1}{2} [\varpi^{j+1}(x) + \varpi^j(x)], j = m+1(1)M. \end{cases}$$

Lemma 5.5 (Semi-discrete Minimum Principle). *Assume that $[\mathbb{L}_{\varepsilon, \mu}^M w^{j+1}(x)]$ is the discrete operator given in Eq. (5.10) and $\Xi(x, t_{j+1})$ is any mesh function satisfying $\Xi(x, t_{j+1})|_{\partial\Delta} \geq 0$ and $[\mathbb{L}_{\varepsilon, \mu}^M \Xi(x, t_{j+1})]|_{\Delta} \leq 0, \forall x \in \Delta$, then $\Xi(x, t_{j+1}) \geq 0, \forall x \in \overline{\Delta}_x$.*

Proof. Let $(\chi^*, t_{j+1}) \in \{(x, t_{j+1}) : x \in \overline{\Delta}_x\}$ be such that

$$\Xi(\chi^*, t_{j+1}) = \min_{x \in \overline{\Delta}_x} \Xi(x, t_{j+1}),$$

and let $\Xi(\chi^*, t_{j+1}) < 0$. Then $(\chi^*, t_{j+1}) \notin \{(0, t_{j+1}), (1, t_{j+1})\}$, also $\Xi_x(\chi^*, t_{j+1}) = 0$ and $\Xi_{xx}(\chi^*, t_{j+1}) \geq 0$. By using this assumption and Eq. (5.10), we arrive on $\mathbb{L}_{\varepsilon, \mu}^M \Xi(x, t_{j+1}) > 0$, which is a contradiction to the hypothesis, $\mathbb{L}_{\varepsilon, \mu}^M \Xi(x, t_{j+1}) \leq 0$. Hence, our assumption is wrong so $\Xi(\chi^*, t_{j+1}) \geq 0$ and $\Rightarrow \Xi(x, t_{j+1}) \geq 0, \forall x \in \overline{\Delta}_x$. $\square$

Definition 5.1. (*Priyadarshana et al., 2022*) *The global and local errors associated with the time semi-discretization of Eq. (5.10) are defined as*

$$E^j = \max_{j+1 \leq T/k} \|w^{j+1}(x) - W^{j+1}(x)\|, \quad e^{j+1} = |w^{j+1}(x) - W^{j+1}(x)|, \quad (5.11)$$

respectively. Here, the local error measures the contribution of a single time step to the global error. At each semi-discretization stage, it is computed using the exact

solution $w^{j+1}(x)$ rather than the numerical approximation $W^{j+1}(x)$. The temporal order of convergence of the Crank-Nicolson scheme follows from its consistency and stability, as established in the subsequent lemmas.

Lemma 5.6. (*Estimate of local error*). Suppose that $\left\| \frac{\partial^k w(x, t)}{\partial t^k} \right\| \leq C, (x, t) \in \overline{\Delta}, k = 0, 1, 2$. The error estimate in temporal direction for sufficiently large constant C is

$$\|e_{j+1}\| \leq C(k^3).$$

Proof. Using Crank-Nicolson time discretization and the fourth-order Taylor expansions from Eqs. (5.7)-(5.8), we obtain

$$\frac{w^{j+1}(x) - w^j(x)}{k} = \frac{\partial w^{j+1/2}(x)}{\partial t} + O(k^3) \quad (5.12)$$

Using Eq. (5.12) into Eq. (5.1), we get

$$\mathbb{L}_{\varepsilon, \mu} w^{j+1}(x) = w_t^{j+1} + O(k^3).$$

Similarly, applying the semi-discrete minimum operator to e_{k+1} yields

$$\mathbb{L}_{\varepsilon, \mu} e^{j+1}(x) = O(k^3).$$

Then, by applying Lemma 5.5, the LTE can be estimated as:

$$\|e_{j+1}\|_{\infty} \leq C(k^3).$$

□

Lemma 5.7 (*Estimate of global error*). The global error E^j at the (j) th time level satisfies the bound $\|E^j\| \leq C(k^2), \quad \forall j \leq T/k$.

Proof. Following the same reasoning as in Chapter 3, Lemma 3.7, we conclude that the temporal semi-discretization scheme attains second-order accuracy. □

To assess the proposed method, it is essential to understand the behavior of the exact solution's derivatives in the semi-discretized form of Eq. (5.9). After suitable

rearrangements, the characteristic equation of its homogeneous part is obtained as:

$$\varepsilon \lambda^2 + \mu a^{j+1}(x) \lambda - \beta^{j+1}(x) = 0, \quad (5.13)$$

where $\beta^{j+1}(x) = b^{j+1}(x) + 2/k$ and Eq. (5.13) has two roots λ_1 and λ_2 , where

$$\lambda_1 = -\frac{\mu a^{j+1}(x)}{2\varepsilon} + \sqrt{\left(\frac{\mu a^{j+1}(x)}{2\varepsilon}\right)^2 + \frac{\beta^{j+1}(x)}{\varepsilon}} \geq 0,$$

$$\lambda_2 = -\frac{\mu a^{j+1}(x)}{2\varepsilon} - \sqrt{\left(\frac{\mu a^{j+1}(x)}{2\varepsilon}\right)^2 + \frac{\beta^{j+1}(x)}{\varepsilon}} \leq 0.$$

These roots represent the boundary layer behavior of the solution in the neighborhood of $x = 0$ and $x = 1$ (Negero, 2023; Ayele et al., 2023). Let $\varrho_1 = -\max_{x \in [0,1]} \lambda_2(x)$ and $\varrho_2 = \min_{x \in [0,1]} \lambda_1(x)$, we have three cases: (i) when $\frac{\mu^2}{\varepsilon} \rightarrow 0$, as $\varepsilon \rightarrow 0$, $\varrho_1 = \varrho_2 = \sqrt{\frac{\beta^{j+1}}{\varepsilon}}$, where $0 < \beta^* < \beta^{j+1}$ and (ii) when $\frac{\varepsilon}{\mu^2} \rightarrow 0$, as $\mu \rightarrow 0$, $\varrho_1 = \varrho_2 = \frac{\mu a^*}{\varepsilon}$ and $\varrho_2 = 0$, where $0 < a^* < a^{j+1}$, (iii) when $\varrho_2 \ll \varrho_1$, the solutions exhibit a stronger boundary layer along $x = 0$ and $x = 1$ (Negero, 2023).

Next, we establish the semi-discrete bound for the solution $w^{j+1}(x)$ of Eq. (5.9).

Lemma 5.8. *The solution $w^{j+1}(x)$ of Eq. (5.9) satisfies the derivative bound*

$$\left| \frac{d^q w^{j+1}(x)}{dx^q} \right| \leq C \left(1 + \varrho_1^q e^{-\tau \varrho_1 x} + \varrho_2^q e^{-\tau \varrho_2 (1-x)} \right), \quad q = 0(1)r, r > 1 \quad \text{and} \quad 0 < \tau < 1. \quad (5.14)$$

Proof. The detailed proof of this lemma can be found in Chapter 4, Lemma 3.8. $\square$

5.4.2 Spatial Semi-discretization

We discretized the spatial domain $[0, 1]$ into N equal number of mesh elements as:

$\Lambda_x^N = \{x_i = ih, i = 0(1)N, x_0 = 0, x_N = 1, h = 1/N\}$. Integrating Eq. (5.9), in the

interval $[x_{i-1}, x_{i+1}]$, $i = 1(1)N - 1$, we obtain

$$\begin{aligned}
& \frac{\varepsilon}{2} \int_{x_{i-1}}^{x_{i+1}} [w_{xx}^{j+1} + w_{xx}^j](x) dx + \frac{\mu}{2} \int_{x_{i-1}}^{x_{i+1}} [a^{j+1}(x)w^{j+1}(x) + a^j(x)w^j(x)]' dx - \\
& \frac{1}{2} \int_{x_{i-1}}^{x_{i+1}} [b^{j+1}(x)w^{j+1}(x) + b^j(x)w^j(x)] dx = \frac{1}{k} \int_{x_{i-1}}^{x_{i+1}} [w^{j+1}(x) - w^j(x)] dx + \\
& \frac{1}{2} \int_{x_{i-1}}^{x_{i+1}} [\varpi^{j+1}(x) + \varpi^j(x)] dx - \frac{1}{2} \int_{x_{i-1}}^{x_{i+1}} [c^{j+1}(x)w^{j+1-m}(x) + c^j(x)w^{j-m}(x)] dx \\
& + \frac{\mu}{2} \int_{x_{i-1}}^{x_{i+1}} [(a'(x))^{j+1}w^{j+1}(x) + (a'(x))^jw^j(x)] dx, \\
\Rightarrow & \frac{\varepsilon}{2} \left[(w_x(x_{i+1}))^{j+1} - (w_x(x_{i-1}))^{j+1} + (w_x(x_{i+1}))^j - (w_x(x_{i-1}))^j \right] + \\
& \frac{\mu}{2} [a^{j+1}(x_{i+1})w^{j+1}(x_{i+1}) - a^{j+1}(x_{i-1})w^{j+1}(x_{i-1}) + a^j(x_{i+1})w^j(x_{i+1}) - a^j(x_{i-1})w^j(x_{i-1})] \\
& = \int_{x_{i-1}}^{x_{i+1}} \Theta^j(x) dx,
\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}
\Theta^j(x) dx = & \frac{1}{2} [b^{j+1}(x)w^{j+1}(x) + b^j(x)w^j(x)] + \frac{1}{k} [w^{j+1}(x) - w^j(x)] + \frac{1}{2} [\varpi^{j+1}(x) + \varpi^j(x)] - \\
& \frac{1}{2} [c^{j+1}(x)w^{j+1-m}(x) + c^j(x)w^{j-m}(x)] + \frac{\mu}{2} [(a'(x))^{j+1}w^{j+1}(x) + (a'(x))^jw^j(x)].
\end{aligned}$$

In Eq. (5.15), the definite integral is approximated using 1/3rd-Simpson's rule. With the notation $w^j(x_i) = W_i^j$, Eq. (5.15) simplifies to:

$$\begin{aligned}
& \frac{\varepsilon}{2} \left[(W_x)_{i+1}^{j+1} - (W_x)_{i-1}^{j+1} + (W_x)_{i+1}^j - (W_x)_{i-1}^j \right] + \\
& \frac{\mu}{2} [a_{i+1}^{j+1}W_{i+1}^{j+1} - a_{i-1}^{j+1}W_{i-1}^{j+1} + a_{i+1}^jW_{i+1}^j - a_{i-1}^jW_{i-1}^j] \\
& = \frac{h}{3} [\Theta_{i+1}^j + 4\Theta_i^j + \Theta_{i-1}^j].
\end{aligned} \tag{5.16}$$

The first derivatives of $w^{j+1}(x)$ and $w^j(x)$ with respect to x at the grid point x_i , $i = 1, 2, \dots, N - 1$, are approximated as follows:

$$\begin{aligned}
(W_x)_i^{j+1} & \approx \frac{W_{i+1}^{j+1} - W_{i-1}^{j+1}}{2h}, \\
(W_x)_{i+1}^{j+1} & \approx \frac{3W_{i+1}^j - 4W_i^{j+1} + W_{i-1}^{j+1}}{2h}, \\
(W_x)_{i-1}^{j+1} & \approx \frac{-W_{i+1}^{j+1} + 4W_i^{j+1} - 3W_{i-1}^{j+1}}{2h},
\end{aligned} \tag{5.17}$$

The expression in Eq. (5.17) is similarly applied at the j th time level. By substituting Eq. (5.17) into Eq. (5.16) and performing simplification, we obtain the relation:

$$\begin{aligned} & \frac{\varepsilon}{h} [W_{i+1}^{j+1} - 2W_i^{j+1} + W_{i-1}^{j+1} + W_{i+1}^j - 2W_i^j + W_{i-1}^j] + \\ & \frac{\mu}{2} [a_{i+1}^{j+1}W_{i+1}^{j+1} - a_{i-1}^{j+1}W_{i-1}^{j+1} + a_{i+1}^jW_{i+1}^j - a_{i-1}^jW_{i-1}^j] \\ & = \frac{h}{3} [\Theta_{i+1}^j + 4\Theta_i^j + \Theta_{i-1}^j]. \end{aligned} \quad (5.18)$$

To account for the influence of the perturbation parameter, an exponential fitting factor $\sigma(\rho)$ is applied to the ε -dependent term in Eq. (5.18) as follows:

$$\begin{aligned} & \frac{\varepsilon\sigma(\rho)}{h} [W_{i+1}^{j+1} - 2W_i^{j+1} + W_{i-1}^{j+1} + W_{i+1}^j - 2W_i^j + W_{i-1}^j] + \\ & \frac{\mu}{2} [a_{i+1}^{j+1}W_{i+1}^{j+1} - a_{i-1}^{j+1}W_{i-1}^{j+1} + a_{i+1}^jW_{i+1}^j - a_{i-1}^jW_{i-1}^j] \\ & = \frac{h}{3} [\Theta_{i+1}^j + 4\Theta_i^j + \Theta_{i-1}^j]. \end{aligned} \quad (5.19)$$

Let $\rho = \frac{h}{\varepsilon}$ and taking the limit of Eq. (5.19) as $h \rightarrow 0$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{\sigma(\rho)}{\rho} [W_{i+1}^{j+1} - 2W_i^{j+1} + W_{i-1}^{j+1} + W_{i+1}^j - 2W_i^j + W_{i-1}^j] + \\ & \lim_{h \rightarrow 0} \frac{\mu}{2} [a_{i+1}^{j+1}W_{i+1}^{j+1} - a_{i-1}^{j+1}W_{i-1}^{j+1} + a_{i+1}^jW_{i+1}^j - a_{i-1}^jW_{i-1}^j] = 0. \end{aligned} \quad (5.20)$$

5.4.3 Derivation of the Fitting Factor

From the theory of singular perturbations (Negero, 2023; Ayele et al., 2023), the solution of Eq. (5.10) can be written as

$$W^{j+1}(x) = \begin{cases} W_0^{j+1}(x) + [W_0^{j+1}(0) - \Upsilon_l^{j+1}(0)] \exp\left(-\frac{\mu a^{j+1}(0)}{\varepsilon}(x)\right), & \text{at Left Layer,} \\ W_0^{j+1}(x) + [W_0^{j+1}(1) - \Upsilon_r^{j+1}(1)] \exp\left(\frac{\mu a^{j+1}(1)}{\varepsilon}(1-x)\right), & \text{at Right Layer,} \end{cases}$$

where $W_0^{j+1}(x)$ is the solution of reduced problem (Negero, 2023; Singh et al., 2022).

By expanding $a(x)$ in a Taylor series about $x = 0$ and $x = 1$, and assuming a

sufficiently small h , the evaluation of Eq. (5.20) at $x = x_i = h$ yields:

$$W_i^{j+1} = \begin{cases} (W_0)_i^{j+1} + [W_0^{j+1}(0) - \Upsilon_l^{j+1}(0)] \exp\left(-\frac{\mu a^{j+1}(0)}{\varepsilon}(ih)\right), & \text{left layer,} \\ (W_0)_i^{j+1} + [W_0^{j+1}(1) - \Upsilon_r^{j+1}(1)] \exp\left(\frac{\mu a^{j+1}(1)}{\varepsilon}(1-ih)\right), & \text{right layer.} \end{cases} \quad (5.21)$$

For left layer, substituting Eq. (5.21) into Eq. (5.20) and taking $\rho = \frac{h}{\varepsilon}$

$$\begin{aligned} \frac{\sigma}{\rho} \left[e^{(-\mu a^{j+1}(0)\rho)} - 2 + e^{(\mu a^{j+1}(0)\rho)} \right] &= -\frac{\mu}{2} a^{j+1}(0) \left[e^{(-\mu a^{j+1}(0)\rho)} + e^{(\mu a^{j+1}(0)\rho)} \right], \\ \Rightarrow -\frac{\sigma}{\rho} \left[e^{\left(\frac{\mu a^{j+1}(0)\rho}{2}\right)} - e^{\left(-\frac{\mu a^{j+1}(0)\rho}{2}\right)} \right]^2 &= \\ &= -\frac{\mu}{2} a^{j+1}(0) \left[e^{\left(\frac{\mu a^{j+1}(0)\rho}{2}\right)} - e^{\left(-\frac{\mu a^{j+1}(0)\rho}{2}\right)} \right], \end{aligned}$$

After some arrangements, the left and right layers are formed accordingly. With $\sigma(\rho)$ identified and the limit $h \rightarrow 0$ applied to both sides, we arrive:

$$\sigma(\rho) = \begin{cases} \frac{\rho \mu a^{j+1}(0)}{2} \coth\left(\frac{\mu a^{j+1}(0)\rho}{2}\right), & \text{for } x = 0, \\ \frac{\rho \mu a^{j+1}(1)}{2} \coth\left(\frac{\mu a^{j+1}(1)\rho}{2}\right), & \text{for } x = 1. \end{cases} \quad (5.22)$$

where $\rho = h/\varepsilon$. Substituting Eq. (5.22) into (5.19) and solving for Θ_i^j , we arrive at:

$$\left\{ \begin{aligned} &\left[\frac{\sigma(\rho)}{\rho} - \frac{\mu}{2} a_{i-1}^{j+1} - \frac{h}{6} b_{i-1}^{j+1} - \frac{h}{3k} - \frac{h}{6} \mu(a')_{i-1}^{j+1} \right] W_{i-1}^{j+1} + \\ &\left[-\frac{2\sigma(\rho)}{\rho} - \frac{2h}{3} b_i^{j+1} + \frac{4h}{3k} - \frac{2h}{3} \mu(a')_i^{j+1} \right] W_i^{j+1} + \\ &\left[\frac{\sigma(\rho)}{\rho} + \frac{\mu}{2} a_{i+1}^{j+1} - \frac{h}{6} b_{i+1}^{j+1} - \frac{h}{3k} - \frac{h}{6} \mu(a')_{i+1}^{j+1} \right] W_{i+1}^{j+1} = \\ &\left[-\frac{\sigma(\rho)}{\rho} + \frac{\mu}{2} a_{i-1}^j + \frac{h}{6} b_{i-1}^j - \frac{h}{3k} + \frac{h}{6} \mu(a')_{i-1}^j \right] W_{i-1}^j + \\ &\left[\frac{2\sigma(\rho)}{\rho} + \frac{2h}{3} b_i^j - \frac{4h}{3k} + \frac{2h}{3} \mu(a')_i^j \right] W_i^j + \\ &\left[-\frac{\sigma(\rho)}{\rho} - \frac{\mu}{2} a_{i+1}^j + \frac{h}{6} b_{i+1}^j - \frac{h}{3k} + \frac{h}{6} \mu(a')_{i+1}^j \right] W_{i+1}^j + \\ &\frac{h}{6} [\varpi_{i+1}^{j+1} + \varpi_{i+1}^j] + \frac{2h}{3} [\varpi_i^{j+1} + \varpi_i^j] + \frac{h}{6} [\varpi_{i-1}^{j+1} + \varpi_{i-1}^j] - \end{aligned} \right. \quad (5.23)$$

$$\begin{cases} \frac{h}{6} [c_{i-1}^{j+1} W_{i-1}^{j+1-m} + c_{i-1}^j W_{i-1}^{j-m}] - \frac{2h}{3} [c_i^{j+1} W_i^{j+1-m} + c_i^j W_i^{j-m}] \\ -\frac{h}{6} [c_{i+1}^{j+1} W_{i+1}^{j+1-m} + c_{i+1}^j W_{i+1}^{j-m}]. \end{cases}$$

Now, Eq. (5.23) can be expressed as a three-term recurrence relation as:

$$\begin{cases} \mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1} = A_i^- W_{i-1}^{j+1} + A_i^c W_i^{j+1} + A_i^+ W_{i+1}^{j+1} = \\ B_i^- W_{i-1}^j + B_i^c W_i^j + B_i^+ W_{i+1}^j + F_j^j, \\ W_i^{j+1} = \Upsilon_b(x_i, t_{j+1}), \quad i = 1, \dots, N-1, \quad -(m+1) \leq j \leq -1, \\ W_0^{j+1} = \Upsilon_l(t_{j+1}), \quad W_{N+1}^{j+1} = \Upsilon_r(t_{j+1}), \quad 0 \leq j \leq M-1, \end{cases} \quad (5.24)$$

for $i = 0, 1, \dots, N$ and $j = 0, 1, \dots, M$ where

$$\begin{cases} A_i^- = \frac{\sigma(\rho)}{\rho} - \frac{\mu}{2} a_{i-1}^{j+1} - \frac{h}{6} b_{i-1}^{j+1} - \frac{h}{3k} - \frac{h}{6} \mu (a')_{i-1}^{j+1}, \\ A_i^c = -\frac{2\sigma(\rho)}{\rho} - \frac{2h}{3} b_i^{j+1} - \frac{4h}{3k} - \frac{2h}{3} \mu (a')_i^{j+1}, \\ A_i^+ = \frac{\sigma(\rho)}{\rho} + \frac{\mu}{2} a_{i+1}^{j+1} - \frac{h}{6} b_{i+1}^{j+1} - \frac{h}{3k} - \frac{h}{6} \mu (a')_{i+1}^{j+1}, \\ B_i^- = -\frac{\sigma(\rho)}{\rho} + \frac{\mu}{2} a_{i-1}^j + \frac{h}{6} b_{i-1}^j - \frac{h}{3k} + \frac{h}{6} \mu (a')_{i-1}^j, \\ B_i^c = \frac{2\sigma(\rho)}{\rho} + \frac{2h}{3} b_i^j - \frac{4h}{3k} + \frac{2h}{3} \mu (a')_i^j, \\ B_i^+ = -\frac{\sigma(\rho)}{\rho} - \frac{\mu}{2} a_{i+1}^j + \frac{h}{6} b_{i+1}^j - \frac{h}{3k} + \frac{h}{6} \mu (a')_{i+1}^j, \end{cases} \quad (5.25)$$

and the right hand side is given by:

$$F_i^j = \begin{cases} -\frac{h}{6} [c_{i-1}^{j+1} \Upsilon_b(x_{i-1}, t_{j+1}) + 2c_i^{j+1} \Upsilon_b(x_i, t_{j+1}) + c_{i+1}^{j+1} \Upsilon_b(x_{i+1}, t_{j+1})] \\ -\frac{h}{6} [c_{i-1}^j \Upsilon_b(x_{i-1}, t_j) + 2c_i^j \Upsilon_b(x_i, t_j) + c_{i+1}^j \Upsilon_b(x_{i+1}, t_j)] + \\ \frac{h}{6} [\varpi_{i-1}^{j+1} + 2\varpi_i^{j+1} + \varpi_{i+1}^{j+1}] + \frac{h}{6} [\varpi_{i-1}^j + 2\varpi_i^j + \varpi_{i+1}^j], \text{ for } t_j < m, j = 0(1)M-1, \\ -\frac{h}{6} [c_{i-1}^{j+1} W(x_{i-1}, t_{j+1-m}) + 2c_i^{j+1} W(x_i, t_{j+1-m}) + c_{i+1}^{j+1} W(x_{i+1}, t_{j+1-m})] \\ -\frac{h}{6} [c_{i-1}^j W(x_{i-1}, t_{j-m}) + 2c_i^j W(x_i, t_{j-m}) + c_{i+1}^j W(x_{i+1}, t_{j-m})] + \\ \frac{h}{6} [\varpi_{i-1}^{j+1} + 2\varpi_i^{j+1} + \varpi_{i+1}^{j+1}] + \frac{h}{6} [\varpi_{i-1}^j + 2\varpi_i^j + \varpi_{i+1}^j], \text{ for } t_j > m, j = 0(1)M-1. \end{cases}$$

To obtain the numerical solution of Eq. (5.1), the desired finite difference scheme formulated in Eq. (5.24) is identified as a fitted operator FDM, derived using 1/3rd-Simpson's integration rule. By multiplying both sides of Eq. (5.23) by -1 , the resulting co-

efficient matrix is modified such that its row sums remain non-negative and its off-diagonal entries become non-positive, thereby satisfying the requirements for an M -matrix structure. For sufficiently small mesh sizes, it follows from the tridiagonal systems defined in Eqs. (5.24) and (5.25) that

$$|A_i^c| \geq |A_i^-| + |A_i^+|. \quad (5.26)$$

This inequality ensures that the coefficient matrix associated with the discrete operator $\mathbb{L}_{\varepsilon,\mu}^{N,M}$ is irreducibly diagonally dominant and, consequently, non-singular. Hence, the matrix qualifies as an M -matrix, guaranteeing stability and the existence of a unique solution. With the imposed boundary conditions, the inverse of this matrix can then be efficiently employed to compute the numerical solution of Eq. (5.24).

5.5 Convergence Analysis

The boundedness of the continuous solution and its derivatives has been established. Next, the temporal discretization errors are evaluated followed by an analysis of spatial and overall error estimates to verify the scheme's stability and consistency.

Lemma 5.9 (Discrete Minimum Principle). *Assume $\Xi^{j+1}(x)$ be a mesh function satisfying, $\Xi(0, t_{j+1}) \geq 0$, $\Xi(1, t_{j+1}) \geq 0$, and $\mathbb{L}_{\varepsilon,\mu}^{N,M}\Xi(x_i, t_{j+1}) \leq 0$ on $\Delta_x^N \times \Delta_t^M$. Then $\Xi(x_i, t_j) \geq 0$ at each point $\overline{\Delta}_x^N \times \overline{\Delta}_t^M$, where Δ_x^N and Δ_t^M are discretized domain.*

Proof. Let $(x_\vartheta, t_{j+1}) \in \{(x_i, t_{j+1}) : i = 1(1)N\}$ such that $\Xi^{j+1}(x_\vartheta) = \min_{1 \leq i \leq N} \Xi^{j+1}(x_i)$, and let $\Xi^{j+1}(x_\vartheta) < 0$. From this $(x_\vartheta, t_{j+1}) \notin \{(0, t_{j+1}), (1, t_{j+1})\}$, which implies that $(x_\vartheta, t_{j+1}) \in \{(x_i, t_{j+1}) : i = 1(1)N - 1\}$. Then, from Eq. (5.24), we have

$$\mathbb{L}_{\varepsilon,\mu}^{N,M}\Xi_\vartheta^{j+1} = A_i^-\Xi_{\vartheta-1}^{j+1} + A_i^c\Xi_\vartheta^{j+1} + A_i^+\Xi_{\vartheta+1}^{j+1} \geq 0,$$

since $|A_i^-| > 0$, $|A_i^c| > 0$ and $|A_i^+| > 0$, which is a contradiction. As a result, $\Xi^{j+1}(x_\vartheta) \geq 0$. Therefore $\Xi^{j+1}(x_i) \geq 0, \forall (x_i, t_{j+1}) \in \{(x_i, t_{j+1}) : i = 1(1)N - 1\}$. $\square$

From Lemma 5.9, the discrete operator $\mathbb{L}_{\varepsilon,\mu}^{N,M}$ satisfies the discrete minimum principle, ensuring the monotonicity of the associated coefficient matrix. Moreover, its

irreducible diagonal dominance confirms that it is an M -matrix, thereby guaranteeing the existence and uniqueness of the discrete solution.

Lemma 5.10 (Estimating Discrete Uniform Stability). *The solution W_i^{j+1} of the discrete scheme in Eq. (5.24) on $\mathbb{L}_{\varepsilon,\mu}^{N,M}$ gratifies the following estimate:*

$$\|W_i^{j+1}\| \leq \zeta^{-1} \|\mathbb{L}_{\varepsilon,\mu}^{N,M} W_i^{j+1}\| + \max \{|\Upsilon_l(t_{j+1})|, |\Upsilon_r(t_{j+1})|, |\Upsilon_b(x_i, t_{j+1})|\}, \quad (5.27)$$

for $i = 1(1)N - 1$ and $0 < \zeta \leq b(x, t) + c(x, t)$.

Proof. Let $\Gamma = \zeta^{-1} \|\mathbb{L}_{\varepsilon,\mu}^{N,M} W_i^{j+1}\| + \max \{|\Upsilon_l(t_{j+1})|, |\Upsilon_r(t_{j+1})|, |\Upsilon_b(x_i, t_{j+1})|\}$ and describe the barrier functions as:

$$(\nabla^\pm)_i^{j+1} = \Gamma \pm W_i^{j+1}. \quad (5.28)$$

The discrete function $\nabla^\pm(x_i, t_{j+1})$ on the boundary and interval functions are

$$\begin{aligned} \nabla^\pm(0, t_{j+1}) &= \Gamma + \max \{|\Upsilon_l(t_{j+1})|, |\Upsilon_r(t_{j+1})|, |\Upsilon_b(0, t_{j+1})|\} \pm \Upsilon_l(t_{j+1}) \geq 0, \\ \nabla^\pm(1, t_{j+1}) &= \Gamma + \max \{|\Upsilon_l(t_{j+1})|, |\Upsilon_r(t_{j+1})|, |\Upsilon_b(1, t_{j+1})|\} \pm \Upsilon_r(t_{j+1}) \geq 0, \\ \nabla^\pm(x_i, 0) &= \Gamma + \max \{|\Upsilon_l(0)|, |\Upsilon_r(0)|, |\Upsilon_b(x_i, 0)|\} \pm \Upsilon_b(x_i, 0) \geq 0. \end{aligned}$$

Additionally, on the discretized domain, it is given as

$$\begin{aligned} \mathbb{L}_{\varepsilon,\mu}^{N,M} (\nabla^\pm)_i^{j+1} &= \Gamma \pm W_i^{j+1} = A_i^- (\Gamma \pm W_{i-1}^{j+1}) + A_i^c (\Gamma \pm W_i^{j+1}) + A_i^+ (\Gamma \pm W_{i+1}^{j+1}), \\ &= (A_i^- + A_i^c + A_i^+) \Gamma \pm H_i^j \geq 0, \end{aligned}$$

where H_i^j is right hand side of Eq. (5.24). Using Lemma 5.9, the proof is immediate. $\square$

To demonstrate the parameter-uniform convergence of the discrete scheme in Eq. (5.24), let $T_i(h)$ denote the LTE associated with the proposed method. The expression for the LTE is formulated as:

$$T_i(h) = A_i^- W_{i-1}^{j+1} + A_i^c W_i^{j+1} + A_i^+ W_{i+1}^{j+1} - (B_i^- W_{i-1}^j + B_i^c W_i^j + B_i^+ W_{i+1}^j + F) \quad (5.29)$$

To calculate the spatial truncation error, use Eq. (5.24) at $x_\iota, \iota = i, i \pm 1$ in Eq. (5.29).

$$T_i(h) = \begin{cases} [A_i^- W_{i-1}^{j+1} - h(\varepsilon W_{xx}^{j+1} + \mu a^{j+1}(x) W_x^{j+1} - \beta^{j+1}(x) W^{j+1})_{i-1}] + \\ [A_i^c W_i^{j+1} - h(\varepsilon W_{xx}^{j+1} + \mu a^{j+1}(x) W_x^{j+1} - \beta^{j+1}(x) W^{j+1})_i] + \\ [A_i^+ W_{i+1}^{j+1} - h(\varepsilon W_{xx}^{j+1} + \mu a^{j+1}(x) W_x^{j+1} - \beta^{j+1}(x) W^{j+1})_{i+1}] + \\ [B_i^- W_{i-1}^j - h(\varepsilon W_{xx}^j + \mu a^j(x) W_x^j - \beta^j(x) W^j)_{i-1}] + \\ [B_i^c W_i^j - h(\varepsilon W_{xx}^j + \mu a^j(x) W_x^j - \beta^j(x) W^j)_i] + \\ [B_i^+ W_{i+1}^j - h(\varepsilon W_{xx}^j + \mu a^j(x) W_x^j - \beta^j(x) W^j)_{i+1}] \end{cases} \quad (5.30)$$

Replacing a Taylor's series expansion of

$$\begin{aligned} W_{i-1}^{j+1} &\cong W_i^{j+1} - \frac{h}{1!} \frac{dW_i^{j+1}}{dx} + \frac{h^2}{2!} \frac{d^2 W_i^{j+1}}{dx^2} - \frac{h^3}{3!} \frac{d^3 W_i^{j+1}}{dx^3} + \frac{h^4}{4!} \frac{d^4 W_i^{j+1}}{dx^4} + O(h^5), \\ W_{i+1}^{j+1} &\cong W_i^{j+1} + \frac{h}{1!} \frac{dW_i^{j+1}}{dx} + \frac{h^2}{2!} \frac{d^2 W_i^{j+1}}{dx^2} + \frac{h^3}{3!} \frac{d^3 W_i^{j+1}}{dx^3} + \frac{h^4}{4!} \frac{d^4 W_i^{j+1}}{dx^4} + O(h^5), \\ W_{i-1}^j &\cong W_i^j - \frac{h}{1!} \frac{dW_i^j}{dx} + \frac{h^2}{2!} \frac{d^2 W_i^j}{dx^2} - \frac{h^3}{3!} \frac{d^3 W_i^j}{dx^3} + \frac{h^4}{4!} \frac{d^4 W_i^j}{dx^4} + O(h^5), \\ W_{i+1}^j &\cong W_i^j + \frac{h}{1!} \frac{dW_i^j}{dx} + \frac{h^2}{2!} \frac{d^2 W_i^j}{dx^2} + \frac{h^3}{3!} \frac{d^3 W_i^j}{dx^3} + \frac{h^4}{4!} \frac{d^4 W_i^j}{dx^4} + O(h^5), \\ (W_x)_{i-1}^{j+1} &\cong (W_x)_i^{j+1} - \frac{h}{1!} (W_{xx})_i^{j+1} + \frac{h^2}{2!} (W_{xxx})_i^{j+1} - \frac{h^3}{3!} (W_{xxxx})_i^{j+1} + O(h^4), \\ (W_x)_{i+1}^{j+1} &\cong (W_x)_i^{j+1} + \frac{h}{1!} (W_{xx})_i^{j+1} + \frac{h^2}{2!} (W_{xxx})_i^{j+1} + \frac{h^3}{3!} (W_{xxxx})_i^{j+1} + O(h^4), \\ (W_x)_{i-1}^j &\cong (W_x)_i^j - \frac{h}{1!} (W_{xx})_i^j + \frac{h^2}{2!} (W_{xxx})_i^j - \frac{h^3}{3!} (W_{xxxx})_i^j + O(h^4), \\ (W_x)_{i+1}^j &\cong (W_x)_i^j + \frac{h}{1!} (W_{xx})_i^j + \frac{h^2}{2!} (W_{xxx})_i^j + \frac{h^3}{3!} (W_{xxxx})_i^j + O(h^4), \\ (W_{xx})_{i-1}^{j+1} &\cong (W_{xx})_i^{j+1} - \frac{h}{1!} (W_{xxx})_i^{j+1} + \frac{h^2}{2!} (W_{xxxx})_i^{j+1} + O(h^3), \\ (W_{xx})_{i+1}^{j+1} &\cong (W_{xx})_i^{j+1} + \frac{h}{1!} (W_{xxx})_i^{j+1} + \frac{h^2}{2!} (W_{xxxx})_i^{j+1} + O(h^3), \\ (W_{xx})_{i-1}^j &\cong (W_{xx})_i^j - \frac{h}{1!} (W_{xxx})_i^j + \frac{h^2}{2!} (W_{xxxx})_i^j + O(h^3), \\ (W_{xx})_{i+1}^j &\cong (W_{xx})_i^j + \frac{h}{1!} (W_{xxx})_i^j + \frac{h^2}{2!} (W_{xxxx})_i^j + O(h^3), \end{aligned} \quad (5.31)$$

into Eq. (5.30), we have

$$\begin{aligned} T_i(h) &= [\xi_0^{j+1} W_i^{j+1} + \xi_0^j W_i^j] + [\xi_1^{j+1} W_{x,i}^{j+1} + \xi_1^j W_{x,i}^j] + \\ &[\xi_2^{j+1} W_{xx,i}^{j+1} + \xi_2^j W_{xx,i}^j] + [\xi_3^{j+1} W_{xxx,i}^{j+1} + \xi_3^j W_{xxx,i}^j] + \text{Residual terms}, \end{aligned} \quad (5.32)$$

where the coefficients are given as:

$$\begin{aligned}
\xi_0^{j+1} &= (A_i^- + A_i^c + A_i^+) + h(\beta_{i-1}^{j+1} + \beta_i^{j+1} + \beta_{i+1}^{j+1}), \\
\xi_0^j &= -(B_i^- + B_i^c + B_i^+) + h(\beta_{i-1}^j + \beta_i^j + \beta_{i+1}^j), \\
\xi_1^{j+1} &= h(A_i^+ - A_i^-) - h\mu(a_{i-1}^{j+1} + a_i^{j+1} + a_{i+1}^{j+1}) - h^2(\beta_{i-1}^{j+1} - \beta_{i+1}^{j+1}), \\
\xi_1^j &= h(B_i^- - B_i^+) - h\mu(a_{i-1}^j + a_i^j + a_{i+1}^j) + h^2(\beta_{i+1}^j - \beta_{i-1}^j), \\
\xi_2^{j+1} &= \frac{h^2}{2}(A_i^- + A_i^+) - 3h\varepsilon + h^2\mu(a_{i-1}^{j+1} - a_{i+1}^{j+1}) + \frac{h^3}{2}(\beta_{i-1}^{j+1} + \beta_{i+1}^{j+1}), \\
\xi_2^j &= -\frac{h^2}{2}(A_i^+ + A_i^-) - 3h\varepsilon + h^2\mu(a_{i-1}^j - a_{i+1}^j) + \frac{h^3}{2}(\beta_{i-1}^j + \beta_{i+1}^j), \\
\xi_3^{j+1} &= \frac{h^3}{6}(A_i^+ - A_i^-) - \frac{h^3}{2}\mu(a_{i-1}^{j+1} + a_{i+1}^{j+1}) - \frac{h^4}{6}(\beta_{i-1}^{j+1} - \beta_{i+1}^{j+1}), \quad \text{and} \\
\xi_3^j &= \frac{h^3}{6}(B_i^- - B_i^+) - \frac{h^3}{2}\mu(a_{i-1}^j + a_{i+1}^j) - \frac{h^4}{6}(\beta_{i+1}^j - \beta_{i-1}^j).
\end{aligned}$$

Substituting and simplifying the values of $A_i^-, A_i^c, A_i^+, B_i^-, B_i^c, B_i^+$ and restricting the expansion of the coefficients to the first term gives $\xi_0^{j+1} = \xi_0^j = \xi_1^{j+1} = \xi_1^j = 0$ and

$$\begin{cases} \xi_2^{j+1} = \frac{h^2\sigma(\rho)}{\rho} - 3h\varepsilon - \frac{h^3}{6}\left(11(a')_i^{j+1} + 5\beta_i^{j+1}\right) + O(h^3), \\ \xi_2^j = \frac{h^2\sigma(\rho)}{\rho} - 3h\varepsilon - \frac{h^3}{6}\left(11(a')_i^j + 5\beta_i^j\right) + O(h^3). \end{cases} \quad (5.33)$$

Hence, employing the bounded nature of the coefficient functions, we derive:

$$|T_i(h)| \leq \xi_2^{j+1}(W_{xx})_i^{j+1} + \xi_2^j(W_{xx})_i^j \leq h\varepsilon(\sigma - 3)(W_{xx})_i^{j+1}, \quad \text{since } \rho = \frac{h}{\varepsilon}. \quad (5.34)$$

Using the power series expansion of $\coth(\psi)$, the following inequality is obtained:

$$\coth(\psi) = \frac{1}{\psi} + \frac{\psi}{3} - \frac{\psi^3}{45} + O(\psi^5) \Rightarrow \psi \coth(\psi) = 1 + \frac{\psi^2}{3} + O(\psi^4),$$

where $\psi = \frac{\rho\mu a^{j+1}(x_i)}{2}$. Now $|\sigma - 3| = \left| 1 + \frac{\left(\frac{\rho\mu a^{j+1}(x_i)}{2}\right)^2}{3} - 3 \right| \leq C\rho^2\mu^2$. From

this, Eq. (5.34) reduced to

$$|T_i(h)| \leq h\varepsilon(C\rho^2\mu^2)(W_{xx})_i^{j+1} = Ch^3\frac{\mu^2}{\varepsilon}(W_{xx})_i^{j+1}. \quad (5.35)$$

Thus, considering the ratio of μ^2/ε and applying Lemma (5.8), we have

$$|T_i(h)| \leq \begin{cases} Ch^3 \frac{\mu}{\varepsilon} & \text{when } \mu^2 \leq \varepsilon, \\ Ch^3 \frac{\mu^2}{\varepsilon} & \text{when } \varepsilon \leq \mu^2. \end{cases} \quad (5.36)$$

Lemma 5.11. *Let the solutions to Eqs. (5.1) and (5.24) be $w(x_i, t_{j+1})$ and $\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1}$, respectively. Next, in the spatial direction, the proposed scheme satisfies the estimation (Ayele et al., 2023):*

$$|\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1} - w(x_i, t_{j+1})| \leq \begin{cases} Ch^2 & \text{when } \mu^2 \leq \varepsilon, \\ Ch & \text{when } \varepsilon \leq \mu^2. \end{cases} \quad (5.37)$$

Proof. For $i = 1(1)N$, let's construct a barrier function as

$$\Theta^\pm(x_i, t_{j+1}) = T_i(h) \pm (\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1} - w(x_i, t_{j+1})).$$

Now from boundary conditions, $\Theta^\pm(x_0, t_{j+1}) \geq 0$, $\Theta^\pm(x_N, t_{j+1}) \geq 0$ and $\mathbb{L}_{\varepsilon, \mu}^{N, M} \Theta^\pm(x_i, t_{j+1}) \leq 0$ for $i = 1(1)N - 1$. Hence, by applying Lemma 5.9, the required estimate can be achieved. $\square$

Consequently, the established bound leads to the main result presented in the following theorem.

Theorem 5.1. *Let $w(x_i, t_{j+1})$ and W_i^{j+1} be solutions of the continuous problem (5.1) and the discrete problem (5.24), respectively. Then, the error estimate for the fully discrete scheme is given by*

$$\max_{0 \leq i \leq N, 0 \leq j \leq M} \|w(x_i, t_{j+1}) - W_i^{j+1}\| \leq \begin{cases} C(h^2 + k^2) & \text{when } \mu^2 \leq \varepsilon, \\ C(h + k^2) & \text{when } \varepsilon \leq \mu^2, \end{cases} \quad (5.38)$$

where C is a constant unaffected by any perturbation parameter.

Proof. The proof of this theorem initiates by examining the left-hand side of Eq. (5.38). Applying the triangle inequality to the semi-discrete solution $W^{j+1}(x_i)$ and incorporating the error estimates provided in Lemmas 5.11 and 5.7, the desired result is obtained (Mohye et al., 2023). $\square$

Consequently, the proposed method achieves a second-order convergence rate when $\mu^2/\varepsilon \leq 1$, exhibiting uniform convergence that is independent of the perturbation parameters. In contrast, in the regime $\varepsilon/\mu^2 \ll 1$, the scheme attains first-order convergence, reflecting the dominance of spatial discretization effects over temporal variations. This demonstrates the method's robustness across varying perturbation scales and its adaptability to both parameter-sensitive and parameter-insensitive regimes.

5.6 Numerical Examples, Results and Discussion

To verify the theoretical results experimentally, we solve the following examples.

Example 5.1. Consider TP-SPPPs presented in [Negero \(2023\)](#); [Singh et al. \(2022\)](#).

$$\begin{cases} \varepsilon \frac{\partial^2 w}{\partial x^2} + \mu(1+x) \frac{\partial w}{\partial x} - w(x,t) - \frac{\partial w}{\partial t} = \\ \quad -w(x, t - \tau) + 16x^2(1-x)^2, (x,t) \in (0,1) \times (0,2], \\ w(0,t) = 0, \quad w(1,t) = 0, \quad t \in (0,2], \\ w(x,t) = 0, \quad (x,t) \in [0,1] \times [-\tau,0]. \end{cases} \quad (5.39)$$

Example 5.2. Consider TP-SPPPs involving delay terms in [Negero \(2023\)](#).

$$\begin{cases} \varepsilon \frac{\partial^2 w}{\partial x^2} + \mu(1+x-x^2+t^2) \frac{\partial w}{\partial x} - (1+5xt)w(x,t) - \frac{\partial w}{\partial t} = \\ \quad -w(x, t - \tau) + (x-x^2)(e^t - 1), (x,t) \in (0,1) \times (0,2], \\ w(0,t) = 0, \quad w(1,t) = 0, \quad t \in (0,2], \\ w(x,t) = 0, \quad (x,t) \in [0,1] \times [-\tau,0]. \end{cases} \quad (5.40)$$

Example 5.3. Consider TP-SPPPs with a time delay $\tau = 1$ as in [Singh et al. \(2022\)](#).

$$\begin{cases} \varepsilon \frac{\partial^2 w}{\partial x^2} + \mu(1+x^2) \frac{\partial w}{\partial x} - (1+5xt)w(x,t) - \frac{\partial w}{\partial t} = \\ \quad -w(x, t - \tau) + 16x^2(1-x)^2, (x,t) \in (0,1) \times (0,2], \\ w(0,t) = 0, \quad w(1,t) = 0, \quad t \in (0,2], \\ w(x,t) = 0, \quad (x,t) \in [0,1] \times [-\tau,0]. \end{cases} \quad (5.41)$$

As the considered examples have no exact solution, we calculate the scheme's

maximum pointwise error $E_{\varepsilon,\mu}^{N,M}$ and rate of convergence $\overline{R}_{\varepsilon,\mu}^{N,M}$ using the double mesh principle (Doolan et al., 1980) as given in Chapter 3, Section 3.6. Although the proposed method is designed to accommodate both small and large time delays τ , due to limitations in space and scope, we present examples only for the case of a large delay. Nevertheless, the approach is equally applicable to either scenario (Daba et al., 2024). In the examples considered here, the time delay is set as $\tau = 1$.

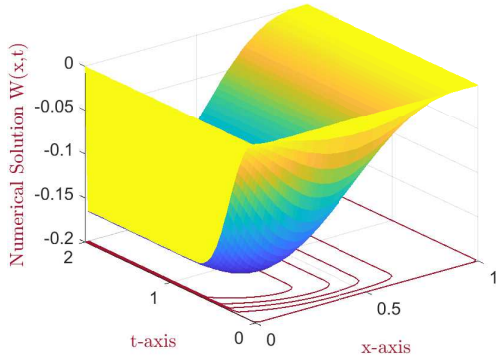
Table 5.1: Computed $E_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ for Example 5.1 at $\mu = 10^{-9}$ and comparison with Sumit et al. (2020); Negro (2023).

ε $\downarrow$	$N = 32$ $M = 8$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$	$N = 1024$ $M = 256$
Present Method						
10^{-2}	$7.6693e-04$ 2.0107	$1.9032e-04$ 1.9997	$4.7589e-05$ 2.0010	$1.1889e-05$ 2.0000	$2.9723e-06$ 2.0000	$7.4306e-07$ —
10^{-4}	$1.2817e-03$ 2.1496	$2.8886e-04$ 2.1416	$6.5466e-05$ 2.0180	$1.6164e-05$ 2.0057	$4.0250e-06$ 2.0008	$1.0057e-06$ —
10^{-6}	$1.3615e-03$ 1.9062	$3.6324e-04$ 1.9573	$9.3536e-05$ 2.0138	$2.3162e-05$ 2.1542	$5.2037e-06$ 2.1790	$1.1491e-06$ —
10^{-8}	$1.3617e-03$ 1.9053	$3.6351e-04$ 1.9518	$9.3962e-05$ 1.9758	$2.3888e-05$ 1.9883	$6.0206e-06$ 1.9967	$1.5086e-06$ —
10^{-10}	$1.3617e-03$ 1.9053	$3.6351e-04$ 1.9518	$9.3964e-05$ 1.9757	$2.3890e-05$ 1.9878	$6.0231e-06$ 1.9939	$1.5122e-06$ —
10^{-12}	$1.3617e-03$ 1.9053	$3.6351e-04$ 1.9518	$9.3964e-05$ 1.9757	$2.3890e-05$ 1.9878	$6.0231e-06$ 1.9939	$1.5122e-06$ —
$E^{N,M}$	$1.3617e-03$	$3.6351e-04$	$9.3964e-05$	$2.3890e-05$	$6.0231e-06$	$1.5122e-06$
$\overline{R}^{N,M}$	1.9053	1.9518	1.9757	1.9878	1.9939	—
Result in Sumit et al. (2020)(Uniform S-mesh)						
$E^{N,M}$	$4.3817e-2$	$1.6750e-02$	$7.4019e-3$	$3.7490e-3$	$1.9008e-3$	$9.5719e-4$
$\overline{R}^{N,M}$	1.3873	1.1781	0.9813	0.9799	0.9898	—
Result in Negro (2023)(Fitted cubic spline in tension method)						
$E^{N,M}$	$3.2186e-03$	$7.9382e-04$	$1.9871e-04$	$4.9629e-05$	$1.2404e-05$	$3.1012e-06$
$\overline{R}^{N,M}$	2.0195	1.9981	2.0014	2.0004	1.9999	—

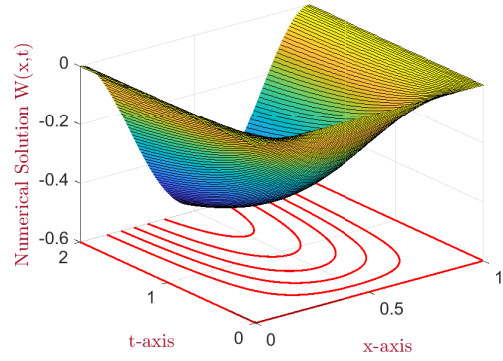
The computed maximum pointwise error $E_{\varepsilon,\mu}^{N,M}$ and rate of convergence $\overline{R}_{\varepsilon,\mu}^{N,M}$ for Examples 5.1-5.3 are presented in Tables 5.1-5.8 with different values of ε, μ, N and M . Tables 5.1-5.5, and 5.7 compare numerical results to those given in (Sumit et al., 2020; Negro, 2023; Govindarao et al., 2019; Singh et al., 2022), by fixing or varying ε or μ . This comparison indicates that, once compared to studies found in the literature, the suggested approach provides a more accurate solution. Consequently, tabulated numerical results confirm that our theoretical estimates in Lemma 5.7 and Theorem 5.1 can be achieved by the proposed scheme, that is 1/3rd Simpson's scheme. Tables 5.5 and 5.7 display the error for Examples 5.2 and 5.3, respectively, for the

Table 5.2: Computed maximum pointwise error and rate of convergence for Example 5.1 at $\mu = 10^{-4}$ and comparison with Govindarao et al. (2019) for $N = M$.

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
Present Method					
10^{-4}	$1.2721e-03$	$2.8911e-04$	$6.5457e-05$	$1.6144e-05$	$4.0406e-06$
	2.1375	2.1430	2.01955	1.99836	
10^{-6}	$1.3591e-03$	$3.6573e-04$	$9.4783e-05$	$2.3704e-05$	$5.4139e-06$
	1.8938	1.9481	1.9995	2.1304	
10^{-8}	$1.3606e-03$	$3.6879e-04$	$9.7117e-05$	$2.5534e-05$	$9.0849e-06$
	1.8834	1.9250	1.9273	1.4909	
10^{-10}	$1.3606e-03$	$3.6879e-04$	$9.7117e-05$	$2.5534e-05$	$9.0851e-06$
	1.8834	1.9250	1.9273	1.4908	
10^{-12}	$1.3606e-03$	$3.6879e-04$	$9.7117e-05$	$2.5534e-05$	$9.0851e-06$
	1.8834	1.9250	1.9273	1.4908	
Result in Govindarao et al. (2019)(Standard Shishkin mesh)($\mu = 10^{-4}$)					
$E^{N,M}$	$2.8286e-3$	$1.4280e-3$	$7.1747e-4$	$3.5960e-4$	$1.8002e-4$
$\bar{R}^{N,M}$	0.9861	0.9930	0.9965	0.9982	
Result in Govindarao et al. (2019)(Bakhvalov-Shishkin mesh)($\mu = 10^{-4}$)					
$E^{N,M}$	$1.8789e-3$	$9.9691e-4$	$5.0033e-4$	$2.5064e-4$	$1.2544e-4$
$\bar{R}^{N,M}$	0.9891	0.9945	0.9973	0.9986	



(a) $N = M = 64, \varepsilon_1 = 10^{-8}, \mu = 1$



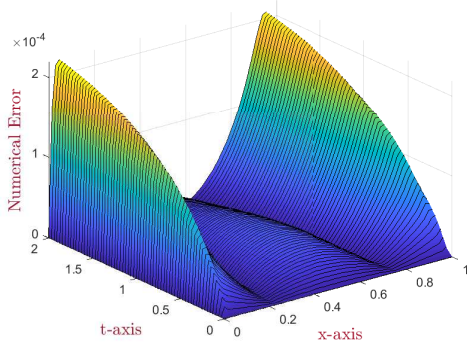
(b) $N = M = 128, \varepsilon_1 = 10^{-12}, \mu = 10^{-10}$

Figure 5.1: The numerical solution profile for Example 5.1.

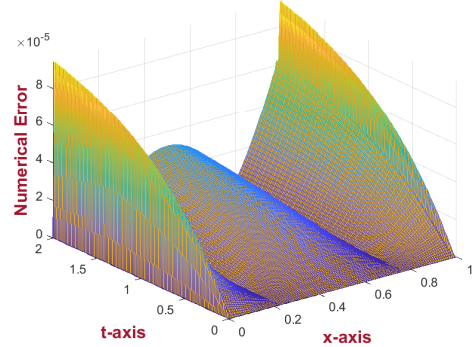
case $\varepsilon/\mu^2 \ll 1$ by taking a fixed value of μ and varying the values of ε . According to the theoretical analysis, our method is expected to be first-order convergent in space under these conditions and provide a robust validation of the method's performance. We have conducted additional numerical experiments to verify this behavior, ensuring that the spatial convergence rate dominates over the temporal direction. As we double the number of mesh points, as we observe in Tables 5.6 and 5.8, errors decrease, and

Table 5.3: Maximum pointwise error, convergence rate and CPU time(in seconds) for Example 5.1 with $\varepsilon = 10^{-4}$ and $N = M$.

μ	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
Present Method					
10^{-4}	$1.2721e-03$	$2.8911e-04$	$6.5457e-05$	$1.6144e-05$	$4.0406e-06$
	2.1375	2.1430	2.0195	1.9984	
10^{-6}	$1.2693e-03$	$2.8796e-04$	$6.5267e-05$	$1.6096e-05$	$4.0083e-06$
	2.1401	2.1414	2.0197	2.0056	
10^{-8}	$1.2693e-03$	$2.8795e-04$	$6.5265e-05$	$1.6096e-05$	$4.0080e-06$
	2.1401	2.1415	2.0196	2.0057	
10^{-10}	$1.2693e-03$	$2.8795e-04$	$6.5265e-05$	$1.6096e-05$	$4.0080e-06$
	2.1401	2.1414	2.0196	2.0057	
10^{-12}	$1.2693e-03$	$2.8795e-04$	$6.5265e-05$	$1.6096e-05$	$4.0080e-06$
	2.1401	2.1414	2.0196	2.0057	
$CPU(s)$	0.52	1.72	12.20	117.92	910.86
Result in Govindarao et al. (2019)(Standard Shishkin mesh)($\varepsilon = 10^{-4}$)					
$E^{N,M}$	$2.8183e-3$	$1.4253e-3$	$7.1638e-4$	$3.5904e-4$	$1.7973e-4$
$\bar{R}^{N,M}$	0.9835	0.9925	0.9965	0.9983	
Result in Govindarao et al. (2019)(Bakhvalov-Shishkin mesh)($\varepsilon = 10^{-4}$)					
$E^{N,M}$	$1.9653e-3$	$9.9001e-4$	$4.9686e-4$	$2.4891e-4$	$1.2531e-4$
$\bar{R}^{N,M}$	0.9892	0.9946	0.9973	0.9986	



(a) $N = M = 64, \varepsilon_1 = 10^{-8}, \mu = 1$



(b) $N = M = 128, \varepsilon_1 = 10^{-12}, \mu = 10^{-10}$

Figure 5.2: Surface plots for numerical Error for Example 5.1.

as we approach downward in each column, errors become constant. These results confirm parameter uniformity and second-order accuracy in the relevant cases. All tables reveal that convergence is independent of perturbation parameters, and that the maximum absolute error decreases as the number of mesh points increases. The time taken by the CPU(in seconds) to deliver the outputs (maximum point-wise error and convergence rate) is also provided, which shows the efficiency of our proposed

Table 5.4: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}^{N,M}$ for Example 5.1 at $\varepsilon = 2^{-10}$ and comparison with Singh et al. (2022) for varying values of μ .

μ $\downarrow$	$N = M = 16$	$N = M = 32$	$N = M = 64$	$N = M = 128$	$N = M = 256$
Present Method					
2^{-10}	$3.8671e-03$ 2.1889	$8.4813e-04$ 2.0235	$2.0860e-04$ 1.9826	$5.2782e-05$ 1.9673	$1.3498e-05$
2^{-14}	$3.8321e-03$ 2.1974	$8.3554e-04$ 2.0205	$2.0594e-04$ 2.0132	$5.1017e-05$ 1.9933	$1.2814e-05$
2^{-18}	$3.8301e-03$ 2.1979	$8.3478e-04$ 2.0203	$2.0578e-04$ 2.0151	$5.0910e-05$ 2.0002	$1.2726e-05$
2^{-22}	$3.8299e-03$ 2.1979	$8.3474e-04$ 2.0203	$2.0577e-04$ 2.0152	$5.0904e-05$ 2.0007	$1.2720e-05$
2^{-26}	$3.8299e-03$ 2.1979	$8.3473e-04$ 2.0203	$2.0577e-04$ 2.0152	$5.0903e-05$ 2.0007	$1.2720e-05$
2^{-30}	$3.8299e-03$ 2.1979	$8.3473e-04$ 2.0203	$2.0577e-04$ 2.0152	$5.0903e-05$ 2.0007	$1.2720e-05$
$E^{N,M}$	$3.8671e-03$	$8.4813e-04$	$2.0860e-04$	$5.2782e-05$	$1.3498e-05$
$\overline{R}^{N,M}$	2.1889	2.0235	1.9826	1.9673	
Result in Singh et al. (2022)(Crank-Nicolson+Cubic B-Spline on S- mesh)					
$E^{N,M}$	$2.5202e-02$	$9.9355e-03$	$3.3543e-03$	$9.0485e-04$	$2.4036e-04$
$\overline{R}^{N,M}$	1.3429	1.5666	1.8903	1.9125	

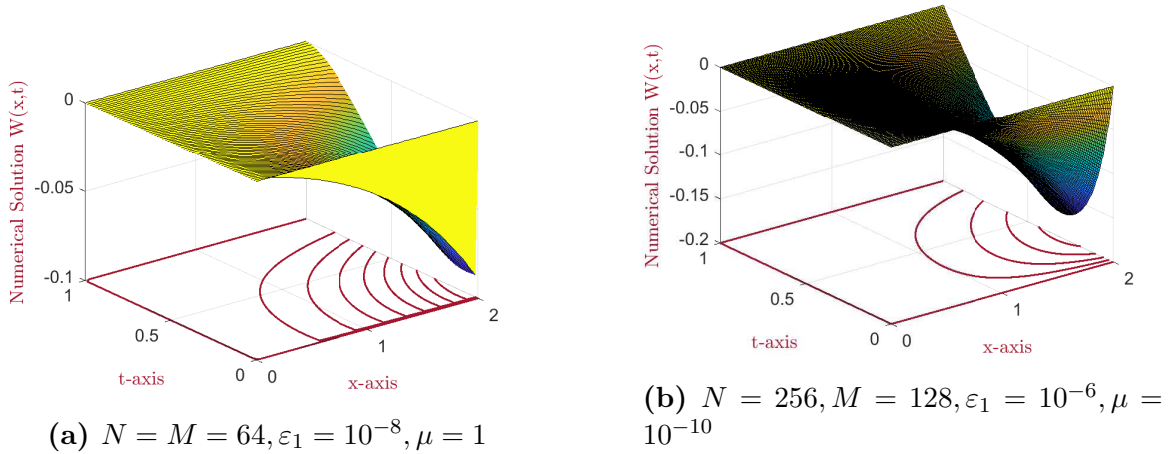


Figure 5.3: The numerical solution profile for Example 5.2.

scheme. The numerical solution surface plots for the test problems (5.39)-(5.41) are presented in Figures 5.1, 5.3 and 5.4, respectively with different values of N, M, ε and μ . The problem displays left and right boundary layers based on the size of ε or μ . In addition, Figures 5.5, 5.6 and 5.7 show the log-log plot of the maximum pointwise errors for Examples 5.1, 5.2 and 5.3 respectively. Figure 5.5a depict the log-

Table 5.5: Computed $E_{\varepsilon,\mu}^{N,M}, \bar{R}_{\varepsilon,\mu}^{N,M}$, $E^{N,M}, \bar{R}^{N,M}$ and CPU time (in seconds) for Example 5.2 at $\mu = 10^{-9}$ and comparison with Sumit et al. (2020); Negro (2023).

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$\downarrow$	$M = 8$	$M = 16$	$M = 32$	$M = 64$	$M = 128$	$M = 256$
Present Method						
10^{-2}	$2.6587e-03$	$1.4769e-03$	$7.7335e-04$	$3.9503e-04$	$1.9957e-04$	$1.0029e-04$
	0.84815	0.93338	0.96916	0.98507	0.99272	—
10^{-4}	$2.7254e-03$	$1.5152e-03$	$7.9322e-04$	$4.0520e-04$	$2.0470e-04$	$1.0287e-04$
	0.84696	0.93372	0.96909	0.98512	0.99269	—
10^{-6}	$2.7261e-03$	$1.5156e-03$	$7.9342e-04$	$4.0530e-04$	$2.0475e-04$	$1.0290e-04$
	0.8469	0.9337	0.9691	0.9851	0.9926	—
10^{-8}	$2.7261e-03$	$1.5156e-03$	$7.9342e-04$	$4.0530e-04$	$2.0475e-04$	$1.0290e-04$
	0.8469	0.9337	0.9691	0.9851	0.9926	—
10^{-10}	$2.7261e-03$	$1.5156e-03$	$7.9342e-04$	$4.0530e-04$	$2.0475e-04$	$1.0290e-04$
	0.8469	0.9337	0.9691	0.9851	0.9926	—
10^{-12}	$2.7261e-03$	$1.5156e-03$	$7.9342e-04$	$4.0530e-04$	$2.0475e-04$	$1.0290e-04$
	0.8469	0.9337	0.9691	0.9851	0.9926	—
CPU(s)	0.38	0.78	3.69	26.83	299.73	4195.14
Result in Negro (2023)(Fitted cubic spline in tension method)						
10^{-2}	$2.7509e-03$	$1.7964e-03$	$1.0154e-03$	$5.3861e-04$	$2.7722e-04$	$1.4062e-04$
10^{-4}	$2.8224e-03$	$1.8422e-03$	$1.0409e-03$	$5.5203e-04$	$2.8411e-04$	$1.4410e-04$
10^{-6}	$2.8230e-03$	$1.8426e-03$	$1.0412e-03$	$5.5216e-04$	$2.8418e-04$	$1.4414e-04$
10^{-8}	$2.8231e-03$	$1.8426e-03$	$1.0412e-03$	$5.5216e-04$	$2.8418e-04$	$1.4414e-04$
10^{-10}	$2.8231e-03$	$1.8426e-03$	$1.0412e-03$	$5.5216e-04$	$2.8418e-04$	$1.4414e-04$
10^{-12}	$2.8231e-03$	$1.8426e-03$	$1.0412e-03$	$5.5216e-04$	$2.8418e-04$	$1.4414e-04$
Result in Sumit et al. (2020)(Uniform S-mesh)						
$E^{N,M}$	$1.110e-2$	$5.0838e-03$	$2.4640e-3$	$1.2162e-3$	$6.0457e-4$	$3.0142e-4$
$\bar{R}^{N,M}$	1.1265	1.0449	1.0185	1.0084	1.0041	—

Table 5.6: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}^{N,M}$ for Example 5.2 at $\mu = 10^{-9}$ after Extrapolation.

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$\downarrow$	$M = 8$	$M = 16$	$M = 32$	$M = 64$	$M = 128$	$M = 256$
10^{-2}	$2.9513e-04$	$6.9354e-05$	$1.6731e-05$	$4.1042e-06$	$1.0161e-06$	$2.5276e-07$
	2.0893	2.0515	2.0274	2.0141	2.0072	—
10^{-4}	$3.0339e-04$	$7.1229e-05$	$1.7184e-05$	$4.2151e-06$	$1.0435e-06$	$2.5959e-07$
	2.0906	2.0514	2.0274	2.0141	2.0071	—
10^{-6}	$3.0347e-04$	$7.1245e-05$	$1.7188e-05$	$4.2161e-06$	$1.0718e-06$	$2.5965e-07$
	2.0907	2.0514	2.0274	1.9759	2.0454	—
10^{-8}	$3.0347e-04$	$7.1246e-05$	$1.7188e-05$	$4.2161e-06$	$1.0438e-06$	$2.5965e-07$
	2.0907	2.0514	2.0274	2.0141	2.0072	—
10^{-10}	$3.0347e-04$	$7.1246e-05$	$1.7188e-05$	$4.2161e-06$	$1.0438e-06$	$2.5965e-07$
	2.0907	2.0514	2.0274	2.0141	2.0072	—
10^{-12}	$3.0347e-04$	$7.1246e-05$	$1.7188e-05$	$4.2161e-06$	$1.0438e-06$	$2.5965e-07$
	2.0907	2.0514	2.0274	2.0141	2.0072	—
$E^{2N,2M}$	$3.0347e-04$	$7.1246e-05$	$1.7188e-05$	$4.2161e-06$	$1.0438e-06$	$2.5965e-07$
$\bar{R}^{2N,2M}$	2.0907	2.0514	2.0274	2.0141	2.0072	—

log plot of maximum pointwise errors for Example 5.1 taking the values in Table 5.1.

A comparison of error between Sumit et al. (2020); Negro (2023) and the present

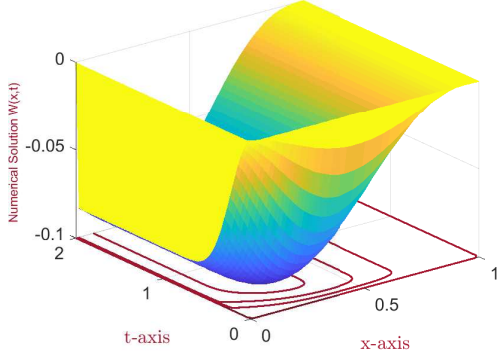
Table 5.7: Maximum pointwise error, convergence rate and CPU time(in seconds) for Example 5.3 with $\varepsilon = 2^{-10}$ and varying μ values.

μ $\downarrow$	$(N, M) = 16$	$(N, M) = 32$	$(N, M) = 64$	$(N, M) = 128$	$(N, M) = 256$
Present Method					
2^{-10}	$2.7430e-03$	$6.7362e-04$	$3.5820e-04$	$1.8625e-04$	$9.5499e-05$
	2.0257	0.91117	0.94352	0.96368	
2^{-14}	$2.7265e-03$	$6.6712e-04$	$3.5677e-04$	$1.8479e-04$	$9.4023e-05$
	2.031	0.90295	0.94911	0.97480	
2^{-18}	$2.7255e-03$	$6.6673e-04$	$3.5668e-04$	$1.8470e-04$	$9.3930e-05$
	2.0313	0.90247	0.94945	0.97553	
2^{-22}	$2.7254e-03$	$6.6671e-04$	$3.5668e-04$	$1.8469e-04$	$9.3924e-05$
	2.0313	0.90243	0.94952	0.97554	
2^{-26}	$2.7254e-03$	$6.6670e-04$	$3.5667e-04$	$1.8469e-04$	$9.3924e-05$
	2.0314	0.90245	0.94948	0.97554	
2^{-30}	$2.7254e-03$	$6.6670e-04$	$3.5667e-04$	$1.8469e-04$	$9.3924e-05$
	2.0314	0.90245	0.94948	0.97554	
$CPU(s)$	0.21	0.40	1.41	9.01	104.61
Result in Singh et al. (2022)(Crank-Nicolson+Cubic B-Spline)					
$E_{\varepsilon,\mu}^{N,M}$	$8.0066e-03$	$3.8811e-03$	$1.4850e-03$	$4.7138e-04$	$1.3316e-04$
$R_{\varepsilon,\mu}^{N,M}$	1.04471	1.3860	1.6524	1.8268	

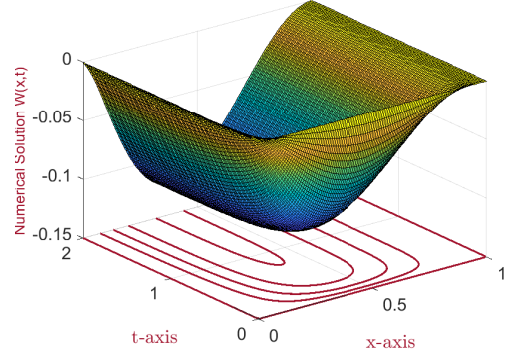
Table 5.8: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}^{N,M}$ for Example 5.3 at $\varepsilon = 10^{-10}$ after Extrapolation.

μ $\downarrow$	$(N, M) = 16$	$(N, M) = 32$	$(N, M) = 64$	$(N, M) = 128$	$(N, M) = 256$
2^{-10}	$1.5314e-03$	$3.4039e-04$	$7.7411e-05$	$1.9129e-05$	$4.9227e-06$
	2.1696	2.1366	2.0168	1.9582	
2^{-14}	$1.5177e-03$	$3.3676e-04$	$7.6716e-05$	$1.8849e-05$	$4.7074e-06$
	2.1721	2.1341	2.0250	2.0015	
2^{-18}	$1.5168e-03$	$3.3654e-04$	$7.6674e-05$	$1.8832e-05$	$4.6949e-06$
	2.1722	2.1340	2.0256	2.0040	
2^{-22}	$1.5168e-03$	$3.3652e-04$	$7.6671e-05$	$1.8831e-05$	$4.6941e-06$
	2.1723	2.1339	2.0256	2.0042	
2^{-26}	$1.5168e-03$	$3.3652e-04$	$7.6671e-05$	$1.8831e-05$	$4.6940e-06$
	2.1723	2.1339	2.0256	2.0042	
2^{-30}	$1.5168e-03$	$3.3652e-04$	$7.6671e-05$	$1.8831e-05$	$4.6940e-06$
	2.1723	2.1339	2.0256	2.0042	
$E_{\varepsilon,\mu}^{N,M}$	$1.5314e-03$	$3.4039e-04$	$7.7411e-05$	$1.9129e-05$	$4.9227e-06$
$\bar{R}^{N,M}$	2.1696	2.1366	2.0168	1.9582	

method using a log-log plot is shown in Figure 5.5b, and it can be noticed that the proposed method yields better accuracy. Figure 5.2 reveal numerical error plots

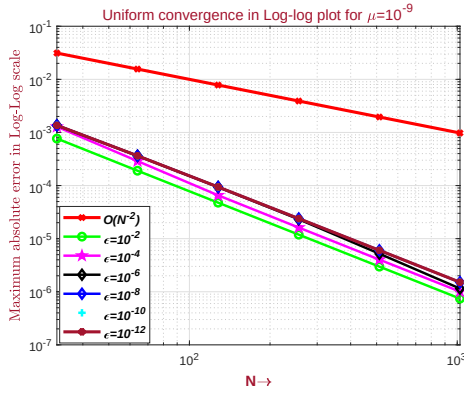


(a) $N = M = 64, \varepsilon_1 = 2^{-8}, \mu = 1$

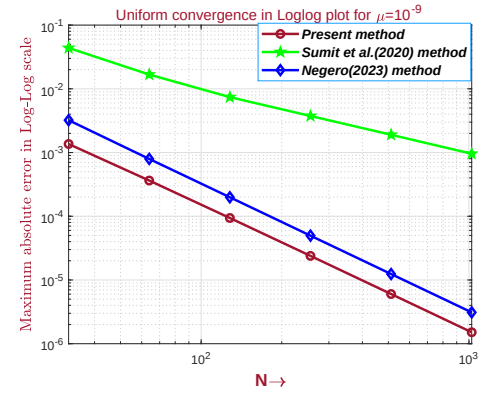


(b) $N = M = 128, \varepsilon_1 = 2^{-10}, \mu = 2^{-10}$

Figure 5.4: The numerical solution profile for Example 5.3.

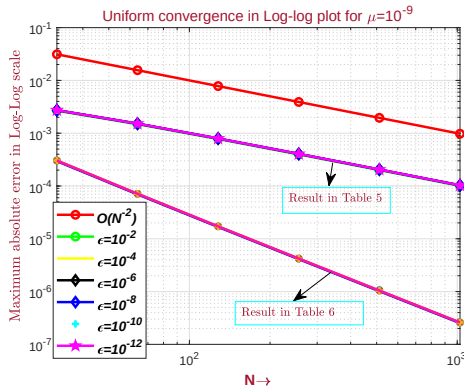


(a) Table 5.1

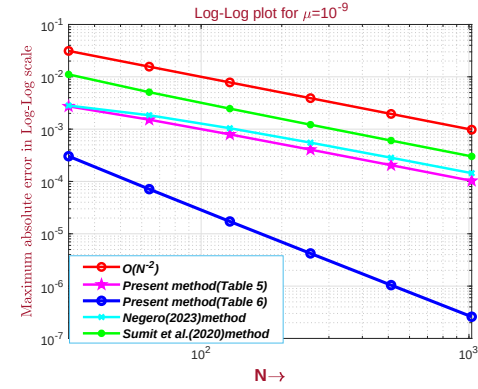


(b) Comparison of the log-log plot with results from the existing literature

Figure 5.5: The Log-log plot for Example 5.1.



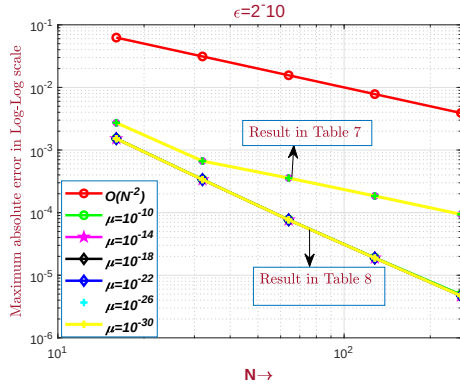
(a) Tables 5.5 and 5.6



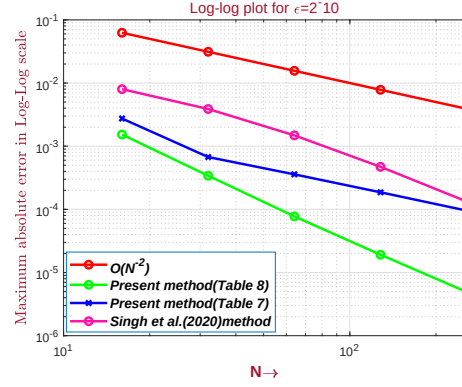
(b) Comparison of the log-log plot with results from the existing literature

Figure 5.6: The Log-log plot for Example 5.2.

for Examples 5.1. Figures 5.2a and 5.2b demonstrate high rises in the error plot, indicating boundary layers near endpoints $x = 0$ and $x = 1$ due to abrupt mesh size changes. For $\mu = 1$ the considered problem belongs to the class of one-parameter



(a) Tables 5.7 and 5.8



(b) Comparison of the log-log plot with results from the existing literature

Figure 5.7: The Log-log plot for Example 5.3.

time-delayed convection-diffusion parabolic problems and a boundary layer of width $O(\epsilon)$ arises in the vicinity of $x = 0$ as shown in Figures 5.1a, 5.3a, and 5.4a.

5.7 Conclusion

In this chapter, we addressed a TP-SPPPs with time delay by employing a reliable numerical integration approach. The Crank–Nicolson method was utilized for the temporal discretization, ensuring second-order accuracy in time, while the 1/3rd Simpson’s rule, enhanced with an appropriate fitting factor, was applied for the spatial discretization. The presence of two perturbation parameters induces twin boundary layers along the lateral boundaries of the rectangular domain. This characteristic behavior was clearly illustrated through a series of carefully chosen numerical examples. The theoretical analysis confirms that the proposed scheme maintains uniform convergence, independent of the perturbation parameters. Furthermore, the numerical results demonstrate that the method achieves higher accuracy compared to existing techniques reported in the literature.

Chapter 6

Fitted Non-Polynomial Cubic Spline Method for Two-Parameter Singularly Perturbed Parabolic Problems with Discontinuous Coefficients

This chapter advances the numerical solutions of TP-SPPPs by incorporating discontinuous convection coefficients and source terms, an aspect not addressed in the preceding chapters, which were limited to smooth parameter dependent formulations. These discontinuities, considered here induce complex boundary-interior layer interactions, posing challenges to classical discretization schemes designed for smooth data. To address this, a uniform scheme is formulated by combining the *implicit Euler method in time with a cubic spline in compression for space*. This hybrid approach enhances stability even under discontinuities. A detailed convergence analysis confirms the scheme's uniform reliability, and numerical experiments demonstrate its accuracy, robustness, and parameter-uniform convergence.

6.1 Introduction

Solutions of TP-SPPPs present large gradients when the perturbation parameters approaches to zero. These problems feature multiscale behavior, sharp gradients, and strong boundary or interior layers, making both analysis and numerical approximation highly challenging. When the convection coefficient and source term are discontinuous, the problem becomes even more intricate. Such discontinuities, representing *abrupt physical transitions like material interfaces or shocks, often produce non-smooth solutions and interior layers*. Standard numerical schemes relying on smoothness assumptions generally fail near these points, showing loss of accuracy. Hence, developing parameter-uniform numerical methods that maintain stability and accuracy across the entire domain, including discontinuous regions, is essential.

Although considerable progress has been achieved, most studies on discontinuous data focus on two-parameter SPODEs rather than parabolic ones. Notable contributions include adaptive-mesh and hybrid approaches proposed by researchers such as [Janani & Tamilselvan \(2019\)](#); [Prabha et al. \(2019\)](#); [Chandru et al. \(2017\)](#); [Roy & Jha \(2023\)](#). For parabolic cases, [Chandru et al. \(2018\)](#) presented a backward Euler upwind scheme on an adaptive mesh, while [Kumar & Kumari \(2021\)](#) employed a Crank–Nicolson and upwind difference method on a Shishkin mesh. More recently, [Singh et al. \(2023\)](#) developed a trigonometric B-splines with adaptive spatial meshes to capture boundary layers. However, such methods often require prior knowledge of layer locations or thicknesses, limiting their broader applicability.

To develop the proposed scheme, the time-dependent TP-SPPPs are discretized using the implicit Euler method on a uniform temporal mesh, while the spatial domain is treated with the cubic spline in tension technique on a uniform grid. Compared with standard FDMs, spline based approaches offer notable benefits, once the solution is computed, all information needed for smooth interpolation between nodes is readily available ([Khan et al., 2004](#)). Moreover, splines provide a more practical and convenient framework for handling BVPs ([Kumar & Gupta, 2010](#)), especially when the solution is required at multiple points within the interval $[a, b]$. These advantages motivate the use of cubic splines in tension for the problem addressed in this work.

The chapter is structured as follows: Section 6.2 presents the problem formulation, while Section 6.3 discusses key properties of the continuous solution. Section 6.4 outlines the construction of the proposed numerical scheme, and Section 6.5 provides its convergence analysis. Section 6.6 includes numerical experiments validating the method, and Section 6.7 concludes with a summary of the main results.

6.2 Definition of the Problem

Consider a parabolic reaction-convection-diffusion problem governed by two perturbation parameters, featuring discontinuous convection coefficients and source terms.

$$\left\{ \begin{array}{l} \mathbb{L}_{\varepsilon, \mu} w(x, t) \equiv \varepsilon \frac{\partial^2 w}{\partial x^2}(x, t) + \mu a(x, t) \frac{\partial w}{\partial x}(x, t) - b(x, t) w(x, t) - \\ c(x, t) \frac{\partial w}{\partial t}(x, t) = \varpi(x, t), (x, t) \in \Omega^- \cup \Omega^+, \\ w(0, t) = \vartheta(t) \quad \text{in } \Omega_l, \quad w(1, t) = \xi(t) \quad \text{in } \Omega_r, \\ w(x, 0) = \phi(x) \quad \text{in } \Omega_b, \\ |[a](\varphi)| \leq C, \quad |[\varpi](\varphi)| \leq C, \\ a(x, t) \leq -\alpha_1 < 0, \quad (x, t) \in \Omega^-, \\ a(x, t) \geq \alpha_2 > 0, \quad (x, t) \in \Omega^+, \\ b(x, t) \geq \beta > 0, \quad c(x, t) \geq c > 0 \end{array} \right. \quad (6.1)$$

Here, $0 < \varepsilon \ll 1, 0 \leq \mu \leq 1$ are perturbation parameters. The domains are specified in Figure 6.1 as follows: $\Omega = \Omega \times (0, T], \Omega^- = \Omega^- \times (0, T], \Omega^+ = \Omega^+ \times (0, T], \mathfrak{M} = \overline{\Omega} \setminus \Omega$, where $\Omega = (0, 1), \Omega^- = (0, \varphi), \Omega^+ = (\varphi, 1)$ and the boundaries are defined as $\Omega_l = \{(0, t) : 0 \leq t \leq T\}, \Omega_r = \{(1, t) : 0 \leq t \leq T\}, \Omega_b = \Omega_b^- \cup \Omega_b^+$, where, $\Omega_b^- = [0, \varphi] \times \{t = 0\}$ and $\Omega_b^+ = [\varphi, 1] \times \{t = 0\}$. We also denote $\mathfrak{M} = \Omega_b \cup \Omega_l \cup \Omega_r$ and $\mathfrak{M}^\pm = \{(\varphi, t) : 0 \leq t \leq T\}$. The terms $a(x, t), \varpi(x, t)$ are smooth enough in $(\Omega^- \cup \Omega^+) \cup \Omega$ and $b(x, t), c(x, t)$ are continuous in Ω . The inequalities $|[a](\varphi)| \leq C, |[\varpi](\varphi)| \leq C$ in Eq.(6.1) are used to bound the magnitude of the jump discontinuities of the convection coefficient and source term at an interior discontinuity point φ . Both terms $a(x, t)$ and $\varpi(x, t)$ has a discontinuity at $\varphi \in \Omega$, symbolized by $[\psi](\varphi, t) = [\psi](\varphi+, t) - [\psi](\varphi-, t)$. If $a(x, t)$ changes sign across the discontinuity, a solution

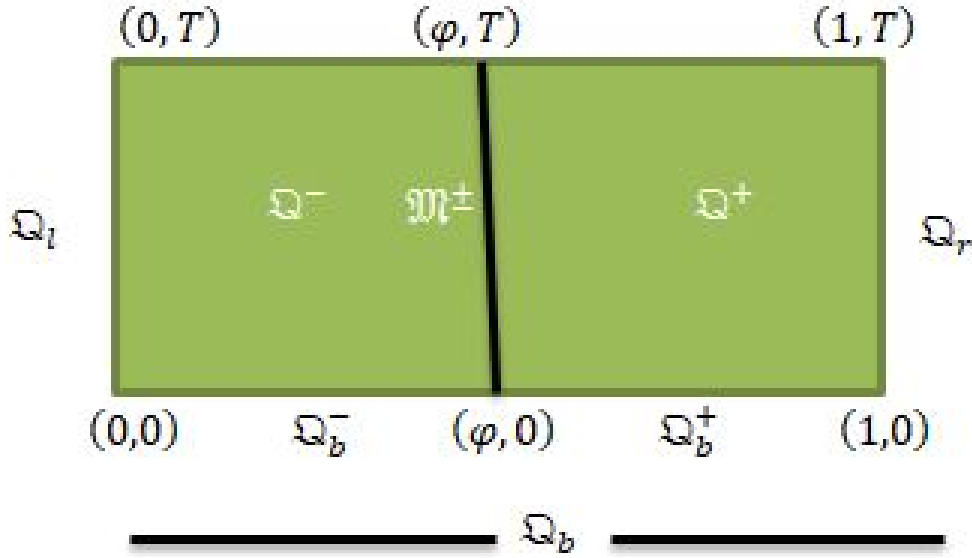


Figure 6.1: Domain $\Omega = [0, 1] \times (0, T]$.

exist for convection-diffusion problems with discontinuous first derivative coefficient. The discontinuities in convection and source terms generate interior layers on either side of the discontinuity at $x = \varphi$. The small diffusion parameter causes two boundary layers in the outcome of Eq. (6.1) and the sign of $a(x, t)$ and the ratio between diffusion and the square of convection parameter determine the behavior of layers.

6.3 Properties of the Continuous Solution

This section gives the comparison principle, the stability estimate and the bounds of Eq. (6.1). For Eq. (6.1) to have existence and uniqueness of the solution, we assume that the coefficients are Hölder continuous and also impose proper compatibility conditions at the corner points $(0, 0)$, $(1, 0)$ and at the transition corner point $(\varphi, 0)$ (Ladyzhenskaia et al., 1968). Necessarily the following conditions are satisfied:

Lemma 6.1 (Existence of Solution). *The solution $w(x, t)$ of Eq. (6.1) satisfies $w \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$.*

Proof. Let w_1 and w_2 be solutions to the following SPDEs (Kumar & Kumari, 2021):

$$\begin{aligned} (\varepsilon w_{1xx} + \mu a_1 w_{1x} - b w_1 - c w_{1t} = \varpi) (x, t), (x, t) \in \mathfrak{Q}^- \quad \text{and} \\ (\varepsilon w_{2xx} + \mu a_2 w_{2x} - b w_2 - c w_{2t} = \varpi) (x, t), (x, t) \in \mathfrak{Q}^+, \end{aligned}$$

where $a_1, a_2 \in C^2(\mathfrak{Q})$ and extended functions meet the following conditions:

$$\begin{aligned} a_1(x, t) = a(x, t), \quad (x, t) \in \mathfrak{Q}^-, \quad \text{ensuring that} \quad a_1(x, t) < 0, (x, t) \in \mathfrak{Q} \\ a_2(x, t) = a(x, t), \quad (x, t) \in \mathfrak{Q}^+, \quad \text{ensuring that} \quad a_2(x, t) > 0, (x, t) \in \mathfrak{Q}. \end{aligned}$$

Look at the subsequent functions:

$$w(x, t) = \begin{cases} w_1(x, t) + (w(0, t) - w_1(0, 1)) \Xi_1(x, t) + \Lambda_1 \Xi_2(x, t), & (x, t) \in \mathfrak{Q}^-, \\ w_2(x, t) + \Lambda_2 \Xi_1(x, t) + (w(1, t) - w_2(0, 1)) \Xi_2(x, t), & (x, t) \in \mathfrak{Q}^+, \end{cases}$$

where Ξ_1 and Ξ_2 are solutions of the following TP-SPPPs:

$$\begin{aligned} (\varepsilon \Xi_{1xx} + \mu a_1 \Xi_{1x} - b \Xi_1 - c \Xi_{1t}) (x, t) = 0, \Xi_1(0, t) = 1, \Xi_1(1, t) = 0, \Xi_1(x, 0) = 0, \\ (\varepsilon \Xi_{2xx} + \mu a_2 \Xi_{2x} - b \Xi_2 - c \Xi_{2t}) (x, t) = 0, \Xi_2(0, t) = 0, \Xi_2(1, t) = 1, \Xi_2(x, 0) = 0. \end{aligned}$$

Now $w(x, t)$ satisfies Eq.(6.1) in $\mathfrak{Q}^- \cup \mathfrak{Q}^+$ for constants Λ_1, Λ_2 , so that $w(x, t) \in C^1(\mathfrak{Q})$. On $(0, 1), 0 < \Xi_i(x, t) < 1, i = 1, 2$ on $\mathfrak{Q}$ (Kumar & Kumari, 2021; Chandru et al., 2018, 2019). Consequently, $\Xi_1(x, t), \Xi_2(x, t)$ cannot attain extremum at any interior point within the domain. Therefore, it follows that $\Xi_1'(x, t) < 0, \Xi_2'(x, t) > 0, (x, t) \in \mathfrak{Q}$. The constants Λ_1, Λ_2 are determined by imposing continuity of $w(x, t)$ and its first derivative across the interface point $x = \varphi$. That is $w(\varphi-, t) = w(\varphi+, t), \quad w_x(\varphi-, t) = w_x(\varphi+, t)$. Substituting the auxiliary Ξ_1 and Ξ_2 into these matching conditions leads to a 2×2 linear system for Λ_1, Λ_2 . In matrix form,

$$\begin{bmatrix} \Xi_2(\varphi, t) & -\Xi_1(\varphi, t) \\ \Xi_2'(\varphi, t) & -\Xi_1'(\varphi, t) \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} = \begin{bmatrix} w(\varphi+, t) - w(\varphi-, t) \\ w_x(\varphi+, t) - w_x(\varphi-, t) \end{bmatrix}.$$

The constants exist uniquely provided that the determinant of the coefficient matrix is nonzero. In other words, a nonzero determinant confirms the linear independence

of the system, ensuring a single, well-defined solution for unknown variables. That is,

$$\begin{vmatrix} \Xi_2(\varphi, t) & -\Xi_1(\varphi, t) \\ \Xi'_2(\varphi, t) & -\Xi'_1(\varphi, t) \end{vmatrix} \neq 0 \Rightarrow \left(\Xi'_2 \Xi_1 - \Xi_2 \Xi'_1 \right) (\varphi, t) > 0,$$

which holds due to the monotonic properties of $\Xi_1(\varphi, t), \Xi_2(\varphi, t)$. Hence, solving this system yields unique constants Λ_1, Λ_2 , ensuring $w(x, t)$ is continuous with a continuous first derivative across the interface. $\square$

Next, we derive the discontinuity conditions at $x = \varphi$ using the integral representation of the governing PDE, given by:

Lemma 6.2. *At the discontinuity point $x = \varphi$, the solution $w(x, t)$ together with its spatial derivative remain continuous, i.e.,*

$$w(\varphi^-, t) = w(\varphi^+, t) \quad \text{and} \quad \frac{\partial w}{\partial x}(\varphi^-, t) = \frac{\partial w}{\partial x}(\varphi^+, t).$$

Proof. Consider integrating Eq. (6.1) with respect to x over the narrow interval $(\varphi - \varrho, \varphi + \varrho)$. Let the jump of any function I at $x = \varphi$ be denoted by $[I]_\varphi := I(\varphi^+, t) - I(\varphi^-, t)$. Fix $t > 0$ and integrate Eq. (6.1) term by term to obtain

$$\varepsilon \int_{\varphi - \varrho}^{\varphi + \varrho} w_{xx} dx = \varepsilon [w_x]_{x=\varphi - \varrho}^{x=\varphi + \varrho} = \varepsilon (w_x(\varphi + \varrho, t) - w_x(\varphi - \varrho, t)).$$

For the first-order convective term, we employ the distributional identity valid for piecewise smooth $a(x, t)$ (possibly discontinuous at $x = \varphi$):

$$\mu \int_{\varphi - \varrho}^{\varphi + \varrho} (a(x, t) w_x(x, t)) dx = \mu [a(x, t) w(x, t)]_{x=\varphi - \varrho}^{x=\varphi + \varrho} - \mu \int_{\varphi - \varrho}^{\varphi + \varrho} a_x(x, t) w(x, t) dx,$$

where the integral involving a_x is regular on each subinterval, while any jump in a at $x = \varphi$ is represented by the boundary term. Collecting all contributions, we obtain

$$\begin{aligned} & \varepsilon (w_x(\varphi + \varrho, t) - w_x(\varphi - \varrho, t)) + \mu (aw(\varphi + \varrho, t) - aw(\varphi - \varrho, t)) \\ & - \mu \int_{\varphi - \varrho}^{\varphi + \varrho} a_x w dx - \int_{\varphi - \varrho}^{\varphi + \varrho} bw dx - c \int_{\varphi - \varrho}^{\varphi + \varrho} w_t dx = \int_{\varphi - \varrho}^{\varphi + \varrho} \varpi dx. \end{aligned}$$

Now let $\varrho \rightarrow 0^+$. Assuming a, b, c , and ϖ are bounded in a neighborhood of $x = \varphi$, and that w, w_t remain bounded (i.e., there are no delta-type singularities in the source),

the integrals of bounded functions over the vanishing interval tend to zero:

$$\lim_{\varrho \rightarrow 0} \int_{\varphi-\varrho}^{\varphi+\varrho} a_x w \, dx = \lim_{\varrho \rightarrow 0} \int_{\varphi-\varrho}^{\varphi+\varrho} b w \, dx = \lim_{\varrho \rightarrow 0} c \int_{\varphi-\varrho}^{\varphi+\varrho} w_t \, dx = \lim_{\varrho \rightarrow 0} \int_{\varphi-\varrho}^{\varphi+\varrho} \varpi \, dx = 0.$$

Thus, only the jump terms remain, leading to the general jump relation $\varepsilon[w_x]_\varphi + \mu[aw]_\varphi = 0$, or equivalently,

$$\varepsilon(w_x(\varphi^+, t) - w_x(\varphi^-, t)) + \mu(a(\varphi^+, t)w(\varphi^+, t) - a(\varphi^-, t)w(\varphi^-, t)) = 0. \quad (6.2)$$

If a is continuous at $x = \varphi$, then $[aw]_\varphi = a(\varphi, t)[w]_\varphi$, and the jump condition gives

$$\varepsilon[w_x]_\varphi + \mu a(\varphi, t)[w]_\varphi = 0.$$

Furthermore, if w itself is continuous across the interface (e.g., in the absence of concentrated sources or sinks and with ϖ containing no Dirac-type term), then $[w]_\varphi = 0$. Hence, $\varepsilon[w_x]_\varphi = 0$, implying $[w_x]_\varphi = 0$ for all $\varepsilon > 0$. Therefore, both w and its first spatial derivative w_x are continuous across the discontinuity point. $\square$

The operator $\mathbb{L}_{\varepsilon, \mu}$ in Eq. (6.1) satisfies a continuous comparison principle on $\overline{\Omega}$.

Lemma 6.3 (Comparison Principle). *If $w(x, t) \in C^{(0)}(\overline{\Omega}) \cap C^{(1)}(\Omega) \cap C^{(2)}(\Omega^- \cup \Omega^+)$ such that $w(0, t) \geq 0, w(x, 0) \geq 0, w(1, t) \geq 0, [w'](\varphi, t) = [w'](\varphi^+, t) - [w'](\varphi^-, t) \geq 0$, and $\mathbb{L}_{\varepsilon, \mu} w(x, t) \geq 0, \forall (x, t) \in \Omega$, then $w(x, t) \leq 0, \forall (x, t) \in \overline{\Omega}$.*

Proof. Let $\mathcal{V}$ defined on $\overline{\Omega}$ be such that

$$w(x, t) = \exp(-\alpha \mu |x - \varphi| / 2\varepsilon) \mathcal{V}(x, t), \quad \alpha = \min\{\alpha_1, \alpha_2\}.$$

For $x \in \Omega^-$, let

$$w(x, t) = \exp\left(-\frac{\alpha_1 \mu (\varphi - x)}{2\varepsilon}\right) \mathcal{V}(x, t),$$

whose first and second spatial derivatives are given by:

$$\begin{aligned} w_x(x, t) &= \exp(-\alpha_1 \mu (\varphi - x) / 2\varepsilon) \left[\frac{\mu \alpha_1}{2\varepsilon} \mathcal{V}(x, t) + \mathcal{V}_x(x, t) \right], \quad \text{and} \\ w_{xx}(x, t) &= \exp(-\alpha_1 \mu (\varphi - x) / 2\varepsilon) \left[\mathcal{V}_{xx}(x, t) + \frac{\mu \alpha_1}{\varepsilon} \mathcal{V}_x(x, t) + \frac{\mu^2 \alpha_1^2}{4\varepsilon^2} \mathcal{V}(x, t) \right]. \end{aligned} \quad (6.3)$$

By substituting Eq. (6.3) into Eq. (6.1), we obtain:

$$\begin{aligned} \mathbb{L}_{\varepsilon,\mu} w(x, t) = & e^{(-\alpha_1\mu(\varphi-x)/2\varepsilon)} \left(\varepsilon \frac{\partial^2 \mathcal{V}}{\partial x^2} + \mu(\alpha_1 + a_1) \frac{\partial \mathcal{V}}{\partial x} + \right. \\ & \left. \left(\frac{\alpha_1\mu^2}{2\varepsilon} \left(\frac{\alpha_1}{2} + a_1 \right) - b \right) \mathcal{V} - c \frac{\partial \mathcal{V}}{\partial t} \right) (x, t), \quad x \in \Omega^-. \end{aligned} \quad (6.4)$$

Similarly, for $x \in \Omega^+$, we define

$$\begin{aligned} w(x, t) &= \exp\left(-\frac{\alpha_2\mu(x-\varphi)}{2\varepsilon}\right) \mathcal{V}(x, t), \\ w_x(x, t) &= \exp\left(-\frac{\alpha_2\mu(x-\varphi)}{2\varepsilon}\right) \left[\mathcal{V}_x(x, t) - \frac{\mu\alpha_2}{2\varepsilon} \mathcal{V}(x, t) \right], \\ w_{xx}(x, t) &= \exp\left(-\frac{\alpha_2\mu(x-\varphi)}{2\varepsilon}\right) \left[\mathcal{V}_{xx}(x, t) - \frac{\mu\alpha_2}{\varepsilon} \mathcal{V}_x(x, t) + \frac{\mu^2\alpha_2^2}{4\varepsilon^2} \mathcal{V}(x, t) \right]. \end{aligned} \quad (6.5)$$

Substituting Eq. (6.5) into Eq. (6.1) yields:

$$\begin{aligned} \mathbb{L}_{\varepsilon,\mu} w(x, t) = & e^{(-\alpha_2\mu(x-\varphi)/2\varepsilon)} \left(\varepsilon \frac{\partial^2 \mathcal{V}}{\partial x^2} + \mu(a_2 - \alpha_2) \frac{\partial \mathcal{V}}{\partial x} + \right. \\ & \left. \left(\frac{\alpha_2\mu^2}{2\varepsilon} \left(\frac{\alpha_2}{2} - a_2 \right) - b \right) \mathcal{V} - c \frac{\partial \mathcal{V}}{\partial t} \right) (x, t), \quad x \in \Omega^+. \end{aligned} \quad (6.6)$$

Let $(\chi, \tau) \in \overline{\Omega}$ be the point where $\mathcal{V}$ attains its maximum. If $\mathcal{V}(\chi, \tau) \leq 0$, the result follows immediately. Otherwise, assume $\mathcal{V}(\chi, \tau) > 0$. By the maximum principle, the point (χ, τ) must lie either in $(\Omega^- \cup \Omega^+)$ or at the interface (φ, τ) . Accordingly, we consider the following cases:

(i) If $(\chi, \tau) \in \Omega^-$, then

$$\begin{aligned} \mathbb{L}_{\varepsilon,\mu} w(\chi, \tau) = & e^{-\frac{\alpha_1\mu(\varphi-\chi)}{2\varepsilon}} \left(\varepsilon \frac{\partial^2 \mathcal{V}}{\partial x^2} + \mu(\alpha_1 + a) \frac{\partial \mathcal{V}}{\partial x} \right. \\ & \left. + \left[\frac{\alpha_1\mu^2}{2\varepsilon} \left(\frac{\alpha_1}{2} + a_1 \right) - b \right] \mathcal{V} - c \frac{\partial \mathcal{V}}{\partial t} \right) (\chi, \tau) < 0, \end{aligned} \quad (6.7)$$

since at the maximum point we have $\mathcal{V}_{xx}(\chi, \tau) < 0$, $\mathcal{V}_x(\chi, \tau) = \mathcal{V}_t(\chi, \tau) = 0$, $\mathcal{V}(\chi, \tau) > 0$, and $a_1 \leq \alpha_1 < 0$.

(ii) If $(\chi, \tau) \in \mathfrak{Q}^+$, then

$$\begin{aligned} \mathbb{L}_{\varepsilon, \mu} w(\chi, \tau) = e^{-\frac{\alpha_2 \mu (\chi - \varphi)}{2\varepsilon}} & \left(\varepsilon \frac{\partial^2 \mathcal{V}}{\partial x^2} + \mu(a_2 - \alpha_2) \frac{\partial \mathcal{V}}{\partial x} \right. \\ & \left. + \left[\frac{\alpha_2 \mu^2}{2\varepsilon} \left(\frac{\alpha_2}{2} - a_2 \right) - b \right] \mathcal{V} - c \frac{\partial \mathcal{V}}{\partial t} \right) (\chi, \tau) < 0. \end{aligned} \quad (6.8)$$

Equations (6.7) and (6.8) contradict the assumption that $\mathbb{L}_{\varepsilon, \mu} w(x, t) \geq 0$ for all $(x, t) \in (\mathfrak{Q}^- \cup \mathfrak{Q}^+)$.

(iii) The remaining possibility is $\chi = \varphi$. At this point, the continuity condition gives $[\mathcal{V}] = [w] = 0$ at (φ, τ) , and

$$[w_x] = [\mathcal{V}_x] - \left(\frac{\alpha_1 + \alpha_2}{2\varepsilon} \right) \mathcal{V}(\varphi, \tau).$$

Since $\mathcal{V}$ attains a maximum at φ , we have $\mathcal{V}_x(\varphi^-, \tau) \geq 0$ and $\mathcal{V}_x(\varphi^+, \tau) \leq 0$, implying $[\mathcal{V}_x](\varphi, \tau) \leq 0$, and hence $[w_x](\varphi, \tau) \leq 0$, which again contradicts the initial assumption.

Lemma (6.3) immediately implies the following stability result, ensuring the uniqueness of the solution. $\square$

Lemma 6.4. *Let $w(x, t)$ be the solution of Eq. (6.1), then*

$$\|w\|_{\overline{\mathfrak{Q}}} \leq \mathbb{C} \max \{ \|\vartheta\|_{\mathfrak{Q}_l}, \|\xi\|_{\mathfrak{Q}_r}, \|\phi\|_{\mathfrak{Q}_b} \} + \frac{1}{\mathcal{S}} \|\mathbb{L}_{\varepsilon, \mu} w\|_{(\mathfrak{Q}^- \cup \mathfrak{Q}^+)}, \quad (6.9)$$

where $\mathcal{S} = \min \left\{ \frac{\alpha_1}{\varphi}, \frac{\alpha_2}{1-\varphi} \right\}$.

Proof. The proof utilizes Lemma 6.3 with barrier functions as:

$$\mathcal{X}^{\pm}(x, t) = C\mathbb{K} \pm w(x, t), \quad \mathbb{K} = \begin{cases} -\Gamma - \frac{x\|\mathbb{L}_{\varepsilon, \mu} w\|}{\mathcal{S}\varphi}, & \text{for } (x, t) \in \mathfrak{Q}^- \cup (\varphi, t), \\ -\Gamma - \frac{(1-x)\|\mathbb{L}_{\varepsilon, \mu} w\|}{\mathcal{S}(1-\varphi)}, & \text{for } (x, t) \in \mathfrak{Q}^+, \end{cases} \quad (6.10)$$

where $\Gamma = \max \{ \|\vartheta\|_{\mathfrak{Q}_l}, \|\xi\|_{\mathfrak{Q}_r}, \|\phi\|_{\mathfrak{Q}_b} \}$. For appropriately chosen C , the functions $\mathcal{X}^{\pm}(x, t) \in C^0(\overline{\mathfrak{Q}})$ satisfy $\mathcal{X}^{\pm}(x, t) \leq 0$ in $\mathbb{K}_0$, for $(x, t) \in (\mathbb{K}^- \cup \mathbb{K}^+)$, we have $\mathbb{L}_{\varepsilon, \mu} \mathcal{X}^{\pm}(x, t) \geq 0$. Moreover, since $\mathcal{X}^{\pm}(x, t) \in C^1(\mathfrak{Q})$, the conditions

$$[\mathcal{X}^{\pm}](\varphi, t) = \pm[w](\varphi, t) = 0 \quad \text{and} \quad [\mathcal{X}_x^{\pm}](\varphi, t) = \frac{\|\mathbb{L}_{\varepsilon, \mu} w\|}{\mathcal{S}\varphi} + \frac{\|\mathbb{L}_{\varepsilon, \mu} w\|}{\mathcal{S}(1-\varphi)} \geq 0$$

hold. By comparison principle, it follows that $\mathcal{X}^\pm(x, t) \leq 0, \forall(x, t) \in \overline{\Omega}$. $\square$

6.4 Formulation of the Numerical Scheme

In this section, we develop a robust numerical framework to approximate Eq. (6.1). The temporal derivative is discretized using the implicit Euler scheme, while the spatial domain is treated with the cubic spline in tension method on a uniform mesh, providing higher-order accuracy and smoothness.

6.4.1 Temporal Semi-discretization

A temporal mesh $\Omega^{M_t} = \{t_j = jk \mid j = 0, 1, \dots, M, k = T/M\}$ is constructed by uniformly partitioning the time interval $[0, T]$ into M subintervals. Employing the implicit Euler method, we derive a numerical scheme for Eq. (6.1). Consequently, the semi-discretized form of Eq. (6.1) over $\Omega^\varphi = (\Omega^- \cup \Omega^+) \times \Omega^{M_t}$ is given by:

$$\begin{aligned} \varepsilon (W_{xx})^{j+1}(x) + \mu a^{j+1}(x) (W_x)^{j+1}(x) - \left(b^{j+1}(x) + \frac{c^{j+1}(x)}{k} \right) W^{j+1}(x) \\ = \varpi^{j+1}(x) - \frac{c^j(x)}{k} W^j(x), \quad j = 0(1)M-1 \\ W^{j+1}(0) = \vartheta(t_{j+1}), \quad W^{j+1}(1) = \xi(t_{j+1}), \quad W^0(x) = \phi(x), x \in \Omega, \end{aligned} \quad (6.11)$$

where $W^{j+1}(x)$ is the approximation of $w(x, t_{j+1})$. Rewrite Eq. (6.11) as:

$$\begin{cases} \mathbb{L}_{\varepsilon, \mu}^M W^{j+1}(x) := z(x, t_{j+1}), x \in \Omega^\varphi, & 0 \leq j \leq M-1, \\ W^{j+1}(0) = \vartheta(t_{j+1}), \quad W^{j+1}(1) = \xi(t_{j+1}), & 0 \leq j \leq M, \\ W^0(x) = \phi(x), x \in \Omega, \end{cases} \quad (6.12)$$

where

$$\mathbb{L}_{\varepsilon, \mu}^M W^{j+1}(x) = \varepsilon (W_{xx})^{j+1}(x) + \mu a^{j+1}(x) (W_x)^{j+1}(x) - \left(b^{j+1}(x) + \frac{c^{j+1}(x)}{k} \right) W^{j+1}(x)$$

and $z(x, t_{j+1}) = \varpi^{j+1}(x) - \frac{c^j(x)}{k} W^j(x)$.

We define the LTE of Eq.(6.12) by $\mathfrak{e}_{j+1} = \mathbb{L}_{\varepsilon, \mu} \overline{W}^{j+1} - z(x, t_{j+1})$, where $\overline{W}^{j+1}$ represents the numerical approximation of Eq. (6.12).

Theorem 6.1. *The local truncation error and global error are defined by*

$$\mathfrak{e}_{j+1}, \mathfrak{E}_j = \sum_{m=1}^j \mathfrak{e}_m,$$

respectively. These errors are bounded by

$$\begin{aligned} \|\mathfrak{e}_{j+1}\| &\leq Ck^2, \\ \|\mathfrak{E}_j\| &\leq Ck, \quad j \leq \frac{T}{k}, \end{aligned} \tag{6.13}$$

where $\mathfrak{E}_j$ shows the contribution of LTEs $\mathfrak{e}_m$ up to the j th time level.

Proof. Refer to the idea given in [Clavero et al. \(2005\)](#) for the first part. Now, we prove the bound of $\|\mathfrak{E}_j\|$ at the j th time level, $\mathfrak{E}_j$ is given by $\sum_{m=1}^j \mathfrak{e}_m$, and thus

$$\begin{aligned} \|\mathfrak{E}_j\| &= \left\| \sum_{m=1}^j \mathfrak{e}_m \right\|, \quad j \leq \frac{T}{k}, \\ &\leq \|\mathfrak{e}_1\| + \|\mathfrak{e}_2\| + \dots + \|\mathfrak{e}_m\|, \\ &\leq Cjk^2 \leq Ck. \end{aligned}$$

That is, the convergence rate in time is of first order. □

6.4.2 Spatial Semi-discretization and Non-Polynomial Cubic Spline Approach

Let $x_0 = 0, x_N = 1, x_i = ih$, where N is the number of subintervals in spatial domain. In (x_i, x_{i+1}) , the function $\mathcal{A}^{j+1}(x, \theta) \in C^2[0, 1]$ interpolates $W^{j+1}(x_i) = W(x_i, t_{j+1})$ at the mesh points x_i and depends on a parameter $\theta > 0$. This parameter controls the stiffness(tension) of the spline and regulates the contribution of curvature in the interpolation function $\mathcal{A}^{j+1}(x, \theta)$. As $\theta \rightarrow 0$, this function reduces to cubic spline on $[0, 1]$, and is known as a parameteric cubic spline function. In (x_i, x_{i+1}) , $\mathcal{A}^{j+1}(x, \theta) = \mathcal{A}^{j+1}(x)$ satisfies the following DE:

$$\begin{aligned} \frac{d^2 \mathcal{A}^{j+1}(x_i)}{dx^2} + \theta \mathcal{A}^{j+1}(x_i) &= \left(\frac{x_{i+1} - x}{h} \right) \left[\frac{d^2 \mathcal{A}^{j+1}(x_i)}{dx^2} + \theta \mathcal{A}^{j+1}(x_i) \right] + \\ &\quad \left(\frac{x - x_i}{h} \right) \left[\frac{d^2 \mathcal{A}^{j+1}(x_{i+1})}{dx^2} + \theta \mathcal{A}^{j+1}(x_{i+1}) \right], \end{aligned} \tag{6.14}$$

where $\mathcal{A}^{j+1}(x_i) = W^{j+1}(x_i)$ and $\theta > 0$ is called a cubic spline under compression. Solving Eq. (6.14) and finding the arbitrary constants using interpolation conditions $\mathcal{A}^{j+1}(x_{i+1}) = W^{j+1}(x_{i+1})$, $\mathcal{A}^{j+1}(x_i) = W^{j+1}(x_i)$ and after writing $\lambda = h\theta^{\frac{1}{2}}$:

$$\begin{aligned} \mathcal{A}^{j+1}(x) = & -\frac{h^2}{\lambda^2 \sin \lambda} \left[\mathfrak{M}_{i+1} \sin \left(\frac{\lambda(x - x_i)}{h} \right) + \mathfrak{M}_i \sin \left(\frac{\lambda(x_{i+1} - x)}{h} \right) \right] + \\ & \frac{h^2}{\lambda^2} \left[\left(\frac{x - x_i}{h} \right) \left(\mathfrak{M}_{i+1} + \frac{\lambda^2}{h^2} \mathcal{A}^{j+1}(x_{i+1}) \right) + \left(\frac{x_{i+1} - x}{h} \right) \left(\mathfrak{M}_i + \frac{\lambda^2}{h^2} \mathcal{A}^{j+1}(x_i) \right) \right] \end{aligned} \quad (6.15)$$

Differentiating Eq. (6.15) and letting $x \rightarrow x_i$, we have

$$\frac{\mathcal{A}^{j+1}(x_{i+})}{dx} = \frac{W^{j+1}(x_{i+1}) - W^{j+1}(x_i)}{h} + \frac{h}{\lambda^2} \left[\left(1 - \frac{\lambda}{\sin \lambda} \right) \mathfrak{M}_{i+1} - (1 - \lambda \cot \lambda) \mathfrak{M}_i \right] \quad (6.16)$$

By analyzing the interval (x_{i-1}, x_i) and following the same steps, we obtain:

$$\frac{\mathcal{A}^{j+1}(x_{i-})}{dx} = \frac{W^{j+1}(x_i) - W^{j+1}(x_{i-1})}{h} + \frac{h}{\lambda^2} \left[(1 - \lambda \cot \lambda) \mathfrak{M}_i - \left(1 - \frac{\lambda}{\sin \lambda} \right) \mathfrak{M}_{i-1} \right] \quad (6.17)$$

By setting Eq. (6.16) equal to Eq. (6.17) at x_i , we get

$$\begin{aligned} \frac{W^{j+1}(x_i) - W^{j+1}(x_{i-1})}{h} + \frac{h}{\lambda^2} \left[(1 - \lambda \cot \lambda) \mathfrak{M}_i - \left(1 - \frac{\lambda}{\sin \lambda} \right) \mathfrak{M}_{i-1} \right] = \\ \frac{W^{j+1}(x_{i+1}) - W^{j+1}(x_i)}{h} + \frac{h}{\lambda^2} \left[\left(1 - \frac{\lambda}{\sin \lambda} \right) \mathfrak{M}_{i+1} - (1 - \lambda \cot \lambda) \mathfrak{M}_i \right] \end{aligned} \quad (6.18)$$

Next, Eq. (6.18) results in a tri-diagonal system

$$h^2 (\lambda_1 \mathfrak{M}_{i-1} + 2\lambda_2 \mathfrak{M}_i + \lambda_1 \mathfrak{M}_{i+1}) = W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}, \quad (6.19)$$

where $\lambda_1 = \frac{1}{\lambda^2} \left(\frac{\lambda}{\sin \lambda} - 1 \right)$, $\lambda_2 = \frac{1}{\lambda^2} (1 - \lambda \cot \lambda)$, $\mathfrak{M}_i = \frac{d^2 \mathcal{A}^{j+1}(x_i)}{dx^2}$ and $\lambda_1 + \lambda_2 = \frac{1}{2}$, $i = 1(1)N - 1$. Using Taylor expansion of $\sin \lambda$ and $\cos \lambda$, it can be shown that as $\lambda \rightarrow 0$, the coefficients satisfy $\lambda_1 \rightarrow \frac{1}{6}$ and $\lambda_2 \rightarrow \frac{2}{3}$. Consequently, the cubic spline in tension reduces to the classical cubic spline formulation, ensuring consistency of the proposed method. The continuity condition provided by Eq. (6.19) guarantees the smoothness of the first-order derivatives of spline $\mathcal{A}^{j+1}(x, \theta)$ at the interior nodes.

Eq. (6.12) at $x = x_q, q = i - 1, i, i + 1$ can be expressed as:

$$\varepsilon \mathfrak{M}_q = -\mu a^{j+1}(x_q) (W_x(x_q))^{j+1} + \left(b^{j+1}(x_q) + \frac{c^{j+1}(x_q)}{k} \right) W^{j+1}(x_q) + \mathcal{R}^{j+1}(x_q), \quad (6.20)$$

where $\mathcal{R}^{j+1}(x_q) = \varpi^{j+1}(x_q) - \frac{c^j(x_q)}{k} U^j(x_q)$. Now, Eq. (6.20) at $q = i, i \pm 1$, becomes:

$$\begin{cases} \varepsilon \mathfrak{M}_i = -\mu a^{j+1}(x_i) (W_x(x_i))^{j+1} + \left(b^{j+1}(x_i) + \frac{c^{j+1}(x_i)}{k} \right) W^{j+1}(x_i) + \mathcal{R}^{j+1}(x_i), \\ \varepsilon \mathfrak{M}_{i-1} = -\mu a^{j+1}(x_{i-1}) (W_x(x_{i-1}))^{j+1} + \left(b^{j+1}(x_{i-1}) + \frac{c^{j+1}(x_{i-1})}{k} \right) W^{j+1}(x_{i-1}) + \mathcal{R}^{j+1}(x_{i-1}), \\ \varepsilon \mathfrak{M}_{i+1} = -\mu a^{j+1}(x_{i+1}) (W_x(x_{i+1}))^{j+1} + \left(b^{j+1}(x_{i+1}) + \frac{c^{j+1}(x_{i+1})}{k} \right) W^{j+1}(x_{i+1}) + \mathcal{R}^{j+1}(x_{i+1}). \end{cases} \quad (6.21)$$

Now, we apply finite difference approximation to approximate $\frac{dW^{j+1}(x_q)}{dx}$ in Eq. (6.21):

$$\begin{cases} \frac{dW^{j+1}(x_{i-1})}{dx} \approx \frac{-W_{i+1}^{j+1} + W U_i^{j+1} - 3W_{i-1}^{j+1}}{2h}, \\ \frac{dW^{j+1}(x_i)}{dx} \approx \frac{W_{i+1}^{j+1} - W_{i-1}^{j+1}}{2h}, \\ \frac{dW^{j+1}(x_{i+1})}{dx} \approx \frac{3W_{i+1}^{j+1} - 4W_i^{j+1} + W_{i-1}^{j+1}}{2h}. \end{cases} \quad (6.22)$$

Now, using Eq. (6.22), Eq. (6.21) reduced to:

$$\begin{cases} \varepsilon \mathfrak{M}_i = \frac{\mu}{2h} a_i^{j+1} W_{i-1}^{j+1} + \left(b_i^{j+1} + \frac{c_i^{j+1}}{k} \right) W_i^{j+1} - \frac{\mu}{2h} a_i^{j+1} W_{i+1}^{j+1} + \mathcal{R}_i^{j+1}, \\ \varepsilon \mathfrak{M}_{i-1} = \left[\frac{3\mu}{2h} a_{i-1}^{j+1} + \left(b_{i-1}^{j+1} + \frac{c_{i-1}^{j+1}}{k} \right) \right] W_{i-1}^{j+1} - \frac{2\mu}{h} a_{i-1}^{j+1} W_i^{j+1} + \frac{\mu}{2h} a_{i-1}^{j+1} W_{i+1}^{j+1} + \mathcal{R}_{i-1}^{j+1}, \\ \varepsilon \mathfrak{M}_{i+1} = -\frac{\mu}{2h} a_{i+1}^{j+1} W_{i-1}^{j+1} + \frac{2\mu}{h} a_{i+1}^{j+1} W_i^{j+1} + \left[-\frac{3\mu}{2h} a_{i+1}^{j+1} + \left(b_{i+1}^{j+1} + \frac{c_{i+1}^{j+1}}{k} \right) \right] W_{i+1}^{j+1} + \mathcal{R}_{i+1}^{j+1}. \end{cases} \quad (6.23)$$

By replacing Eq. (6.23) into Eq. (6.19), we obtain

$$\begin{aligned} & \left[-\frac{\varepsilon}{h^2} + \frac{3\lambda_1}{2h} \mu a_{i-1}^{j+1} + \frac{\lambda_2}{h} \mu a_i^{j+1} - \frac{\lambda_1}{2h} \mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i-1}^{j+1} + \frac{c_{i-1}^{j+1}}{k} \right) \right] W_{i-1}^{j+1} + \\ & \left[\frac{2\varepsilon}{h^2} - \frac{2\lambda_1}{h} \mu a_{i-1}^{j+1} + \frac{2\lambda_1}{h} \mu a_{i+1}^{j+1} + 2\lambda_2 \left(b_i^{j+1} + \frac{c_i^{j+1}}{k} \right) \right] W_i^{j+1} + \\ & \left[-\frac{\varepsilon}{h^2} + \frac{\lambda_1}{2h} \mu a_{i-1}^{j+1} - \frac{\lambda_2}{h} \mu a_i^{j+1} - \frac{3\lambda_1}{2h} \mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i+1}^{j+1} + \frac{c_{i+1}^{j+1}}{k} \right) \right] W_{i+1}^{j+1} = \\ & - \left[\lambda_1 \varpi_{i-1}^{j+1} + 2\lambda_2 \varpi_i^{j+1} + \lambda_1 \varpi_{i+1}^{j+1} \right] + \lambda_1 \frac{c_{i-1}^j}{k} W_{i-1}^j + 2\lambda_2 \frac{c_i^j}{k} W_i^j + \lambda_1 \frac{c_{i+1}^j}{k} W_{i+1}^j. \end{aligned} \quad (6.24)$$

In order to handle the influence of the perturbation parameter, the exponentially fitting factor $\delta(\rho)$ is multiplied on the term containing ε as

$$\begin{aligned}
& \left[-\frac{\delta(\rho)\varepsilon}{h^2} + \frac{3\lambda_1}{2h}\mu a_{i-1}^{j+1} + \frac{\lambda_2}{h}\mu a_i^{j+1} - \frac{\lambda_1}{2h}\mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i-1}^{j+1} + \frac{c_{i-1}^{j+1}}{k} \right) \right] W_{i-1}^{j+1} + \\
& \left[\frac{2\delta(\rho)\varepsilon}{h^2} - \frac{2\lambda_1}{h}\mu a_{i-1}^{j+1} + \frac{2\lambda_1}{h}\mu a_{i+1}^{j+1} + 2\lambda_2 \left(b_i^{j+1} + \frac{c_i^{j+1}}{k} \right) \right] W_i^{j+1} + \\
& \left[-\frac{\delta(\rho)\varepsilon}{h^2} + \frac{\lambda_1}{2h}\mu a_{i-1}^{j+1} - \frac{\lambda_2}{h}\mu a_i^{j+1} - \frac{3\lambda_1}{2h}\mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i+1}^{j+1} + \frac{c_{i+1}^{j+1}}{k} \right) \right] W_{i+1}^{j+1} = \\
& -\lambda_1 \left[\varpi_{i-1}^{j+1} - \frac{c_{i-1}^j}{k} W_{i-1}^j \right] - 2\lambda_2 \left[\varpi_{i-1}^{j+1} - \frac{c_i^j}{k} W_i^j \right] - \lambda_1 \left[\varpi_{i+1}^{j+1} - \frac{c_{i+1}^j}{k} W_{i+1}^j \right]
\end{aligned} \tag{6.25}$$

To guarantee that the numerical solution of Eq. (6.25) attains uniform convergence to the continuous problem Eq. (6.1), it becomes essential to determine an appropriate fitting factor $\delta(\rho)$. This factor plays a pivotal role in accurately capturing the boundary layer behavior and stabilizing the scheme. With this in mind, the corresponding three-term recurrence relation for Eq. (6.25) can be expressed as follows:

$$\begin{aligned}
\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1} &\equiv \mathfrak{Y}_i^- W_{i-1}^{j+1} + \mathfrak{Y}_i^c W_i^{j+1} + \mathfrak{Y}_i^+ W_{i+1}^{j+1} = \mathcal{H}_i^j, \\
W^{j+1}(0) &= \vartheta^{j+1}(0), \quad W^{j+1}(1) = \xi^{j+1}(1), \quad 0 \leq j \leq M, \\
W_i^0 &= \phi(x_i), \quad x \in [0, 1],
\end{aligned} \tag{6.26}$$

$i = 1(1)N - 1, j = 1(1)M - 1$, where

$$\begin{cases}
\mathfrak{Y}^- = -\frac{\delta(\rho)\varepsilon}{h^2} + \frac{3\lambda_1}{2h}\mu a_{i-1}^{j+1} + \frac{\lambda_2}{h}\mu a_i^{j+1} - \frac{\lambda_1}{2h}\mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i-1}^{j+1} + \frac{c_{i-1}^{j+1}}{k} \right), \\
\mathfrak{Y}^c = \frac{2\delta(\rho)\varepsilon}{h^2} - \frac{2\lambda_1}{h}\mu a_{i-1}^{j+1} + \frac{2\lambda_1}{h}\mu a_{i+1}^{j+1} + 2\lambda_2 \left(b_i^{j+1} + \frac{c_i^{j+1}}{k} \right), \\
\mathfrak{Y}^+ = -\frac{\delta(\rho)\varepsilon}{h^2} + \frac{\lambda_1}{2h}\mu a_{i-1}^{j+1} - \frac{\lambda_2}{h}\mu a_i^{j+1} - \frac{3\lambda_1}{2h}\mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i+1}^{j+1} + \frac{c_{i+1}^{j+1}}{k} \right), \\
\mathcal{H}_i^j = -\lambda_1 \left[\varpi_{i-1}^{j+1} - \frac{c_{i-1}^j}{k} W_{i-1}^j \right] - 2\lambda_2 \left[\varpi_{i-1}^{j+1} - \frac{c_i^j}{k} W_i^j \right] - \\
\lambda_1 \left[\varpi_{i+1}^{j+1} - \frac{c_{i+1}^j}{k} W_{i+1}^j \right].
\end{cases} \tag{6.27}$$

The grid points corresponding to the line $(x_{N/2}, t_{j+1})$ lie within the subregions $\mathfrak{M}^{N\pm} = \mathfrak{M}^\pm \cap \mathfrak{Q}^N$ as illustrated in Figure 6.1. This location coincides with the discontinuity interface $(x_{N/2}, t_{j+1}) = (\varphi, t_{j+1})$, where conventional central difference lose their ac-

curacy. To accurately capture the behavior of the solution across this interface and preserve second-order accuracy, we employ a second-order hybrid difference operator. This operator blends forward and backward finite differences on either side of the discontinuity to enforce smooth flux transmission and maintain numerical stability. Accordingly, the discrete formulation at the discontinuity point is expressed as:

$$\mathbb{L}_{\varepsilon, \mu}^{N, M} W_{\frac{N}{2}}^{j+1} = \frac{-W_{\frac{N}{2}+2}^{j+1} + 4W_{\frac{N}{2}+1}^{j+1} - 3W_{\frac{N}{2}}^{j+1}}{2h} - \frac{W_{\frac{N}{2}-2}^{j+1} - 4W_{\frac{N}{2}-1}^{j+1} + 3W_{\frac{N}{2}}^{j+1}}{2h} = 0. \quad (6.28)$$

This formulation effectively bridges the solution values on both sides of the interface, ensuring continuity of the numerical flux and enhancing the overall accuracy of the scheme in the presence of discontinuities. Tri-diagonal systems, when combined with exponential fitting factors, have been shown to provide uniform convergence, which is critical for accurately capturing sharp gradients and discontinuities.

6.4.3 Derivation of the Fitting Factor

To ensure that Eq. (6.26) uniformly converges to the solution of Eq. (6.1), it is essential to determine the fitting factor $\delta(\rho)$. Given that the fitting factor is time-independent, Eq. (6.25) can be reformulated to solve for $\delta(\rho)$ in Eq. (6.26).

$$\begin{aligned} & -\frac{\delta(\rho)\varepsilon}{h^2} [W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}] + \\ & \left[\frac{3\lambda_1}{2h} \mu a_{i-1}^{j+1} + \frac{\lambda_2}{h} \mu a_i^{j+1} - \frac{\lambda_1}{2h} \mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i-1}^{j+1} + \frac{c_{i-1}^{j+1}}{k} \right) \right] W_{i-1}^{j+1} + \\ & \left[-\frac{2\lambda_1}{h} \mu a_{i-1}^{j+1} + \frac{2\lambda_1}{h} \mu a_i^{j+1} + 2\lambda_2 \left(b_i^{j+1} + \frac{c_i^{j+1}}{k} \right) \right] W_i^{j+1} + \\ & \left[\frac{\lambda_1}{2h} \mu a_{i-1}^{j+1} - \frac{\lambda_2}{h} \mu a_i^{j+1} - \frac{3\lambda_1}{2h} \mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i+1}^{j+1} + \frac{c_{i+1}^{j+1}}{k} \right) \right] W_{i+1}^{j+1} = \mathcal{H}_i^j. \end{aligned} \quad (6.29)$$

By Multiplying both sides of Eq. (6.29) by h and considering $\lim_{h \rightarrow 0}$, we obtain:

$$-\frac{\delta(\rho)}{\rho} [W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}] + \mu a_0 (\lambda_1 + \lambda_2) [W_{i-1}^{j+1} - W_{i+1}^{j+1}] = 0, \quad (6.30)$$

where $\rho = h/\varepsilon$. When the boundary layer is on the left of the domain, from the theory of singular perturbation (O'Malley, 1974), the solution of Eq. (6.12) is of the form:

$$W^{j+1}(x) = W_0^{j+1}(x) + \frac{a^{j+1}(0)}{a^{j+1}(x)} (\vartheta^{j+1}(0) - W_0^{j+1}(0)) e^{\left(-\int_0^x \left(-\frac{\mu a^{j+1}(x)}{\varepsilon}\right) dx\right)}, \quad (6.31)$$

where $W_0^{j+1}(x)$ is solution of the reduced form of Eq. (6.12). By limiting the Taylor series expansion of $a^{j+1}(x)$ to its first term around $x = 0$ and then taking the limit as $h \rightarrow 0$, we end up with:

$$\lim_{h \rightarrow 0} W^{j+1}(ih) = W_0^{j+1}(0) + (\vartheta^{j+1}(0) - W_0^{j+1}(0)) e^{(-a^{j+1}(0)\mu(\frac{1}{\varepsilon} - i\rho))} + O(\varepsilon). \quad (6.32)$$

By substituting Eq. (6.32) into (6.30) and carrying out the necessary simplifications, we derive the explicit expression for the fitting factor as follows:

$$\delta_0(\rho) = \rho\mu(\lambda_1 + \lambda_2)a^{j+1}(0) \coth\left(\frac{\rho\mu a^{j+1}(0)}{2}\right). \quad (6.33)$$

If a boundary layer is at the right boundary, the corresponding formulation becomes:

$$W^{j+1}(x) = W_0^{j+1}(x) + \frac{a^{j+1}(1)}{a^{j+1}(x)} (\xi^{j+1}(1) - W_0^{j+1}(1)) e^{\left(-\int_x^1 \left(-\frac{\mu a^{j+1}(x)}{\varepsilon}\right) dx\right)}. \quad (6.34)$$

Expanding $a^{j+1}(x)$ in a Taylor series about the line $x = 1$ and retaining terms up to the required order, we obtain

$$a^{j+1}(x) = a^{j+1}(1) + (x - 1)a_x^{j+1}(1) + \frac{(x - 1)^2}{2!}a_{xx}^{j+1}(1) + \dots \quad (6.35)$$

Inserting Eq. (6.35) into Eq. (6.34) and letting $h \rightarrow 0$ systematically simplifies the expression, yielding the refined form:

$$\lim_{h \rightarrow 0} U^{j+1}(ih) = U_0^{j+1}(1) + (\xi^{j+1}(1) - U_0^{j+1}(1)) e^{(-a^{j+1}(1)\mu(\frac{1}{\varepsilon} - i\rho))} + O(\varepsilon). \quad (6.36)$$

After making adjustments and substituting Eq. (6.36) into Eq (6.30), we obtain

$$\delta_1(\rho) = \rho\mu(\lambda_1 + \lambda_2)a^{j+1}(N) \coth\left(\frac{\rho\mu a^{j+1}(N)}{2}\right). \quad (6.37)$$

Using the relations in Eqs. (6.37) and (6.33), we derive a tailored variable fitting factor for the discontinuous convection coefficient $a(x, t)$, yielding the expression:

$$\delta_\varphi(\rho) = \begin{cases} \rho\mu(\lambda_1 + \lambda_2)a_1^{j+1}(i) \coth\left(\frac{\rho\mu a_1^{j+1}(i)}{2}\right) & \text{if } 0 \leq i \leq N/2 \\ \rho\mu(\lambda_1 + \lambda_2)a_2^{j+1}(i) \coth\left(\frac{\rho\mu a_2^{j+1}(i)}{2}\right) & \text{if } N/2 \leq i < N. \end{cases} \quad (6.38)$$

6.5 Convergence Analysis

Before proving convergence, we first outline the key properties of the discrete operator $\mathbb{L}_{\varepsilon, \mu}^{N, M}$, which underpin the subsequent analysis.

Lemma 6.5 (Discrete Comparison Principle). *Let $\mathbb{W}_i^{j+1}$ be a comparison function such that if $W_0^{j+1} \leq \mathbb{W}_0^{j+1}$ and $W_N^{j+1} \leq \mathbb{W}_N^{j+1}$, then $\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1} \leq \mathbb{L}_{\varepsilon, \mu}^{N, M} \mathbb{W}_i^{j+1}$, $i = 1(1)N - 1$ implies that $W_i^{j+1} \leq \mathbb{W}_i^{j+1}$, $i = 0(1)N$.*

Proof. The matrix $\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1}$ has a size of $(N + 1) \times (N + 1)$ with its entries for $i = 1(1)N - 1$ which are $\mathfrak{Y}^- < 0$, $\mathfrak{Y}^c > 0$, and $\mathfrak{Y}^+ < 0$. As a result, the coefficient matrix meets an M -matrix criteria, ensuring the existence of a positive inverse matrix. This confirms both the existence and uniqueness of the discrete solution. $\square$

Lemma 6.6. *At $(j+1)$ th time level, the solution W_i^{j+1} of Eq. (6.1) fulfills the estimate provided below and is bounded.*

$$\|W_i^{j+1}\| \leq \frac{\|\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1}\|}{\mathfrak{K}} + \max\{|\vartheta(t_{j+1})|, |\xi(t_{j+1})|\}, \quad (6.39)$$

where $\mathfrak{K} = \min\left\{\frac{\alpha_1}{\varphi}, \frac{\alpha_2}{1-\varphi}\right\}$.

Proof. Let $\mathbb{D} = \frac{\|\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1}\|}{\mathfrak{K}} + \max\{|\vartheta(t_{j+1})|, |\xi(t_{j+1})|\}$ and define barrier function

$$(\mathfrak{V}^\pm)_i^{j+1} = \mathbb{D} \pm W_i^{j+1}. \quad (6.40)$$

The discrete function $(\mathfrak{V}^\pm)_i^{j+1}$ on the boundaries:

$$\begin{aligned} \mathfrak{V}^\pm(0, t_{j+1}) &= \mathbb{D} \pm \vartheta(t_{j+1}) = \frac{\|\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1}\|}{\mathfrak{K}} + \max\{|\vartheta(t_{j+1})|, |\xi(t_{j+1})|\} \pm \vartheta(t_{j+1}) \geq 0, \\ \mathfrak{V}^\pm(1, t_{j+1}) &= \mathbb{D} \pm \xi(t_{j+1}) = \frac{\|\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1}\|}{\mathfrak{K}} + \max\{|\vartheta(t_{j+1})|, |\xi(t_{j+1})|\} \pm \xi(t_{j+1}) \geq 0, \end{aligned}$$

and on the discretized domain, we have

$$\begin{aligned}
\mathbb{L}_{\varepsilon, \mu}^{N, M} (\mathfrak{W}^\pm)_i^{j+1} \mathbb{D} \pm W_i^{j+1} \\
&= \mathfrak{Y}^- (\mathbb{D} \pm W_{i-1}^{j+1}) + \mathfrak{Y}^c (\mathbb{D} \pm W_i^{j+1}) + \mathfrak{Y}^+ (\mathbb{D} \pm W_{i+1}^{j+1}), \\
&= (\mathfrak{Y}^- + \mathfrak{Y}^c + \mathfrak{Y}^+) \mathbb{D} \pm \mathcal{H}_i^j, \\
&= 2\beta (\lambda_1 + \lambda_2) \mathbb{D} \pm \mathcal{H}_i^j \geq 0,
\end{aligned} \tag{6.41}$$

where $\beta = \max \left(b_l^{j+1} + \frac{c_l^{j+1}}{k} \right)$ for $l = i, i \pm 1$. Lemma 6.5 provides an immediate proof. Consequently, the method maintains uniform stability in the maximum norm. $\square$

Local Truncation Error($\mathcal{LTE}$).

Lemma 6.7. *The discrete solution adheres to the subsequent error bound:*

$$\|W^{j+1}(x_i) - W_i^{j+1}\| \leq Ch^2. \tag{6.42}$$

Proof. The $\mathcal{LTE}$ in space discretization is given as:

$$\begin{aligned}
\mathcal{LTE} = & \delta(\rho)\varepsilon \left[\frac{W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}}{h^2} - (W_{xx})_i^{j+1} \right] + \\
& a_{i-1}^{j+1} \lambda_1 \left[\frac{-3W_{i-1}^{j+1} + 4W_i^{j+1} - W_{i+1}^{j+1}}{2h} - (W_x)_{i-1}^{j+1} \right] + \\
& 2a_i^{j+1} \lambda_2 \left[\frac{W_{i+1}^{j+1} - W_{i-1}^{j+1}}{2h} - (W_x)_i^{j+1} \right] + \\
& a_{i+1}^{j+1} \lambda_1 \left[\frac{W_{i-1}^{j+1} - 4W_i^{j+1} + 3W_{i+1}^{j+1}}{2h} - (W_x)_{i+1}^{j+1} \right]
\end{aligned} \tag{6.43}$$

Using Taylor's expansion of the terms W_{i+1}^{j+1} and W_{i-1}^{j+1} , we have:

$$W_{i\pm 1}^{j+1} = W_i^{j+1} \pm h (W_x)_i^{j+1} + \frac{h^2}{2!} (W_{xx})_i^{j+1} \pm \frac{h^3}{3!} (W_{xxx})_i^{j+1} + \frac{h^4}{4!} (W_{xxxx})_i^{j+1} + \dots \tag{6.44}$$

With Eq. (6.44) substituted into Eq. (6.43), we now have

$$\begin{aligned}
\mathcal{LTE} \leq & \max_{x_{i-1} \leq x \leq x_{i+1}} \left[\frac{\delta(\rho)_i \varepsilon h^2}{12} |(W_{xxxx})_i^{j+1}| \right] + \\
& \max_{x_{i-1} \leq x \leq x_{i+1}} \left[\frac{\lambda_2 a_i^{j+1} h^2}{3} |(W_{xxx})_i^{j+1}| \right].
\end{aligned} \tag{6.45}$$

Using $\mathcal{J} = \rho\mu(\lambda_1 + \lambda_2)a^{j+1}(i) \coth\left(\frac{\rho\mu a^{j+1}(i)}{2}\right)$ in Eq. (6.45) where $\rho = h/\varepsilon$, we have:

$$\begin{aligned}
|\mathcal{LTE}| &\leq \max_{x_{i-1} \leq x \leq x_{i+1}} \left[\mathcal{J} h^3 12 \left| (W_{xxx})_i^{j+1} \right| \right] + \\
&\quad \max_{x_{i-1} \leq x \leq x_{i+1}} \left[\frac{\lambda_2 a_i^{j+1} h^2}{3} \left| (W_{xx})_i^{j+1} \right| \right], \\
&\leq \max_{x_{i-1} \leq x \leq x_{i+1}} \left[\frac{\lambda_2 a_i^{j+1} h^2}{3} \left| (W_{xx})_i^{j+1} \right| \right] + O(h^3), \\
&\leq O(h^2), \quad i = 1(1)N - 1.
\end{aligned} \tag{6.46}$$

As a result, the stability estimate of scheme is implied, and boundedness of $\mathcal{LTE}$ in space discretization is guaranteed. Now, Eq. (6.26) can be written as

$$\left[-\frac{\delta(\rho)\varepsilon}{h^2} + \bar{w}_i \right] W_{i-1}^{j+1} + \left[2\frac{\delta(\rho)\varepsilon}{h^2} + \bar{f}_i \right] W_i^{j+1} + \left[-\frac{\delta(\rho)\varepsilon}{h^2} + \bar{c}_i \right] W_{i+1}^{j+1} + \mathfrak{X}_i + \mathcal{LTE}_i = 0, \tag{6.47}$$

where

$$\begin{cases} \bar{w}_i = \frac{3\lambda_1}{2h} \mu a_{i-1}^{j+1} + \frac{\lambda_2}{h} \mu a_i^{j+1} - \frac{\lambda_1}{2h} \mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i-1}^{j+1} + \frac{c_{i-1}^{j+1}}{k} \right), \\ \bar{f}_i = -\frac{2\lambda_1}{h} \mu a_{i-1}^{j+1} + \frac{2\lambda_1}{h} \mu a_{i+1}^{j+1} + 2\lambda_2 \left(b_i^{j+1} + \frac{c_i^{j+1}}{k} \right), \\ \bar{c}_i = \frac{\lambda_1}{2h} \mu a_{i-1}^{j+1} - \frac{\lambda_2}{h} \mu a_i^{j+1} - \frac{3\lambda_1}{2h} \mu a_{i+1}^{j+1} + \lambda_1 \left(b_{i+1}^{j+1} + \frac{c_{i+1}^{j+1}}{k} \right), \\ \mathfrak{X}_i = \lambda_1 \left[\varpi_{i-1}^{j+1} - \frac{c_{i-1}^j}{k} W_{i-1}^j \right] + 2\lambda_2 \left[\varpi_{i-1}^{j+1} - \frac{c_i^j}{k} W_i^j \right] + \\ \quad \lambda_1 \left[\varpi_{i+1}^{j+1} - \frac{c_{i+1}^j}{k} W_{i+1}^j \right]. \end{cases} \tag{6.48}$$

By applying the boundary conditions to Eq. (6.47), the resulting system can be expressed in a compact matrix form as follows:

$$[\mathbb{S} + \mathbb{I}] U + \mathcal{K} + \mathcal{LTE}_i = 0, \tag{6.49}$$

where

$$\mathbb{S} = \text{trid} \left[-\frac{\delta(\rho)\varepsilon}{h^2}, 2\frac{\delta(\rho)\varepsilon}{h^2}, -\frac{\delta(\rho)\varepsilon}{h^2} \right], \mathbb{I} = \text{trid} [\bar{w}_i, \bar{f}_i, \bar{c}_i]$$

are $(N-1) \times (N-1)$ tridiagonal matrices,

$$\mathcal{K} = \left[\mathfrak{X}_1 + \left[-\frac{\delta(\rho)\varepsilon}{h^2} + \bar{w}_1 \right] \vartheta_0^{j+1}, \mathfrak{X}_2, \dots, \mathfrak{X}_{N-2}, \mathfrak{X}_{N-1} + \left[-\frac{\delta(\rho)\varepsilon}{h^2} + \bar{c}_{N-1} \right] \xi_N^{j+1} \right],$$

and $W = [W_1^{j+1}, \dots, W_{N-1}^{j+1}]^T$ are component vectors of Eq. (6.49).

Let $\bar{W} = [\bar{W}_1^{j+1}, \bar{W}_2^{j+1}, \dots, \bar{W}_{N-1}^{j+1}] \cong W$ satisfy the following expression:

$$[\mathbb{S} + \mathbb{I}] \bar{W} + \mathcal{K} = 0. \quad (6.50)$$

Subtracting Eq. (6.49) from Eq. (6.50), we obtain:

$$[\mathbb{S} + \mathbb{I}] \mathbb{E} = \mathcal{L} \mathcal{T} \mathcal{E}_i, \quad (6.51)$$

where $\mathbb{E} = \bar{W} - W = [\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{N-1}]^T$ is the discretization error. Let $|a_s^{j+1}| \leq \mathcal{C}_1$, $|b_s^{j+1}| \leq \mathcal{C}_2$ and $|c_s^{j+1}| \leq \mathcal{C}_3$, $s = i, i \pm 1$, where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are positive constants. Multiplying Eq. (6.47) by h and let $\bar{r}_{ij}$ be the (i, j) th element of the matrix $\mathbb{I}$; then for $i = 1(1)N - 1$:

$$\begin{aligned} |\mathbb{I}_{i,i+1}| &= |\bar{c}_i| \leq -\mu \mathcal{C}_1 (\lambda_1 + \lambda_1) + h \lambda_1 \left(\mathcal{C}_2 + \frac{\mathcal{C}_3}{k} \right), \quad i = 1(1)N - 2, \\ |\mathbb{I}_{i,i-1}| &= |\bar{w}_i| \leq \mu \mathcal{C}_1 (\lambda_1 + \lambda_1) + h \lambda_1 \left(\mathcal{C}_2 + \frac{\mathcal{C}_3}{k} \right), \quad i = 2(1)N - 1. \end{aligned} \quad (6.52)$$

Thus, for adequately small h , that is as $h \rightarrow 0$, we obtain:

$$\begin{cases} -\frac{\delta(\rho)}{\rho} + |\mathbb{I}_{i,i+1}| \neq 0, & i = 1(1)N - 2, \\ -\frac{\delta(\rho)}{\rho} + |\mathbb{I}_{i,i-1}| \neq 0, & i = 2(1)N - 1, \end{cases} \quad \rho = h/\varepsilon. \quad (6.53)$$

Let the matrix $(\mathbb{S} + \mathbb{I})$ has $\mathfrak{L}_i$ as the sum of its i th row elements, we get:

$$\begin{aligned} \mathfrak{L}_1 &= \sum_{m=1}^{N-1} \mathbb{R}_{1,j} = \frac{\delta(\rho)}{\rho} - \frac{3\lambda_1}{2} \mu a_{i-1}^{j+1} - \lambda_2 \mu a_i^{j+1} + \frac{\lambda_1}{2} \mu a_{i+1}^{j+1} + h (2\lambda_2 \beta_i^{j+1} + \lambda_1 \beta_{i+1}^{j+1}), \quad i = 1, \\ \mathfrak{L}_i &= \sum_{m=2}^{N-2} \mathbb{R}_{i,j} = h (\lambda_1 \beta_{i-1}^{j+1} + 2\lambda_2 \beta_i^{j+1} + \lambda_2 \beta_{i+1}^{j+1}) + O(h^2) = \beta_i^{j+1} + O(h^2), \quad i = 2(1)N - 2, \\ \mathfrak{L}_{N-1} &= \sum_{m=1}^{N-1} \mathbb{R}_{N-1,j} = \frac{\delta(\rho)}{\rho} - \frac{\lambda_1}{2} \mu a_{i-1}^{j+1} + \lambda_2 \mu a_i^{j+1} + \frac{3\lambda_1}{2} \mu a_{i+1}^{j+1} + \\ &\quad h (\lambda_1 \beta_{i-1}^{j+1} + 2\lambda_2 \beta_i^{j+1}), \quad i = N - 1, \end{aligned} \quad (6.54)$$

where $\beta_i^{j+1} = (b_i^{j+1} + c_i^{j+1}/k)$. The matrix $(\mathbb{S} + \mathbb{I})$ is irreducible if the matrix satisfies

Eq. (6.53) (Daba & Duressa, 2022). The matrix $(\mathbb{S} + \mathbb{I})$ is monotone for sufficiently small h . Hence, $(\mathbb{S} + \mathbb{I})^{-1}$ exist and $(\mathbb{S} + \mathbb{I})^{-1} \geq 0$. We obtain from Eq. (6.51) that:

$$\mathbb{E} = [\mathbb{S} + \mathbb{I}]^{-1} \mathcal{L}\mathcal{T}\mathcal{E}_i. \quad (6.55)$$

Eq. (6.55) can be expressed as follows by using the matrix norm

$$\|\mathbb{E}\| \leq \|[\mathbb{S} + \mathbb{I}]^{-1}\| \mathcal{L}\mathcal{T}\mathcal{E}. \quad (6.56)$$

Assume that the (m, n) th elements of $(\mathbb{S} + \mathbb{I})^{-1}$ are $\hat{r}_{m,n}$, and we define

$$\|(\hat{r}_{m,n})^{-1}\| = \max_{1 \leq m \leq N-1} \sum_{n=1}^{N-1} |(\hat{r}_{m,n})^{-1}|.$$

Given that $(\hat{r}_{m,n})^{-1} \geq 0$, we can derive:

$$\sum_{n=1}^{N-1} \hat{r}_{m,n} \mathfrak{L}_i = 1 \Rightarrow \sum_{n=1}^{N-1} \hat{r}_{m,n} \leq \frac{1}{\min_{0 \leq n \leq N-1} \mathfrak{L}_i} \leq \frac{1}{|\beta_i^{j+1}|}, m = 1(1)N-1. \quad (6.57)$$

Thus, from Eqs.(6.47), (6.56), (6.57) and (6.46), we obtain:

$$\mathbb{E}_i = \sum_{n=1}^{N-1} \hat{r}_{m,n} \mathcal{L}\mathcal{T}\mathcal{E}_i \leq \frac{O(h^2)}{|\beta_i^{j+1}|}, \quad m = 1(1)N-1. \quad (6.58)$$

For this reason, $\|\mathbb{E}\| \leq O(h^2)$, indicating that the method exhibits second-order convergence in the spatial direction during the semi-discretization process. $\square$

Theorem 6.2. *Let $w(x_i, t_{j+1})$ and W_i^{j+1} are solution of Eq. (6.1) and approximate solution Eq. (6.26), respectively. Next, we have the fully discrete scheme error estimate which is given by*

$$\sup_{0 < \epsilon, \mu < 1} \max_{0 < i, j \leq N, M} \|w(x_i, t_{j+1}) - W_i^{j+1}\|_{\infty} \leq C (k + h^2). \quad (6.59)$$

Proof. The required error estimate, which ensures first-order accuracy in time and second-order accuracy in space, is established by systematically combining the results of Theorem 6.1 with the local error analysis provided in Lemma 6.7. This integration forms the basis for demonstrating the overall convergence of the proposed scheme. $\square$

6.6 Numerical Examples, Results and Discussion

In this section, some numerical experiments are conducted on four examples to show the applicability of the proposed scheme. The exact solution of the problem is not available for comparison. So we calculate the error using the double mesh principle (Doolan et al., 1980). The pointwise maximum error $E_{\varepsilon,\mu}^{N,M}$ and the corresponding convergence rate $\bar{R}_{\varepsilon,\mu}^{N,M}$ are defined in Chapter 3, Section 3.6. To investigate the behavior of layer phenomena under different conditions, we consider four separate cases that encompass all possible sign variations between $a(x, t)$ and $\varpi(x, t)$. Drawing inspiration from the illustrative examples and rigorous error analysis discussed in Kumar & Kumari (2021), we construct a set of representative test problems.

Example 6.1. *Chandru et al. (2019); Singh et al. (2023).* In this problem, we take $\Omega = (0, 1)$ and $\varphi = 0.5$

$$\begin{cases} \varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a(x, t) \frac{\partial w}{\partial x} - (1 + e^x)w - \frac{\partial w}{\partial t} = \varpi(x, t), & (x, t) \in \Omega^- \cup \Omega^+, \\ w(0, t) = 0 \quad \text{on } \Omega_l, \quad w(1, t) = 0 \quad \text{on } \Omega_r, \\ w(x, 0) = 0 \quad \text{on } \Omega_b, \end{cases}$$

$$\text{where } a(x, t) = \begin{cases} -(1 + x - x^2), & (x, t) \in \Omega^- \\ (1 + x - x^2), & (x, t) \in \Omega^+ \end{cases}, \quad \varpi(x, t) = \begin{cases} -2t(1 + x^2), & (x, t) \in \Omega^- \\ 2t(1 + x^2), & (x, t) \in \Omega^+ \end{cases}.$$

Example 6.2. *(Chandru et al., 2018).* We focus on the second illustrative example, which involves a problem characterized by non-smooth input data.

$$\begin{cases} \varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a(x, t) \frac{\partial w}{\partial x} - (1 + e^x)w - \frac{\partial w}{\partial t} = \varpi(x, t), & (x, t) \in \Omega^- \cup \Omega^+, \\ w(0, t) = 0 \quad \text{on } \Omega_l, \quad w(1, t) = 0 \quad \text{on } \Omega_r, \\ w(x, 0) = 0 \quad \text{on } \Omega_b, \end{cases}$$

$$\text{where } a(x, t) = \begin{cases} (x + 2), & 0 \leq x \leq 0.5, \\ -(2x + 3), & 0.5 < x \leq 1, \end{cases}, \quad \varpi(x, t) = \begin{cases} t(2x + 1), & 0 \leq x \leq 0.5, \\ -t(3x + 4), & 0.5 < x \leq 1 \end{cases}.$$

For illustrations (6.3) and (6.4), only the coefficients, source functions, and initial conditions are provided to fully specify the parabolic IBVP described in (6.1).

Example 6.3. Let us take Eq. (6.1) and its coefficients to be in Chandru et al. (2018)

$$a(x, t) = \begin{cases} -(1 + e^{-xt}), & 0 \leq x \leq 0.5, \\ (2 + x + t), & 0.5 < x \leq 1, \end{cases}, \varpi(x, t) = \begin{cases} (e^{t^2} - 1)(1 + xt), & 0 \leq x \leq 0.5, \\ -t^2(2 + x), & 0.5 < x \leq 1, \end{cases}$$

$$b(x, t) = 2 + xt, c(x, t) = 1, \quad w(0, t) = 0 = w(1, t) \text{ and } w(x, 0) = 0.$$

Example 6.4. Let us take Eq. (6.1) and its coefficients to be in Chandru et al. (2018)

$$a(x, t) = \begin{cases} (1 + x - x^2 + t), & (x, t) \in \mathfrak{Q}^-, \\ -(1 + 3xt), & (x, t) \in \mathfrak{Q}^+, \end{cases}, \varpi(x, t) = \begin{cases} (e^t - 1)(1 + x), & (x, t) \in \mathfrak{Q}^-, \\ t(x - 2), & (x, t) \in \mathfrak{Q}^+, \end{cases}$$

$$b(x, t) = 1 + x + t, c(x, t) = 1 \quad w(0, t) = 0 = w(1, t) \text{ and } w(x, 0) = 0.$$

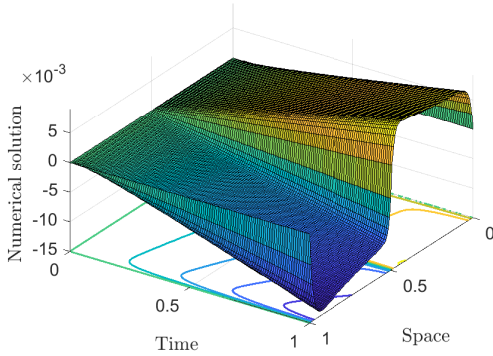
Table 6.1: Computed $E_{\varepsilon, \mu}^{N, M}$ and $\bar{R}_{\varepsilon, \mu}^{N, M}$ when $\mu = 2^{-4}$ for Example 6.1.

Mesh points count: N equals M						
$\varepsilon \downarrow$	16	32	64	128	256	512
2^0	1.9460e-02	1.4727e-02	9.8114e-03	5.7649e-03	3.1369e-03	1.6573e-03
	4.0205e-01	5.8593e-01	7.6716e-01	8.7796e-01	9.2050e-01	—
2^{-2}	3.7203e-02	2.2243e-02	1.2298e-02	6.5594e-03	3.4347e-03	1.7763e-03
	7.4207e-01	8.5493e-01	9.0679e-01	9.3338e-01	9.5131e-01	—
2^{-4}	4.4987e-02	2.5013e-02	1.3410e-02	7.0215e-03	3.6197e-03	1.8487e-03
	8.4683e-01	8.9937e-01	9.3346e-01	9.5591e-01	9.6936e-01	—
2^{-6}	4.8237e-02	2.6584e-02	1.4097e-02	7.2966e-03	3.7278e-03	1.8907e-03
	8.5958e-01	9.1517e-01	9.5009e-01	9.6890e-01	9.7940e-01	—
2^{-8}	4.9256e-02	2.7424e-02	1.4433e-02	7.4428e-03	3.7896e-03	1.9143e-03
	8.4486e-01	9.2607e-01	9.5545e-01	9.7380e-01	9.8523e-01	—
2^{-10}	4.9365e-02	2.7629e-02	1.4632e-02	7.5114e-03	3.8199e-03	1.9262e-03
	8.3731e-01	9.1706e-01	9.6197e-01	9.7555e-01	9.8778e-01	—
2^{-12}	4.9368e-02	2.7644e-02	1.4667e-02	7.5563e-03	3.8304e-03	1.9329e-03
	8.3661e-01	9.1439e-01	9.5682e-01	9.8018e-01	9.8673e-01	—
2^{-14}	4.9368e-02	2.7644e-02	1.4668e-02	7.5606e-03	3.8387e-03	1.9341e-03
	8.3661e-01	9.1439e-01	9.5682e-01	9.8018e-01	9.8896e-01	—
2^{-16}	4.9368e-02	2.7644e-02	1.4668e-02	7.5606e-03	3.8390e-03	1.9344e-03
	8.3661e-01	9.1439e-01	9.5682e-01	9.8018e-01	9.8884e-01	—
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-30}	4.9368e-02	2.7644e-02	1.4668e-02	7.5606e-03	3.8390e-03	1.9344e-03
	8.3661e-01	9.1439e-01	9.5682e-01	9.8018e-01	9.8884e-01	—
$E_{\varepsilon, \mu}^{N, M}$	4.9368e-02	2.7644e-02	1.4668e-02	7.5606e-03	3.8390e-03	1.9344e-03
$\bar{R}_{\varepsilon, \mu}^{N, M}$	8.3661e-01	9.1439e-01	9.5682e-01	9.8018e-01	9.8884e-01	—

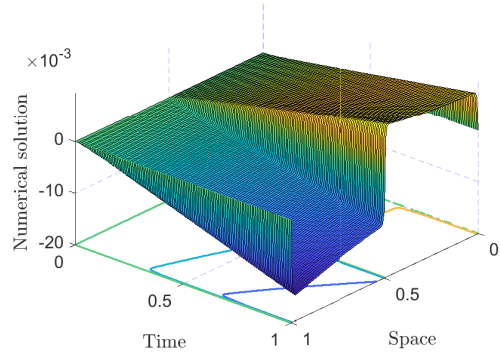
The maximum pointwise absolute errors $E_{\varepsilon, \mu}^{N, M}$ and order of convergence $\bar{R}_{\varepsilon, \mu}^{N, M}$ for Example 6.1- 6.4 are displayed in Tables 6.1-6.4 using various values of ε, μ . Notably, in every example, there is an abrupt change in neighborhood of $x = 0.5$ due to discontinuity in convection coefficients and source terms. In Tables 6.1 and 6.4, we analyze a constant μ while adjusting ε , demonstrating results that exhibit nearly first-order uniform convergence. We take a fixed value of ε and vary μ to present

Table 6.2: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for $\varepsilon = 2^{-12}$ for Example 6.2.

For equal values of mesh points: $N = M$						
$\mu \downarrow$	16	32	64	128	256	512
2^0	$8.1573e-02$	$4.8402e-02$	$2.5717e-02$	$1.3251e-02$	$6.7289e-03$	$3.3910e-03$
	$7.5303e-01$	$9.1234e-01$	$9.5662e-01$	$9.7766e-01$	$9.8866e-01$	—
2^{-2}	$9.2766e-02$	$5.0196e-02$	$2.6171e-02$	$1.3372e-02$	$6.7601e-03$	$3.3988e-03$
	$8.8602e-01$	$9.3960e-01$	$9.6875e-01$	$9.8410e-01$	$9.9202e-01$	—
2^{-4}	$9.4033e-02$	$5.0576e-02$	$2.6278e-02$	$1.3400e-02$	$6.7563e-03$	$3.3970e-03$
	$8.9471e-01$	$9.4460e-01$	$9.7162e-01$	$9.8793e-01$	$9.9197e-01$	—
2^{-6}	$9.4376e-02$	$5.0684e-02$	$2.6295e-02$	$1.3365e-02$	$6.7555e-03$	$3.3952e-03$
	$8.9689e-01$	$9.4674e-01$	$9.7633e-01$	$9.8433e-01$	$9.9257e-01$	—
2^{-8}	$9.4453e-02$	$5.0624e-02$	$2.6150e-02$	$1.3364e-02$	$6.7506e-03$	$3.3949e-03$
	$8.9978e-01$	$9.5301e-01$	$9.6846e-01$	$9.8526e-01$	$9.9165e-01$	—
2^{-10}	$9.4309e-02$	$5.0443e-02$	$2.6151e-02$	$1.3363e-02$	$6.7486e-03$	$3.3948e-03$
	$9.0274e-01$	$9.4779e-01$	$9.6862e-01$	$9.8558e-01$	$9.9126e-01$	—
2^{-12}	$9.4194e-02$	$5.0368e-02$	$2.6151e-02$	$1.3363e-02$	$6.7480e-03$	$3.3947e-03$
	$9.0313e-01$	$9.4564e-01$	$9.6862e-01$	$9.8571e-01$	$9.9118e-01$	—
2^{-14}	$9.4158e-02$	$5.0347e-02$	$2.6151e-02$	$1.3362e-02$	$6.7478e-03$	$3.3947e-03$
	$9.0318e-01$	$9.4504e-01$	$9.6873e-01$	$9.8565e-01$	$9.9113e-01$	—
2^{-16}	$9.4148e-02$	$5.0342e-02$	$2.6151e-02$	$1.3362e-02$	$6.7478e-03$	$3.3947e-03$
	$9.0317e-01$	$9.4490e-01$	$9.6873e-01$	$9.8565e-01$	$9.9113e-01$	—
2^{-18}	$9.4145e-02$	$5.0340e-02$	$2.6151e-02$	$1.3362e-02$	$6.7478e-03$	$3.3947e-03$
	$9.0319e-01$	$9.4484e-01$	$9.6873e-01$	$9.8565e-01$	$9.9113e-01$	—
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-24}	$9.4145e-02$	$5.0340e-02$	$2.6151e-02$	$1.3362e-02$	$6.7478e-03$	$3.3947e-03$
	$9.0318e-01$	$9.4484e-01$	$9.6873e-01$	$9.8565e-01$	$9.9113e-01$	—
$E_{\varepsilon,\mu}^{N,M}$	$9.4453e-02$	$5.0684e-02$	$2.6295e-02$	$1.3400e-02$	$6.7601e-03$	$3.3988e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$8.9807e-01$	$9.4674e-01$	$9.7256e-01$	$9.8712e-01$	$9.9202e-01$	—



(a) $\varepsilon = 2^{-4}, \mu = 1$



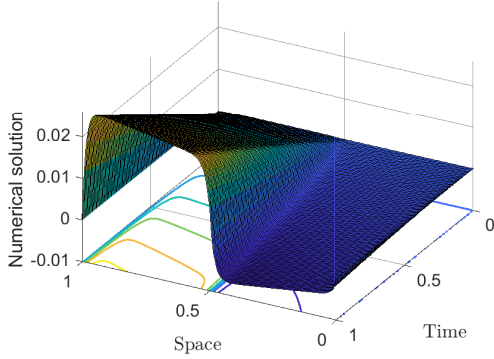
(b) $\varepsilon = 2^{-12}, \mu = 2^{-12}$

Figure 6.2: The numerical solution profile for Example 6.1.

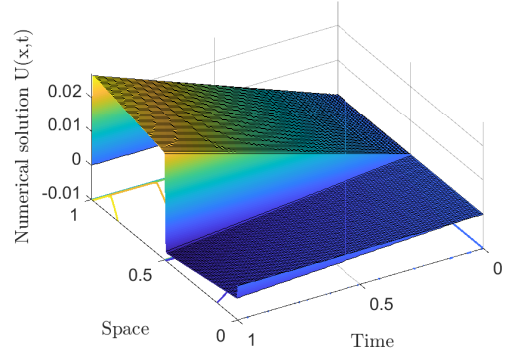
almost first-order uniform convergent results in Tables 6.2 and 6.3. These four tables' numerical results support our theoretical assertions that the suggested scheme yields uniform outcomes in both scenarios. The tables show that, in both cases, for fixed N, M , the maximum pointwise absolute error is constant at a given small ε or μ .

Table 6.3: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ with $\varepsilon = 2^{-8}$ for Example 6.3.

$N = M$						
$\mu \downarrow$	16	32	64	128	256	512
2^0	$3.7484e-02$	$2.0606e-02$	$1.0889e-02$	$5.6228e-03$	$2.8631e-03$	$1.4459e-03$
	$8.6321e-01$	$9.2019e-01$	$9.5351e-01$	$9.7371e-01$	$9.8561e-01$	—
2^{-2}	$4.0906e-02$	$2.1744e-02$	$1.1240e-02$	$5.7237e-03$	$2.8898e-03$	$1.4522e-03$
	$9.1170e-01$	$9.5198e-01$	$9.7362e-01$	$9.8598e-01$	$9.9273e-01$	—
2^{-4}	$4.2102e-02$	$2.2153e-02$	$1.1342e-02$	$5.7474e-03$	$2.8948e-03$	$1.4535e-03$
	$9.2639e-01$	$9.6583e-01$	$9.8069e-01$	$9.8945e-01$	$9.9393e-01$	—
2^{-6}	$4.2736e-02$	$2.2209e-02$	$1.1373e-02$	$5.7531e-03$	$2.8960e-03$	$1.4537e-03$
	$9.4431e-01$	$9.6553e-01$	$9.8320e-01$	$9.9028e-01$	$9.9433e-01$	—
2^{-8}	$4.2794e-02$	$2.2266e-02$	$1.1378e-02$	$5.7547e-03$	$2.8962e-03$	$1.4538e-03$
	$9.4257e-01$	$9.6860e-01$	$9.8343e-01$	$9.9058e-01$	$9.9433e-01$	—
2^{-10}	$4.2802e-02$	$2.2277e-02$	$1.1380e-02$	$5.7550e-03$	$2.8963e-03$	$1.4538e-03$
	$9.4212e-01$	$9.6905e-01$	$9.8361e-01$	$9.9060e-01$	$9.9438e-01$	—
2^{-12}	$4.2804e-02$	$2.2280e-02$	$1.1380e-02$	$5.7551e-03$	$2.8963e-03$	$1.4538e-03$
	$9.4200e-01$	$9.6925e-01$	$9.8359e-01$	$9.9063e-01$	$9.9438e-01$	—
2^{-14}	$4.2804e-02$	$2.2281e-02$	$1.1380e-02$	$5.7551e-03$	$2.8963e-03$	$1.4538e-03$
	$9.4193e-01$	$9.6931e-01$	$9.8359e-01$	$9.9063e-01$	$9.9438e-01$	—
$\vdots$	$\vdots$			$\vdots$		$\vdots$
2^{-24}	$4.2804e-02$	$2.2281e-02$	$1.1380e-02$	$5.7551e-03$	$2.8963e-03$	$1.4538e-03$
	$9.4193e-01$	$9.6931e-01$	$9.8359e-01$	$9.9063e-01$	$9.9438e-01$	—
$E_{\varepsilon,\mu}^{N,M}$	$4.2804e-02$	$2.2281e-02$	$1.1380e-02$	$5.7551e-03$	$2.8963e-03$	$1.4538e-03$
$\overline{R}_{\varepsilon,\mu}^{N,M}$	$9.4193e-01$	$9.6931e-01$	$9.8359e-01$	$9.9063e-01$	$9.9438e-01$	—



(a) $\varepsilon = 2^{-4}, \mu = 2^{-2}$



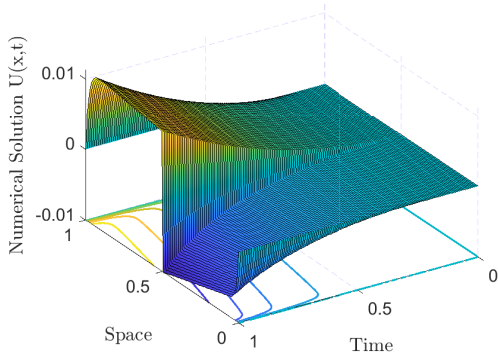
(b) $\varepsilon = 2^{-12}, \mu = 2^{-18}$

Figure 6.3: The numerical solution profile for Example 6.2.

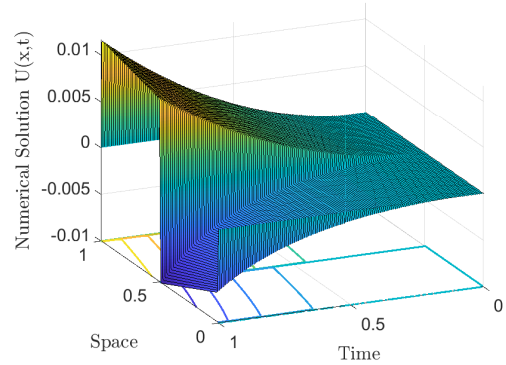
This demonstrates that the approach is independent of ε and μ . Table 6.5 compare numerical results to those given in Chandru et al. (2018, 2019), by fixing or varying ε or μ . This comparison reveals that, when evaluated against existing studies in the literature, the proposed approach yields a more precise solution. Every result shown in the tables was acquired by taking $N = M$. The solution profiles in the form of surface-contour plots are provided in Figures 6.2, 6.3, 6.4, and 6.5 for Examples 6.1, 6.2, 6.3, and 6.4, respectively to illustrate how the gradient of solution steepens at the point of

Table 6.4: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for $\mu = 2^{-10}$ and various values of ε for Example 6.4.

For equal mesh points in both directions						
$\varepsilon \downarrow$	16	32	64	128	256	512
2^0	$1.1487e-02$	$1.0042e-02$	$6.8505e-03$	$3.9935e-03$	$2.1448e-03$	$1.1189e-03$
	$1.9396e-01$	$5.5177e-01$	$7.7856e-01$	$8.9681e-01$	$9.3876e-01$	—
2^{-2}	$2.5450e-02$	$1.5559e-02$	$8.4794e-03$	$4.4495e-03$	$2.3003e-03$	$1.1781e-03$
	$7.0992e-01$	$8.7572e-01$	$9.3032e-01$	$9.5182e-01$	$9.6536e-01$	—
2^{-4}	$3.2209e-02$	$1.7363e-02$	$9.0907e-03$	$4.6828e-03$	$2.3901e-03$	$1.2124e-03$
	$8.9145e-01$	$9.3355e-01$	$9.5702e-01$	$9.7030e-01$	$9.7921e-01$	—
2^{-6}	$3.4363e-02$	$1.8239e-02$	$9.4273e-03$	$4.8172e-03$	$2.4416e-03$	$1.2317e-03$
	$9.1383e-01$	$9.5211e-01$	$9.6865e-01$	$9.8037e-01$	$9.8718e-01$	—
2^{-8}	$3.5276e-02$	$1.8709e-02$	$9.6213e-03$	$4.8915e-03$	$2.4692e-03$	$1.2425e-03$
	$9.1495e-01$	$9.5943e-01$	$9.7595e-01$	$9.8623e-01$	$9.9080e-01$	—
2^{-10}	$3.5937e-02$	$1.8866e-02$	$9.7252e-03$	$4.9261e-03$	$2.4834e-03$	$1.2481e-03$
	$9.2968e-01$	$9.5599e-01$	$9.8128e-01$	$9.8813e-01$	$9.9258e-01$	—
2^{-12}	$3.6573e-02$	$1.9011e-02$	$9.7482e-03$	$4.9507e-03$	$2.4904e-03$	$1.2512e-03$
	$9.4394e-01$	$9.6363e-01$	$9.7750e-01$	$9.9126e-01$	$9.9307e-01$	—
2^{-14}	$3.6721e-02$	$1.9177e-02$	$9.7742e-03$	$4.9538e-03$	$2.4966e-03$	$1.2526e-03$
	$9.3723e-01$	$9.7233e-01$	$9.8044e-01$	$9.8857e-01$	$9.9504e-01$	—
2^{-16}	$3.6735e-02$	$1.9207e-02$	$9.8141e-03$	$4.9548e-03$	$2.4970e-03$	$1.2534e-03$
	$9.3552e-01$	$9.6870e-01$	$9.8603e-01$	$9.8863e-01$	$9.9435e-01$	—
2^{-18}	$3.6735e-02$	$1.9208e-02$	$9.8180e-03$	$4.9627e-03$	$2.4969e-03$	$1.2535e-03$
	$9.3545e-01$	$9.6821e-01$	$9.8430e-01$	$9.9099e-01$	$9.9418e-01$	—
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-24}	$3.6735e-02$	$1.9208e-02$	$9.8180e-03$	$4.9629e-03$	$2.4969e-03$	$1.2535e-03$
	$9.3545e-01$	$9.6821e-01$	$9.8425e-01$	$9.9105e-01$	$9.9418e-01$	—
$E_{\varepsilon,\mu}^{N,M}$	$3.6735e-02$	$1.9208e-02$	$9.8180e-03$	$4.9629e-03$	$2.4969e-03$	$1.2535e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$9.3545e-01$	$9.6821e-01$	$9.8425e-01$	$9.9105e-01$	$9.9418e-01$	—



(a) $\varepsilon = 2^{-8}, \mu = 1$



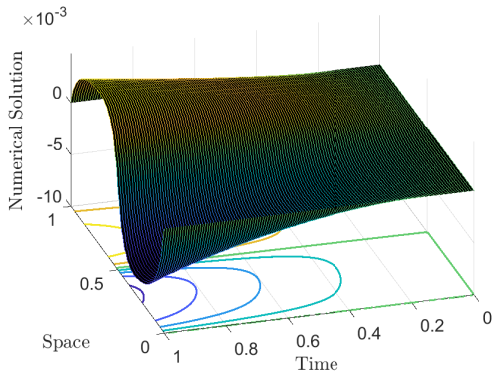
(b) $\varepsilon = 2^{-12}, \mu = 2^{-8}$

Figure 6.4: The numerical solution profile for Example 6.3.

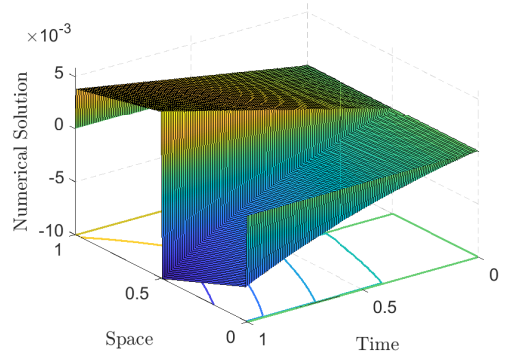
discontinuity when ε or μ decreases. Due to the occurrence of ε, μ and discontinuous data in the convection and source terms, the surface plots of the numerical solution show the presence of layers(both boundary and interior). We notice a decrement in the layer width as we decrease μ or ε (see Figures 6.2a, 6.3a, 6.4a and 6.5a). The

Table 6.5: Comparison of $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 6.1-6.4.

$(N = M) \rightarrow$	64	128	256	512
Present Method for Example 6.1				
$E_{\varepsilon,\mu}^{N,M}$	$1.4668e-02$	$7.5606e-03$	$3.8390e-03$	$1.9344e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$9.5682e-01$	$9.8018e-01$	$9.8884e-01$	—
Result in Chandru et al. (2019)(Upwind scheme on S-Mesh)				
$E_{\varepsilon,\mu}^{N,M}$	$8.2743e-2$	$5.2731e-2$	$3.0348e-2$	$1.647e-2$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$6.4997e-01$	$7.9705e-01$	$8.8105e-01$	—
Present Method for Example 6.2				
$E_{\varepsilon,\mu}^{N,M}$	$2.6295e-02$	$1.3400e-02$	$6.7601e-03$	$3.3988e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$9.7256e-01$	$9.8712e-01$	$9.9202e-01$	—
Result in Chandru et al. (2018)(Upwind scheme on S-Mesh)				
$E_{\varepsilon,\mu}^{N,M}$	$2.3788e-02$	$1.8358e-02$	$1.2613e-02$	$8.0788e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$3.7381e-01$	$5.4151e-01$	$6.4270e-01$	—
Present Method for Example 6.3				
$E_{\varepsilon,\mu}^{N,M}$	$1.1380e-02$	$5.7551e-03$	$2.8963e-03$	$1.4538e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$9.8359e-01$	$9.9063e-01$	$9.9438e-01$	—
Result in Chandru et al. (2019)(Upwind scheme on S-Mesh)				
$E_{\varepsilon,\mu}^{N,M}$	$1.3387e-02$	$1.1631e-02$	$8.6796e-03$	$5.8905e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$2.0280e-01$	$4.2236e-01$	$5.5922e-01$	—
Present Method for Example 6.4				
$E_{\varepsilon,\mu}^{N,M}$	$9.8180e-03$	$4.9629e-03$	$2.4969e-03$	$1.2535e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$9.8425e-01$	$9.9105e-01$	$9.9418e-01$	—
Result in Chandru et al. (2019)(Upwind scheme on S-Mesh)				
$E_{\varepsilon,\mu}^{N,M}$	$2.1843e-02$	$1.7869e-02$	$1.2829e-02$	$7.8367e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	$2.8973e-01$	$4.7804e-01$	$7.1108e-01$	—

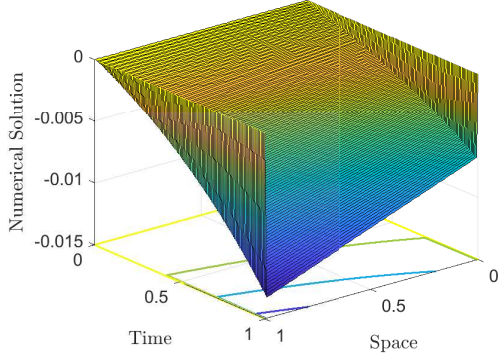


(a) $\varepsilon = 1, \mu = 2^{-6}$

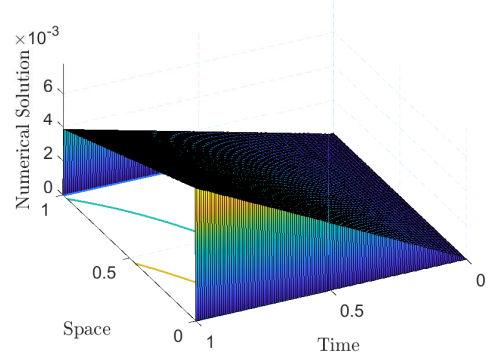


(b) $\varepsilon = 2^{-16}, \mu = 2^{-16}$

Figure 6.5: The numerical solution profile for Example 6.4.

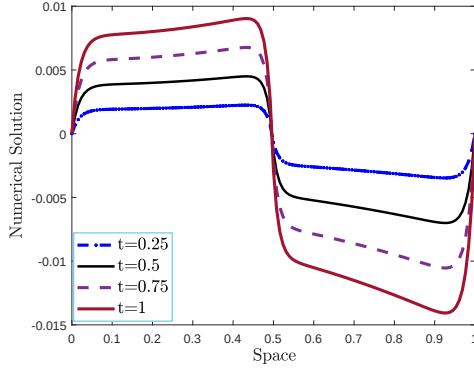


(a) Source term is $\varpi(x, t) = (e^t - 1)(1 + x)$

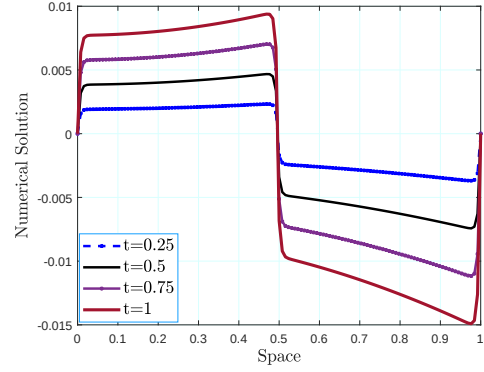


(b) Source term is $\varpi(x, t) = t(x - 2)$

Figure 6.6: Numerical solution for Example 6.4 with $\varepsilon = 2^{-16}$ and $\mu = 2^{-16}$.

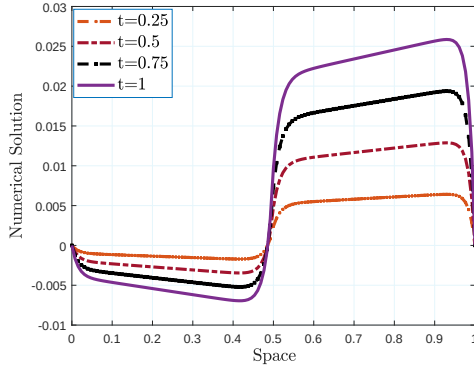


(a) $\varepsilon = 2^{-4}, \mu = 1$

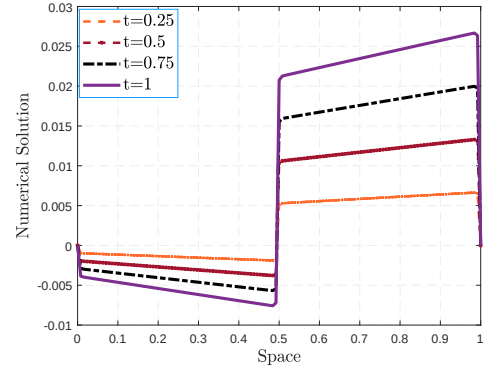


(b) $\varepsilon = 2^{-8}, \mu = 2^{-12}$

Figure 6.7: Line plots for Example 6.1.



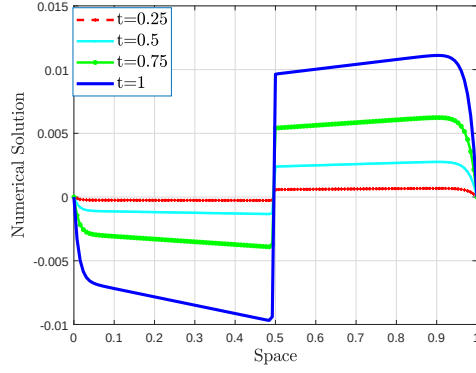
(a) $\varepsilon = 2^{-4}, \mu = 2^{-2}$



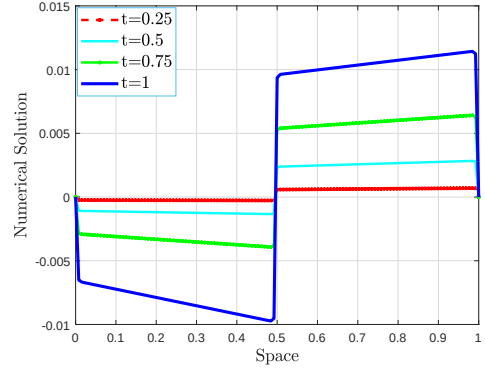
(b) $\varepsilon = 2^{-12}, \mu = 2^{-18}$

Figure 6.8: Line plots for Example 6.2 at different time-levels.

line plots for Examples 6.1, 6.2, 6.3, and 6.4 at various time steps while maintaining $N = 128 = M$ are represented in Figures 6.7, 6.8, 6.9, and 6.10, which show that the maximum error occurs near the boundary and interior layer regions. The observation

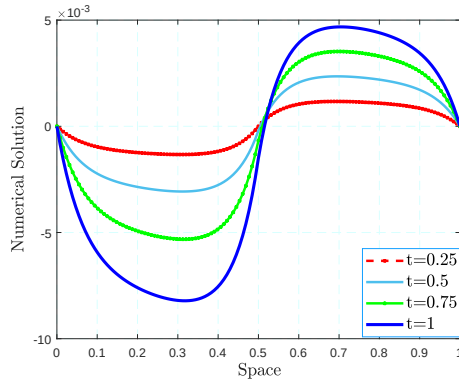


(a) $\varepsilon = 2^{-8}, \mu = 1$

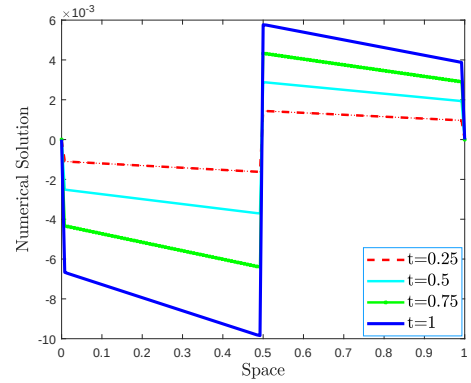


(b) $\varepsilon = 2^{-12}, \mu = 2^{-8}$

Figure 6.9: Line plots for Example 6.3 at different time-levels.

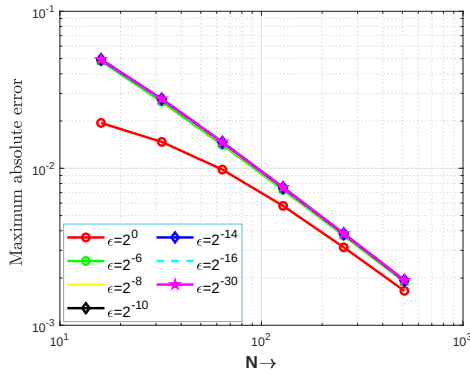


(a) $\varepsilon = 1, \mu = 2^{-6}$

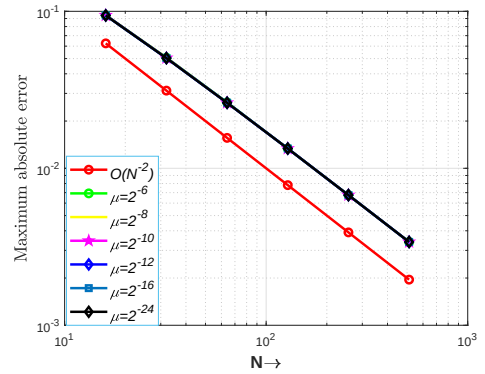


(b) $\varepsilon = 2^{-16}, \mu = 2^{-16}$

Figure 6.10: Line plots for Example 6.4 at different time-levels.



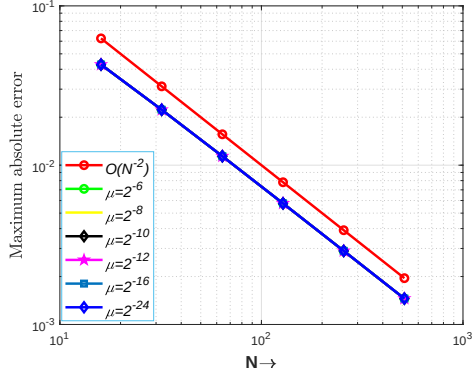
(a) Log-log plot for Example 6.1



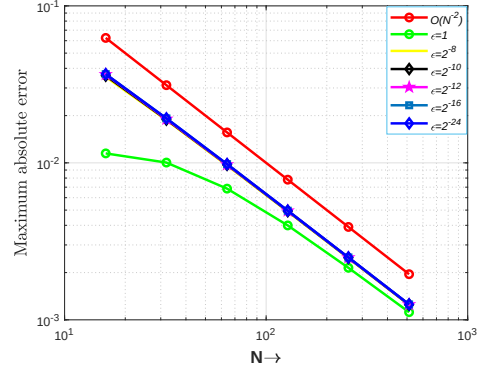
(b) Log-log plot for Example 6.2

Figure 6.11: Line plots for Example 6.4 at different time-levels.

of a sharper layer as we decrease the values of ε and μ is confirmed, which helps us understand the influence of parameter values and how they affect the formation of layers. Every line plots and surface plots are created for fixed values of $N = M = 128$. Based on the research conducted by O'Riordan & Shishkin (2004), it was found that

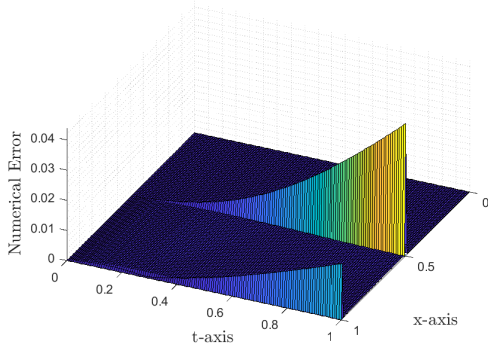


(a) Log-log plot for Example 6.3

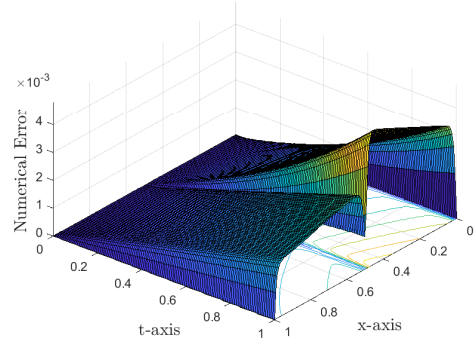


(b) Log-log plot for Example 6.4

Figure 6.12: The Log-log plot of maximum absolute error.

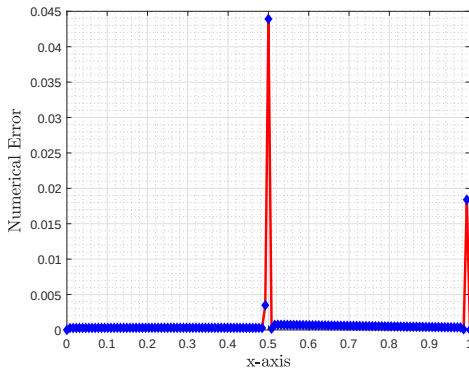


(a) $N = M = 128, \varepsilon = 2^{-18}, \mu = 2^{-8}$

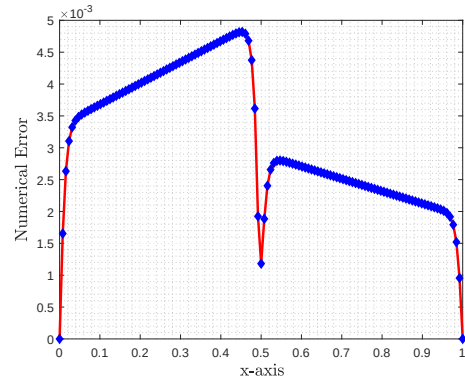


(b) $N = M = 128, \varepsilon = 2^{-68}, \mu = 2^{-18}$

Figure 6.13: Numerical error corresponding to Examples 6.2 and 6.4, respectively.



(a) $N = M = 128, \varepsilon = 2^{-18}, \mu = 2^{-8}$



(b) $N = M = 128, \varepsilon = 2^{-68}, \mu = 2^{-18}$

Figure 6.14: Numerical error corresponding to Examples 6.2 and 6.4, respectively.

the solution may contain an interior layer solely when the discontinuity is present in the convection-coefficient ($\varpi(x, t)$ is continuous). This numerical insight is crucial in understanding the behavior of solutions in the presence of discontinuities. It is evident from Figure 6.6 that the solution might only contain the boundary layers

if the discontinuity is in the convection-coefficient alone. The discontinuous source functions in Example 6.4 are replaced with smooth function $\varpi(x, t) = (e^t - 1)(1 + x)$ and $\varpi(x, t) = t(x - 2)$, respectively, to draw Figures 6.6a and 6.6b. For this examples, the solution surface plot at figure 6.6 observes only the right and left boundary layers. In this context, the discontinuity in convection coefficient may only cause a shift in the layer within the solution. Despite the discontinuities in the convection coefficients, the interior layer phenomena might not manifest if the source functions are considered smooth. Surface plots of the error provide a clear depiction of the solution's behavior across the spatial and temporal domains, particularly in regions where sharp gradients or boundary layers occur. Figures 6.13a and 6.13b show the surface error plots for Examples 6.2 and 6.4, illustrating that the highest errors occur near the boundary and interior regions, as indicated by the prominent spikes. Similarly, Figures 6.14a and 6.14b present the line error plots for the same examples, confirming that the maximum errors are concentrated in these regions. Furthermore, to illustrate the practical validity of the expected rate of convergence, we have presented the error plots on a Log-Log scale in Figures 6.11 and 6.12, corresponding to Examples 6.1-6.4.

6.7 Conclusion

In this chapter, a fitted numerical scheme was formulated to address TP-SPPPs with discontinuous convection coefficients and source terms. The method captures both boundary and interior layers arising from the interplay of discontinuities and small perturbation parameters. The proposed approach combines the implicit Euler method for temporal discretization with the cubic spline in compression technique for spatial approximation on a uniform mesh. Analytical results confirm first-order accuracy in time and second-order accuracy in space, with uniform convergence concerning both perturbation parameters. Numerical experiments on benchmark problems validate these findings, revealing clear parabolic boundary layers and sharp interior layers near discontinuities. Comparative results further demonstrate that the scheme delivers high accuracy across various parameter ranges. To further enhance spatial accuracy and improve convergence rates beyond the current method, Chapter 7 introduces a refined numerical framework based on a nonstandard finite difference approach.

Chapter 7

Uniformly Convergent Numerical Methods for Two-Parameter Singularly Perturbed Parabolic Problems with Non-Smooth Data

Building upon the fitted spline-based numerical framework established in Chapter 6, this chapter develops uniformly convergent scheme for TP-SPPPs involving discontinuous convection coefficients and source terms. Such problems exhibit pronounced boundary and interior layers arising from the interplay between coefficient discontinuities and small perturbation parameters, which complicate their numerical approximation. To address these challenges, the time derivative is discretized using the implicit Euler method, ensuring temporal stability, while the spatial domain is treated using a nonstandard finite difference scheme specifically tailored to resolve steep layer structures. To further increase accuracy without additional mesh refinement, Richardson extrapolation is incorporated. Numerical experiments demonstrate strong agreement with theoretical predictions and confirm uniform convergence with respect to both perturbation parameters. Overall, the method provides an efficient and reliable extension of the framework developed in Chapter 6, offering improved accuracy and convergence properties for problems featuring discontinuous data.

7.1 Introduction

Two-parameter singularly perturbed parabolic problems with discontinuous data pose substantial analytical and computational difficulties. Jumps in convection coefficients or source terms typically generate sharp boundary and interior layers whose positions and magnitudes depend on both perturbation parameters. These layers can shift abruptly as the parameters vary, making uniform resolution extremely challenging. Classical discretization techniques often struggle to capture such fine-scale structures, leading to deterioration in accuracy and, in many cases, numerical instabilities.

In response to the broader question of why instabilities arise in finite difference schemes (FDS) for DEs, [Mickens \(1993, 2005, 2020\)](#) systematically investigated the underlying mechanisms and proposed the principle of exact FDS. An exact scheme is one whose discrete formulation reproduces the qualitative and in certain cases the exact solution structure of the original DE when evaluated at discrete points. Mickens introduced a set of six guiding rules that form the foundation of nonstandard finite difference schemes (NSFDS). These rules modify classical discretizations in ways that preserve positivity, enforce correct asymptotic behavior, and align the discrete dynamics more closely with the continuous model. Subsequent works [Mickens \(1993, 2005, 2020\)](#) further solidified the theoretical basis for NSFDS and highlighted their advantages in achieving improved consistency, stability, and convergence properties, particularly for problems exhibiting non-smooth or highly sensitive behavior.

The motivation behind the nonstandard finite difference method (NS-FDM) is therefore not to obtain exact solutions in a literal sense, but to enhance the fidelity of numerical schemes by incorporating structural features of the continuous problem directly into the discretization. Although exactness is not expected from the resulting schemes, numerous studies have shown that their accuracy and qualitative reliability often surpass those of standard methods, especially in stiff or layer-dominated regimes ([Munyakazi, 2015](#); [Duressa, 2021](#); [Mickens, 2020](#)).

In light of these considerations, this chapter develops a uniformly convergent numerical strategy tailored to the complexities of TP-SPPPs with discontinuities. Extending the methods in Chapter 6, the method employs the implicit Euler scheme for time

discretization. Spatially, a problem adapted NS-FDM is constructed to capture steep layer profiles generated by coefficient jumps. To further elevate accuracy without increasing mesh density, Richardson extrapolation is incorporated, enabling enhanced parameter-uniform performance. Together, these components form a coherent computational approach for addressing the intricate layer structures and discontinuous features inherent to such problems.

The chapter is structured as follows: Section 7.2 presents the problem formulation. Section 7.3 outlines the construction of the proposed numerical scheme, and Section 7.4 provides its convergence analysis. Section 7.5 includes numerical experiments validating the method, and Section 7.6 concludes with a summary of the main results.

7.2 Definition of the Problem

We consider the problem defined in Chapter 6, Section 6.2 and assume the solution properties established in Section 6.3.

7.3 Formulation of the Numerical Scheme

This section presents the computational technique employed to solve the TP-SPPPs characterized by discontinuous convection coefficients and source terms. The proposed approach utilizes the Implicit Euler scheme for the temporal discretization, ensuring stability and accuracy over time, while the nonstandard finite difference framework introduced by Mickens' is adopted for spatial discretization, offering superior performance in capturing the intricate layer behaviors inherent to such problems.

7.3.1 Temporal Semi-discretization

The interval $[0, T]$ is divided into M uniform partitions, forming the mesh $\Omega^{M_t} = \{t_j = jk : j = 1(1)M, k = T/M\}$. Then, on $\Omega^\varphi \times \Omega^{M_t}$, Eq. (6.1) is semi-discretized as

$$\begin{aligned} & -c^{j+1}(x)D_t^-W^{j+1}(x) + \varepsilon (W^{j+1})_{xx}(x) + \mu a^{j+1}(x) (W^{j+1})_x(x) - \\ & b^{j+1}(x)W^{j+1}(x) = \varpi^{j+1}(x), \quad x \in \Omega^\varphi, 0 \leq j \leq M-1, \end{aligned}$$

where $W^{j+1}(x)$ is the approximation of $w(x, t_{j+1})$, $D_t^- = (W^{j+1}(x) - W^j(x)) / k$ is backward difference operator. Rewrite the above equation as

$$\begin{cases} \mathbb{L}_{\varepsilon, \mu}^M W^{j+1}(x) = \mathcal{G}^{j+1}(x), & x \in \Omega^\varphi, \quad 0 \leq j \leq M-1, \\ W^{j+1}(0) = \vartheta(t_{j+1}), \quad W^{j+1}(1) = \xi(t_{j+1}), \\ W^0(x) = \phi(x), \quad x \in \overline{\Omega}, \end{cases} \quad (7.1)$$

where the operator $\mathbb{L}_{\varepsilon, \mu}^M$ is defined as

$$\mathbb{L}_{\varepsilon, \mu}^M \equiv \varepsilon \frac{d^2}{dx^2} + \mu a^{j+1}(x) \frac{d}{dx} - \left(b^{j+1}(x) + \frac{c^{j+1}(x)}{k} \right) I, \quad \text{and}$$

$$\mathcal{G}^{j+1}(x) = \varpi^{j+1}(x) - \frac{c^{j+1}(x)}{k} W^j(x).$$

The LTE in the temporal semi-discretization of Eq. (7.1) is given by $\mathcal{E}_{j+1} = W^{j+1}(x) - \overline{U}(x)$, where $\overline{U}(x)$ is the computed solution of

$$\begin{cases} \mathbb{L}_{\varepsilon, \mu} \overline{U}(x) = \varpi(x, t_{j+1}) - \frac{c(x, t_{j+1})}{k} W(x, t_j), & x \in \Omega^\varphi, \\ \overline{U}(0) = \vartheta(t_{j+1}), \quad \overline{U}(1) = \xi(t_{j+1}), & 0 \leq j \leq M-1, \\ \overline{U}(x) = \phi(x), & x \in \overline{\Omega}, \end{cases} \quad (7.2)$$

The next lemma is the bounds of the LTE $\mathcal{E}_{j+1}$, which is

Lemma 7.1. *The LTE and global error are defined as $\mathcal{E}_{j+1}$ and $\mathbb{E}_j = \sum_{i=1}^j \mathcal{E}_i$, respectively, and are bounded by :*

$$\begin{aligned} \|\mathcal{E}_{j+1}\| &\leq Ck^2, \\ \|\mathbb{E}_j\| &\leq Ck, \quad j \leq \frac{T}{k}, \end{aligned} \quad (7.3)$$

where $\mathbb{E}_j$ represents the cumulative effect of LTE up to the j th time level.

Proof. The derivation of the first component of Eq. (7.3) follows the approach in Singh et al. (2023), while the second is established in Chapter 6, Lemma 6.1. $\square$

7.3.2 Determination of the Denominator Function

Beginning with the equation given below, this section is dedicated to constructing a robust and accurate finite difference method for the considered problem:

$$\varepsilon (W^{j+1})_{xx}(x) + \mu a^{j+1}(x) (W^{j+1})_x(x) - \zeta^{j+1}(x) W^{j+1}(x) = \mathcal{G}^{j+1}(x), \quad (7.4)$$

where $\zeta^{j+1}(x) = b(x, t_{j+1}) + \frac{c(x, t_{j+1})}{k}$ and $\mathcal{G}^{j+1}(x) = \varpi(x, t_{j+1}) - \frac{c(x, t_{j+1})}{k} W^j(x)$. Subsequently, we introduce the time-level index and investigate the corresponding homogeneous problems with constant coefficients associated with Eq. (7.4), laying the foundation for the analytical and numerical development of the proposed scheme.

$$\varepsilon W_{xx}(x) + \mu a(x) W_x(x) - \zeta(x) W(x) = 0, \quad \text{where} \quad \zeta(x) = b(x) + \frac{d(x)}{k}. \quad (7.5)$$

$$\varepsilon W_{xx}(x) + \mu a(x) W_x(x) = 0 \quad (7.6)$$

Eq. (7.5) admits two distinct linearly independent solutions, $e^{\mathcal{C}_1 x}$ and $e^{\mathcal{C}_2 x}$, where

$$\mathcal{C}_{1,2} = \frac{-\mu a \pm \sqrt{(\mu a)^2 + 4\varepsilon \zeta}}{2\varepsilon} \quad (7.7)$$

A uniform mesh with a spatial step size h is employed over the domain $[0, 1]$, ensuring consistent discretization and balanced resolution across the entire interval.

$$\Omega_x^N = \begin{cases} x_i = x_0 + ih, & i = 1(1)N - 1, \\ x_0 = 0, x_{\frac{N}{2}} = \varphi, x_N = 1, h = \frac{1}{N}. \end{cases}$$

At every mesh point x_i , the discrete approximation of the continuous function $W(x)$ is written as W_i . Our aim is to derive a corresponding discrete relation at x_i that retains the structural form of the general solution given in Eq. (7.5), stated as:

$$W_i = C_1 \exp(\mathcal{C}_1 x_i) + C_2 \exp(\mathcal{C}_2 x_i). \quad (7.8)$$

In constructing the second-order difference equations, the formulation is guided by principles of DE theory, ensuring that the discrete model accurately reflects the qualitative behavior of its continuous counterpart. This approach follows the frame-

work proposed by [Mickens \(1993\)](#), which emphasizes the development of nonstandard schemes that maintain consistency and correct dynamic properties of the original DEs.

$$\begin{vmatrix} W_{i-1} & W_{1,i-1} & W_{2,i-1} \\ W_i & W_{1,i} & W_{2,i-1} \\ W_{i+1} & W_{1,i+1} & W_{2,i-1} \end{vmatrix} = \begin{vmatrix} W_{i-1} & \exp(\mathcal{C}_1 x_{i-1}) & \exp(\mathcal{C}_2 x_{i-1}) \\ W_i & \exp(\mathcal{C}_1 x_i) & \exp(\mathcal{C}_2 x_i) \\ W_{i+1} & \exp(\mathcal{C}_1 x_{i+1}) & \exp(\mathcal{C}_2 x_{i+1}) \end{vmatrix} = 0, \quad (7.9)$$

$$\begin{vmatrix} W_{i-1} & \exp(\mathcal{C}_1 x_i - \mathcal{C}_1 h) & \exp(\mathcal{C}_2 x_i - \mathcal{C}_2 h) \\ W_i & \exp(\mathcal{C}_1 x_i) & \exp(\mathcal{C}_2 x_i) \\ W_{i+1} & \exp(\mathcal{C}_1 x_i + \mathcal{C}_1 h) & \exp(\mathcal{C}_2 x_i + \mathcal{C}_2 h) \end{vmatrix} = 0,$$

$$\exp(\mathcal{C}_1 x_i + \mathcal{C}_2 x_i) \begin{vmatrix} W_{i-1} & \exp(-\mathcal{C}_1 h) & \exp(-\mathcal{C}_2 h) \\ W_i & 1 & 1 \\ W_{i+1} & \exp(\mathcal{C}_1 h) & \exp(\mathcal{C}_2 h) \end{vmatrix} = 0.$$

By computing the determinant and simplifying the expressions through the application of double-angle identities for hyperbolic functions, we obtain

$$-\exp\left(\frac{\mu a}{2\varepsilon}\right) W_{i+1} + 2 \cosh\left(\frac{h\sqrt{(\mu a)^2 + 4\varepsilon\zeta}}{2\varepsilon}\right) W_i - \exp\left(-\frac{\mu a}{2\varepsilon}\right) W_{i-1} = 0, \quad (7.10)$$

which is exact scheme for Eq. (7.6) in the sense that Eq. (7.10) has same general solution Eq. (7.8) as Eq. (7.5). We substitute denominator function for h^2 to construct the nonstandard FDM. It is not straightforward to extract the denominator function from Eq. (7.10) in this manner. However, in the case of Eq. (7.6), the layer behaviors of the solutions to Eq. (6.1) and that of Eq. (7.5) are similar ([Patidar, 2008](#); [Gupta et al., 2019](#)). Thus, for Eq. (7.6), we have the exact scheme of the form:

$$-\exp\left(\frac{\mu a h}{2\varepsilon}\right) W_{i+1} + \left(\exp\left(\frac{\mu a h}{2\varepsilon}\right) + \exp\left(-\frac{\mu a h}{2\varepsilon}\right)\right) W_i - \exp\left(-\frac{\mu a h}{2\varepsilon}\right) W_{i-1} = 0. \quad (7.11)$$

Multiplying both sides of Eq. (7.11) by $\exp\left(\frac{\mu a h}{2\varepsilon}\right)$, we obtain:

$$\exp\left(\frac{\mu a h}{\varepsilon}\right) W_{i+1} - \left(\exp\left(\frac{\mu a h}{\varepsilon}\right) + 1\right) W_i + W_{i-1} = 0. \quad (7.12)$$

By adding and subtracting $W_{i+1} + W_i$, and after some manipulations, we obtain

$$W_{i+1} - 2W_i + W_{i-1} + \left(\exp\left(\frac{\mu ah}{\varepsilon}\right) - 1 \right) (W_{i+1} - W_i) = 0. \quad (7.13)$$

From Eq. (7.13), we have

$$\varepsilon \frac{W_{i+1} - 2W_i + W_{i-1}}{\mathfrak{J}^2} + \mu a \frac{W_{i+1} - W_i}{h} = 0, \quad (7.14)$$

where $\mathfrak{J}^2$ is given by:

$$\mathfrak{J}^2(h, \varepsilon, \mu) \equiv \mathfrak{J}^2 = \frac{h\varepsilon}{\mu a} \left(\exp\left(\frac{\mu ah}{\varepsilon}\right) - 1 \right). \quad (7.15)$$

We can express the variable coefficients using the denominator function as follows:

$$\mathfrak{J}_i^2(h, \varepsilon, \mu) \equiv \mathfrak{J}_i^2 = \frac{h\varepsilon}{\mu a_i} \left(\exp\left(\frac{\mu a_i h}{\varepsilon}\right) - 1 \right), \quad \text{for } i = 1, 2. \quad (7.16)$$

7.3.3 Non-Standard Finite Difference Methods

The concept of exact and optimal FDMs serves as the fundamental basis for the development of non-standard discrete modeling techniques. The pioneering framework for constructing such NS-FDMs was established by [Mickens \(1993\)](#), who proposed a systematic set of design principles to ensure both accuracy and consistency with the underlying continuous model. By adhering to these principles, one can derive a discrete formulation that is inherently different from classical FDMs, often preserving essential dynamical properties that standard methods fail to capture. In constructing such a discrete model, special attention is given to the denominator functions associated with the highest-order discrete derivatives. Unlike traditional approaches that rely on simple step-size ratios, NS-FDMs employ nonlinear or functional representations of step sizes, as recommended by Mickens' rules. These modified denominator functions introduce flexibility and stability into the discrete structure, allowing the scheme to handle singular perturbations and boundary layer behaviors effectively. To illustrate this construction process and highlight its computational essence, we now

consider Eq. (7.1) evaluated at a grid point (x_i, t_{j+1}) , as demonstrated below:

$$\begin{aligned} & \varepsilon (W_{xx}) (x_i, t_{j+1}) + \mu a(x_i, t_{j+1}) (W_x) (x_i, t_{j+1}) - \\ & \left(b(x_i, t_{j+1}) + \frac{c(x_i, t_{j+1})}{k} \right) W(x_i, t_{j+1}) = \varpi(x_i, t_{j+1}) - \frac{c(x_i, t_{j+1})}{k} W(x_i, t_j). \end{aligned} \quad (7.17)$$

At the grid point (x_i, t_{j+1}) , Eq. (7.17) can be expressed in the form:

$$\begin{aligned} & \varepsilon (W_{xx}) (x_i, t_{j+1}) + \mu a(x_i, t_{j+1}) (W_x) (x_i, t_{j+1}) - \zeta(x_i, t_{j+1}) W(x_i, t_{j+1}) = \\ & \varpi(x_i, t_{j+1}) - \frac{c(x_i, t_{j+1})}{k} W(x_i, t_j), \end{aligned} \quad (7.18)$$

where $\zeta(x_i, t_{j+1}) = b(x_i, t_{j+1}) + \frac{c(x_i, t_{j+1})}{k}$. Motivated by Eq. (7.14), we can employ the NS-FDM to approximate Eq. (7.18).

$$\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1} = \begin{cases} \varepsilon \frac{W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}}{\mathfrak{J}_{1,i}^2} + \mu a_i^{j+1} \left(\frac{W_i^{j+1} - W_{i-1}^{j+1}}{h} \right) - \\ \quad \zeta_i^{j+1} W_i^{j+1} = \varpi_i^{j+1} - \frac{c_i^{j+1}}{k} W_i^j, & \text{for } 0 \leq i \leq \varphi, \\ \varepsilon \frac{W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}}{\mathfrak{J}_{2,i}^2} + \mu a_i^{j+1} \left(\frac{W_{i+1}^{j+1} - W_i^{j+1}}{h} \right) - \\ \quad \zeta_i^{j+1} W_i^{j+1} = \varpi_i^{j+1} - \frac{c_i^{j+1}}{k} W_i^j, & \text{for } \varphi < i \leq N-1, \end{cases} \quad (7.19)$$

under the subsequent boundary and initial conditions:

$$\begin{cases} W_i^0 = \phi(x_i), & i = 1(1)N-1, j = 0(1)M \\ W^{j+1}(0) = \vartheta(t_{j+1}), & W^{j+1}(N) = \xi(t_{j+1}), & t_{j+1} \in [0, T], \\ D_x^- W_{\frac{N}{2}}^{j+1} = D_x^+ W_{\frac{N}{2}}^{j+1}, & x = \varphi, \end{cases} \quad (7.20)$$

where the variable denominator function Eq. (7.16) is given as

$$\mathfrak{J}_i^2(h, \varepsilon, \mu) \equiv \mathfrak{J}_i^2 = \begin{cases} \frac{h\varepsilon}{\mu a_{1,i}^{j+1}} \left(\exp \left(\frac{\mu a_{1,i}^{j+1} h}{\varepsilon} \right) - 1 \right) & \text{for } i = 0 \leq i \leq \varphi, \\ \frac{h\varepsilon}{\mu a_{2,i}^{j+1}} \left(\exp \left(\frac{\mu a_{2,i}^{j+1} h}{\varepsilon} \right) - 1 \right) & \text{for } i = \varphi < i \leq N. \end{cases} \quad (7.21)$$

The matrix representation of the discrete conditions in Eq. (7.19), together with the scheme in Eq. (7.20), is presented below.

$$\mathcal{M}\mathcal{F} = \mathcal{T}, \quad i = 1(1)N - 1, j = 0(1)M, \quad (7.22)$$

where $\mathcal{F}$ and $\mathcal{T}$ are column vectors, and $\mathcal{M}$ is an $(N-1) \times (N-1)$ tridiagonal matrix, defined as:

$$\mathcal{M} = (\mathcal{M}_{i,\ell}), \quad \ell = i-1, i, i+1, \quad (7.23)$$

where the nonzero entries of the tridiagonal matrix $\mathcal{M}$ are defined as follows:

$$\begin{aligned} \mathcal{M}_{i,i-1} &= \begin{cases} \frac{\varepsilon}{\mathfrak{J}_{1,i}^2} - \frac{\mu}{h} a_i^{j+1}, & i = 2, 3, \dots, \frac{N}{2}, \\ \frac{\varepsilon}{\mathfrak{J}_{2,i}^2}, & i = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N-1, \end{cases} \\ \mathcal{M}_{i,i} &= \begin{cases} -\left(\frac{2\varepsilon}{\mathfrak{J}_{1,i}^2} - \frac{\mu}{h} a_i^{j+1} + b_i^{j+1} + \frac{c_i^{j+1}}{k} \right), & i = 1, 2, \dots, \frac{N}{2}, \\ -\left(\frac{2\varepsilon}{\mathfrak{J}_{2,i}^2} + \frac{\mu}{h} a_i^{j+1} + b_i^{j+1} + \frac{c_i^{j+1}}{k} \right), & i = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N-1, \end{cases} \\ \mathcal{M}_{i,i+1} &= \begin{cases} \frac{\varepsilon}{\mathfrak{J}_{2,i}^2}, & i = 1, 2, \dots, \frac{N}{2}, \\ \frac{\varepsilon}{\mathfrak{J}_{2,i}^2} + \frac{\mu}{h} a_i^{j+1}, & i = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N-1. \end{cases} \end{aligned}$$

The elements of column vectors $\mathcal{F}$ and $\mathcal{T}$ are given as:

$$\begin{cases} \mathcal{T}_0^{j+1} = \vartheta(t_{j+1}), \\ \mathcal{T}_i^{j+1} = \varpi_i^{j+1} - \frac{c_i^{j+1}}{k} W_i^j, & i = 1(1)N - 1, \\ \mathcal{T}_N^{j+1} = \xi(t_{j+1}), \\ \mathcal{F} = [\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{N-1}, \mathcal{F}_N]^T, \end{cases} \quad (7.24)$$

Before discussing the convergence results of the proposed scheme, it is essential to first describe several key properties satisfied by $\mathbb{L}_{\varepsilon,\mu}^{N,M} W_i^{j+1}$, as defined in Eq. (7.19).

Lemma 7.2. *Let $\tilde{\Phi}_i^{j+1}$ be any mesh function that satisfies $\tilde{\Phi}_0^{j+1} \geq 0, \tilde{\Phi}_{x_N}^{j+1} \geq 0$, and $D_x^+ \tilde{\Phi}_{x_{\frac{N}{2}}}^{j+1} - D_x^- \tilde{\Phi}_{x_{\frac{N}{2}}}^{j+1} \leq 0$. Then, $\mathbb{L}_{\varepsilon,\mu}^{N,M} \tilde{\Phi}_i^{j+1} \leq 0$ for all $i \in \{1(1)N\} \setminus \{\frac{N}{2}\}$ implies that*

$$\tilde{\Phi}_i^{j+1} \geq 0, \forall x_i \in \Omega^N.$$

Proof. Let χ_i and τ_1 be indices such that $\tilde{\Phi}^{\tau_1}(\chi_i) = \min_{\forall(i,j)} \tilde{\Phi}_i^{j+1} < 0$ for $\tilde{\Phi}_i^{j+1} \in (\Omega^- \cup \Omega^+)$. Evidently, we have $(\chi_i, \tau_1) \neq \{(0, \tau_1), (1, \tau_1)\}$ because $\tilde{\Phi}(x, t) \geq 0$ for $x = 0, 1$. From this, we have $\tilde{\Phi}^{\tau_1+1}(\chi_{i+1}) - \tilde{\Phi}^{\tau_1+1}(\chi_i) \geq 0$ and $\tilde{\Phi}^{\tau_1+1}(\chi_i) - \tilde{\Phi}^{\tau_1+1}(\chi_{i-1}) \leq 0$. Next, three distinct scenarios are considered:

(a) For $(\chi_i, \tau_1) \in \Omega^-$,

$$\begin{aligned} \mathbb{L}_{\varepsilon, \mu}^{N, M} \tilde{\Phi}^{\tau_1}(\chi_i) &= \frac{\varepsilon}{\mathfrak{J}_i^2} \left(\tilde{\Phi}^{\tau_1+1}(\chi_{i+1}) - 2\tilde{\Phi}^{\tau_1+1}(\chi_i) + \tilde{\Phi}^{\tau_1+1}(\chi_{i-1}) \right) + \\ &\quad \mu a_1^{\tau_1+1}(\chi_i) \left(\frac{\tilde{\Phi}^{\tau_1+1}(\chi_i) - \tilde{\Phi}^{\tau_1+1}(\chi_{i-1})}{h} \right) - \zeta^{\tau_1+1}(\chi_i) \tilde{\Phi}^{\tau_1+1}(\chi_i), \\ &= \frac{\varepsilon}{\mathfrak{J}_i^2} \left(\left(\tilde{\Phi}^{\tau_1+1}(\chi_{i+1}) - \tilde{\Phi}^{\tau_1+1}(\chi_i) \right) - \left(\tilde{\Phi}^{\tau_1+1}(\chi_i) - \tilde{\Phi}^{\tau_1+1}(\chi_{i-1}) \right) \right) + \\ &\quad \frac{\mu a_1^{\tau_1+1}(\chi_i)}{h} \left(\tilde{\Phi}^{\tau_1+1}(\chi_i) - \tilde{\Phi}^{\tau_1+1}(\chi_{i-1}) \right) - \zeta^{\tau_1+1}(\chi_i) \tilde{\Phi}^{\tau_1+1}(\chi_i), \\ &> 0. \quad \text{since } a_1^{j+1}(x_i) < 0. \end{aligned}$$

(b) For $(\chi_i, \tau_1) \in \Omega^+$,

$$\begin{aligned} \mathbb{L}_{\varepsilon, \mu}^{N, M} \tilde{\Phi}^{\tau_1}(\chi_i) &= \frac{\varepsilon}{\mathfrak{J}_i^2} \left(\tilde{\Phi}^{\tau_1+1}(\chi_{i+1}) - 2\tilde{\Phi}^{\tau_1+1}(\chi_i) + \tilde{\Phi}^{\tau_1+1}(\chi_{i-1}) \right) + \\ &\quad \mu a_2^{\tau_1+1}(\chi_i) \left(\frac{\tilde{\Phi}^{\tau_1+1}(\chi_{i+1}) - \tilde{\Phi}^{\tau_1+1}(\chi_i)}{h} \right) - \zeta^{\tau_1+1}(\chi_i) \tilde{\Phi}^{\tau_1+1}(\chi_i), \\ &= \frac{\varepsilon}{\mathfrak{J}_i^2} \left(\left(\tilde{\Phi}^{\tau_1+1}(\chi_{i+1}) - \tilde{\Phi}^{\tau_1+1}(\chi_i) \right) - \left(\tilde{\Phi}^{\tau_1+1}(\chi_i) - \tilde{\Phi}^{\tau_1+1}(\chi_{i-1}) \right) \right) + \\ &\quad \frac{\mu a_2^{\tau_1+1}(\chi_i)}{h} \left(\tilde{\Phi}^{\tau_1+1}(\chi_{i+1}) - \tilde{\Phi}^{\tau_1+1}(\chi_i) \right) - \zeta^{\tau_1+1}(\chi_i) \tilde{\Phi}^{\tau_1+1}(\chi_i), \\ &> 0. \quad \text{since } a_2^{j+1}(x_i) > 0. \end{aligned}$$

(c) From Lemma 6.2, when $(\chi_i, \tau_1) = (\varphi, \tau_1)$, $D_x^+ \tilde{\Phi}_{x_{\frac{N}{2}}}^{\tau_1+1} > 0$ and $D_x^- \tilde{\Phi}_{x_{\frac{N}{2}}}^{\tau_1+1} < 0$, so

$$D_x^+ \tilde{\Phi}_{x_{\frac{N}{2}}}^{\tau_1+1} - D_x^- \tilde{\Phi}_{x_{\frac{N}{2}}}^{\tau_1+1} > 0.$$

Therefore, the assumption $\tilde{\Phi}^{\tau_1+1}(\chi_i) < 0, \forall(\chi_i, \tau_1)$ is incorrect in all cases because we have a contradiction. Consequently, $\tilde{\Phi}^{\tau_1+1}(\chi_i) > 0$ suggests that $\tilde{\Phi}_i^{j+1} \geq 0$. $\square$

The following lemma, linked to Lemma 7.2, provides a uniform stability estimate.

Lemma 7.3. *At time level t_j , let $\mathfrak{S}_{i,j+1}$ be the numerical solution of Eq. (7.22). Then*

$$|\mathfrak{S}_{i,j+1}| \leq C \max_{1 \leq i \leq N-1} \{|\mathfrak{S}_{0,j+1}|, |\mathfrak{S}_{N,j+1}|, \|\mathbb{L}_{\varepsilon,\mu}^{N,M} \mathfrak{S}_{i,j+1}\|\}. \quad (7.25)$$

Proof. We use Lemma 7.2 and barrier functions to prove this lemma. Let

$$\mathbb{k}_{i,j+1}^{\pm} = \max_{1 \leq i \leq N-1} \{|\mathfrak{S}_{0,j+1}|, |\mathfrak{S}_{N,j+1}|, \|\mathbb{L}_{\varepsilon,\mu}^{N,M} \mathfrak{S}_{i,j+1}\|\} \pm \mathfrak{S}_{i,j+1}.$$

Clearly, $\mathbb{k}_{i,j+1}^{\pm} \geq 0$ for $i = 0, N$ and $\mathbb{L}_{\varepsilon,\mu}^{N,M} \mathbb{k}_{i,j+1}^{\pm} \leq 0$ for $i \in \{1(1)N-1\} \setminus \{\frac{N}{2}\}$. Moreover, $(D_x^+ - D_x^-) \mathbb{k}_{i,j+1}^{\pm} = 0$ for $i = \frac{N}{2}$. From Lemma 7.2, we have $\mathbb{k}_{i,j+1}^{\pm} \geq 0$. $\square$

7.4 Convergence Analysis

This section presents the stability and convergence analysis of the scheme in Eq. (7.22). The following lemma provides a key tool for establishing its uniform convergence.

Lemma 7.4. *Let the interval $[0, 1]$ be partitioned uniformly into N subintervals of equal width $h = 1/N$, and let $x_i = ih$ for $i = 1, 2, \dots, N-1$. Then, for any fixed mesh and any positive integer κ , the following limits hold:*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \max_{1 < i \leq N-1} \varepsilon^{-\frac{\kappa}{2}} \exp\left(-\frac{Cx_i}{\sqrt{\varepsilon}}\right) &= 0, \quad \text{for } i = 1(1)\frac{N}{2}, \\ \lim_{\varepsilon \rightarrow 0} \max_{1 < i \leq N-1} \varepsilon^{-\frac{\kappa}{2}} \exp\left(-\frac{C(1-x_i)}{\sqrt{\varepsilon}}\right) &= 0, \quad \text{for } i = \frac{N}{2}(1)N-1, \end{aligned} \quad (7.26)$$

where $C > 0$ is a constant independent of ε and N .

Proof. Consider the expression

$$\Psi_i(\varepsilon) = \varepsilon^{-\frac{\kappa}{2}} \exp\left(-\frac{Cx_i}{\sqrt{\varepsilon}}\right), \quad \text{for } i = 1, 2, \dots, N-1.$$

We examine the asymptotic behavior of $\Psi_i(\varepsilon)$ as $\varepsilon \rightarrow 0$. By taking the natural logarithm of $\Psi_i(\varepsilon)$ gives

$$\ln \Psi_i(\varepsilon) = -\frac{\kappa}{2} \ln \varepsilon - \frac{Cx_i}{\sqrt{\varepsilon}}.$$

As $\varepsilon \rightarrow 0^+$, $-\frac{\kappa}{2} \ln \varepsilon \rightarrow +\infty$ and $-\frac{Cx_i}{\sqrt{\varepsilon}} \rightarrow -\infty$. However, since the function $1/\sqrt{\varepsilon}$ grows asymptotically faster than $|\ln \varepsilon|$ as $\varepsilon \rightarrow 0$, the second term dominates.

Consequently, $\ln \Psi_i(\varepsilon) \rightarrow -\infty \Rightarrow \Psi_i(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. To check the uniformity with respect to the mesh points, since $x_i = ih \geq h > 0$ for all $i \geq 1$, we have $\exp\left(-\frac{Cx_i}{\sqrt{\varepsilon}}\right) \leq \exp\left(-\frac{Ch}{\sqrt{\varepsilon}}\right)$. Therefore, $\max_{1 \leq i \leq N-1} \Psi_i(\varepsilon) \leq \varepsilon^{-\frac{\kappa}{2}} \exp\left(-\frac{Ch}{\sqrt{\varepsilon}}\right)$. As $\varepsilon \rightarrow 0$, the right-hand side tends to zero due to the exponential decay dominating any algebraic growth in $\varepsilon^{-\frac{\kappa}{2}}$. Hence,

$$\lim_{\varepsilon \rightarrow 0} \max_{1 \leq i \leq N-1} \varepsilon^{-\frac{\kappa}{2}} \exp\left(-\frac{Cx_i}{\sqrt{\varepsilon}}\right) = 0.$$

For the second limit, again consider

$$\psi_i(\varepsilon) = \varepsilon^{-\frac{\kappa}{2}} \exp\left(-\frac{C(1-x_i)}{\sqrt{\varepsilon}}\right), \quad \text{for } i = 1, 2, \dots, N-1.$$

Let $\xi_i = 1 - x_i$. Since $\xi_i \in (0, 1)$ and $\xi_i \geq h$, it follows that

$$\psi_i(\varepsilon) \leq \varepsilon^{-\frac{\kappa}{2}} \exp\left(-\frac{Ch}{\sqrt{\varepsilon}}\right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Hence,

$$\lim_{\varepsilon \rightarrow 0} \max_{1 \leq i \leq N-1} \varepsilon^{-\frac{\kappa}{2}} \exp\left(-\frac{C(1-x_i)}{\sqrt{\varepsilon}}\right) = 0.$$

Both limits vanish as $\varepsilon \rightarrow 0$. Therefore, for any fixed mesh and $\kappa \in \mathbb{Z}^+$, the exponential decay terms involving either x_i or $1 - x_i$ dominate the growth in $\varepsilon^{-\frac{\kappa}{2}}$. $\square$

The analysis below concentrates on the spatial variable x , highlighting how spatial discretization governs the solution's behavior across the domain and affects both accuracy and layer resolution.

Theorem 7.1 (Error estimation in spatial discretization). *Let $w^{j+1}(x_i)$ and W_i^{j+1} are solution of Eqs. (7.1) and (7.19), respectively. The local truncation error of spatial variable for proposed approach is given by:*

$$|\mathbb{L}_{\varepsilon, \mu}^{N, M}(W^{j+1}(x_i) - W_i^{j+1})| \leq CN^{-1}, \quad (7.27)$$

where N and C are mesh number and sufficiently large constant.

Proof. In the spatial discretization on $\Omega^- = [0, \varphi] \times [0, T]$, the local truncation error

can be expressed as follows:

$$\begin{aligned}
\mathbb{L}_{\varepsilon,\mu}^{N,M} (W^{j+1}(x_i) - W_i^{j+1}) &= \mathbb{L}_{\varepsilon,\mu}^M W^{j+1}(x_i) - \mathbb{L}_{\varepsilon,\mu}^{N,M} W_i^{j+1}, \\
&= \varepsilon \frac{d^2 W^{j+1}(x_i)}{dx^2} + \mu a_1^{j+1}(x_i) \frac{dW^{j+1}(x_i)}{dx} - \zeta^{j+1}(x_i) W^{j+1}(x_i) - \\
&\quad \left(\varepsilon \left(\frac{W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}}{\mathfrak{J}_i^2} \right) + \mu a_{1,i}^{j+1} \left(\frac{W_i^{j+1} - W_{i-1}^{j+1}}{h} \right) - \zeta_i^{j+1} \right), \quad (7.28) \\
&= \varepsilon \frac{d^2 W^{j+1}(x_i)}{dx^2} + \mu a_1^{j+1}(x_i) \frac{dW^{j+1}(x_i)}{dx} - \\
&\quad \frac{\varepsilon}{\mathfrak{J}_i^2} (W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1}) - \frac{\mu a_1^{j+1}(x_i)}{h} (W_i^{j+1} - W_{i-1}^{j+1}).
\end{aligned}$$

The terms W_{i-1}^{j+1} and W_{i+1}^{j+1} have the following Taylor series expansions:

$$\begin{aligned}
W_{i-1}^{j+1} &= W_i^{j+1} - \frac{h}{1!} \frac{dW^{j+1}(x_i)}{dx} + \frac{h^2}{2!} \frac{d^2 W^{j+1}(x_i)}{dx^2} - \frac{h^3}{3!} \frac{d^3 W^{j+1}(x_i)}{dx^3} + \frac{h^4}{4!} \frac{d^4 W^{j+1}(\varsigma_2)}{dx^4}, \\
W_{i+1}^{j+1} &= W_i^{j+1} + \frac{h}{1!} \frac{dW^{j+1}(x_i)}{dx} + \frac{h^2}{2!} \frac{d^2 W^{j+1}(x_i)}{dx^2} + \frac{h^3}{3!} \frac{d^3 W^{j+1}(x_i)}{dx^3} + \frac{h^4}{4!} \frac{d^4 W^{j+1}(\varsigma_2)}{dx^4}, \quad (7.29)
\end{aligned}$$

where $\varsigma_1 \in (x_{i-1}, x_i)$ and $\varsigma_2 \in (x_i, x_{i+1})$. By adding the first two equations in Eq. (7.29), we obtain

$$W_{i-1}^{j+1} - 2W_i^{j+1} + W_{i+1}^{j+1} = h^2 \frac{d^2 W^{j+1}(x_i)}{dx^2} + \frac{h^4}{12} \frac{d^4 W^{j+1}(x_i)}{dx^4} + \dots \quad (7.30)$$

Using the first-order Taylor series expansion of Eq. (7.29), we obtain

$$W_i^{j+1} - W_{i-1}^{j+1} = h \frac{dW^{j+1}(x_i)}{dx} - \frac{h^2}{2} \frac{d^2 W^{j+1}(x_i)}{dx^2} + \frac{h^3}{6} \frac{d^3 W^{j+1}(x_i)}{dx^3} + \dots \quad (7.31)$$

Substituting the expressions from Eqs. (7.30) and (7.31) into their respective positions in Eq. (7.28), and simplifying the resulting terms, yields the relation:

$$\begin{aligned}
\mathbb{L}_{\varepsilon,\mu}^{N,M} (W^{j+1}(x_i) - W_i^{j+1}) &= \varepsilon \frac{d^2 W^{j+1}(x_i)}{dx^2} - \frac{\varepsilon}{\mathfrak{J}_i^2} \left[h^2 \frac{d^2 W^{j+1}(x_i)}{dx^2} + \frac{h^4}{12} \frac{d^4 W^{j+1}(x_i)}{dx^4} \right] \\
&\quad + \frac{h}{2} \mu a_1^{j+1}(x_i) \frac{d^2 W^{j+1}(x_i)}{dx^2} - \frac{h^2}{6} \mu a_1^{j+1}(x_i) \frac{d^3 W^{j+1}(x_i)}{dx^3}. \quad (7.32)
\end{aligned}$$

Utilizing a truncated Taylor series expansion for $\mathfrak{J}_i^2, i = 1(1)N/2$, we obtain:

$$\begin{aligned}
\mathfrak{J}_i^2 &= \frac{h\varepsilon}{\mu a_{1,i}^{j+1}} \left(\exp \left(\frac{\mu a_{1,i}^{j+1} h}{\varepsilon} \right) - 1 \right), i = 1(1)N/2, \\
&= \frac{h\varepsilon}{\mu a_{1,i}^{j+1}} \left(1 + \frac{h\mu}{\varepsilon} a_{1,i}^{j+1} + \frac{h^2\mu^2}{2\varepsilon^2} (a_{1,i}^{j+1})^2 + \frac{h^3\mu^3}{6\varepsilon^3} (a_{1,i}^{j+1})^3 + \frac{h^4\mu^4}{24\varepsilon^4} (a_{1,i}^{j+1})^4 + \dots - 1 \right), \\
&= \frac{h\varepsilon}{\mu a_{1,i}^{j+1}} \left(\frac{h\mu}{\varepsilon} a_{1,i}^{j+1} + \frac{h^2\mu^2}{2\varepsilon^2} (a_{1,i}^{j+1})^2 + \frac{h^3\mu^3}{6\varepsilon^3} (a_{1,i}^{j+1})^3 + \frac{h^4\mu^4}{24\varepsilon^4} (a_{1,i}^{j+1})^4 + \dots \right), \\
&= h^2 + \frac{h^3\mu}{2\varepsilon} a_{1,i}^{j+1} + \frac{h^4\mu^2}{6\varepsilon^2} (a_{1,i}^{j+1})^2 + \frac{h^5\mu^3}{24\varepsilon^3} (a_{1,i}^{j+1})^3 + \dots, \\
\Rightarrow \frac{1}{\mathfrak{J}_i^2} &= \frac{1}{h^2 \left(1 + \frac{h\mu}{2\varepsilon} a_{1,i}^{j+1} + \frac{h^2\mu^2}{6\varepsilon^2} (a_{1,i}^{j+1})^2 + \frac{h^3\mu^3}{24\varepsilon^3} (a_{1,i}^{j+1})^3 + \dots \right)}.
\end{aligned}$$

By making a simplifications and adjusting the values $\frac{1}{\mathfrak{J}_i^2}$ in Eq. (7.32), we now obtain:

$$\begin{aligned}
\mathbb{L}_{\varepsilon,\mu}^{N,M} (W^{j+1}(x_i) - W_i^{j+1}) &= \\
&h \left(\mu a_1^{j+1}(x_i) \frac{d^2 W^{j+1}(x_i)}{dx^2} \right) + h^2 \left(\frac{5\mu^2}{12\varepsilon} (a_1^{j+1}(x_i))^2 \frac{d^2 W^{j+1}(x_i)}{dx^2} \right) - \\
&h^2 \left(\frac{\mu}{6} a_1^{j+1}(x_i) \frac{d^3 W^{j+1}(x_i)}{dx^3} \right) - h^2 \left(\varepsilon \frac{d^4 W^{j+1}(x_i)}{dx^4} \right) + O(h^3).
\end{aligned} \tag{7.33}$$

By combining the result of Lemma 7.26 with the derivative bounds, and noting that for sufficiently small mesh sizes h the relation $h > h^2 > h^3$ holds, the discretization error can be sharpened. Using these observations, we obtain:

$$|\mathbb{L}_{\varepsilon,\mu}^{N,M} (W^{j+1}(x_i) - W_i^{j+1})| \leq Ch = CN^{-1}, \quad \text{for } i = 1(1)\frac{N}{2}, \tag{7.34}$$

where C remains constant regardless of mesh sizes and perturbation parameters. We arrive at the following outcome using Lemma 7.26's uniform stability.

$$|W^{j+1}(x_i) - W_i^{j+1}| \leq Ch = CN^{-1}, \quad \text{for } i = 1(1)\frac{N}{2}, \tag{7.35}$$

□

Remark 7.1. On $\mathfrak{Q}^+ = (\varphi, 1] \times [0, T]$, a comparable convergence analysis can be carried out.

The convergence of the fully discretized scheme follows directly from combining

Lemma 7.1 with Theorem 7.1, which together provide the foundation for its stability and accuracy. The complete convergence proof is given in the theorem below.

Theorem 7.2. *Let W_i^{j+1} denote the solution to the discrete problem in Eq. (7.19) with its corresponding discrete boundary conditions in Eq. (7.20), and $W^{j+1}(x_i)$ the solution to the problem in Eq. (6.1). The total error bound then satisfies:*

$$\sup_{0 < \epsilon, \mu < 1} \left[\max_{0 \leq i \leq N, 0 \leq j \leq M} |W_i^{j+1} - W^{j+1}(x_i)| \right] \leq C(h + k) = C(N^{-1} + k). \quad (7.36)$$

Proof. Let W_i^{j+1} be the numerical solution of the discrete problem (7.19)–(7.20), and let $W^{j+1}(x_i)$ denote the corresponding exact solution of the semi-discrete problem (6.1) evaluated at the mesh points. The total error can be decomposed as

$$|W_i^{j+1} - W^{j+1}(x_i)| \leq |W_i^{j+1} - \bar{W}_i^{j+1}| + |\bar{W}_i^{j+1} - W^{j+1}(x_i)|,$$

where $\bar{W}_i^{j+1}$ represents the exact solution of the continuous problem projected onto the discrete mesh. By Lemma 7.1, the implicit Euler time-stepping introduces a truncation error of order $\mathcal{O}(k)$. Hence,

$$\max_{0 \leq i \leq N, 0 \leq j \leq M} |\bar{W}_i^{j+1} - W^{j+1}(x_i)| \leq C k.$$

Using Theorem 7.1, we have

$$\max_{0 \leq i \leq N, 0 \leq j \leq M} |W_i^{j+1} - \bar{W}_i^{j+1}| \leq C h \leq C N^{-1}.$$

Combining the above two inequalities yields

$$\max_{0 \leq i \leq N, 0 \leq j \leq M} |W_i^{j+1} - W^{j+1}(x_i)| \leq C(h + k).$$

Since the constants are independent of ϵ and μ , we obtain

$$\sup_{0 < \epsilon, \mu < 1} \left[\max_{0 \leq i \leq N, 0 \leq j \leq M} |W_i^{j+1} - W^{j+1}(x_i)| \right] \leq C(h + k) = C(N^{-1} + k).$$

Therefore, the fully discrete formulation attains parameter-uniform convergence, delivering first-order accuracy in both temporal and spatial directions. $\square$

Richardson Extrapolation

In this section, Richardson extrapolation is utilized to boost both the accuracy and convergence of the computed solution, applied post-processing rather than modifying the original numerical scheme. This approach not only refines the final results but also offers a practical mechanism to assess the scheme's robustness across varying perturbation parameters. Furthermore, Richardson extrapolation provides insights into the error structure, helping to identify dominant error terms and guide mesh refinement strategies. By applying Eq. (7.36), we obtain:

$$|W_i^{j+1} - W(x_i, t_{j+1})| \leq C(N^{-1} + k), \quad (7.37)$$

where W_i^{j+1} and $W(x_i, t_{j+1})$ are approximate and exact solutions, respectively. Let $\Pi_{x,t}^{N,M} \subset \Pi_{x,t}^{2N,2M}$ where $\Pi_{x,t}^{N,M}$ is the mesh obtained from h, k and $\Pi_{x,t}^{2N,2M}$ is the mesh obtained by bisecting h and k . Let the numerical solution obtained from $\Pi_{x,t}^{2N,2M}$ is denoted by $\ddot{W}_i^{j+1}$. For the mesh $(x_i, t_{j+1}) \in \Pi_{x,t}^{N,M}$ and from Eq. (7.36), we get:

$$W_i^{j+1} - W(x_i, t_{j+1}) \leq C(N^{-1} + k) + \mathbf{R}^{N,M}. \quad (7.38)$$

For the mesh $(\tilde{x}_i, \tilde{t}_{j+1}) \in \Pi_{x,t}^{2N,2M}$, where $\tilde{x}_{i+1} - \tilde{x}_i = \tilde{h} = \frac{h}{2}$, $\tilde{x}_i \in \Pi_x^{2N}$ and $\tilde{t}_{j+1} - \tilde{t}_j = \tilde{k} = \frac{k}{2}$, $\tilde{t}_{i+1} \in \Pi_t^{2M}$, we have:

$$\ddot{W}_i^{j+1} - W(x_i, t_{j+1}) \leq C \left(\frac{h}{2} + \frac{k}{2} \right) + \mathbf{R}^{2N,2M}, \quad (7.39)$$

where $\mathbf{R}^{N,M}$ and $\mathbf{R}^{2N,2M}$ are remainder terms with truncation error of order two both in space and time directions. Combining Eq. (7.38) and Eq. (7.39) gives us

$$W(x_i, t_{j+1}) - \left(2\ddot{W}_i^{j+1} - W(x_i, t_{j+1}) \right) \leq C(h^2 + k^2) = C(N^{-2} + k^2),$$

it results in that

$$(W_i^{j+1})_{extrap} = 2\ddot{W}_i^{j+1} - W_i^{j+1}, \quad (7.40)$$

is also an extrapolated numerical solution. Richardson extrapolation is formulated in Eqs. (7.37)–(7.40) to improve accuracy without changing the numerical method. In

turn, the following lemma gives summary of error bound for extrapolated solution.

Lemma 7.5. *Let W_i^{j+1} denote the solution of Eq. (6.1), and $(W_i^{j+1})_{extrap}$ represent the extrapolated solution. The resulting error bound is then given by:*

$$\sup_{0 < \varepsilon \leq 1} \max_{0 \leq i \leq N, 0 \leq j \leq M} \left| (W_i^{j+1})_{extrap} - W(x_i, t_{j+1}) \right| \leq C (h^2 + k^2). \quad (7.41)$$

Thus, the first-order uniformly convergent method is upgraded to second-order accuracy through Richardson extrapolation.

7.5 Numerical Examples, Results and Discussion

We validate the proposed approach through several numerical experiments. Since the exact solutions are unknown, the maximum pointwise error for each ε, μ is estimated using the double-mesh technique (Farrell et al., 2000; Chandru et al., 2017) as

$$\begin{aligned} \mathbf{E}_{\varepsilon, \mu}^{N, M} &= \max_{0 \leq i, j \leq N, M} \left| W_{i, j}^{N, M} - W_{i, j}^{2N, 2M} \right|, \\ (\mathbf{E}_{\varepsilon, \mu}^{N, M})_{extrap} &= \max_{0 \leq i, j \leq N, M} \left| (W_{i, j}^{N, M})_{extrap} - (W_{i, j}^{2N, 2M})_{extrap} \right|, \end{aligned}$$

prior to and subsequent to extrapolation, respectively. The numerical convergence rate $\overline{R}_{\varepsilon, \mu}^{N, M}$ before and after extrapolation is defined as:

$$\overline{R}_{\varepsilon, \mu}^{N, M} = \frac{\log(\mathbf{E}_{\varepsilon, \mu}^{N, M}) - \log(\mathbf{E}_{\varepsilon, \mu}^{2N, 2M})}{\log(2)}, \quad (\overline{R}_{\varepsilon, \mu}^{N, M})_{extrap} = \frac{\log((\mathbf{E}_{\varepsilon, \mu}^{N, M})_{extrap} / (\mathbf{E}_{\varepsilon, \mu}^{2N, 2M})_{extrap})}{\log(2)}.$$

Example 7.1. (Singh et al., 2023). Consider the following TP-SPPPs:

$$\begin{cases} \varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a(x, t) \frac{\partial w}{\partial x} - (1 + e^x)w - \frac{\partial w}{\partial t} = \varpi(x, t), & (x, t) \in \mathfrak{Q}^- \cup \mathfrak{Q}^+, \\ w(0, t) = 0 \quad \text{on} \quad \mathfrak{Q}_l, \quad w(1, t) = 0 \quad \text{on} \quad \mathfrak{Q}_r, \\ w(x, 0) = 0 \quad \text{on} \quad \mathfrak{Q}_b, \end{cases} \quad (7.42)$$

$$\text{where } a(x, t) = \begin{cases} -(1 + x - x^2), & (x, t) \in \mathfrak{Q}^- \\ (1 + x - x^2), & (x, t) \in \mathfrak{Q}^- \end{cases}, \quad \varpi(x, t) = \begin{cases} -2t(1 + x^2), & (x, t) \in \mathfrak{Q}^- \\ 2t(1 + x^2), & (x, t) \in \mathfrak{Q}^- \end{cases}.$$

For Examples 7.2 and 7.3, only the coefficients, source functions, and initial conditions are provided to fully specify the parabolic problem described in Eq. (6.1).

Example 7.2. Consider Eq. (6.1) and its coefficients as Kumar & Kumari (2021).

$$a(x, t) = \begin{cases} -(1 + e^{-xt}), & 0 \leq x \leq 0.5, \\ (2 + x + t), & 0.5 < x \leq 1, \end{cases}, \varpi(x, t) = \begin{cases} (e^{t^2} - 1)(1 + xt), & 0 \leq x \leq 0.5, \\ -t^2(2 + x), & 0.5 < x \leq 1, \end{cases}$$

$$b(x, t) = 2 + xt, c(x, t) = 1, \quad w(0, t) = 0 = w(1, t) \text{ and } w(x, 0) = 0.$$

Example 7.3. Consider Eq. (6.1) and its coefficients as Chandru et al. (2019).

$$a(x, t) = \begin{cases} (1 + x - x^2 + t), & (x, t) \in \mathfrak{Q}^-, \\ -(1 + 3xt), & (x, t) \in \mathfrak{Q}^+, \end{cases}, \varpi(x, t) = \begin{cases} (e^t - 1)(1 + x), & (x, t) \in \mathfrak{Q}^-, \\ t(x - 2), & (x, t) \in \mathfrak{Q}^+, \end{cases}$$

$$b(x, t) = 1 + x + t, c(x, t) = 1 \quad w(0, t) = 0 = w(1, t) \text{ and } w(x, 0) = 0.$$

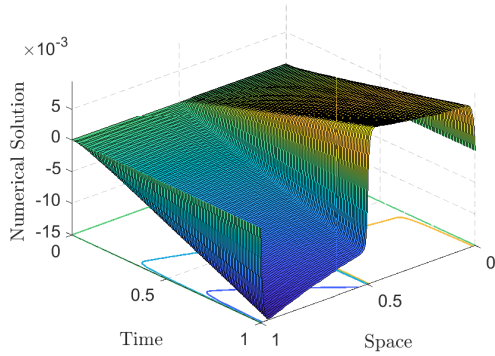
Table 7.1: Computed $E_{\varepsilon, \mu}^{N, M}$ and $\overline{R}_{\varepsilon, \mu}^{N, M}$ when $\mu = 2^{-8}$ for Example 7.1.

Number of spatial and temporal intervals for $N = M$						
ε	Extrapolation	$(N, M) = 16$	$(N, M) = 32$	$(N, M) = 64$	$(N, M) = 128$	$(N, M) = 256$
$\downarrow$	After/Before					
2^{-18}	After	$4.7645e - 03$	$1.5136e - 03$	$4.2892e - 04$	$1.1434e - 04$	$2.9528e - 05$
		1.6543	1.8192	1.9074	1.9532	—
		$5.2112e - 02$	$2.8469e - 02$	$1.4905e - 02$	$7.6291e - 03$	$3.8597e - 03$
2^{-20}	Before	0.87222	0.93360	0.96621	0.98302	—
		$4.7645e - 03$	$1.5136e - 03$	$4.2892e - 04$	$1.1434e - 04$	$2.9532e - 05$
		1.6543	1.8192	1.9074	1.9532	—
2^{-22}	Before	$5.2112e - 02$	$2.8469e - 02$	$1.4905e - 02$	$7.6291e - 03$	$3.8600e - 03$
		0.87222	0.93360	0.96621	0.98291	—
		$4.7645e - 03$	$1.5136e - 03$	$4.2892e - 04$	$1.1434e - 04$	$2.9532e - 05$
2^{-24}	After	1.6543	1.8192	1.9074	1.9530	—
		$5.2112e - 02$	$2.8469e - 02$	$1.4905e - 02$	$7.6291e - 03$	$3.8600e - 03$
		0.87222	0.93360	0.96621	0.98291	—
2^{-32}	Before	$4.7645e - 03$	$1.5136e - 03$	$4.2892e - 04$	$1.1434e - 04$	$2.9532e - 05$
		1.6543	1.8192	1.9074	1.9530	—
		$5.2112e - 02$	$2.8469e - 02$	$1.4905e - 02$	$7.6291e - 03$	$3.8600e - 03$
$\vdots$	$\vdots$	0.87222	0.93360	0.96621	0.98291	—
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$E_{\varepsilon, \mu}^{N, M}$	After	$4.7645e - 03$	$1.5136e - 03$	$4.2892e - 04$	$1.1434e - 04$	$2.9532e - 05$
		1.6543	1.8192	1.9074	1.9530	—
		$5.2112e - 02$	$2.8469e - 02$	$1.4905e - 02$	$7.6291e - 03$	$3.8600e - 03$
$\overline{R}_{\varepsilon, \mu}^{N, M}$	Before	0.87222	0.93360	0.96621	0.98291	—
		$E_{\varepsilon, \mu}^{N, M}$	$\overline{R}_{\varepsilon, \mu}^{N, M}$	$E_{\varepsilon, \mu}^{N, M}$	$\overline{R}_{\varepsilon, \mu}^{N, M}$	$E_{\varepsilon, \mu}^{N, M}$
		$\overline{R}_{\varepsilon, \mu}^{N, M}$	$\overline{R}_{\varepsilon, \mu}^{N, M}$	$\overline{R}_{\varepsilon, \mu}^{N, M}$	$\overline{R}_{\varepsilon, \mu}^{N, M}$	$\overline{R}_{\varepsilon, \mu}^{N, M}$

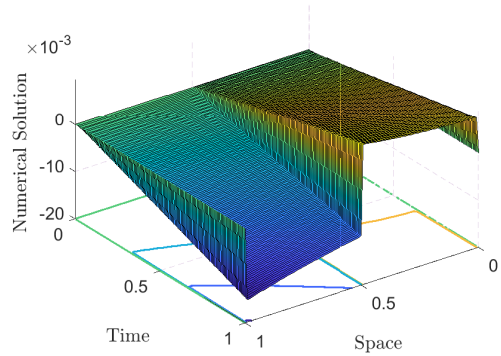
We present the numerical outcomes summarized in Tables 7.1-7.6, obtained for various combinations of the parameters ε, μ, N , and M . The results consistently

Table 7.2: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 7.1 for $\mu = 2^{-12}$ and $\varphi = 0.5$.

Number of mesh points for $N \neq M$						
ε	Extrapolation	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
$\downarrow$	After/Before	$M = 10$	$M = 20$	$M = 40$	$M = 80$	$M = 160$
2^{-24}	After	$9.4886e-03$ 1.4810	$3.3993e-03$ 1.7274	$1.0266e-03$ 1.8596	$2.8288e-04$ 1.9286	$7.4309e-05$ —
	Before	$7.7805e-02$ 0.82495	$4.3921e-02$ 0.90809	$2.3405e-02$ 0.95288	$1.2091e-02$ 0.97610	$6.1465e-03$ —
2^{-26}	After	$9.4886e-03$ 1.4810	$3.3993e-03$ 1.7274	$1.0266e-03$ 1.8596	$2.8288e-04$ 1.9286	$7.4309e-05$ —
	Before	$7.7805e-02$ 0.82495	$4.3921e-02$ 0.90809	$2.3405e-02$ 0.95288	$1.2091e-02$ 0.97610	$6.1465e-03$ —
$\vdots$		$\vdots$		$\vdots$		$\vdots$
2^{-32}	After	$9.4886e-03$ 1.4810	$3.3993e-03$ 1.7274	$1.0266e-03$ 1.8596	$2.8288e-04$ 1.9286	$7.4309e-05$ —
	Before	$7.7805e-02$ 0.82495	$4.3921e-02$ 0.90809	$2.3405e-02$ 0.95288	$1.2091e-02$ 0.97610	$6.1465e-03$ —
$E_{\varepsilon,\mu}^{N,M}$	After	$4.7645e-03$	$1.5136e-03$	$4.2892e-04$	$1.1434e-04$	$7.4309e-05$
$\bar{R}_{\varepsilon,\mu}^{N,M}$		1.4810	1.7274	1.8596	1.9286	—
$E_{\varepsilon,\mu}^{N,M}$	Before	$7.7805e-02$	$4.3921e-02$	$2.3405e-02$	$1.2091e-02$	$6.1465e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$		0.82495	0.90809	0.95288	0.97610	—
Results in Singh et al. (2023)						
$E_{\varepsilon,\mu}^{N,M}$	After	$2.7518e-02$	$9.2992e-03$	$2.7288e-03$	$7.4791e-04$	$2.0015e-04$
$\bar{R}_{\varepsilon,\mu}^{N,M}$		1.565	1.769	1.867	1.902	—
$E_{\varepsilon,\mu}^{N,M}$	Before	$1.0870e-01$	$6.7954e-02$	$3.8587e-02$	$2.0648e-02$	$1.0695e-02$
$\bar{R}_{\varepsilon,\mu}^{N,M}$		0.678	0.816	0.902	0.949	—



(a) $N = M = 128, \mu = 2^{-2}, \varepsilon = 2^{-6}$



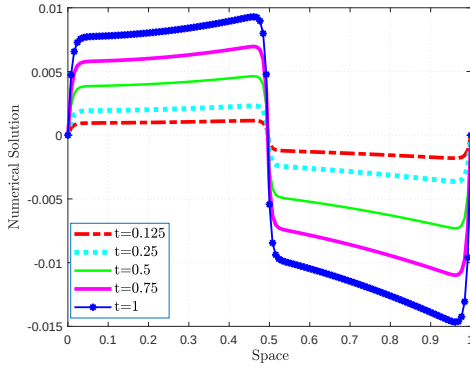
(b) $N = M = 128, \mu = 2^{-8}, \varepsilon = 2^{-12}$

Figure 7.1: Solution profiles for Example 7.1.

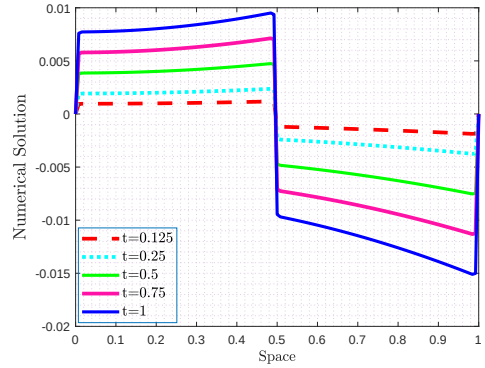
reveal that the maximum absolute error decreases as the spatial and temporal mesh sizes are refined, indicating the stability and accuracy of the proposed numerical scheme. Furthermore, it is observed that the absolute error remains nearly constant even as the perturbation parameters ε and μ approach zero, confirming the uniform nature of convergence. The corresponding numerical evidence demonstrates nearly

Table 7.3: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 7.2 for $\varepsilon = 10^{-8}$ and $\varphi = 0.5$.

Count of space and time intervals						
μ	Extrapolation	$N = M = 16$	$N = M = 32$	$N = M = 64$	$N = M = 128$	$N = M = 256$
$\downarrow$	After/Before					
10^{-4}	After	$1.1508e-03$	$2.8346e-04$	$6.9955e-05$	$1.7347e-05$	$4.3171e-06$
		2.0214	2.0186	2.0117	2.0066	—
	Before	$4.3684e-02$	$2.2639e-02$	$1.1520e-02$	$5.8097e-03$	$2.9173e-03$
		0.94829	0.97467	0.98761	0.99383	—
10^{-6}	After	$1.1498e-03$	$2.8319e-04$	$6.9884e-05$	$1.7329e-05$	$4.3128e-06$
		2.0215	2.0187	2.0118	2.0065	—
	Before	$4.3685e-02$	$2.2640e-02$	$1.1520e-02$	$5.8097e-03$	$2.9173e-03$
		0.94826	0.97473	0.98761	0.99383	—
10^{-8}	After	$1.1497e-03$	$2.8319e-04$	$6.9884e-05$	$1.7329e-05$	$4.3130e-06$
		2.0214	2.0187	2.0118	2.0064	—
	Before	$4.3685e-02$	$2.2640e-02$	$1.1520e-02$	$5.8097e-03$	$2.9173e-03$
		0.94826	0.97473	0.98761	0.99383	—
10^{-10}	After	$1.1497e-03$	$2.8319e-04$	$6.9884e-05$	$1.7329e-05$	$4.3130e-06$
		2.0214	2.0187	2.0118	2.0064	—
	Before	$4.3685e-02$	$2.2640e-02$	$1.1520e-02$	$5.8097e-03$	$2.9173e-03$
		0.94826	0.97473	0.98761	0.99383	—
10^{-12}	After	$1.1497e-03$	$2.8319e-04$	$6.9884e-05$	$1.7329e-05$	$4.3130e-06$
		2.0214	2.0187	2.0118	2.0064	—
	Before	$4.3685e-02$	$2.2640e-02$	$1.1520e-02$	$5.8097e-03$	$2.9173e-03$
		0.94826	0.97473	0.98761	0.99383	—
$E_{\varepsilon,\mu}^{N,M}$	After	$1.1508e-03$	$2.8346e-04$	$6.9955e-05$	$1.7347e-05$	$4.3171e-06$
$\bar{R}_{\varepsilon,\mu}^{N,M}$		2.0214	2.0186	2.0117	2.0066	—
$E_{\varepsilon,\mu}^{N,M}$	Before	$4.3685e-02$	$2.2640e-02$	$1.1520e-02$	$5.8097e-03$	$2.9173e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$		0.94826	0.97473	0.98761	0.99383	—



(a) $N = M = 128, \mu = 2^{-2}, \varepsilon = 2^{-6}$



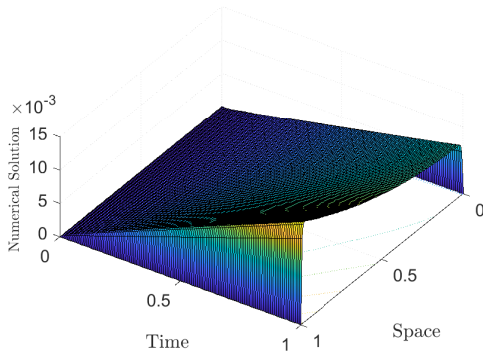
(b) $N = M = 128, \mu = 2^{-8}, \varepsilon = 2^{-12}$

Figure 7.2: Solution plots for Example 7.1.

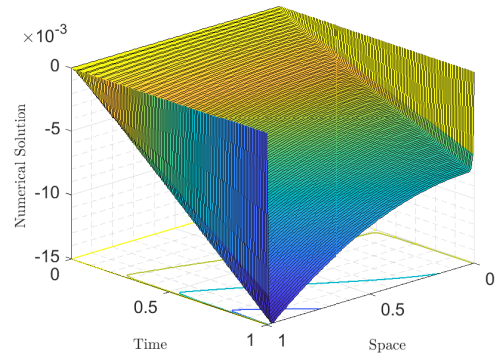
first-order uniform convergence in the spatial direction prior to extrapolation. To further enhance accuracy, Richardson's extrapolation technique is applied, and the extrapolated results, also shown in Table 7.1 confirm that the scheme attains almost second-order uniform convergence in both space and time. In Table 7.2, we fix μ and vary ε for $M \neq N$, again verifying first-order uniform convergence before extrapolation

Table 7.4: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ for Example 7.2 for $\mu = 2^{-12}$.

Space-time interval count						
ε	Extrapolation	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
$\downarrow$		$M = 10$	$M = 20$	$M = 40$	$M = 80$	$M = 160$
2^{-24}	After	$3.0814e-03$ 1.4810	$7.5084e-04$ 1.7274	$1.8297e-04$ 1.8596	$4.4969e-05$ 1.9286	$1.1133e-05$ —
	Before	$6.7565e-02$ 0.82495	$3.5656e-02$ 0.90809	$1.8294e-02$ 0.95288	$9.2617e-03$ 0.97610	$4.6593e-03$ —
2^{-26}	After	$3.0814e-03$ 1.4810	$7.5084e-04$ 1.7274	$1.8297e-04$ 1.8596	$4.4969e-05$ 1.9286	$1.1133e-05$ —
	Before	$6.7565e-02$ 0.82495	$3.5656e-02$ 0.90809	$1.8294e-02$ 0.95288	$9.2617e-03$ 0.97610	$4.6593e-03$ —
2^{-28}	After	$3.0814e-03$ 1.4810	$7.5084e-04$ 1.7274	$1.8297e-04$ 1.8596	$4.4969e-05$ 1.9286	$1.1133e-05$ —
	Before	$6.7565e-02$ 0.82495	$3.5656e-02$ 0.90809	$1.8294e-02$ 0.95288	$9.2617e-03$ 0.97610	$4.6593e-03$ —
2^{-30}	After	$3.0814e-03$ 1.4810	$7.5084e-04$ 1.7274	$1.8297e-04$ 1.8596	$4.4969e-05$ 1.9286	$1.1133e-05$ —
	Before	$6.7565e-02$ 0.82495	$3.5656e-02$ 0.90809	$1.8294e-02$ 0.95288	$9.2617e-03$ 0.97610	$4.6593e-03$ —
2^{-32}	After	$3.0814e-03$ 1.4810	$7.5084e-04$ 1.7274	$1.8297e-04$ 1.8596	$4.4969e-05$ 1.9286	$1.1133e-05$ —
	Before	$6.7565e-02$ 0.82495	$3.5656e-02$ 0.90809	$1.8294e-02$ 0.95288	$9.2617e-03$ 0.97610	$4.6593e-03$ —
$E_{\varepsilon,\mu}^{N,M}$ $\overline{R}_{\varepsilon,\mu}^{N,M}$	After	$3.0814e-03$ 1.4810	$7.5084e-04$ 1.7274	$1.8297e-04$ 1.8596	$4.4969e-05$ 1.9286	$7.4309e-05$ —
	Before	$6.7565e-02$ 0.82495	$3.5656e-02$ 0.90809	$1.8294e-02$ 0.95288	$9.2617e-03$ 0.97610	$6.1465e-03$ —
Results in Singh et al. (2023)						
$E_{\varepsilon,\mu}^{N,M}$ $\overline{R}_{\varepsilon,\mu}^{N,M}$	After	$2.7518e-02$ 1.565	$9.2992e-03$ 1.769	$2.7288e-03$ 1.867	$7.4791e-04$ 1.902	$2.0015e-04$ —
	Before	$1.0870e-01$ 0.678	$6.7954e-02$ 0.816	$3.8587e-02$ 0.902	$2.0648e-02$ 0.949	$1.0695e-02$ —



(a) $\varpi(x, t) = -2t(1 + x^2)$



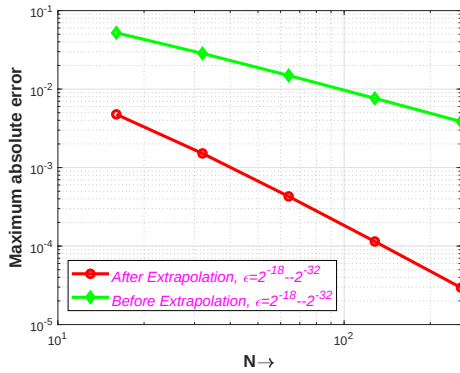
(b) $\varpi(x, t) = 2t(1 + x^2)$

Figure 7.3: Numerical solutions for Example 7.1 with different source terms.

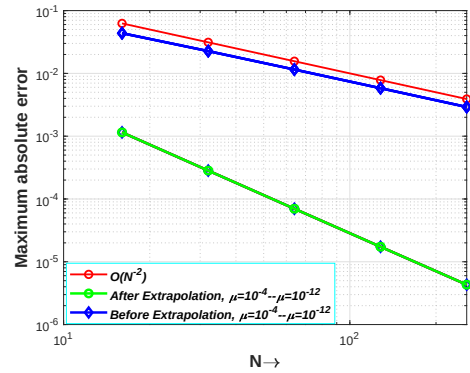
and second-order convergence after extrapolation. This behavior indicates that the method maintains its accuracy uniformly with respect to ε and μ . Similarly, for

Table 7.5: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 7.3 for $\varepsilon = 10^{-6}$ and $\varphi = 0.5$.

Quantity of divisions in spatial and temporal domains						
μ	Extrapolation	$(N, M) = 16$	$(N, M) = 32$	$(N, M) = 64$	$(N, M) = 128$	$(N, M) = 256$
$\downarrow$	After/Before					
10^{-2}	After	$2.1271e-03$	$6.1819e-04$	$1.6797e-04$	$4.3835e-05$	$1.1198e-05$
	Before	1.7828	1.8798	1.9380	1.9688	—
10^{-4}	After	$3.6778e-02$	$1.9250e-02$	$9.8452e-03$	$4.9781e-03$	$2.5030e-03$
	Before	0.93398	0.96737	0.98383	0.99194	—
10^{-6}	After	$2.1576e-03$	$6.2753e-04$	$1.6940e-04$	$4.4020e-05$	$1.1220e-05$
	Before	1.7817	1.8893	1.9442	1.9721	—
10^{-8}	After	$3.6854e-02$	$1.9270e-02$	$9.8503e-03$	$4.9794e-03$	$2.5032e-03$
	Before	0.93546	0.96812	0.98420	0.99220	—
10^{-10}	After	$2.1579e-03$	$6.2762e-04$	$1.6942e-04$	$4.4025e-05$	$1.1221e-05$
	Before	1.7817	1.8893	1.9442	1.9721	—
10^{-12}	After	$3.6854e-02$	$1.9270e-02$	$9.8502e-03$	$4.9793e-03$	$2.5032e-03$
	Before	0.93546	0.96813	0.98421	0.99217	—
$E_{\varepsilon,\mu}^{N,M}$	After	$2.1579e-03$	$6.2762e-04$	$1.6942e-04$	$4.4025e-05$	$1.1221e-05$
$\bar{R}_{\varepsilon,\mu}^{N,M}$	Before	1.7817	1.8893	1.9442	1.9721	—
$E_{\varepsilon,\mu}^{N,M}$	Before	$3.6854e-02$	$1.9270e-02$	$9.8502e-03$	$4.9793e-03$	$2.5032e-03$
$\bar{R}_{\varepsilon,\mu}^{N,M}$		0.93546	0.96813	0.98421	0.99217	—



(a) Example 7.1



(b) Example 7.2

Figure 7.4: Log-log plots for Example 7.1 and Example 7.2.

Example 7.2, the results reported in Tables 7.3 and 7.4 demonstrate consistent uniform convergence both before and after applying extrapolation. For Example 7.3, the computed values of $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ in Tables 7.5 and 7.6 for a range of ε, μ , and

Table 7.6: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\overline{R}_{\varepsilon,\mu}^{N,M}$ when $\mu = 2^{-10}$ for Example 7.3.

Equal space-time divisions for $N = M$						
$\varepsilon \downarrow$	16	32	64	128	256	512
2^0	$2.6618e-02$	$1.5899e-02$	$8.5886e-03$	$4.4925e-03$	$2.3171e-03$	$1.1846e-03$
	0.74347	0.88844	0.93490	0.95520	0.96792	—
2^{-2}	$3.2688e-02$	$1.7491e-02$	$9.1496e-03$	$4.7091e-03$	$2.4003e-03$	$1.2162e-03$
	0.90215	0.93483	0.95826	0.97224	9.8083	—
2^{-4}	$3.4666e-02$	$1.8331e-02$	$9.4749e-03$	$4.8324e-03$	$2.4471e-03$	$1.2339e-03$
	0.91924	0.95210	0.97137	0.98167	0.98785	—
2^{-6}	$3.5305e-02$	$1.8769e-02$	$9.6337e-03$	$4.8969e-03$	$2.4732e-03$	$1.2437e-03$
	0.91152	0.96219	0.97622	0.98549	0.99174	—
2^{-8}	$3.6175e-02$	$1.8870e-02$	$9.7350e-03$	$4.9273e-03$	$2.4864e-03$	$1.2487e-03$
	0.93890	0.95484	0.98238	0.98674	0.99363	—
2^{-10}	$3.6687e-02$	$1.9095e-02$	$9.7485e-03$	$4.9522e-03$	$2.4929e-03$	$1.2516e-03$
	0.94207	0.96994	0.97711	0.99024	0.99405	—
2^{-12}	$3.6815e-02$	$1.9230e-02$	$9.8070e-03$	$4.9538e-03$	$2.4968e-03$	$1.2530e-03$
	0.93693	0.97147	0.98528	0.98846	0.99469	—
2^{-14}	$3.6844e-02$	$1.9263e-02$	$9.8414e-03$	$4.9692e-03$	$2.4969e-03$	$1.2535e-03$
	0.93560	0.96890	0.98585	0.99288	0.99418	—
2^{-16}	$3.6847e-02$	$1.9268e-02$	$9.8491e-03$	$4.9778e-03$	$2.5011e-03$	$1.2535e-03$
	0.93534	0.96814	0.98448	0.99295	0.99660	—
2^{-18}	$3.6847e-02$	$1.9269e-02$	$9.8499e-03$	$4.9793e-03$	$2.5031e-03$	$1.2547e-03$
	0.93527	0.96810	0.98417	9.9223	0.99637	—
2^{-20}	$3.6847e-02$	$1.9269e-02$	$9.8499e-03$	$4.9793e-03$	$2.5033e-03$	$1.2551e-03$
	0.93527	0.96810	0.98417	0.99211	0.99603	—
$\vdots$			$\vdots$		$\vdots$	$\vdots$
2^{-28}	$3.6847e-02$	$1.9269e-02$	$9.8499e-03$	$4.9793e-03$	$2.5033e-03$	$1.2551e-03$
	0.93527	0.96810	0.98417	0.99211	0.99603	—
$E_{\varepsilon,\mu}^{N,M}$	$3.6847e-02$	$1.9269e-02$	$9.8499e-03$	$4.9793e-03$	$2.5033e-03$	$1.2551e-03$
$\overline{R}_{\varepsilon,\mu}^{N,M}$	0.93527	0.96810	0.98417	0.99211	0.99603	—
Result in Chandru et al. (2018)(Upwind scheme on piecewise uniform mesh)						
$E_{\varepsilon,\mu}^{N,M}$			$2.18430e-02$	$1.78687e-02$	$1.28289e-02$	$7.83669e-03$
$\overline{R}_{\varepsilon,\mu}^{N,M}$			0.289734	0.478036	0.711084	—

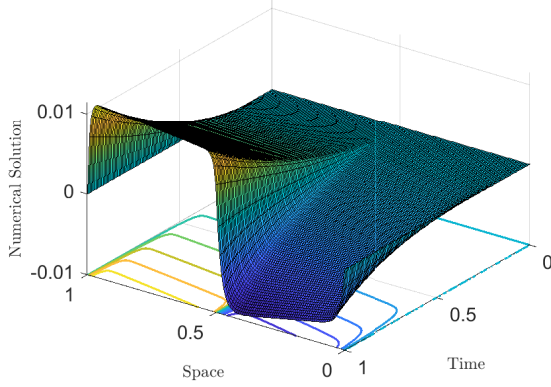
Table 7.7: Computed $E_{\varepsilon,\mu}^{N,M}$ for Example 7.1 by taking $N = M = 128$.

$\mu \downarrow \varepsilon \rightarrow$	2^{-6}	2^{-10}	2^{-14}	2^{-18}	2^{-22}	2^{-26}	2^{-30}
2^{-6}	$7.3047e-03$	$7.5320e-03$	$7.6106e-03$	$7.6284e-03$	$7.6284e-03$	$7.6284e-03$	$7.6284e-03$
2^{-10}	$7.3043e-03$	$7.5276e-03$	$7.5754e-03$	$7.6282e-03$	$7.6293e-03$	$7.6293e-03$	$7.6293e-03$
2^{-14}	$7.3043e-03$	$7.5273e-03$	$7.5721e-03$	$7.6260e-03$	$7.6293e-03$	$7.6293e-03$	$7.6293e-03$
2^{-18}	$7.3042e-03$	$7.5273e-03$	$7.5719e-03$	$7.6258e-03$	$7.6291e-03$	$7.6293e-03$	$7.6293e-03$
2^{-22}	$7.3042e-03$	$7.5273e-03$	$7.5719e-03$	$7.6257e-03$	$7.6291e-03$	$7.6293e-03$	$7.6293e-03$
2^{-26}	$7.3042e-03$	$7.5273e-03$	$7.5719e-03$	$7.6257e-03$	$7.6291e-03$	$7.6293e-03$	$7.6293e-03$
2^{-30}	$7.3042e-03$	$7.5273e-03$	$7.5719e-03$	$7.6257e-03$	$7.6291e-03$	$7.6293e-03$	$7.6293e-03$

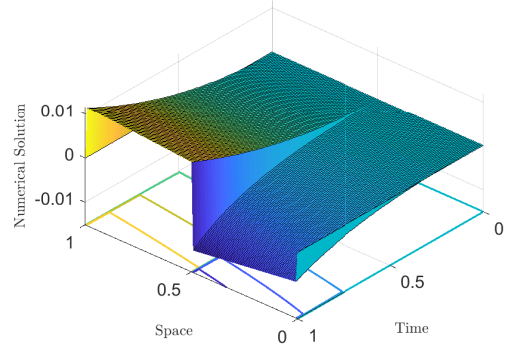
with $\varphi = 0.5$ further validate our theoretical error estimates and convergence results. Moreover, Tables 7.7 and 7.8 reveal that, for fixed mesh parameters N and M , the maximum pointwise errors tend to *stabilize as both ε and μ approach zero*, reinforcing the robustness of the proposed method for highly singularly perturbed regimes. A comparative study with existing results from Singh et al. (2023); Chandru et al.

Table 7.8: Computed $E_{\varepsilon,\mu}^{N,M}$ for Example 7.2 by taking $N = M = 128$.

$\varepsilon \downarrow \mu \rightarrow$	2^0	2^{-4}	2^{-8}	2^{-12}	2^{-16}	2^{-20}	2^{-24}
2^0	$5.6150e-03$	$5.6141e-03$	$5.6141e-03$	$5.6141e-03$	$5.6141e-03$	$5.6141e-03$	$5.6141e-03$
2^{-4}	$5.7592e-03$	$5.7552e-03$	$5.7549e-03$	$5.7548e-03$	$5.7548e-03$	$5.7548e-03$	$5.7548e-03$
2^{-8}	$5.7924e-03$	$5.7929e-03$	$5.7922e-03$	$5.7921e-03$	$5.7921e-03$	$5.7921e-03$	$5.7921e-03$
2^{-12}	$5.7925e-03$	$5.7984e-03$	$5.8002e-03$	$5.7985e-03$	$5.7983e-03$	$5.7983e-03$	$5.7983e-03$
2^{-16}	$5.7925e-03$	$5.7984e-03$	$5.8090e-03$	$5.8091e-03$	$5.8090e-03$	$5.8090e-03$	$5.8090e-03$
2^{-20}	$5.7925e-03$	$5.7984e-03$	$5.8090e-03$	$5.8096e-03$	$5.8096e-03$	$5.8096e-03$	$5.8096e-03$
2^{-24}	$5.7925e-03$	$5.7984e-03$	$5.8090e-03$	$5.8096e-03$	$5.8097e-03$	$5.8097e-03$	$5.8097e-03$

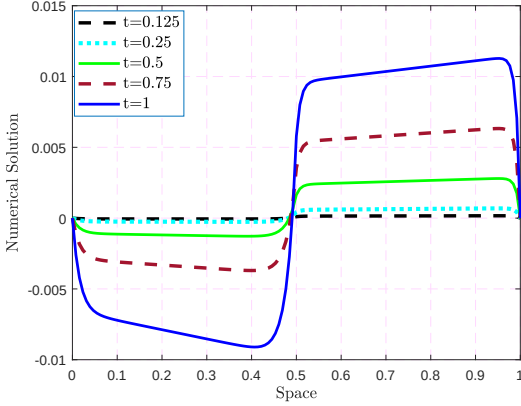


(a) $\mu = 2^{-8}, \varepsilon = 2^{-2}$

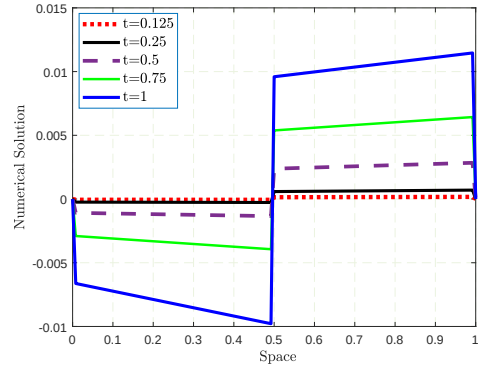


(b) $\mu = 2^{-12}, \varepsilon = 2^{-24}$

Figure 7.5: Numerical solutions for Example 7.2 with $N = M = 128$.



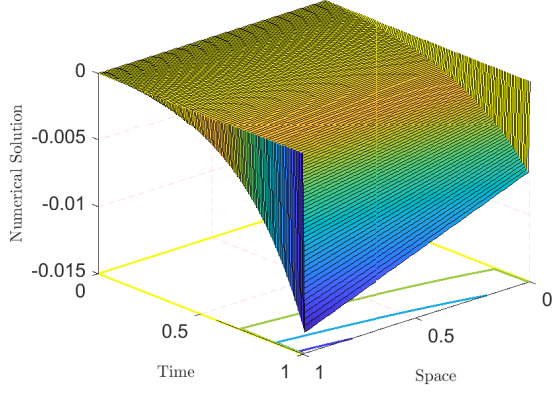
(a) $\mu = 2^{-8}, \varepsilon = 2^{-2}$



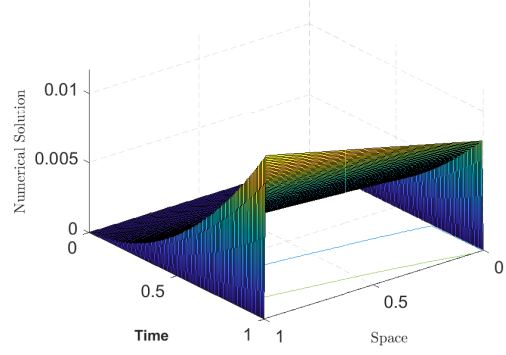
(b) $\mu = 2^{-12}, \varepsilon = 2^{-24}$

Figure 7.6: Line plot for Example 7.2 with $N = M = 128$.

(2018), shown in Tables 7.2, 7.4, and 7.6, indicates that our approach consistently provides higher accuracy. The solution profiles are shown as surface-contour plots in Figures 7.1, 7.5, and 7.9 for Examples 7.1, 7.2, and 7.3 respectively, illustrating the steepening of the solution gradient near points of discontinuity as ε or μ decreases. Due to the presence of ε, μ and discontinuities, these surface plots reveal boundary and interior layers in the numerical solution. We notice a decrement in the layer

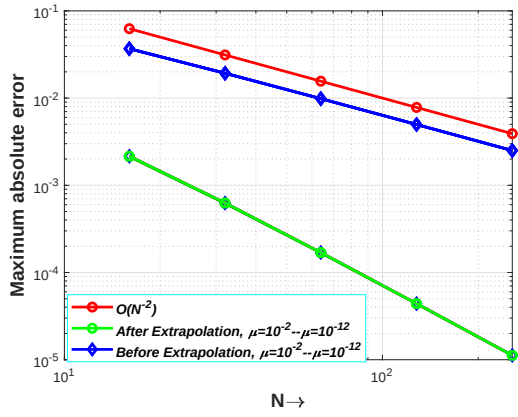


(a) $\varpi(x, t) = (e^{t^2} - 1)(1 + xt)$

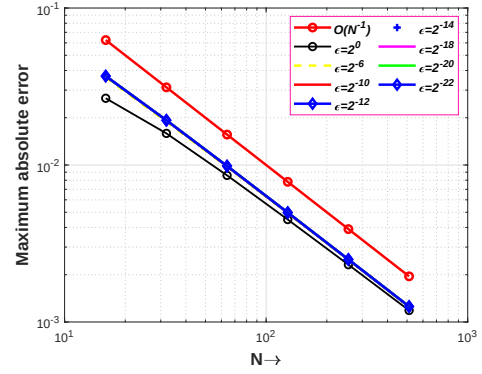


(b) $\varpi(x, t) = -t^2(x + 2)$

Figure 7.7: Numerical solution for Example 7.2 with different source terms.

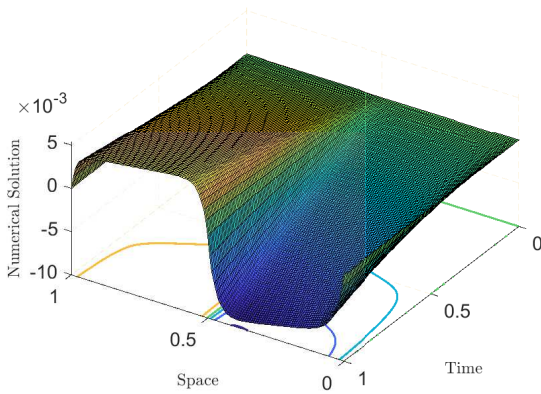


(a) Table 7.5

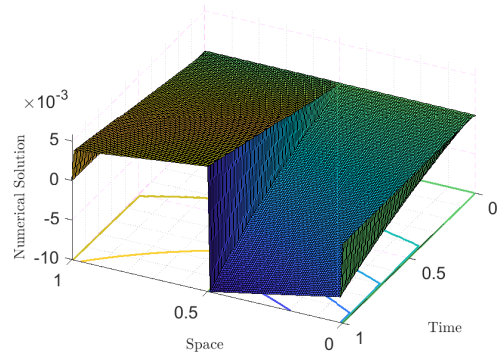


(b) Table 7.6

Figure 7.8: Log-log plot for Example 7.3.



(a) $\mu = 2^{-4}, \varepsilon = 2^0$



(b) $\mu = 2^{-14}, \varepsilon = 2^{-26}$

Figure 7.9: Solution profiles for Example 7.3 for $N = M = 128$.

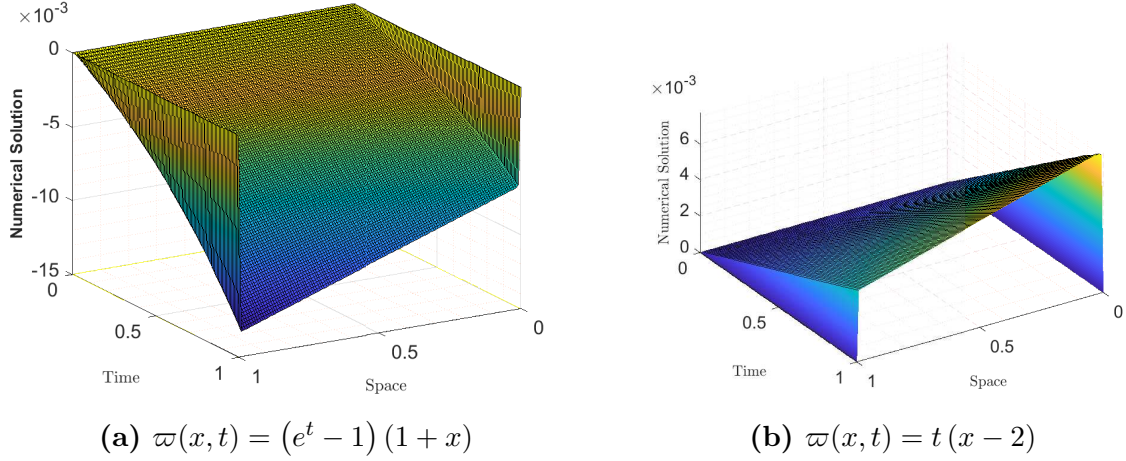


Figure 7.10: For Example 7.3, the numerical solution with $\varepsilon = 2^{-10}, \mu = 2^{-12}$.

width as we decrease μ or ε (see Figures 7.1a, 7.5a, and 7.9a). The line plots for Examples 7.1, 7.2, and 7.3 at various time steps while maintaining $N = 128 = M$ are represented in Figures 7.2 and 7.6, which show that the maximum error occurs near the boundary and interior layer regions. The observation of a sharper layer as we decrease the values of ε and μ is confirmed, which helps us understand the influence of parameter values and how they affect the formation of layers. The research conducted in O’Riordan & Shishkin (2004), it was found that the solution may contain an interior layer solely when the discontinuity is present in the convection-coefficient ($\varpi(x, t)$) is continuous. This numerical insight is crucial in understanding the behavior of solutions in the presence of discontinuities. It is evident from Figures 7.3 and 7.10 that the solution might only contain the boundary layers if the discontinuity is in the convection-coefficient alone ($\varpi(x, t)$ is smooth). To generate Figures 7.3a–7.3b, 7.7a–7.7b, and 7.10a–7.10b, the discontinuous source terms in Examples 7.1, 7.2, and 7.3, respectively, are replaced by smooth functions. For all modified cases, the surface plots in Figures 7.3, 7.7 and 7.10 exhibit only left and right boundary layers. This indicates that a discontinuous convection coefficient primarily causes layer shifting, while interior layers may not form when the source term is smooth. Furthermore, to illustrate the practical validity of the expected rate of convergence, we have presented the error plots on a Log-log scale in Figures 7.4 and 7.8, corresponding to Examples 7.1–7.3, respectively.

7.6 Conclusion

In this chapter, we developed a parameter-uniform scheme for the $1D$ time-dependent convection-diffusion problem with two small perturbation parameters. The equation, featuring both parabolic and interior layers, was discretized using the implicit Euler method in time and a NS-FDM in space. Our analysis confirmed the scheme's consistency, stability, and uniform convergence on standard meshes. Numerical experiments (Tables 7.1-7.6) validate the theory and align well with earlier studies [Chandru et al. \(2018\)](#); [Singh et al. \(2023\)](#). Richardson extrapolation further improved accuracy without modifying the core scheme, delivering first-order uniform convergence before extrapolation and second-order afterward. Overall, the method demonstrates highly accurate results and standing competitive with leading approaches for SPPPs involving multiple small parameters and discontinuous data.

Chapter 8

Mesh-Fitted Cubic Spline Method for Two-Parameter Singularly Perturbed Parabolic Problems with Discontinuous Convection and Source Terms

Building on the uniform convergence framework established in Chapters 6 and 7, this chapter presents a mesh-adapted numerical scheme for TP-SPPPs with discontinuous convection coefficients and source terms. Such discontinuities, combined with small perturbation parameters, generate strong boundary and interior layers that challenge standard numerical methods and often undermine both stability and accuracy. To address these difficulties, a Shishkin-type mesh is employed to refine the spatial grid in regions of steep gradients. The method couples the implicit Euler scheme in time with a cubic spline in compression for space. A rigorous analysis confirms its stability and parameter-uniform convergence, delivering first-order accuracy in time and second-order accuracy up to logarithmic factor in space under the maximum norm, independent of the perturbation parameters. Extensive numerical tests verify these theoretical results, showing that the scheme is uniformly convergent across a broad range of parameter values.

8.1 Introduction

The tension spline was originally introduced to suppress the unwanted inflection points often produced by cubic splines when fitting curves that represent the geometric shape of physical structures, such as ship hulls (Pruess, 1976). Its name comes from the tension parameter (denoted by θ in this work), which governs the stability and consistency of the resulting formulation. Over the past two decades, numerous researchers (Aziz & Khan, 2002; Khan et al., 2004; Bawa, 2005; Kanth & Murali, 2018) have employed tension splines to SPPs. Such problems arise frequently in models related to fluid dynamics, heat and mass transfer in chemical engineering, quantum mechanics (Pramanik, 2014), elasticity, plate and shell theories, oil and gas reservoir simulation, and magneto hydrodynamic flows (Gupta et al., 2019). Time-dependent and two-parameter SPPs also appear in chemical reactor theory (O'Malley, 1974) and in boundary-layer phenomena influenced by suction or injection of fluids (Schlichting & Gersten, 2016; Schlichting et al., 1979).

It is well known that solutions to SPPs exhibit sharp variations within boundary layers while remaining smooth in the outer region. This multi-scale behavior causes standard numerical methods to lose accuracy and become computationally inefficient. To address these limitations, researchers have developed schemes that remain uniformly accurate, independent of the perturbation parameter, and capable of capturing the inherent singularity. Among the most effective approaches are specially constructed fitted methods. For example, Singh et al. (2023); Chandru et al. (2018, 2019) employed fitted mesh techniques to resolve both boundary and interior layers.

Splines offer a straightforward and highly effective framework for handling BVPs, making them an attractive computational tool (Kumar & Gupta, 2010). This motivates the present chapter, where we focus on the use of cubic splines in tension. Unlike classical cubic splines, which may introduce unwanted oscillations, cubic splines in tension incorporate a controllable tension parameter that enhances stability, suppresses spurious inflection points, and allows the curve to adapt more faithfully to sharp gradients. These features make them particularly suitable for problems exhibiting boundary or interior layers. In this study, the numerical solution is produced through

a fitted mesh approach: an implicit Euler scheme is applied in time, while the spatial domain is discretized using a cubic spline in tension on Shishkin-type. The resulting method is tailored to capture the singular perturbation structure uniformly, and to the best of our knowledge, this formulation is applied here for the first time to the problem under investigation.

The chapter begins with the problem formulation in Section 8.2. Section 8.3 introduces the numerical scheme, and Section 8.4 analyzes its convergence. Numerical results are reported in Section 8.5, and Section 8.6 concludes the chapter.

8.2 Definition of the Problem

This chapter focuses on the numerical treatment of TP-SPPPs, building on the model problem formulated in Chapter 6, Section 6.2. The corresponding computational domain is depicted in Figure 6.1. The problem (6.1) admits two notable limiting cases: the reaction-diffusion case ($\mu = 0$) and the convection-diffusion case ($\mu = 1$), both of which can emerge as particular forms of a general TP-SPPP model. When $\mu = 0$, boundary layers develop symmetrically on both ends of the domain, each with an approximate thickness of $O(\sqrt{\varepsilon})$. In contrast, when $\mu = 1$, the boundary layer forms near either the left or right boundary, depending primarily on the sign of the convection coefficient, and has a characteristic width of $O(\varepsilon)$. If the problem involves a discontinuous source term while $a(x, t) > 0$ (or $a(x, t) < 0$), the solution exhibits weak interior layers for any small ε . However, when the convection coefficient itself is discontinuous such as $a(x, t) < 0$ for $x < \varphi$ and $a(x, t) > 0$ for $x > \varphi$, strong interior layers appear within the domain. A detailed analysis of this particular case $\mu = 1$ can be found in Mukherjee & Natesan (2008), where the sign change of the convection coefficient is shown to shift the location of the layers within the solution.

8.3 Formulation of the Numerical Scheme

This section first applies a time semi-discretization of Eq. (6.1) on a uniform mesh. The resulting system is then discretized in space using cubic splines on a piecewise uniform mesh, improving both accuracy and stability of the overall scheme.

8.3.1 Temporal Semi-discretization

A uniform mesh is employed to determine the nodal points in the temporal domain. The time interval $[0, T]$ is divided into $M = T/k$ equal subintervals using a mesh size of k . Let Ω^{M_t} represent the temporal mesh over $[0, T]$, defined as $\Omega^{M_t} = \{t_j = jk : j = 1(1)M, k = \frac{T}{M}\}$. To advance the numerical solution in time, we apply the implicit Euler method. Accordingly, the semi-discretized form of problem (6.1) is given by:

$$\begin{aligned} -c^{j+1}(x)D_t^-W^{j+1}(x) + \varepsilon (W^{j+1})_{xx}(x) + \mu a^{j+1}(x)(W^{j+1})_x(x) - \\ b^{j+1}(x)W^{j+1}(x) = \varpi^{j+1}(x), \quad x \in \Omega^\varphi, 0 \leq j \leq M-1, \\ W^{j+1}(0) = \vartheta(t_{j+1}), \quad W^{j+1}(1) = \xi(t_{j+1}), 0 \leq j \leq M-1, \\ W^0(x) = \phi(x), \quad x \in \overline{\Omega}, \end{aligned}$$

where $W^{j+1}(x)$ is the approximation of $w(x, t_{j+1})$, $D_t^- = (W^{j+1}(x) - W^j(x))/k$ is backward difference operator. Rewrite the above equation as

$$\begin{cases} \mathbb{L}_{\varepsilon, \mu}^M W^{j+1}(x) = \mathcal{G}^{j+1}(x), & x \in \Omega^\varphi, 0 \leq j \leq M-1, \\ W^{j+1}(0) = \vartheta(t_{j+1}), \quad W^{j+1}(1) = \xi(t_{j+1}), \\ W^0(x) = \phi(x), & x \in \overline{\Omega}, \end{cases} \quad (8.1)$$

where the operator $\mathbb{L}_{\varepsilon, \mu}^M$ is defined as

$$\mathbb{L}_{\varepsilon, \mu}^M \equiv \varepsilon \frac{d^2}{dx^2} + \mu a^{j+1}(x) \frac{d}{dx} - \left(b^{j+1}(x) + \frac{c^{j+1}(x)}{k} \right) I, \quad \text{and}$$

$$\mathcal{G}^{j+1}(x) = \varpi^{j+1}(x) - \frac{c^{j+1}(x)}{k} W^j(x).$$

The LTE in the temporal semi-discretization of Eq. (8.1) is given by $\mathcal{E}_{j+1} = W^{j+1}(x) - \overline{U}(x)$, where $\overline{U}(x)$ is the computed solution of

$$\begin{cases} \mathbb{L}_{\varepsilon, \mu} \overline{U}(x) = \varpi(x, t_{j+1}) - \frac{c(x, t_{j+1})}{k} W(x, t_j), & x \in \Omega^\varphi, 0 \leq j \leq M-1 \\ \overline{U}(0) = \mathcal{B}(t_{j+1}), \quad \overline{U}(1) = \mathcal{R}(t_{j+1}), \\ \overline{U}(x) = \mathcal{I}(x), & x \in \overline{\Omega}, \end{cases} \quad (8.2)$$

The following lemma which estimates LTE $\mathcal{E}_{j+1}$ can be proved by the approach given in [Kumar & Kumari \(2021\)](#).

Lemma 8.1. *The LTE and global error are defined as $\mathcal{E}_{j+1}$ and $\mathbb{E}_j = \sum_{\iota=1}^j \mathcal{E}_\iota$, respectively, and are bounded by :*

$$\begin{aligned} \|\mathcal{E}_{j+1}\| &\leq Ck^2, \\ \|\mathbb{E}_j\| &\leq Ck, \quad j \leq \frac{T}{k}, \end{aligned} \tag{8.3}$$

where $\mathbb{E}_j$ represents the cumulative effect of LTE up to the j th time level.

Proof. The first part of (8.3) follows the approach in [Singh et al. \(2023\)](#), whereas the second is justified in Chapter 6, Lemma 6.1. $\square$

The Solutions

$$\ell_0(x) = -\frac{\mu a^{j+1}(x)}{2\varepsilon} - \sqrt{\left(\frac{\mu a^{j+1}(x)}{2\varepsilon}\right)^2 + \frac{\zeta^{j+1}(x)}{\varepsilon}} < 0, \quad \text{and}$$

$$\ell_1(x) = -\frac{\mu a^{j+1}(x)}{2\varepsilon} + \sqrt{\left(\frac{\mu a^{j+1}(x)}{2\varepsilon}\right)^2 + \frac{\zeta^{j+1}(x)}{\varepsilon}} > 0,$$

of the characteristic equation

$$\varepsilon \ell^2(x) + \mu a^{j+1}(x) \ell(x) - \zeta^{j+1}(x) = 0,$$

where $\zeta^{j+1}(x) = \left(b^{j+1}(x) + \frac{c^{j+1}(x)}{k}\right)$ clarify the solution's layer behavior. More precisely, the rate at which solution decay or rise in layer region is given by $\wp_0 = -\max_{x \in \bar{\Omega}} \ell_0(x)$ and $\wp_1 = \min_{x \in \bar{\Omega}} \ell_1(x)$. Let $\aleph = \min_{\Omega^- \cup \Omega^+} \left\{ \frac{b(x,t)}{a(x,t)} \right\}$ and $\alpha = \min \{\alpha_1, \alpha_1\}$ be given. We have 2 cases:

- When $\alpha \mu^2 \leq \aleph \varepsilon$, both ℓ_0 and ℓ_1 are of $O(\varepsilon^{-1/2})$,
- When $\alpha \mu^2 \geq \aleph \varepsilon$, ℓ_0 and ℓ_1 are of $O(\frac{\mu}{\varepsilon})$ and $O(\mu^{-1})$, respectively.

8.3.2 Spatial Semi-discretization

The considered problem exhibits boundary layers near both endpoints of the spatial domain and an additional interior layer at the point of discontinuity. To accurately

capture these rapid variations in the solution, we employ a priori constructed Shishkin mesh(S-mesh), denoted by $\Omega^N = \{x_i\}_{i=0}^N$ which partitions the interval $[0, 1]$ into six subintervals as follows:

$$[0, 1] = [0, \delta_1] \cup [\delta_1, \varphi - \delta_2] \cup [\varphi - \delta_2, \varphi] \cup [\varphi, \varphi + \delta_3] \cup [\varphi + \delta_3, 1 - \delta_4] \cup [1 - \delta_4, 1].$$

Here, $\delta_1, \delta_2, \delta_3$, and δ_4 are *transition parameters* that determine the location and thickness of the layers. The transition parameters are judiciously chosen to maintain an optimal balance between the effects of boundary and interior layers, ensuring both stability and accuracy of the numerical approximation throughout the computational domain and are defined as follows:

$$\begin{aligned} \delta_1 &= \min \left\{ \frac{\varphi}{4}, \frac{2}{\wp_0} \ln N \right\}, & \delta_2 &= \min \left\{ \frac{\varphi}{4}, \frac{2}{\wp_1} \ln N \right\}, \\ \delta_3 &= \min \left\{ \frac{1-\varphi}{4}, \frac{2}{\wp_1} \ln N \right\}, & \delta_4 &= \min \left\{ \frac{1-\varphi}{4}, \frac{2}{\wp_0} \ln N \right\}, \end{aligned}$$

where N denotes the number of mesh intervals in $[0, 1]$ and $N \geq 8$. The values of $\wp_0$ and $\wp_1$ are given as

$$\wp_0 = \begin{cases} \frac{1}{2} \sqrt{\frac{\alpha \aleph}{\varepsilon}}, & \text{if } \alpha \mu^2 \leq \aleph \varepsilon, \\ \frac{\alpha \mu}{2\varepsilon}, & \text{if } \alpha \mu^2 \geq \aleph \varepsilon, \end{cases} \quad \wp_1 = \begin{cases} \frac{1}{2} \sqrt{\frac{\alpha \aleph}{\varepsilon}}, & \text{if } \alpha \mu^2 \leq \aleph \varepsilon, \\ \frac{\aleph}{2\mu}, & \text{if } \alpha \mu^2 \geq \aleph \varepsilon. \end{cases}$$

Each subintervals $[0, \delta_1], [\varphi - \delta_2, \varphi], [\varphi, \varphi + \delta_3]$ and $[1 - \delta_4, 1]$ contains $N/8$ mesh points and $[\delta_1, \varphi - \delta_2]$ and $[\varphi + \delta_3, 1 - \delta_4]$ contains $N/4$ mesh points. The mesh spacing is given as

$$h_i = x_i - x_{i-1} = \begin{cases} \frac{8\delta_1}{N}, & \text{for } i = 1, 2, \dots, \frac{N}{8}, \\ \frac{4(\varphi - \delta_1 - \delta_2)}{N}, & \text{for } \frac{N}{8} + 1, \frac{N}{8} + 2, \dots, \frac{3N}{8}, \\ \frac{8\delta_2}{N}, & \text{for } \frac{3N}{8} + 1, \frac{3N}{8} + 2, \dots, \frac{N}{2}, \\ \frac{8\delta_3}{N}, & \text{for } \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, \frac{5N}{8}, \\ \frac{4(1 - \varphi - \delta_3 - \delta_4)}{N}, & \text{for } \frac{5N}{8} + 1, \frac{5N}{8} + 2, \dots, \frac{7N}{8}, \\ \frac{8\delta_4}{N}, & \text{for } \frac{7N}{8} + 1, \frac{7N}{8} + 2, \dots, N, \end{cases}$$

and the mesh points are given by

$$x_i = \begin{cases} \frac{8\delta_1}{N}i, & \text{for } i = 0, 1, \dots, \frac{N}{8}, \\ \delta_1 + \frac{4(\varphi - \delta_1 - \delta_2)}{N} \left(i - \frac{N}{8}\right), & \text{for } \frac{N}{8} + 1, \frac{N}{8} + 2, \dots, \frac{3N}{8}, \\ \varphi - \delta_2 + \frac{8\delta_2}{N} \left(i - \frac{3N}{8}\right), & \text{for } \frac{3N}{8} + 1, \frac{3N}{8} + 2, \dots, \frac{N}{2}, \\ \varphi + \frac{8\delta_3}{N} \left(i - \frac{N}{2}\right), & \text{for } \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, \frac{5N}{8}, \\ \varphi + \delta_3 + \frac{4(1 - \varphi - \delta_3 - \delta_4)}{N} \left(i - \frac{5N}{8}\right), & \text{for } \frac{5N}{8} + 1, \frac{5N}{8} + 2, \dots, \frac{7N}{8}, \\ 1 - \delta_4 + \frac{8\delta_4}{N} \left(i - \frac{7N}{8}\right), & \text{for } \frac{7N}{8} + 1, \frac{7N}{8} + 2, \dots, N. \end{cases} \quad (8.4)$$

The interior mesh points are denoted by

$$\Omega_x^N = \left\{x_i : 1 \leq i \leq \frac{N}{2} - 1\right\} \cup \left\{x_i : \frac{N}{2} + 1 \leq i \leq N - 1\right\}.$$

We choose $x_{\frac{N}{2}} = \varphi$ and $\overline{\Omega}_x^N = \{x_i\}_0^N \cup \{\varphi\}$.

8.3.3 Cubic Spline in Tension Method

A function $S(x, t_{j+1})$ interpolates the solution $W(x_i, t_{j+1}) = W_i^{j+1}$ and passes through the points $(x_i, t_{j+1}, W_i^{j+1})$ and $(x_{i+1}, t_{j+1}, W_{i+1}^{j+1})$ such that

$$S(x, t_{j+1}) = -\frac{h_i^2}{\theta^2 \sin(\theta)} \left[\sin\left(\frac{\theta(x - x_i)}{h_i}\right) Y_{i+1}^{j+1} + \sin\left(\frac{\theta(x_{i+1} - x)}{h_i}\right) Y_i^{j+1} \right] + \frac{h_i^2}{\theta^2} \left[\frac{x - x_i}{h_i} \left(Y_{i+1}^{j+1} + \frac{h_i^2}{\theta^2} W_{i+1}^{j+1} \right) + \frac{x_{i+1} - x}{h_i} \left(Y_i^{j+1} - \frac{\theta^2}{h_i^2} W_i^{j+1} \right) \right], \quad (8.5)$$

where $Y_i^{j+1} = (W'')_i^{j+1}$. The first derivatives of $S(x, t_{j+1})$ from the right and left at $x = x_i$ are given respectively by

$$S'(x_i^+, t_{j+1}) = \frac{W_{i+1}^{j+1} - W_i^{j+1}}{h_i} + \frac{h_i}{\theta^2} [(1 - \theta \csc(\theta)) Y_{i+1}^{j+1} - (1 - \theta \cot(\theta)) Y_i^{j+1}] \quad \text{and} \\ S'(x_i^-, t_{j+1}) = \frac{W_i^{j+1} - W_{i-1}^{j+1}}{h_{i-1}} + \frac{h_{i-1}}{\theta^2} [(1 - \theta \cot(\theta)) Y_i^{j+1} - (1 - \theta \csc(\theta)) Y_{i-1}^{j+1}].$$

Then, from the continuity condition, we have $S'(x_i^+, t_{j+1}) = S'(x_i^-, t_{j+1})$ and obtain

$$\lambda_1 h_{i-1} Y_{i-1}^{j+1} + \lambda_2 (h_{i-1} + h_i) Y_i^{j+1} + \lambda_1 h_i Y_{i+1}^{j+1} = \frac{W_{i+1}^{j+1} - W_i^{j+1}}{h_i} - \frac{W_i^{j+1} - W_{i-1}^{j+1}}{h_{i-1}}, \quad (8.6)$$

where $\lambda_1 = \frac{\theta \csc(\theta) - 1}{\theta^2}$, $\lambda_2 = \frac{1 - \theta \cot(\theta)}{\theta^2}$. Application of the L'Hôpital's rule and using small-angle identity, i.e., $\csc \theta - \cot \theta \sim \theta/2$ as $\theta \rightarrow 0$, and $\csc \theta \sim 1/\theta$ gives $\lambda_1 + \lambda_2 = \frac{1}{2}$.

At $x = x_{i,i\pm 1}$ using the notation $Y_i^{j+1} = (W'')_i^{j+1}$, from Eq.(8.1) we can have

$$\begin{aligned} Y_{i-1}^{j+1} &= \frac{1}{\varepsilon} \left\{ -\mu a_{i-1}^{j+1} (W')_{i-1}^{j+1} + \left(b_{i-1}^{j+1} + \frac{c_{i-1}^{j+1}}{k} \right) W_{i-1}^{j+1} + \varpi_{i-1}^{j+1} - \frac{c_{i-1}^j}{k} W_{i-1}^{j+1} \right\}, \\ Y_i^{j+1} &= \frac{1}{\varepsilon} \left\{ -\mu a_i^{j+1} (W')_i^{j+1} + \left(b_i^{j+1} + \frac{c_i^{j+1}}{k} \right) W_i^{j+1} + \varpi_i^{j+1} - \frac{c_i^j}{k} W_i^{j+1} \right\}, \\ Y_{i+1}^{j+1} &= \frac{1}{\varepsilon} \left\{ -\mu a_{i+1}^{j+1} (W')_{i+1}^{j+1} + \left(b_{i+1}^{j+1} + \frac{c_{i+1}^{j+1}}{k} \right) W_{i+1}^{j+1} + \varpi_{i+1}^{j+1} - \frac{c_{i+1}^j}{k} W_{i+1}^{j+1} \right\}. \end{aligned} \quad (8.7)$$

Using Taylor's series approximations for W about $x_z, z = i \pm 1$, we have

$$W^{j+1}(x_{i-1}) \approx W^{j+1}(x_i) - h_{i-1} \frac{dW^{j+1}(x_i)}{dx} + \frac{h_{i-1}^2}{2} \frac{d^2W^{j+1}(x_i)}{dx^2}, \quad (8.8)$$

$$W^{j+1}(x_{i+1}) \approx W^{j+1}(x_i) + h_i \frac{dW^{j+1}(x_i)}{dx} + \frac{h_i^2}{2} \frac{d^2W^{j+1}(x_i)}{dx^2}. \quad (8.9)$$

Multiplying Eq. (8.8) by $\frac{h_i^2}{h_{i-1}^2}$ and subtracting the result from Eq. (8.9), we derive an approximate expression for $(W'')_i^{j+1}$:

$$\begin{aligned} \frac{dW^{j+1}(x_i)}{dx} &\approx \frac{1}{h_i h_{i-1} (h_{i-1} + h_i)} (h_{i-1}^2 W^{j+1}(x_{i-1}) + (h_i^2 - h_{i-1}^2) W^{j+1}(x_i)) + \\ &\quad \frac{1}{h_i h_{i-1} (h_{i-1} + h_i)} (h_{i-1}^2 W^{j+1}(x_{i+1})). \end{aligned} \quad (8.10)$$

Similarly, multiplying Eq.(8.8) by $\frac{h_i}{h_{i-1}}$ and then adding it to Eq.(8.9), we get an approximation for $(W'')_i^{j+1}$:

$$\begin{aligned} \frac{d^2W^{j+1}(x_i)}{dx^2} &\approx \frac{2}{h_i h_{i-1} (h_{i-1} + h_i)} (h_i W^{j+1}(x_{i-1}) + (h_i - h_{i-1}) W^{j+1}(x_i)) \\ &\quad + \frac{2}{h_i h_{i-1} (h_{i-1} + h_i)} (h_{i-1} W^{j+1}(x_{i+1})). \end{aligned} \quad (8.11)$$

Inserting Eqs.(8.10) and (8.11) in

$$\frac{dW^{j+1}(x_{i+1})}{dx} \approx \frac{dW^{j+1}(x_i)}{dx} + h_i \frac{d^2W^{j+1}(x_{i+1})}{dx^2} \quad \text{and}$$

$$\frac{dW^{j+1}(x_{i-1})}{dx} \approx \frac{dW^{j+1}(x_i)}{dx} + h_{i-1} \frac{d^2W^{j+1}(x_{i+1})}{dx^2},$$

we obtain

$$\begin{aligned} (W')_{i-1}^{j+1} &= \frac{1}{h_i h_{i-1} (h_{i-1} + h_i)} \left\{ - (h_i^2 + 2h_{i-1}h_i) W_{i-1}^{j+1} + (h_{i-1} + h_i)^2 W_i^{j+1} - h_{i-1}^2 W_{i+1}^{j+1} \right\}, \\ (W')_i^{j+1} &= \frac{1}{h_i h_{i-1} (h_{i-1} + h_i)} \left\{ -h_i^2 W_{i-1}^{j+1} + (h_i^2 - h_{i-1}^2) W_i^{j+1} + h_{i-1}^2 W_{i+1}^{j+1} \right\}, \\ (W')_{i+1}^{j+1} &= \frac{1}{h_i h_{i-1} (h_{i-1} + h_i)} \left\{ h_i^2 W_{i-1}^{j+1} - (h_{i-1} + h_i)^2 W_i^{j+1} + (h_{i-1}^2 + 2h_{i-1}h_i) W_{i+1}^{j+1} \right\}. \end{aligned} \quad (8.12)$$

Employing cubic spline formulation proposed by [Kumar & Ravi Kanth \(2020\)](#) gives

$$\lambda_1 h_{i-1} Y_{i-1}^{j+1} + \lambda_2 (h_i + h_{i-1}) Y_i^{j+1} + \lambda_1 h_i Y_{i+1}^{j+1} = \frac{W_{i+1}^{j+1} - W_i^{j+1}}{h_i} - \frac{W_i^{j+1} - W_{i-1}^{j+1}}{h_{i-1}}. \quad (8.13)$$

Inserting Eqs. (8.7) and (8.12) into Eq. (8.13) yields a combined expression integrating the spline formulation with the finite difference framework.

$$\mathbb{L}_{\varepsilon, \mu}^{N, M} W_i^{j+1} = r_i^- W_{i-1}^{j+1} + r_i^c W_i^{j+1} + r_i^+ W_{i+1}^{j+1} = F_i^{j+1}, \quad (8.14)$$

$i = 1(1)N - 1$, $j = 0(1)M - 1$, where $\mathbb{L}_{\varepsilon, \mu}^{N, M}$ represents the overall discrete operator that encapsulates the full numerical approximation of the governing problem, and the matrix coefficients r_i are specified by:

$$\left\{ \begin{aligned} r_i^- &= \mu \left[\lambda_1 \left(\frac{h_i + 2h_{i-1}}{h_{i-1} + h_i} \right) a_{i-1}^{j+1} + \lambda_2 \left(\frac{h_i}{h_{i-1}} \right) a_i^{j+1} - \lambda_1 \left(\frac{h_i^2}{h_{i-1} (h_{i-1} + h_i)} \right) a_{i+1}^{j+1} \right. \\ &\quad \left. + \lambda_1 h_{i-1} \left(b_{i-1}^{j+1} + \frac{c_{i-1}^{j+1}}{k} \right) - \frac{\varepsilon}{h_{i-1}}, \right. \\ r_i^c &= -\mu \left[\lambda_1 \left(\frac{h_{i-1} + h_i}{h_i} \right) a_i^{j+1} + \lambda_2 \left(\frac{h_i^2 - h_{i-1}^2}{h_i h_{i-1}} \right) a_i^{j+1} - \lambda_1 \left(\frac{h_{i-1} + h_i}{h_{i-1}} \right) a_{i+1}^{j+1} \right. \\ &\quad \left. + \lambda_2 (h_i + h_{i-1}) \left(b_i^{j+1} + \frac{c_i^{j+1}}{k} \right) + \varepsilon \left(\frac{h_i + h_{i-1}}{h_{i-1} h_i} \right), \right. \\ r_i^+ &= \mu \left[\lambda_1 \left(\frac{h_{i-1}^2}{h_i (h_{i-1} + h_i)} \right) a_{i-1}^{j+1} - \lambda_2 \left(\frac{h_{i-1}}{h_i} \right) a_i^{j+1} - \lambda_1 \left(\frac{h_{i-1} + 2h_i}{(h_{i-1} + h_i)} \right) a_{i+1}^{j+1} \right. \\ &\quad \left. + \lambda_1 h_i \left(b_{i+1}^{j+1} + \frac{c_{i+1}^{j+1}}{k} \right) - \frac{\varepsilon}{h_i}, \right. \\ F_i^{j+1} &= \lambda_1 h_{i-1} \varpi_{i-1}^{j+1} + \lambda_2 (h_{i-1} + h_i) \varpi_i^{j+1} + \lambda_1 h_i \varpi_{i+1}^{j+1} - \lambda_1 h_{i-1} \frac{c_{i-1}^j}{k} W_{i-1}^j \\ &\quad - \lambda_2 (h_{i-1} + h_i) \frac{c_i^j}{k} W_i^j - \lambda_1 h_i \frac{c_{i+1}^j}{k} W_{i+1}^j. \end{aligned} \right.$$

Discrete Scheme at the Discontinuity Point $x = \varphi$

Let the spatial mesh be divided such that $\Omega_x^N = \{x_i : 1 \leq i \leq \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \leq i \leq N - 1\}$, and choose the interface point $x_{\frac{N}{2}} = \varphi$. The extended mesh including the discontinuity is denoted by $\bar{\Omega}_x^N = \{x_i\}_0^N \cup \{\varphi\}$. Define the local mesh sizes near the interface as $h_L = x_{\frac{N}{2}} - x_{\frac{N}{2}-1}$, $h_R = x_{\frac{N}{2}+1} - x_{\frac{N}{2}}$. Then, at the discontinuity node $x = \varphi = x_{\frac{N}{2}}$, the fully discrete cubic spline in tension scheme can be written as

$$r_\varphi^- W_{\frac{N}{2}-1}^{j+1} + r_\varphi^c W_{\frac{N}{2}}^{j+1} + r_\varphi^+ W_{\frac{N}{2}+1}^{j+1} = F_{\frac{N}{2}}^{j+1}.$$

The coefficients r_φ^- , r_φ^c , r_φ^+ , and the right-hand side $F_{\frac{N}{2}}^{j+1}$ are obtained by substituting $i = \frac{N}{2}$, $h_{i-1} = h_L$, and $h_i = h_R$ into the general expressions as follows:

$$\begin{aligned} r_\varphi^- &= \mu \left[\lambda_1 \left(\frac{h_R + 2h_L}{h_L + h_R} \right) a_{\frac{N}{2}-1}^{j+1} + \lambda_2 \left(\frac{h_R}{h_L} \right) a_{\frac{N}{2}}^{j+1} - \lambda_1 \left(\frac{h_R^2}{h_L(h_L + h_R)} \right) a_{\frac{N}{2}+1}^{j+1} \right] \\ &\quad + \lambda_1 h_L \left(b_{\frac{N}{2}-1}^{j+1} + \frac{c_{\frac{N}{2}-1}^{j+1}}{k} \right) - \frac{\varepsilon}{h_L}, \\ r_\varphi^c &= -\mu \left[\lambda_1 \left(\frac{h_L + h_R}{h_R} \right) a_{\frac{N}{2}}^{j+1} + \lambda_2 \left(\frac{h_R^2 - h_L^2}{h_R h_L} \right) a_{\frac{N}{2}}^{j+1} - \lambda_1 \left(\frac{h_L + h_R}{h_L} \right) a_{\frac{N}{2}+1}^{j+1} \right] \\ &\quad + \lambda_2 (h_R + h_L) \left(b_{\frac{N}{2}}^{j+1} + \frac{c_{\frac{N}{2}}^{j+1}}{k} \right) + \varepsilon \left(\frac{h_R + h_L}{h_L h_R} \right), \\ r_\varphi^+ &= \mu \left[\lambda_1 \left(\frac{h_L^2}{h_R(h_L + h_R)} \right) a_{\frac{N}{2}-1}^{j+1} - \lambda_2 \left(\frac{h_L}{h_R} \right) a_{\frac{N}{2}}^{j+1} - \lambda_1 \left(\frac{h_L + 2h_R}{h_L + h_R} \right) a_{\frac{N}{2}+1}^{j+1} \right] \\ &\quad + \lambda_1 h_R \left(b_{\frac{N}{2}+1}^{j+1} + \frac{c_{\frac{N}{2}+1}^{j+1}}{k} \right) - \frac{\varepsilon}{h_R}. \end{aligned}$$

The corresponding right-hand side term is given by

$$\begin{aligned} F_\varphi^{j+1} &= \lambda_1 h_L \varpi_{\frac{N}{2}-1}^{j+1} + \lambda_2 (h_L + h_R) \varpi_{\frac{N}{2}}^{j+1} + \lambda_1 h_R \varpi_{\frac{N}{2}+1}^{j+1} \\ &\quad - \lambda_1 h_L \frac{c_{\frac{N}{2}-1}^j}{k} W_{\frac{N}{2}-1}^j - \lambda_2 (h_L + h_R) \frac{c_{\frac{N}{2}}^j}{k} W_{\frac{N}{2}}^j - \lambda_1 h_R \frac{c_{\frac{N}{2}+1}^j}{k} W_{\frac{N}{2}+1}^j. \end{aligned}$$

For sufficiently small values of h , the resulting matrix remains non-singular, satisfying the condition $|r_i^c| \geq |r_i^-| + |r_i^+|$, which ensures that the matrix is diagonally dominant.

Consequently, it qualifies as an M -matrix, possessing a positive inverse. Therefore, the corresponding system of discrete equations admits a unique solution, which can be efficiently obtained through the matrix inverse method, subject to the prescribed boundary conditions $W^{j+1}(0)$ and $W^{j+1}(1)$.

8.4 Convergence Analysis

In this section, we analyze the stability and uniform convergence of the fully discrete scheme (8.14) and rigorously derive the uniform stability of the scheme, ensuring that the accuracy and convergence of the numerical method remain unaffected even as ε and μ approaches zero.

Lemma 8.2. (*Discrete comparison Principle*). Suppose W_i^{j+1} satisfies

$$r_i^- W_{i-1}^{j+1} + r_i^c W_i^{j+1} + r_i^+ W_{i+1}^{j+1} \geq 0,$$

with $r_i^- \geq 0, r_i^+ \geq 0, r_i^c \geq r_i^+ + r_i^-$ (i.e., diagonally dominant). Then if $W_i^0 \geq 0$ and boundary data $W_0^{j+1}, W_N^{j+1} \geq 0$, it follows that $W_i^{j+1} \geq 0$ for all i, j .

The LTE is defined by substituting the exact solution into the discrete scheme:

$$\begin{aligned} T_i &= \mathbb{L}_{\varepsilon, \mu}^{N, M} W(x_i, t_{j+1}) - F_i^{j+1}, \\ &= r_i^- W^{j+1}(x_{i-1}) + r_i^c W^{j+1}(x_i) + r_i^+ W^{j+1}(x_{i+1}) - \lambda_1 h_{i-1} \mathcal{G}^{j+1}(x_{i-1}) \\ &\quad - \lambda_2 (h_i + h_{i-1}) \mathcal{G}^{j+1}(x_i) - \lambda_1 h_i \mathcal{G}^{j+1}(x_{i+1}) \end{aligned} \quad (8.15)$$

By substituting the expressions of $\mathcal{G}^{j+1}(x_{i-1})$, $\mathcal{G}^{j+1}(x_i)$, and $\mathcal{G}^{j+1}(x_{i+1})$ from Eq.(8.1) into Eq.(8.15), we derive the following relation:

$$\begin{aligned} T_i &= r_i^- W^{j+1}(x_{i-1}) + r_i^c W^{j+1}(x_i) + r_i^+ W^{j+1}(x_{i+1}) \\ &\quad - \lambda_1 h_{i-1} \left\{ \varepsilon \frac{d^2 W^{j+1}(x_{i-1})}{dx^2} + \mu a_{i-1} \frac{dW^{j+1}(x_{i-1})}{dx} - Q_{i-1} W^{j+1}(x_{i-1}) \right\} \\ &\quad - \lambda_2 (h_i + h_{i-1}) \left\{ \varepsilon \frac{d^2 W^{j+1}(x_i)}{dx^2} + \mu a_i \frac{dW^{j+1}(x_i)}{dx} - Q_i W^{j+1}(x_i) \right\} \\ &\quad - \lambda_1 h_i \left\{ \varepsilon \frac{d^2 W^{j+1}(x_{i+1})}{dx^2} + \mu a_{i+1} \frac{dW^{j+1}(x_{i+1})}{dx} - Q_{i+1} W^{j+1}(x_{i+1}) \right\} \end{aligned} \quad (8.16)$$

Using Taylor's series approximation for $W^{j+1}(x_{i-1})$, $W^{j+1}(x_{i+1})$ and their derivatives

in the spatial variables, we have

$$W^{j+1}(x_{i+1}) \approx W^{j+1}(x_i) + h_i \frac{dW^{j+1}(x_i)}{dx} + \frac{h_i^2}{2} \frac{d^2W^{j+1}(x_i)}{dx^2} + \frac{h_i^3}{6} \frac{d^3W^{j+1}(x_i)}{dx^3} + \dots \quad (8.17)$$

$$W^{j+1}(x_{i-1}) \approx W^{j+1}(x_i) - h_{i-1} \frac{dW^{j+1}(x_i)}{dx} + \frac{h_{i-1}^2}{2} \frac{d^2W^{j+1}(x_i)}{dx^2} - \frac{h_{i-1}^3}{6} \frac{d^3W^{j+1}(x_i)}{dx^3} + \dots \quad (8.18)$$

$$\begin{aligned} \frac{dW^{j+1}(x_{i-1})}{dx} &\approx \frac{dW^{j+1}(x_i)}{dx} - h_{i-1} \frac{d^2W^{j+1}(x_i)}{dx^2} + \frac{h_{i-1}^2}{2} \frac{d^3W^{j+1}(x_i)}{dx^3} \\ &\quad - \frac{h_{i-1}^3}{6} \frac{d^4W^{j+1}(x_i)}{dx^4} + \dots \end{aligned} \quad (8.19)$$

$$\frac{dW^{j+1}(x_{i+1})}{dx} \approx \frac{dW^{j+1}(x_i)}{dx} + h_i \frac{d^2W^{j+1}(x_i)}{dx^2} + \frac{h_i^2}{2} \frac{d^3W^{j+1}(x_i)}{dx^3} + \frac{h_i^3}{6} \frac{d^4W^{j+1}(x_i)}{dx^4} + \dots \quad (8.20)$$

$$\frac{d^2W^{j+1}(x_{i+1})}{dx^2} \approx \frac{d^2W^{j+1}(x_i)}{dx^2} + h_i \frac{d^3W^{j+1}(x_i)}{dx^3} + \frac{h_i^2}{2} \frac{d^4W^{j+1}(x_i)}{dx^4} + \dots \quad (8.21)$$

$$\frac{d^2W^{j+1}(x_{i-1})}{dx^2} \approx \frac{d^2W^{j+1}(x_i)}{dx^2} - h_i \frac{d^3W^{j+1}(x_i)}{dx^3} + \frac{h_i^2}{2} \frac{d^4W^{j+1}(x_i)}{dx^4} + \dots \quad (8.22)$$

By inserting the expressions from Eqs. (8.17)-(8.22) into Eq. (8.16), we have

$$\begin{aligned} T_i = & T_{1,i} W^{j+1}(x_i) + T_{2,i} \frac{dW^{j+1}(x_{i+1})}{dx} + T_{3,i} \frac{d^2W^{j+1}(x_{i+1})}{dx^2} + T_{4,i} \frac{d^3W^{j+1}(x_{i+1})}{dx^3} \\ & + T_{5,i} \frac{d^4W^{j+1}(x_{i+1})}{dx^4} + \dots, \end{aligned} \quad (8.23)$$

where

$$T_{1,i} = r_i^- + r_i^c + r_i^+ + \lambda_1 h_{i-1} Q_{i-1} + \lambda_2 (h_{i-1} + h_i) Q_i + \lambda_1 h_i Q_{i+1},$$

$$T_{2,i} = -h_{i-1} r_i^- + h_i r_i^+ - \lambda_1 \mu h_{i-1} a_{i-1} - \lambda_1 h_{i-1}^2 Q_{i-1} -$$

$$\lambda_2 \mu (h_i + h_{i-1}) a_i - \lambda_1 \mu h_i a_{i+1} + \lambda_1 h_i^2 Q_{i+1}$$

$$T_{3,i} = \frac{h_{i-1}^2}{2} r_i^- + \frac{h_i^2}{2} r_i^+ - \varepsilon \lambda_1 h_{i-1} + \lambda_1 \mu h_{i-1}^2 a_{i-1} + \lambda_1 \frac{h_{i-1}^3}{2} Q_{i-1} -$$

$$\varepsilon \lambda_2 (h_i + h_{i-1}) - \varepsilon \lambda_1 h_i - \lambda_1 \mu h_i^2 a_{i+1} + \lambda_1 \frac{h_i^3}{2} Q_{i+1},$$

$$T_{4,i} = -\frac{h_{i-1}^3}{6} r_i^- + \frac{h_i^3}{6} r_i^+ + \varepsilon \lambda_1 h_{i-1}^2 - \lambda_1 \mu \frac{h_{i-1}^2}{2} a_{i-1} - \lambda_1 \frac{h_{i-1}^4}{6} Q_{i-1} -$$

$$\varepsilon \lambda_1 h_i^2 - \lambda_1 \mu \frac{h_i^3}{2} a_{i+1} + \lambda_1 \frac{h_i^4}{6} Q_{i+1},$$

$$T_{5,i} = \frac{h_{i-1}^4}{24} r_i^- + \frac{h_i^4}{24} r_i^+ - \varepsilon \lambda_1 \frac{h_{i-1}^3}{2} + \lambda_1 \mu \frac{h_{i-1}^4}{6} a_{i-1} + \lambda_1 \frac{h_{i-1}^5}{24} Q_{i-1} - \varepsilon \lambda_1 \frac{h_i^2}{2} - \lambda_1 \mu \frac{h_i^4}{6} a_{i+1} + \lambda_1 \frac{h_i^5}{24} Q_{i+1}.$$

By simplifying the expressions and retaining only the leading terms in the coefficient expansions, we assume that the coefficients satisfy the boundedness conditions $|a(x)| \leq A$ and $|Q(x)| \leq Q_{\max}$, while the mesh sizes meet $h_i, h_{i-1} \leq h_{\max}$. Further, let the weighting parameters fulfill the constraint $\lambda_1 + \lambda_2 = \frac{1}{2}$. When the coefficients in Eq. (8.14) of the three-point stencil satisfy the *moment consistency* conditions $r_i^- + r_i^c + r_i^+ = 0$ and $-h_{i-1}r_i^- + h_i r_i^+ = 0$, the scheme is said to be *stencil-consistent*. Under these assumptions, Eq.(8.23) at (x_i, t_{j+1}) satisfies

$$T_i = \sum_{n=1}^5 T_{n,i} \partial_x^{n-1} W(x_i, t_{j+1}) + O(h_i^5),$$

where the coefficients $T_{n,i}$ depend on $h_i, h_{i-1}, \varepsilon, \mu, a(x), Q(x)$, and the spline parameters $r_i^\pm, r_i^c$. The moment consistency conditions of the cubic spline in tension imply $T_{1,i} = T_{2,i} = T_{3,i} = 0$, so that the leading contribution of the LTE is

$$|T_i| \leq C(|T_{4,i}| \|W_{xxx}\| + |T_{5,i}| \|W_{xxxx}\|). \quad (8.24)$$

Lemma 8.3 (Solution decomposition and layer bounds). *The exact solution $w(x, t)$ of problem (3.26) admits the decomposition:*

$$w(x, t) = v(x, t) + u_l(x, t) + u_i(x, t) + u_r(x, t), \quad (8.25)$$

where v is the regular component, u_l and u_r are the left and right boundary layer components, respectively, and u_i denotes the interior layer component at $x = \varphi$. Moreover, for $0 \leq m \leq 4$, there exists a positive constant C , independent of ε and μ , such that

$$\begin{aligned} |v^{(m)}(x, t)| &\leq C, & |u_l^{(m)}(x, t)| &\leq C \varphi_0^m e^{-\varphi_0 x} \\ |u_r^{(m)}(x, t)| &\leq C \varphi_0^m e^{-\varphi_0(1-x)}, \\ |u_i^{(m)}(x, t)| &\leq C \varphi_1^m e^{-\varphi_1|x-\varphi|}. \end{aligned}$$

Proof. The result follows from standard singular perturbation theory for two-parameter

convection-diffusion-reaction problems with discontinuous coefficients. The analysis relies on the sign assumptions imposed on $a(x, t)$, the positivity of the discrete operator coefficients, and the boundedness of the jump conditions across the interior point $x = \varphi$ (see (Chandru et al., 2019; Singh et al., 2023)). $\square$

Lemma 8.4 (Local truncation error bound). *For all interior mesh points $x_i \in \Omega_x^N$, the LTE satisfies*

$$|T_i| \leq \begin{cases} CN^{-2} (\ln N)^2, & x_i \in [0, \delta_1] \cup [\varphi - \delta_2, \varphi] \cup [\varphi, \varphi + \delta_3] \cup [1 - \delta_4, 1], \\ CN^{-2}, & x_i \in [\delta_1, \varphi - \delta_2] \cup [\varphi + \delta_3, 1 - \delta_4]. \end{cases} \quad (8.26)$$

Proof. By combining the derivative bounds given in Lemma 8.3 with Eq. (8.24), the truncation error can be estimated separately over each mesh subregion.

- (i) Boundary layer regions: For $x_i \in [0, \delta_1] \cup [1 - \delta_4, 1]$, the mesh width satisfies $h_i = O\left(\frac{\ln N}{N_{\wp_0}}\right)$ and the fourth derivatives of the left and right layer components are bounded by $|u_l^{(4)}|, |u_r^{(4)}| \leq C_{\wp_0^4}$, which yields $|T_i| \leq C \left(\frac{\ln N}{N}\right)^2 \leq CN^{-2} (\ln N)^2$.
- (ii) Interior layer regions: For $x_i \in [\varphi - \delta_2, \varphi] \cup [\varphi, \varphi + \delta_3]$, the mesh width satisfies $h_i = O\left(\frac{\ln N}{N_{\wp_1}}\right)$, and the fourth derivatives of the interior layer is bounded by $|u_i^{(4)}| \leq C_{\wp_1^4}$, which yields $|T_i| \leq C \left(\frac{\ln N}{N}\right)^2 \leq CN^{-2} (\ln N)^2$.
- (iii) Regular(smooth) regions: For $x_i \in [\delta_1, \varphi - \delta_2] \cup [\varphi + \delta_3, 1 - \delta_4]$ derivatives are bounded and $h_i = O(N^{-1})$, yielding $|T_i| \leq CN^{-2}$.

By aggregating the estimates obtained over all mesh subregions, a uniform spatial truncation error bound is established.

Lemma 8.5. *Let W_i^{j+1} be the numerical solution obtained using the cubic spline in tension method on the six-piece Shishkin mesh. Then, for all $\varepsilon, \mu \in (0, 1]$, the spatial local truncation error satisfies*

$$\|W(x_i, t_{j+1}) - W^{j+1}\| \leq CN^{-2} (\ln N)^2,$$

where C is independent of ε, μ and N . $\square$

The desired result for the complete discrete scheme follows immediately by combining the triangle inequality with Lemmas 3.33, 8.26, and 8.5.

Theorem 8.1. *Let $w(x, t), (x, t) \in \bar{\Omega}$ be the solution of Eq.(6.1) and $W(x_i, t_{j+1}), (x_i, t_{j+1}) \in \Omega^{M_t} \times \Omega^{N_x}$ be the solution of Eq.(8.14). Then, the error estimate for the fully discrete scheme is given by*

$$\sup_{0 < \varepsilon, \mu \leq 1} \max_{i,j} \|w(x_i, t_{j+1}) - W(x_i, t_{j+1})\| \leq C (N^{-2} (\ln N)^2 + k).$$

Therefore, the proposed numerical scheme exhibits uniform convergence, ensuring reliable performance across both smooth and singular solution regimes. The spatial approximation attains second-order accuracy up to a logarithmic factor, whereas the temporal discretization is first-order accurate, thereby validating the consistency and stability of the method.

8.5 Numerical Examples, Results and Discussion

Several illustrative test cases are examined using the proposed scheme to verify the theoretical convergence results. Because the exact solutions are unknown, the maximum pointwise error for each pair (ε, μ) is approximated via the double-mesh technique (Kumar & Kumari, 2021; Singh et al., 2023).

$$E_{\varepsilon, \mu}^{N, M} = \max_{0 \leq i, j \leq N, M} |W_{i, j}^{N, M} - W_{i, j}^{2N, 2M}|,$$

where $W_{i, j}^{N, M}$ are computed numerical solutions obtained on the mesh $\Omega^{N, M} = \Omega^N \times \Omega^M$ with N and M mesh intervals in the spatial and temporal directions respectively. Whereas, $W_{i, j}^{2N, 2M}$ are computed numerical solutions on the mesh $\Omega^{2N, 2M} = \Omega^{2N} \times \Omega^{2M}$ by adding the midpoint $x_{i+1/2} = (x_{i+1} + x_i) / 2$ and $t_{j+1/2} = (t_{i+1} + t_i) / 2$ into the mesh points. For any values of N and M , the uniform maximum pointwise errors are calculated using $E_{\varepsilon, \mu}^{N, M} = \max_{\varepsilon, \mu} E_{\varepsilon, \mu}^{N, M}$, and the corresponding uniform rate of convergence $\bar{R}_{\varepsilon, \mu}^{N, M}$ is:

$$\bar{R}_{\varepsilon, \mu}^{N, M} = \frac{\log(E_{\varepsilon, \mu}^{N, M}) - \log(E_{\varepsilon, \mu}^{2N, 2M})}{\log(2)}.$$

Example 8.1. Following [Chandru et al. \(2018, 2019\)](#), we consider the TP-SPPPs.

$$\begin{cases} \varepsilon \frac{\partial^2 w}{\partial x^2} + \mu a(x, t) \frac{\partial w}{\partial x} - (1 + e^x)w - \frac{\partial w}{\partial t} = \varpi(x, t), (x, t) \in \Omega^- \cup \Omega^+, \\ w(0, t) = 0 \quad \text{on } \Omega_l, \quad w(1, t) = 0 \quad \text{on } \Omega_r, \\ w(x, 0) = 0 \quad \text{on } \Omega_b, \end{cases}$$

$$\text{where } a(x, t) = \begin{cases} -(1 + x - x^2), & (x, t) \in \Omega^- \\ (1 + x - x^2), & (x, t) \in \Omega^+ \end{cases}, \varpi(x, t) = \begin{cases} -2t(1 + x^2), & (x, t) \in \Omega^- \\ 2t(1 + x^2), & (x, t) \in \Omega^+ \end{cases}.$$

Example 8.2. We adopt Eq. (6.1), along with its associated coefficients, as presented in [Kumar & Kumari \(2021\)](#).

$$\begin{aligned} a(x, t) &= \begin{cases} -(1 + e^{-xt}), & 0 \leq x \leq \varphi, \\ (2 + x + t), & \varphi < x \leq 1, \end{cases}, \varpi(x, t) = \begin{cases} (e^{t^2} - 1)(1 + xt), & 0 \leq x \leq \varphi, \\ -t^2(2 + x), & \varphi < x \leq 1, \end{cases} \\ b(x, t) &= 2 + xt, c(x, t) = 1, \quad w(0, t) = 0 = w(1, t) \text{ and } w(x, 0) = 0. \end{aligned}$$

The problems in Examples 8.1 and 8.2 develop strong boundary and interior layers due to the small diffusion parameters ε and μ , along with the presence of non-smooth data. Such layer phenomena commonly appear in applications such as heat transfer in composite media, flow through porous structures, and chemical transport with sudden concentration shifts.

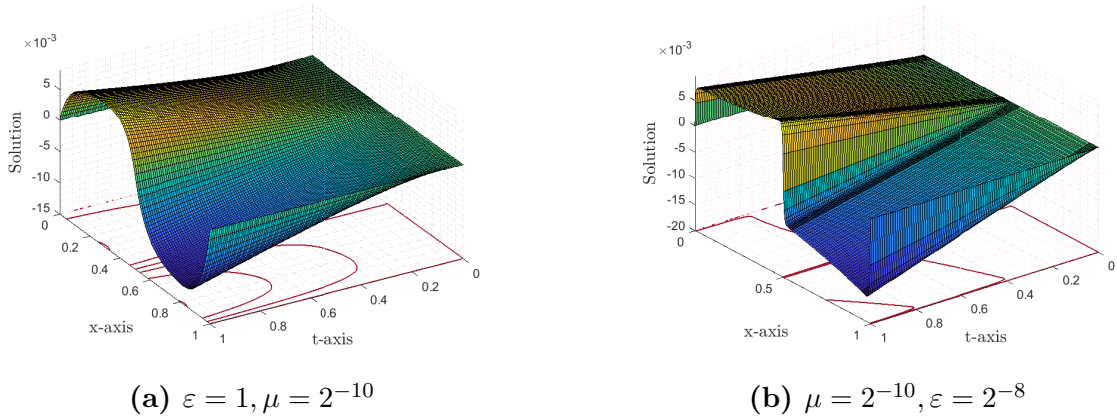


Figure 8.1: The computed solution profiles for Example 8.1 using $N = M = 128$.

Numerical results are summarized in Tables 8.1, 8.2 and 8.5 for various choices of ε , μ , N and M . The data demonstrate that the maximum pointwise error decreases as the mesh is refined, while the error remains essentially insensitive to further reductions in the perturbation parameters, confirming parameter-robust behavior. In

Table 8.1: Computed $E_{\varepsilon,\mu}^{N,M}$, $E^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ when $\mu = 2^{-4}$ for Example 8.1.

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
Present Method					
2^{-0}	$1.3300e-02$	$9.5221e-03$	$5.7104e-03$	$3.1247e-03$	$1.6541e-03$
2^{-2}	$2.1453e-02$	$1.2117e-02$	$6.5107e-03$	$3.4218e-03$	$1.7729e-03$
2^{-4}	$2.4427e-02$	$1.3234e-02$	$6.9725e-03$	$3.6067e-03$	$1.8453e-03$
2^{-6}	$2.5976e-02$	$1.3938e-02$	$7.2532e-03$	$3.7168e-03$	$1.8876e-03$
2^{-8}	$2.7178e-02$	$1.4351e-02$	$7.4179e-03$	$3.7802e-03$	$1.9117e-03$
2^{-10}	$2.7718e-02$	$1.4590e-02$	$7.5105e-03$	$3.8156e-03$	$1.9251e-03$
2^{-12}	$2.7993e-02$	$1.4716e-02$	$7.5618e-03$	$3.8353e-03$	$1.9325e-03$
2^{-14}	$2.8433e-02$	$1.4784e-02$	$7.5884e-03$	$3.8459e-03$	$1.9365e-03$
2^{-16}	$2.8913e-02$	$1.4816e-02$	$7.6016e-03$	$3.8514e-03$	$1.9386e-03$
2^{-18}	$2.9071e-02$	$1.4823e-02$	$7.6079e-03$	$3.8540e-03$	$1.9396e-03$
2^{-20}	$2.9113e-02$	$1.4825e-02$	$7.6091e-03$	$3.8550e-03$	$1.9401e-03$
$E^{N,M}$	$2.9113e-02$	$1.4825e-02$	$7.6091e-03$	$3.8550e-03$	$1.9401e-03$
$\bar{R}^{N,M}$	0.974	0.962	0.981	0.991	—
Result in Chandru et al. (2018)(Uniform S-mesh)					
$E^{N,M}$		$8.27430e-02$	$5.27314e-02$	$3.03482e-02$	$1.64782e-02$
$\bar{R}^{N,M}$		0.650	0.797	0.881	—

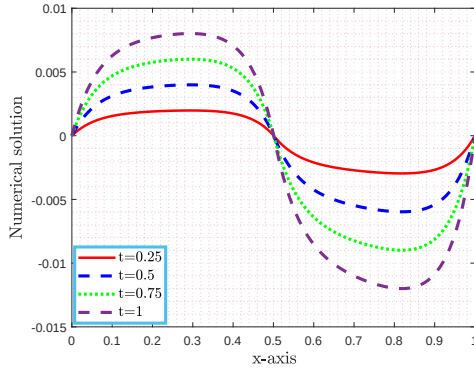
Table 8.2: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ when $\varepsilon = 2^{-8}$ for Example 8.1.

μ	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
2^{-0}	$2.5997e-02$	$1.4063e-02$	$7.3439e-03$	$3.7619e-03$	$1.9074e-03$
2^{-2}	$2.6886e-02$	$1.4309e-02$	$7.4054e-03$	$3.7771e-03$	$1.9109e-03$
2^{-4}	$2.7178e-02$	$1.4351e-02$	$7.4179e-03$	$3.7802e-03$	$1.9117e-03$
2^{-6}	$2.7210e-02$	$1.4370e-02$	$7.4207e-03$	$3.7809e-03$	$1.9119e-03$
2^{-8}	$2.7211e-02$	$1.4373e-02$	$7.4210e-03$	$3.7809e-03$	$1.9119e-03$
2^{-10}	$2.7210e-02$	$1.4374e-02$	$7.4211e-03$	$3.7810e-03$	$1.9119e-03$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	$2.7210e-02$	$1.4374e-02$	$7.4211e-03$	$3.7810e-03$	$1.9119e-03$
$E^{N,M}$	$2.7211e-02$	$1.4374e-02$	$7.4211e-03$	$3.7810e-03$	$1.9119e-03$
$\bar{R}^{N,M}$	0.921	0.954	0.973	0.984	—

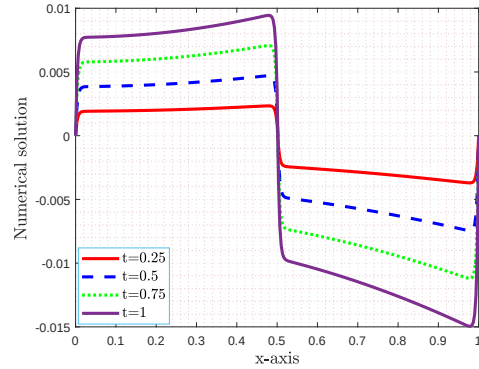
Table 8.1 we fix $\mu = 2^{-4}$ and $\varphi = 0.5$ and vary ε from 1 to 2^{-20} to report $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for Example 8.1; these values indicate nearly first-order, parameter-uniform convergence. Table 8.2 presents a complementary study with ε fixed and varying μ , again showing close to first-order uniform convergence. For Example 8.2, Table 8.5 displays consistent uniform convergence across tested parameter sets. Comparative results in Tables 8.1, 8.3, 8.4, 8.5 and 8.6 against the methods of Kumar & Kumari

Table 8.3: Comparison of $E_{N,M}^{\varepsilon,\mu}$ for Example 8.1 with $\mu = 2^{-4}$.

Number of intervals in space and time				
ε	$N = M = 64$	$N = M = 128$	$N = M = 256$	$N = M = 512$
$\downarrow$				
Present Approach				
2^{-6}	$1.39e-02$	$7.25e-03$	$3.72e-03$	$1.89e-03$
2^{-10}	$1.46e-02$	$7.51e-03$	$3.82e-03$	$1.93e-03$
2^{-14}	$1.48e-02$	$7.59e-03$	$3.85e-03$	$1.94e-03$
2^{-18}	$1.48e-02$	$7.61e-03$	$3.85e-03$	$1.94e-03$
2^{-22}	$1.48e-02$	$7.61e-03$	$3.86e-03$	$1.94e-03$
Results in Chandru et al. (2019)				
2^{-6}	$4.28e-03$	$1.87e-03$	$8.63e-04$	$4.14e-04$
2^{-10}	$3.62e-02$	$1.84e-02$	$8.96e-03$	$5.14e-03$
2^{-14}	$7.63e-02$	$4.69e-02$	$2.62e-02$	$1.39e-02$
2^{-18}	$8.23e-02$	$5.23e-02$	$3.00e-02$	$1.63e-02$
2^{-22}	$8.27e-02$	$5.27e-02$	$3.03e-02$	$1.65e-02$
Results in Kumar (2021)				
2^{-6}	$3.04e-03$	$1.55e-03$	$7.83e-04$	$3.94e-04$
2^{-10}	$1.86e-02$	$1.23e-02$	$7.52e-03$	$4.46e-03$
2^{-14}	$2.98e-02$	$1.80e-02$	$1.08e-02$	$6.35e-03$
2^{-18}	$3.13e-02$	$1.93e-02$	$1.18e-02$	$6.94e-03$
2^{-22}	$3.14e-02$	$1.94e-02$	$1.19e-02$	$6.98e-03$



(a) $\varepsilon = 1, \mu = 2^{-10}$



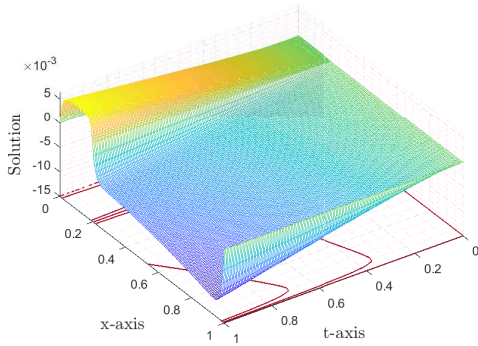
(b) $\mu = 2^{-10}, \varepsilon = 2^{-8}$

Figure 8.2: Numerical line plots illustrating Example 8.1 for several time increments.

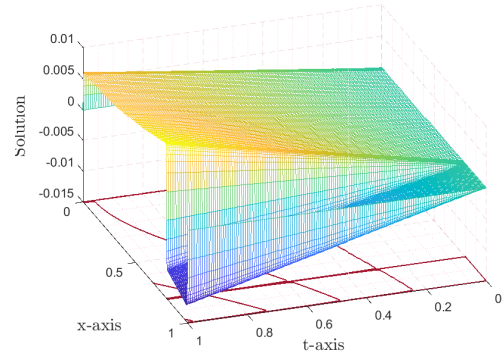
(2021); Singh et al. (2023); Chandru et al. (2019) further confirm that the proposed scheme attains superior accuracy. In Tables 8.4 and 8.6, we present numerical results for $E_{\varepsilon,\mu}^{N,M}, \bar{R}_{\varepsilon,\mu}^{N,M}$, taking different values of ε, μ and φ and these tables confirm our theoretical claims. Finally, both examples exhibit a pronounced solution transition

Table 8.4: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for different ε, μ and φ for Example 8.1.

Number of intervals in space and time						
		$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
φ	(ε, μ)	$M = 10$	$M = 20$	$M = 40$	$M = 80$	$M = 160$
Present Approach						
0.2	$(2^{-6}, 2^{-18})$	$7.1577e-02$	$3.9198e-02$	$2.1401e-02$	$1.133e-02$	$5.8576e-03$
		0.869	0.873	0.918	0.952	—
	$(2^{-12}, 2^{-28})$	$7.7493e-02$	$4.2412e-02$	$2.2892e-02$	$1.1917e-02$	$6.0871e-03$
		0.870	0.890	0.942	0.969	—
	Results in Singh et al. (2023)					
	$(2^{-6}, 2^{-18})$	$7.8576e-02$	$6.9196e-02$	$4.7451e-02$	$2.8080e-02$	$1.5321e-02$
0.183		0.544	0.757	0.874	—	
$(2^{-12}, 2^{-28})$	$1.1880e-01$	$7.3299e-02$	$4.1310e-02$	$2.2011e-02$	$1.1373e-02$	
	0.697	0.827	0.908	0.953	—	
Present Approach						
0.8	$(2^{-6}, 2^{-18})$	$6.1000e-02$	$3.8654e-02$	$2.1423e-02$	$1.1328e-02$	$5.8584e-03$
		0.658	0.851	0.919	0.951	—
	$(2^{-12}, 2^{-28})$	$7.7482e-02$	$4.2405e-02$	$2.2891e-02$	$1.1917e-02$	$6.0871e-03$
		0.870	0.890	0.942	0.969	—
	Results in Singh et al. (2023)					
	$(2^{-6}, 2^{-18})$	$7.8860e-02$	$4.4466e-02$	$2.3437e-02$	$1.2013e-02$	$6.0799e-03$
0.827		0.924	0.964	0.983	—	
$(2^{-12}, 2^{-28})$	$1.4226e-01$	$7.7775e-02$	$4.0788e-02$	$2.0889e-02$	$1.0571e-02$	
	0.871	0.931	0.965	0.983	—	



(a) $\varepsilon = 2^{-4}, \mu = 2^{-8}$ and $\varphi = 0.2$



(b) $\mu = 2^{-8}, \varepsilon = 2^{-10}$ and $\varphi = 0.8$

Figure 8.3: Computed solution profiles for Example 8.1 with $N = 256$ and $M = 160$, illustrating the influence of varying φ values.

near $x = 0.5$, a direct consequence of the discontinuity, which is effectively resolved by the Shishkin mesh and is evident in the accompanying solution plots. The computed solution profiles are illustrated as surface-contour and mesh plots in Figures 8.1, 8.3, 8.4, 8.6, and 8.7 for Examples 8.1 and 8.2, respectively. These figures clearly demonstrate the steep gradients of the numerical solution near the discontinuity points, particularly as the perturbation parameters ε and μ decrease. Such behavior is a

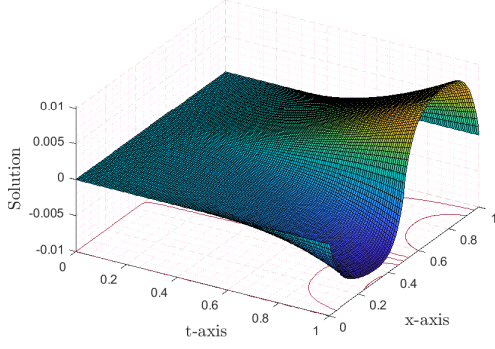
Table 8.5: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ when $\varepsilon = 2^{-8}$ for Example 8.2.

μ	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\downarrow$	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
1	$2.2303e-02$	$1.1082e-02$	$5.6730e-03$	$2.8751e-03$	$1.4485e-03$
2^{-2}	$2.2303e-02$	$1.1071e-02$	$5.6738e-03$	$2.8754e-03$	$1.4485e-03$
2^{-4}	$2.1148e-02$	$1.1066e-02$	$5.6735e-03$	$2.8754e-03$	$1.4485e-03$
2^{-6}	$2.1087e-02$	$1.1063e-02$	$5.6732e-03$	$2.8754e-03$	$1.4485e-03$
2^{-8}	$2.1095e-02$	$1.1064e-02$	$5.6730e-03$	$2.8754e-03$	$1.4485e-03$
2^{-10}	$2.1097e-02$	$1.1065e-02$	$5.6731e-03$	$2.8754e-03$	$1.4485e-03$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-22}	$2.1097e-02$	$1.1065e-02$	$5.6731e-03$	$2.8754e-03$	$1.4485e-03$
2^{-24}	$2.1097e-02$	$1.1065e-02$	$5.6731e-03$	$2.8754e-03$	$1.4485e-03$
$E^{N,M}$	$2.2303e-02$	$1.1082e-02$	$5.6738e-03$	$2.8754e-03$	$1.4485e-03$
$\bar{R}^{N,M}$	1.01	0.966	0.981	0.989	—
Result in Chandru et al. (2018)					
$E^{N,M}$		$1.33874e-02$	$1.16318e-02$	$8.67966e-03$	$5.89059e-03$
$\bar{R}^{N,M}$		0.202	0.422	0.559	—

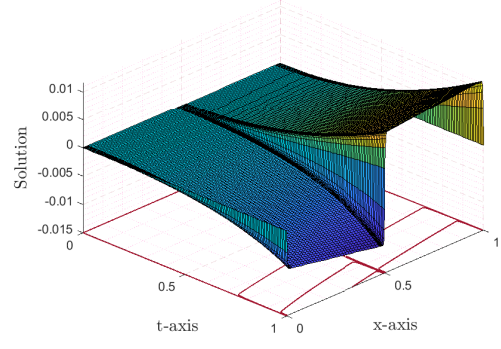
Table 8.6: Computed $E_{\varepsilon,\mu}^{N,M}$ and $\bar{R}_{\varepsilon,\mu}^{N,M}$ for different ε, μ and φ for Example 8.2.

Number of intervals in space and time						
		$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
φ	(ε, μ)	$M = 10$	$M = 20$	$M = 40$	$M = 80$	$M = 160$
Present Approach						
0.2	$(2^{-6}, 2^{-18})$	$5.6359e-02$	$3.1368e-02$	$1.6956e-02$	$8.8405e-03$	$4.5256e-03$
		0.869	0.873	0.918	0.952	—
	$(2^{-12}, 2^{-28})$	$5.8369e-02$	$3.2425e-02$	$1.7407e-02$	$9.0228e-03$	$4.5945e-03$
		0.848	0.897	0.948	0.974	—
Results in Singh et al. (2023)						
	$(2^{-6}, 2^{-18})$	$6.7015e-02$	$6.1676e-02$	$4.3579e-02$	$2.6251e-02$	$1.4462e-02$
		0.120	0.501	0.731	0.860	—
	$(2^{-12}, 2^{-28})$	$3.7093e-02$	$2.8247e-02$	$1.8060e-02$	$9.9259e-03$	$5.2073e-02$
		0.393	0.645	0.864	0.931	—
Present Approach						
0.8	$(2^{-6}, 2^{-18})$	$5.2613e-02$	$3.1062e-02$	$1.7031e-02$	$8.9668e-03$	$4.6131e-03$
		0.760	0.867	0.926	0.959	—
	$(2^{-12}, 2^{-28})$	$5.8369e-02$	$3.2582e-02$	$1.7746e-02$	$9.2488e-03$	$4.7213e-03$
		0.841	0.877	0.940	0.970	—
Results in Singh et al. (2023)						
	$(2^{-6}, 2^{-18})$	$5.6605e-02$	$4.4466e-02$	$1.7632e-02$	$9.0974e-03$	$4.6191e-03$
		0.779	0.904	0.955	0.978	—
	$(2^{-12}, 2^{-28})$	$3.9828e-02$	$3.1977e-02$	$1.9895e-02$	$1.0191e-02$	$5.1934e-03$
		0.317	0.685	0.965	0.973	—

direct indication of the presence of boundary and interior layers, a common feature in SPPs involving discontinuous data. The narrowing of these layers with decreasing ε

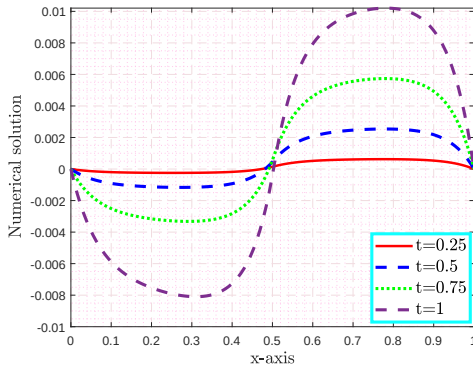


(a) $\varepsilon = 1, \mu = 2^{-12}$

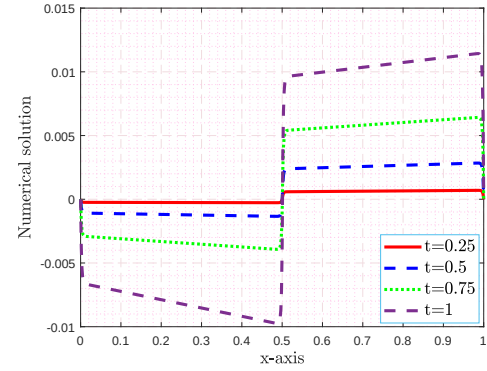


(b) $\mu = 2^{-8}, \varepsilon = 2^{-10}$

Figure 8.4: The computed solution profiles for Example 8.2 using $N = M = 128$.

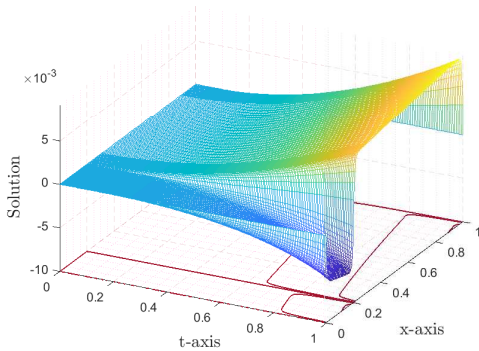


(a) $\varepsilon = 1, \mu = 2^{-12}$

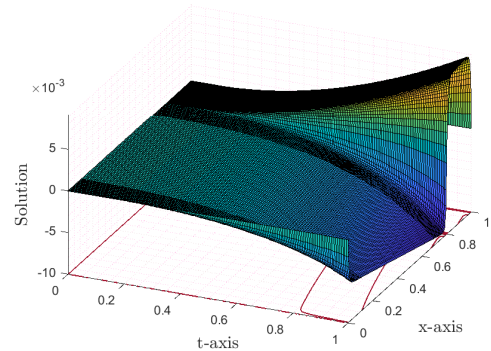


(b) $\mu = 2^{-8}, \varepsilon = 2^{-10}$

Figure 8.5: Numerical line plots illustrating Example 8.2 for several time increments.



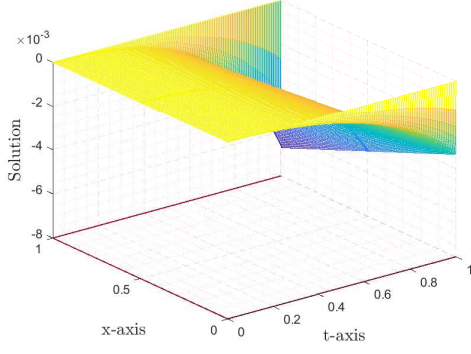
(a) $\varepsilon = 2^{-6}, \mu = 2^{-18}$ and $\varphi = 0.2$



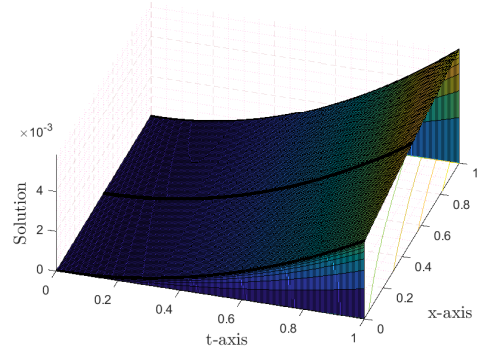
(b) $\mu = 2^{-18}, \varepsilon = 2^{-6}$ and $\varphi = 0.8$

Figure 8.6: Computed solution profiles for Example 8.2 with $N = 256$ and $M = 160$, illustrating the influence of varying φ values.

or μ (see Figures 8.1a and 8.4a) highlights the sensitivity of the solution to the perturbation parameters, emphasizing the necessity of fitted mesh approaches for accurate numerical resolution. In addition, surface and line plots for various discontinuity po-

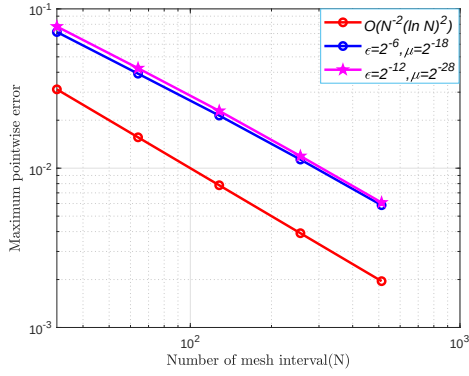


(a) $\varpi(x, t) = (e^{t^2} - 1)(1 + xt)$

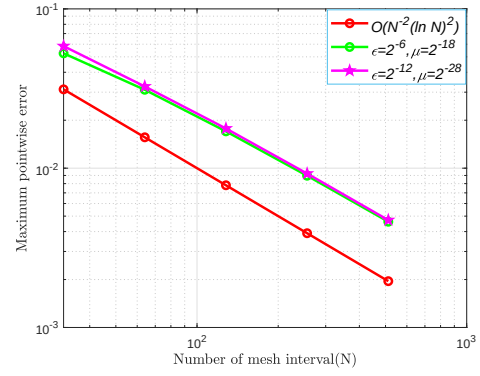


(b) $\varpi(x, t) = -t^2(2 + x)$

Figure 8.7: Computed solution profiles for Example 8.2 with $N = M = 256$, $\varepsilon = 2^{-12}$, $\mu = 2^{-8}$, considering multiple continuous source terms.



(a) Example 8.1



(b) Example 8.2

Figure 8.8: Log-log plot of Maximum pointwise errors.

sitions φ are displayed in Figures 8.3 and 8.6 corresponding to Examples 8.1 and 8.2. The line plots at multiple time levels with $N = M = 128$ (Figures 8.2 and 8.5) indicate that the maximum numerical error predominantly occurs within boundary and interior layer regions, where rapid variations in the solution profile are observed. The plots further confirm that reducing ε and μ produces sharper layer transitions, validating the theoretical predictions regarding the dependence of layer thickness on the perturbation magnitudes. The findings are consistent with earlier results reported in O’Riordan & Shishkin (2004), which concluded that interior layers arise primarily due to discontinuities in the convection coefficient, provided that the source term $\varpi(x, t)$ remains continuous. This numerical evidence enhances our understanding of the interplay between discontinuous coefficients and solution layer structures. As observed in Figure 8.7, if the discontinuity appears solely in the convection coefficient

(with a smooth $\varpi(x, t)$), the resulting solution tends to exhibit only boundary layers. To further analyze this behavior, the discontinuous source functions in Example 8.2 were replaced by smooth functions, for generating Figures 8.7a and 8.7b, respectively. The results show that while the discontinuity in the convection coefficient induces a shift in the layer position, no distinct interior layer forms when the source term is smooth. This observation reveals that discontinuities in the source term are a key factor in determining the interior layers. Moreover, to validate the theoretical convergence behavior of the proposed method, error distributions are depicted on a Log-Log scale in Figures 8.8a and 8.8b for Examples 8.1 and 8.2, respectively. The reference curve being below the computed errors is expected and confirms the presence of strong layers and non-smooth data. It does not contradict uniform convergence; it reflects layer-controlled behavior in the maximum norm.

8.6 Conclusion

In this chapter, an implicit numerical scheme constructed on a predefined Shishkin mesh has been developed for the numerical solution of two-parameter singularly perturbed parabolic problems with discontinuous convection coefficients and source terms. The coexistence of boundary and interior layers leads to highly complex solution structures, setting these problems apart from their smooth-data counterparts. The proposed approach combines the implicit Euler method for time discretization with a cubic spline in compression for spatial approximation on a fitted mesh. Rigorous theoretical analysis, supported by extensive numerical experiments, demonstrates that the scheme is uniformly stable and robust with respect to all perturbation parameters, attaining first-order accuracy in time and second-order accuracy in space up to a logarithmic factor. The graphical illustrations clearly capture boundary layers along the lateral boundaries and pronounced interior layers near the discontinuity point $x = \varphi$, confirming the method's reliability in resolving steep gradients. Overall, the proposed scheme exhibits performance comparable to, and in several cases superior to existing numerical methods, thereby establishing it as a reliable and reliable tool for solving singularly perturbed problems with discontinuous data.

Chapter 9

General Conclusions and Recommendations

This chapter synthesizes the principal findings of the dissertation and delineates promising directions for advancing the present research.

9.1 General Conclusions

This dissertation has explored a challenging class of SPPDEs that incorporate time delays, discontinuous convection coefficients and source terms. The presence of two small perturbation parameters in both the diffusion and convection terms renders these problems highly nontrivial, giving rise to complex multiscale behaviors across spatial and temporal domains. Such problems frequently arise in heat transfer, fluid dynamics, and reaction-diffusion systems, where abrupt physical changes coexist with small-scale processes. By developing both theoretical foundations and computational frameworks for delay-dependent and discontinuous problems, this work contributes meaningfully to the ongoing advancement of multi-parameter singular perturbation theory.

A key theoretical result established in this study is the existence and uniqueness of a sufficiently smooth analytical solution under appropriate smoothness and compatibility conditions, particularly at the spatial boundaries, points of discontinuity, and delay interfaces. This analytical foundation enables the formulation of rigorous asymptotic and numerical analysis. Due to the combined influence of the two perturbation pa-

rameters and discontinuous problem data, the solutions exhibit multiple boundary layers of varying thicknesses, along with interior layers whose locations and intensities depend on the discontinuities and the relative magnitudes of the perturbation parameters. These intricate layer structures reflect the essence of the underlying physical phenomena, regions of rapid local variation smoothly transitioning into broader, globally stable behaviors.

The core contribution of this research lies in the construction and analysis of uniform numerical schemes designed to solve these complex classes of SPPDEs with and without time delay. Six distinct fitted operator and fitted mesh schemes have been formulated and examined. The fitted operator methods, encompassing exponential fitting and nonstandard finite difference formulations, are developed and analyzed in Chapters 3-7, while the fitted mesh methods are presented and studied in Chapter 8. All schemes are systematically built using Rothé’s technique, which discretizes the time variable prior to spatial discretization, ensuring both stability and consistency in the numerical process.

The effectiveness and robustness of the proposed numerical techniques have been thoroughly validated through a series of carefully designed test problems that satisfy the governing equations’ requirements. For each example, numerical errors and corresponding convergence rates are tabulated, and the solution profiles are visualized through mesh, surface, and line plots. Comparative analyses with existing numerical methods in the literature demonstrate that the proposed schemes achieve higher accuracy, uniform convergence, and numerical stability. Among the two categories, the fitted operator methods, notably the exponential fitted operator and nonstandard finite difference schemes exhibit relatively better performance compared to the fitted mesh approaches. Graphical results further illustrate the impact of the perturbation parameters ε, μ and discontinuity point on the formation and evolution of boundary and interior layers, offering valuable physical insight.

In summary, this dissertation presents a comprehensive, step-by-step framework for designing robust numerical schemes applicable to a broad range of delay-dependent, discontinuous SPPs. The proposed algorithms are not only mathematically rigorous and computationally efficient but also accessible for implementation, allowing future

researchers to adapt and extend them to even more complex real-world systems. Ultimately, this study enhances the understanding of multiscale parabolic dynamics and provides a reliable computational foundation for further exploration of SPPs in applied science and engineering.

9.2 Recommendations for the Future Work

This dissertation has developed and rigorously examined numerical schemes for $1D$, time-dependent TP-SPPs with and without time delays and non-smooth data. By integrating Mickens' nonstandard finite difference approach, exponentially fitted operators, and piecewise-uniform Shishkin meshes, the study provides a stable and better accurate computational framework capable of resolving sharp boundary and interior layers while preserving uniform convergence. The resulting methods are broadly applicable to fields such as fluid dynamics, quantum mechanics, reaction–diffusion systems, and semiconductor modeling, where singular perturbations and discontinuities strongly influence system behavior. Looking ahead, several promising directions emerge, including extending the schemes to higher-order models, incorporating more advanced layer-adapted meshes for enhanced resolution, adapting the methodology to nonlinear problems with turning points and fractional-order derivatives, integrating modern computational tools such as machine learning, and developing user-friendly software implementations. Furthermore, other numerical schemes devised to solve the problem under consideration in the future may include the finite element method where complex geometries are involved, higher-order B-spline collocations method to achieve better results.

References

- Adomian, G., & Rach, R. (1983). Nonlinear stochastic differential delay equations. *Journal of Mathematical Analysis and Applications*, 91(1), 94–101.
- Aga, Duressa, G. F., & Degla, G. (2021). Accelerated fitted operator finite difference method for singularly perturbed parabolic reaction-diffusion problems. *Computational Methods for Differential Equations*, 9(3), 886–898.
- Aga, File, G., & Degla, G. (2019). Fitted operator average finite difference method for singularly perturbed parabolic convection-diffusion problem. *International Journal of Engineering and Applied Sciences*, 11(3), 414–427.
- Allen, D., & Southwell, R. (1955). Relaxation methods applied to determine the motion, in two dimensions, of a viscous fluid past a fixed cylinder. *The Quarterly Journal of Mechanics and Applied Mathematics*, 8(2), 129–145.
- Andisso, F. S., & Duressa, G. F. (2024). Graded mesh modified backward finite difference method for two parameters singularly perturbed second-order boundary value problems. *Results in Applied Mathematics*, 22, 100442.
- Ansari, A., Bakr, S., & Shishkin, G. (2007). A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations. *Journal of computational and applied mathematics*, 205(1), 552–566.
- Ayele, M. A., Tiruneh, A. A., & Derese, G. A. (2023). Fitted cubic spline scheme for two-parameter singularly perturbed time-delay parabolic problems. *Results in Applied Mathematics*, 18, 100361.
- Aziz, T., & Khan, A. (2002). A spline method for second-order singularly perturbed boundary-value problems. *Journal of Computational and Applied Mathematics*, 147(2), 445–452.
- Bakhvalov, N. S. (1969). The optimization of methods of solving boundary value problems with a boundary layer. *USSR Computational Mathematics and Mathematical Physics*, 9(4), 139–166.
- Baranowski, J. (2010). Legendre polynomial approximations of time delay systems. In *Proceeding of 12th international phd workshop* (Vol. 1, pp. 15–20).
- Bashier, E. B., & Patidar, K. C. (2011). A novel fitted operator finite difference method for a singularly perturbed delay parabolic partial differential equation. *Applied mathematics and Computation*, 217(9), 4728–4739.

- Baumert, H., Braun, P., Glos, E., Müller, W.-D., & Stoyan, G. (2005). Modelling and computation of water quality problems in river networks. In *Optimization techniques: Proceedings of the 9th ifip conference on optimization techniques warsaw, september 4–8, 1979* (pp. 482–491).
- Bawa, R. K. (2005). Spline based computational technique for linear singularly perturbed boundary value problems. *Applied mathematics and computation*, 167(1), 225–236.
- Bellen, A., & Zennaro, M. (2013). *Numerical methods for delay differential equations*. Numerical Mathematics and Scie.
- Bender, C. M., & Orszag, S. A. (1999). *Asymptotic methods and perturbation theory. (No Title)*.
- Bender, C. M., & Orszag, S. A. (2013). *Advanced mathematical methods for scientists and engineers i: Asymptotic methods and perturbation theory*. Springer Science & Business Media.
- Bhathawala, P., & Verma, A. (1975). A two-parameter singular perturbation solution of one dimension flow through unsaturated porous media. *Appl Math*, 43(5), 380–4.
- Bobisud, L. (1968). Second-order linear parabolic equations with a small parameter. *Archive for Rational Mechanics and Analysis*, 27(5), 385–397.
- Bullo, Duressa, G., & Degla, G. (2019). Higher order fitted operator finite difference method for two-parameter parabolic convection-diffusion problems. *International Journal of Engineering and Applied Sciences*, 11(4), 455–467.
- Bullo, Duressa, G. F., & Degla, G. A. (2021). Robust finite difference method for singularly perturbed two-parameter parabolic convection-diffusion problems. *International Journal of Computational Methods*, 18(02), 2050034.
- Bush, A. (2018). *Perturbation methods for engineers and scientists*. Routledge.
- Chandru, M., Das, P., & Ramos, H. (2018). Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non-smooth data. *Mathematical Methods in the Applied Sciences*, 41(14), 5359–5387.
- Chandru, M., Prabha, T., Das, P., & Shanthi, V. (2019). A numerical method for solving boundary and interior layers dominated parabolic problems with discontinuous convection coefficient and source terms. *Differential Equations and Dynamical Systems*, 27(1), 91–112.
- Chandru, M., Prabha, T., & Shanthi, V. (2017). A parameter robust higher order numerical method for singularly perturbed two parameter problems with non-smooth data. *Journal of Computational and Applied Mathematics*, 309, 11–27.
- Clavero, C., Gracia, J., & Jorge, J. (2005). High-order numerical methods for one-dimensional parabolic singularly perturbed problems with regular layers. *Numerical Methods For Partial Differential Equations: An International Journal*, 21(1), 149–169.

- Daba, I. T., & Duressa, G. F. (2022). Computational method for singularly perturbed parabolic differential equations with discontinuous coefficients and large delay. *Heliyon*, 8(9).
- Daba, I. T., Melesse, W. G., & Kebede, G. D. (2024). Third-degree b-spline collocation method for singularly perturbed time delay parabolic problem with two parameters. *Frontiers in Applied Mathematics and Statistics*, 9, 1260651.
- Das, & Mehrmann, V. (2016). Numerical solution of singularly perturbed convection-diffusion-reaction problems with two small parameters. *BIT Numerical Mathematics*, 56(1), 51–76.
- Das, & Natesan, S. (2012). Higher-order parameter uniform convergent schemes for robin type reaction-diffusion problems using adaptively generated grid. *International Journal of Computational Methods*, 9(04), 1250052.
- Das, & Natesan, S. (2013a). Numerical solution of a system of singularly perturbed convection diffusion boundary value problems using mesh equidistribution technique. *Aust. J. Math. Anal. Appl.*, 10(1), 1–17.
- Das, & Natesan, S. (2013b). A uniformly convergent hybrid scheme for singularly perturbed system of reaction-diffusion robin type boundary-value problems. *Journal of Applied Mathematics and Computing*, 41, 447–471.
- Das, & Natesan, S. (2015). Uniformly convergent hybrid numerical scheme for singularly perturbed delay parabolic convection–diffusion problems on shishkin mesh. *Applied Mathematics and Computation*, 271, 168–186.
- Das, & Natesan, S. (2018). Second-order uniformly convergent numerical method for singularly perturbed delay parabolic partial differential equations. *International Journal of Computer Mathematics*, 95(3), 490–510.
- Doolan, E. P., Miller, J. J., & Schilders, W. H. (1980). *Uniform numerical methods for problems with initial and boundary layers*. Boole Press.
- Duressa, G. F. (2021). Novel approach to solve singularly perturbed boundary value problems with negative shift parameter. *Heliyon*, 7(7).
- Duressa, G. F., Daba, I. T., & Deressa, C. T. (2023). A systematic review on the solution methodology of singularly perturbed differential difference equations. *Mathematics*, 11(5), 1108.
- Duressa, G. F., & Mekonnen, T. B. (2023). An exponentially fitted method for two parameter singularly perturbed parabolic boundary value problems. *Communications of the Korean Mathematical Society*, 38(1), 299–318.
- Eckhaus, W. (2011). *Matched asymptotic expansions and singular perturbations* (Vol. 6). Elsevier.
- Epstein, I. R. (1992). Delay effects and differential delay equations in chemical kinetics. *International Reviews in Physical Chemistry*, 11(1), 135–160.

- Eugene, M. J. J. O., & Shishkin, G. (2012). *Fitted numerical methods for singular perturbation problems: Error estimates in the maximum norm for linear problems in one and two dimensions (revised edition): Error estimates in the maximum norm for linear problems in one and two dimensions*. World Scientific Publishing Company.
- Ewing, R. E. (1983). Problems arising in the modeling of processes for hydrocarbon recovery. In *The mathematics of reservoir simulation* (pp. 3–34). SIAM.
- Farrell, P., Hegarty, A., Miller, J. M., O’Riordan, E., & Shishkin, G. I. (2000). *Robust computational techniques for boundary layers*. Chapman and hall/CRC.
- Friedrichs, K. O., & Wasow, W. R. (1946). Singular perturbations of non-linear oscillations.
- Ganesh Kumar, V., & Phaneendra, K. (2020). Computational technique for two parameter singularly perturbed parabolic convection-diffusion problem. *J. Math. Comput. Sci.*, 10(4), 1251–1261.
- Glizer, V. (2000). Asymptotic solution of a boundary-value problem for linear singularly-perturbed functional differential equations arising in optimal control theory. *Journal of optimization theory and applications*, 106(2), 309–335.
- Govindarao, L., & Mohapatra, J. (2020). Numerical analysis and simulation of delay parabolic partial differential equation involving a small parameter. *Engineering Computations*, 37(1), 289–312.
- Govindarao, L., Sahu, S. R., & Mohapatra, J. (2019). Uniformly convergent numerical method for singularly perturbed time delay parabolic problem with two small parameters. *Iranian Journal of Science and Technology, Transactions A: Science*, 43(5), 2373–2383.
- Gowrisankar, S., & Natesan, S. (2017). ε -uniformly convergent numerical scheme for singularly perturbed delay parabolic partial differential equations. *International Journal of Computer Mathematics*, 94(5), 902–921.
- Gupta, V., Kadalbajoo, M. K., & Dubey, R. K. (2019). A parameter-uniform higher order finite difference scheme for singularly perturbed time-dependent parabolic problem with two small parameters. *International Journal of Computer Mathematics*, 96(3), 474–499.
- Gupta, V., Kumar, M., & Kumar, S. (2018). Higher order numerical approximation for time dependent singularly perturbed differential-difference convection-diffusion equations. *Numerical Methods for Partial Differential Equations*, 34(1), 357–380.
- Hahn, S.-Y., Bignon, J., & Sabonnadiere, J.-C. (1987). An upwind finite element method for electromagnetic field problems in moving media. *International journal for numerical methods in engineering*, 24(11), 2071–2086.
- Hirsch, C. (1990). Numerical computation of internal and external flows. *Computational methods for inviscid and viscous flows*, 2.

- Holmes, M. H. (2007). *Introduction to numerical methods in differential equations*. Springer.
- Holmes, M. H. (2012). *Introduction to perturbation methods* (Vol. 20). Springer Science & Business Media.
- Huan-wen, C. (1984). Some applications of the singular perturbation method to the bending problems of thin plates and shells. *Applied Mathematics and Mechanics*, 5(4), 1449–1457.
- Janani, & Tamilselvan, A. (2019). Hybrid difference scheme for singularly perturbed convection diffusion boundary turning point problems with discontinuous source term. *International Journal of Applied and Computational Mathematics*, 5(3), 88.
- Kadalbajoo, & Arora, P. (2009). B-spline collocation method for the singular-perturbation problem using artificial viscosity. *Computers & Mathematics with Applications*, 57(4), 650–663.
- Kadalbajoo, & Gupta, V. (2010). A brief survey on numerical methods for solving singularly perturbed problems. *Applied mathematics and computation*, 217(8), 3641–3716.
- Kadalbajoo, & Jha, A. (2012). Exponentially fitted cubic spline for two-parameter singularly perturbed boundary value problems. *International Journal of Computer Mathematics*, 89(6), 836–850.
- Kadalbajoo, & Patidar, K. C. (2002). A survey of numerical techniques for solving singularly perturbed ordinary differential equations. *Applied Mathematics and Computation*, 130(2-3), 457–510.
- Kadalbajoo, & Patidar, K. C. (2003). Singularly perturbed problems in partial differential equations: a survey. *Applied Mathematics and Computation*, 134(2-3), 371–429.
- Kadalbajoo, & Reddy, Y. (1989). Asymptotic and numerical analysis of singular perturbation problems: a survey. *Applied Mathematics and computation*, 30(3), 223–259.
- Kadalbajoo, & Yadaw, A. S. (2012). Parameter-uniform finite element method for two-parameter singularly perturbed parabolic reaction-diffusion problems. *International Journal of computational methods*, 9(04), 1250047.
- Kanth, A. R., & Murali, M. K. P. (2018). A numerical technique for solving nonlinear singularly perturbed delay differential equations. *Mathematical Modelling and Analysis*, 23(1), 64–78.
- Kevorkian, J., & Cole, J. D. (2013). *Perturbation methods in applied mathematics* (Vol. 34). Springer Science & Business Media.
- Khan, A., Khan, I., Aziz, T., & Stojanovic, M. (2004). A variable-mesh approximation method for singularly perturbed boundary-value problems using cubic spline in tension. *International Journal of Computer Mathematics*, 81(12), 1513–1518.

- Kuang, Y. (1993). *Delay differential equations: with applications in population dynamics* (Vol. 191). Academic press.
- Kuerten, J. v., Van der Geld, C., & Geurts, B. J. (2011). Turbulence modification and heat transfer enhancement by inertial particles in turbulent channel flow. *Physics of fluids*, 23(12).
- Kumar. (2021). A parameter-uniform scheme for the parabolic singularly perturbed problem with a delay in time. *Numerical Methods for Partial Differential Equations*, 37(1), 626–642.
- Kumar, & Gupta, Y. (2010). Methods for solving singular boundary value problems using splines: a review. *Journal of Applied Mathematics and Computing*, 32(1), 265–278.
- Kumar, & Kumari, P. (2019). A parameter-uniform numerical scheme for the parabolic singularly perturbed initial boundary value problems with large time delay. *Journal of Applied Mathematics and Computing*, 59(1), 179–206.
- Kumar, & Kumari, P. (2020). A parameter-uniform scheme for singularly perturbed partial differential equations with a time lag. *Numerical Methods for Partial Differential Equations*, 36(4), 868–886.
- Kumar, & Kumari, P. (2021). Uniformly convergent scheme for two-parameter singularly perturbed problems with non-smooth data. *Numerical Methods for Partial Differential Equations*, 37(1), 796–817.
- Kumar, & Ravi Kanth, A. (2020). Computational study for a class of time-dependent singularly perturbed parabolic partial differential equation through tension spline. *Computational and Applied Mathematics*, 39(3), 233.
- Kumar, Singh, P., & Mishra, H. K. (2007). A recent survey on computational techniques for solving singularly perturbed boundary value problems. *International Journal of Computer Mathematics*, 84(10), 1439–1463.
- Kumar, Stecher, G., Li, M., Knyaz, C., & Tamura, K. (2018). Mega x: molecular evolutionary genetics analysis across computing platforms. *Molecular biology and evolution*, 35(6), 1547–1549.
- Ladyzhenskaia, O. A., Solonnikov, V. A., & Ural'tseva, N. N. (1968). *Linear and quasi-linear equations of parabolic type* (Vol. 23). American Mathematical Soc.
- Lange, C. G., & Miura, R. M. (1982). Singular perturbation analysis of boundary value problems for differential-difference equations. *SIAM Journal on Applied Mathematics*, 42(3), 502–531.
- LeVeque, R. J. (1998). Finite difference methods for differential equations. *Draft version for use in AMath*, 585(6), 112.
- Li, Z., Qiao, Z., & Tang, T. (2017). *Numerical solution of differential equations: introduction to finite difference and finite element methods*. Cambridge University.

- Linß, T. (2009). *Layer-adapted meshes for reaction-convection-diffusion problems*. Springer.
- Longtin, A., & Milton, J. G. (1988). Complex oscillations in the human pupil light reflex with mixed and delayed feedback. *Mathematical Biosciences*, 90(1-2), 183–199.
- Ma, H., & Duan, Z. (2020). Similarities of flow and heat transfer around a circular cylinder. *Symmetry*, 12(4), 658.
- Mahaffy, J. M., Bélair, J., & Mackey, M. C. (1998). Hematopoietic model with moving boundary condition and state dependent delay: applications in erythropoiesis. *Journal of theoretical biology*, 190(2), 135–146.
- Marchuk, G. I. (1997). *Mathematical modelling of immune response in infectious diseases* (Vol. 395). Springer Science & Business Media.
- Mekonnen, T. B., & Duressa, G. F. (2020). Computational method for singularly perturbed two-parameter parabolic convection-diffusion problems. *Cogent Mathematics & Statistics*, 7(1), 1829277.
- Mickens, R. E. (1993). *Nonstandard finite difference models of differential equations*. world scientific.
- Mickens, R. E. (2005). *Advances in the applications of nonstandard finite difference schemes*. World Scientific.
- Mickens, R. E. (2020). *Nonstandard finite difference schemes: methodology and applications*. World Scientific.
- Miller. (2006). *Applied asymptotic analysis* (Vol. 75). American Mathematical Soc.
- Miller, O’riordan, E., Shishkin, G., & Shishkina, L. (1998). Fitted mesh methods for problems with parabolic boundary layers. In *Mathematical proceedings of the royal irish academy* (pp. 173–190).
- Miller, O’riordan, E., & Shishkin, G. I. (1996). *Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions*. World scientific.
- Miller, Shishkin, G., Koren, B., & Shishkina, L. (2004). Grid approximation of a singularly perturbed boundary value problem modelling heat transfer in the case of flow over a flat plate with suction of the boundary layer. *Journal of computational and applied mathematics*, 166(1), 221–232.
- Mohye, M. A., Munyakazi, J. B., & Dinka, T. G. (2023). A nonstandard fitted operator finite difference method for two-parameter singularly perturbed time-delay parabolic problems. *Frontiers in Applied Mathematics and Statistics*, 9, 1222162.
- Morton, K. W. (1996). Numerical solution of convection-diffusion problems. (*No Title*).

- Mukherjee, K., & Natesan, S. (2008). An efficient numerical scheme for singularly perturbed parabolic problems with interior layer. *Neural, Parallel and Scientific Computations*, 16(3), 405.
- Munyakazi, J. B. (2009). *Higher order numerical methods*. Unpublished doctoral dissertation, Department of Mathematics and Applied Mathematics.
- Munyakazi, J. B. (2015). A robust finite difference method for two-parameter parabolic convection-diffusion problems. *Applied Mathematics & Information Sciences*, 9(6), 2877.
- Munyakazi, J. B., & Patidar, K. C. (2014). Performance of richardson extrapolation on some numerical methods for a singularly perturbed turning point problem whose solution has boundary layers. *Journal of the Korean Mathematical Society*, 51(4), 679–702.
- Nayfeh. (2000). *Perturbation methods* (wiley-vch, weinheim).
- Nayfeh, & Balachandran, B. (2008). *Applied nonlinear dynamics: analytical, computational, and experimental methods*. John Wiley & Sons.
- Negero, N. T. (2023). Fitted cubic spline in tension difference scheme for two-parameter singularly perturbed delay parabolic partial differential equations. *Partial Differential Equations in Applied Mathematics*, 8, 100530.
- Negero, N. T., Duressa, G. F., Rathour, L., & Mishra, V. N. (2023). A novel fitted numerical scheme for singularly perturbed delay parabolic problems with two small parameters. *Partial Differential Equations in Applied Mathematics*, 8, 100546.
- Neill, S. P., & Hashemi, M. R. (2018). *Fundamentals of ocean renewable energy: generating electricity from the sea*. Academic Press.
- Nichols, N. (1989). On the numerical integration of a class of singular perturbation problems. *Journal of optimization theory and applications*, 60(3), 439–452.
- Nohel, J. A., Sattinger, D. H., & Rota, G.-C. (1997). *Selected papers of norman levinson*. Springer Science & Business Media.
- Norkin, S. B., et al. (1973). *Introduction to the theory and application of differential equations with deviating arguments* (Vol. 105). Academic Press.
- O'malley, R. (1967). Two-parameter singular perturbation problems for second-order equations. *Journal of Mathematics and Mechanics*, 16(10), 1143–1164.
- O'Malley, R. E. (1974). *Introduction to singular perturbations* (Tech. Rep.). New York: University of Arizona.
- O'malley, R. E., et al. (1991). *Singular perturbation methods for ordinary differential equations* (Vol. 89). Springer.
- O'Riordan, E., Pickett, M. L., & Shishkin, G. I. (2003). Singularly perturbed problems modeling reaction-convection-diffusion processes. *Computational Methods in Applied Mathematics*, 3(3).

- O’Riordan, E., & Shishkin, G. (2004). Singularly perturbed parabolic problems with non-smooth data. *Journal of computational and applied mathematics*, 166(1), 233–245.
- Patidar, K. C. (2008). A robust fitted operator finite difference method for a two-parameter singular perturbation problem1. *Journal of Difference Equations and Applications*, 14(12), 1197–1214.
- Peiró, J., & Sherwin, S. (2005). Finite difference, finite element and finite volume methods for partial differential equations. In *Handbook of materials modeling: Methods* (pp. 2415–2446). Springer.
- Phaneendra, K., & Lalu, M. (2019a). Gaussian quadrature for two-point singularly perturbed boundary value problems with exponential fitting. *Communications in Mathematics and Applications*(3), 447.
- Phaneendra, K., & Lalu, M. (2019b). Numerical solution of singularly perturbed delay differential equations using gaussian quadrature method. In *Journal of physics: Conference series* (Vol. 1344, p. 012013).
- Phaneendra, K., & Mahesh, G. (2017). Solution of two parameters singular perturbation problem using higher order compact numerical method. *Global and Stochastic Analysis*, 4(2), 225–236.
- Podila, P. C., & Kumar, K. (2020). A new stable finite difference scheme and its convergence for time-delayed singularly perturbed parabolic pdes. *Computational and Applied Mathematics*, 39(3), 140.
- Polak, S., Den Heijer, C., Schilders, W., & Markowich, P. (1987). Semiconductor device modelling from the numerical point of view. *International Journal for Numerical Methods in Engineering*, 24(4), 763–838.
- Prabha, T., Chandru, M., Shanthi, V., & Ramos, H. (2019). Discrete approximation for a two-parameter singularly perturbed boundary value problem having discontinuity in convection coefficient and source term. *Journal of Computational and Applied Mathematics*, 359, 102–118.
- Pramanik, S. (2014). Casson fluid flow and heat transfer past an exponentially porous stretching surface in presence of thermal radiation. *Ain shams engineering journal*, 5(1), 205–212.
- Prandtl, L. (1905). Über flüssigkeitsbewegung bei sehr kleiner reibung. *Verhandl. 3rd Int. Math. Congr. Heidelberg (1904)*, Leipzig.
- Priyadarshana, S., Mohapatra, J., & Pattanaik, S. (2022). Parameter uniform optimal order numerical approximations for time-delayed parabolic convection diffusion problems involving two small parameters. *Computational and Applied Mathematics*, 41(6), 233.
- Pruess, S. (1976). Properties of splines in tension. *Journal of Approximation Theory*, 17(1), 86–96.

- Pudykiewicz, J. (1989). Simulation of the chernobyl dispersion with a 3-d hemispheric tracer model. *Tellus B*, 41(4), 391–412.
- Roos, H.-G. (2008). *Robust numerical methods for singularly perturbed differential equations*. Springer.
- Roos, H.-G. (2012). Robust numerical methods for singularly perturbed differential equations: a survey covering 2008–2012. *International Scholarly Research Notices*, 2012(1), 379547.
- Roos, H.-G., & Uzelac, Z. (2003). The sdfem for a convection-diffusion problem with two small parameters. *Computational Methods in Applied Mathematics*, 3(3), 443–458.
- Roy, N., & Jha, A. (2023). A parameter uniform method for two-parameter singularly perturbed boundary value problems with discontinuous data. *MethodsX*, 10, 102004.
- Salama, A., & Al-Amery, D. (2017). A higher order uniformly convergent method for singularly perturbed delay parabolic partial differential equations. *International Journal of Computer Mathematics*, 94(12), 2520–2546.
- Sastry, S. S. (2012). *Introductory methods of numerical analysis*. PHI Learning Ltd.
- Schlichting, H., & Gersten, K. (2016). *Boundary-layer theory*. Heidelberg, Germany: Springer.
- Schlichting, H., Truckenbrodt, E., & Ramm, H. J. (1979). Aerodynamics of the airplane. (*No Title*).
- Shishkin, G. I. (1989). Approximation of the solutions of singularly perturbed boundary-value problems with a parabolic boundary layer. *USSR Computational Mathematics and Mathematical Physics*, 29(4), 1–10.
- Shishkin, G. I., & Shishkina, L. P. (2008). *Difference methods for singular perturbation problems*. Chapman and Hall/CRC.
- Shivhare, M., Podila, P. C., & Kumar, D. (2021). A uniformly convergent quadratic b-spline collocation method for singularly perturbed parabolic partial differential equations with two small parameters. *Journal of Mathematical Chemistry*, 59(1), 186–215.
- Singh, Choudhary, R., & Kumar, D. (2023). An efficient numerical technique for two-parameter singularly perturbed problems having discontinuity in convection coefficient and source term. *Computational and Applied Mathematics*, 42(1), 62.
- Singh, Kumar, S., & Kumar, M. (2018). A domain decomposition method for solving singularly perturbed parabolic reaction-diffusion problems with time delay. *Numerical Methods for Partial Differential Equations*, 34(5), 1849–1866.
- Singh, Kumari, P., & Kumar, D. (2022). An effective numerical approach for two parameter time-delayed singularly perturbed problems. *Computational and Applied Mathematics*, 41(8), 337.

- Stein, R. B. (1967). Some models of neuronal variability. *Biophysical*, 7(1), 37–68.
- Sumit, Kumar, S., Kuldeep, & Kumar, M. (2020). A robust numerical method for a two-parameter singularly perturbed time delay parabolic problem. *Computational and Applied Mathematics*, 39(3), 209.
- Tefera, D. M., Tiruneh, A. A., & Derese, G. A. (2022). Fitted operator method over gaussian quadrature formula for parabolic singularly perturbed convection-diffusion problem. *Numerical Analysis and Applications*, 15(3), 256–269.
- Teofanov, L., & Roos, H.-G. (2007). An elliptic singularly perturbed problem with two parameters i: Solution decomposition. *Journal of computational and applied mathematics*, 206(2), 1082–1097.
- Tuckwell, H. C. (1989). *Stochastic processes in the neurosciences*. SIAM.
- Van Dyke, M. (1964). Higher approximations in boundary-layer theory part 3. parabola in uniform stream. *Journal of Fluid Mechanics*, 19(1), 145–159.
- Van Dyke, M. (1975). Perturbation methods in fluid mechanics/annotated edition. *NASA STI/Recon Technical Report A*, 75, 46926.
- Van Dyke, M. (1994). Nineteenth-century roots of the boundary-layer idea. *Siam Review*, 36(3), 415–424.
- Van Harten, A., & Schumacher, J. (1978). On a class of partial functional differential equations arising in feed-back control theory. In *North-holland mathematics studies* (Vol. 31, pp. 161–179). Elsevier.
- Verhulst, F. (2006). *Methods and applications of singular perturbations: boundary layers and multiple timescale dynamics* (Vol. 50). Springer Science & Business.
- Vulanović, R. (2001). A higher-order scheme for quasilinear boundary value problems with two small parameters. *Computing*, 67(4).
- Wang, P. K. C. (1963). Asymptotic stability of a time-delayed diffusion system.
- Wasow, W. R. (1942). *On boundary layer problems in the theory of ordinary differential equation*. New York University.
- Woldaregay, M. M., Hunde, T. W., & Mishra, V. N. (2023). Fitted exact difference method for solutions of a singularly perturbed time delay parabolic pde. *Partial Differential Equations in Applied Mathematics*, 8, 100556.
- Wolfshtein, M., Lin, A., & Naot, D. (1975). Models of turbulence. In: *Topics in transport phenomena: Bioprocesses*, 3–45.
- Wu, J. (2012). *Theory and applications of partial functional differential equations* (Vol. 119). Springer Science & Business Media.
- Zahra, W. K., & El Mhlawy, A. M. (2013). Numerical solution of two-parameter singularly perturbed boundary value problems via exponential spline. *Journal of King Saud University-Science*, 25(3), 201–208.
- Zak, S. H., et al. (2003). *Systems and control* (Vol. 198). Oxford university press New York.