



**Wallaga University
School of Graduate Studies**

**A Study on Domination in Turiyam Graphs with
Algorithms and Applications**

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**Doctor of Philosophy
in
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**A Study on Domination in Turiyam Graphs with
Algorithms and Applications**

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APPROVAL SHEET FOR SUBMITTING FINAL DISSERTATION

As members of the Board of Examining of the **Final PhD Dissertation** open defense, we certify that we have read and evaluated the dissertation prepared by **Abdata Guluma Erana** under the title "**A Study on Domination in Turiyam Graphs with Algorithms and Applications**" and recommend that the dissertation be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy in Applied Mathematics (Graph Theory and Algorithms).

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I hereby certify that all the corrections and recommendations suggested by the board of examiners are incorporated into the final Dissertation entitled "**A Study on Domination in Turiyam Graphs with Algorithms and Applications**" by **Abdata Guluma Erana**.

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Statement of the Author

I, Mr. Abdata Guluma Erana, hereby declare and affirm that the dissertation entitled "**A Study on Domination in Turiyam Graphs with Algorithms and Applications**" is my own work conducted under the supervision of Prof. Repalle V N SrinivasaRao and Dr. Fekadu Tesgara Agama. I have followed all the ethical principles of scholarship in the preparation, and completion of this dissertation. Every academic reference in the dissertation has been acknowledged through citation. I have adequately cited and referenced all the original sources. I also declare that I have adhered to all principles of academic honesty and integrity, and I have not misrepresented, fabricated, or falsified any idea, data, fact, or source in my submission. This dissertation is being turned in in partial fulfillment of Wallaga University's Post Graduate Studies degree requirements. I further declare that this dissertation has not been submitted to any other institution anywhere for the award of any academic degree, diploma, or certificate.

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Author Biography

Abdata Guluma Erana was born in 1990 in Nunu Qumba Woreda, located in the East Wallaga Zone of the Oromia Region, Ethiopia. He completed his elementary education at Nunu Gaba Robi Primary School and then continued his secondary education at Nunu Secondary School. For his preparatory studies, he joined Arjo Mokonin Damise Secondary and Preparatory School. In 2009 GC, he enrolled at Hawassa University to pursue his bachelor's degree, which he completed successfully with strong academic performance. After graduating, he began his professional career as a high school mathematics teacher at Gindeberet Senior Secondary School, where he served for two years. Seeking further professional growth, he joined Samara University. With sponsorship from Samara University and the Ethiopian Ministry of Education, he pursued his Master's degree at Wallaga University in 2018 GC and successfully completed his MSc in 2019. Afterward, he returned to Samara University and served as a lecturer in the Department of Mathematics for two years. In 2021, he joined Wallaga University to pursue his Ph.D. studies. He is deeply grateful to Almighty God, whose guidance and strength have sustained him throughout his academic and professional journey.

CERTIFICATE

This is to certify that the dissertation entitled

“A Study on Domination in Turiyam Graphs with Algorithms and Applications”

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for the award of the degree of **Doctor of Philosophy (Ph.D.) in Mathematics** (Graph Theory and Algorithms), has been carried out under our guidance and supervision at **Wollega University**. We hereby affirm that the research work embodied in this dissertation is the original and independent work of the candidate. To the best of our knowledge, the results presented herein have not previously formed the basis for the award of any degree, diploma, associateship, fellowship, or any other similar academic distinction at this or any other institution. We further certify that the dissertation satisfies the academic standards and requirements prescribed for the award of the degree of Doctor of Philosophy and is worthy of being submitted for evaluation.

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Abbreviations and Acronyms

Symbol / Acronym	Meaning
Classical Graph Theory	
G	Classical graph
$V(G)$	Vertex set of a graph
$E(G)$	Edge set of a graph
$N(u)$	Neighborhood of vertex u
$\deg(u)$	Degree of vertex u
$\delta(G)$	Minimum degree of graph G
$\Delta(G)$	Maximum degree of graph G
Fuzzy Environment	
FS	Fuzzy Set
FL	Fuzzy Logic
FG	Fuzzy Graph
DFG	Domination in Fuzzy Graph
Intuitionistic Fuzzy Environment	
IFS	Intuitionistic Fuzzy Set
IFL	Intuitionistic Fuzzy Logic
IFG	Intuitionistic Fuzzy Graph
DIFG	Domination in Intuitionistic Fuzzy Graph
Neutrosophic Environment	
NS	Neutrosophic Set
NL	Neutrosophic Logic
NG	Neutrosophic Graph
DNG	Domination in Neutrosophic Graph
Turiyam Graph Framework	
TG	Turiyam Graph
DTG	Domination in Turiyam Graph
$G_{TS} = (A_{TS}, B_{TS})$	Turiyam graph structure
A_{TS}	Set of Turiyam vertices
B_{TS}	Set of Turiyam arcs

Symbol / Acronym	Meaning
T	Truth membership component
I	Indeterminacy membership component
F	Falsity membership component
L	Limitation membership component
v_{sf}	Vertex score function
e_{sf}	Edge score function
p	Order of Turiyam graph (total node score)
q	Size of Turiyam graph (total edge score)
Domination Concepts	
Dom	Domination
DG	Domination in classical graph / Dominating graph
D	Dominating set
MDS	Minimum dominating set
$\gamma(G)$	Domination number
TD	Total domination
TDS	Total dominating set
TTD	Total Turiyam domination
PD	Paired domination
PDS	Paired dominating set
SD	Split domination
SDS	Split dominating set
ED	Equitable domination
EDS	Equitable dominating set
$\gamma_{ef-TG}(G_{TS})$	Equitable domination number in Turiyam graph
Strong arc	Arc with high membership strength
Weak arc	Arc with low membership strength
Balanced domination	Degree-balanced domination condition
EED	Extremely equitable domination

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Dedication

I lovingly dedicate this work to my beloved wife, Anane Tafera; my dear children, Boharti, Kaku, Boju, and my parents.

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List of Publications/Communications

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4. Split domination in Turiyam graphs with application in health system. *Scientific African* (**Under Review**)

Abstract

This dissertation investigates domination properties in Turiyam graphs, focusing on the development of theoretical results, efficient algorithms, and practical applications in graph theory. Domination theory plays a significant role in network design, communication systems, optimization, and resource allocation, where identifying minimum dominating structures is essential for efficient control and monitoring of complex systems. In this study, various domination parameters in Turiyam graphs, including domination number, total domination, connected domination, and independent domination, are examined in detail. The structural characteristics of Turiyam graphs are analyzed to understand how graph configurations influence domination properties and algorithmic behavior. To address the computational challenges associated with domination problems, efficient graph-theoretic algorithms are developed for determining dominating sets and related parameters in Turiyam graphs. The proposed algorithms are designed using systematic graph construction techniques and optimized procedures to improve computational efficiency and accuracy. Illustrative examples are presented to validate the theoretical findings and demonstrate the effectiveness of the proposed algorithms. The results show that the developed methods provide accurate and computationally efficient solutions for domination-related problems in Turiyam graphs. Comparative analysis with existing graph domination approaches further highlights improvements in performance, scalability, and robustness. The applications discussed in this work emphasize the usefulness of domination concepts in communication networks, social network analysis, facility location problems, and optimization systems, thereby contributing to both theoretical graph theory and practical real-world problem solving.

CHAPTER 1

General Introduction

This chapter presents a coherent overview of the fundamental ideas about classical graph, fuzzy graph, intuitionistic fuzzy graph, neutrosophic graph and their domination. It highlights the theoretical principles and rational behind graph theory. In addition, the chapter defines the research background, formulate the central problem, and specifies the objectives and scope of the study. It further emphasizes the motivation for undertaking this work outlines its anticipated contributions and concludes with a brief description of the dissertation's overall structure.

1.1 Background of the study

Graph theory is a fundamental branch of discrete mathematics that studies relationships between objects through abstract representations known as graphs. A graph consists of a set of vertices (or nodes) and a set of edges (or links) that connect pairs of vertices. The subject originated in 1736 with Leonhard Euler's solution to the Königsberg bridge problem, which is widely recognized as the first formal result in graph theory. Euler's work established the idea that complex structural problems can be analyzed using abstract mathematical representations rather than geometric details.

1.1.1 Classical Graph Theory

Classical graph theory studies structures composed of a finite set of vertices connected by edges, representing pairwise relationships. A classical graph is defined as an ordered pair $G = (V, E)$, where V a nonempty finite set of vertices and $E \subseteq V \times V$ is a set of edges. The origins of classical graph theory date back to Euler's solution of the Königsberg bridge problem in 1736, which demonstrated that structural properties of networks could be studied

abstractly. Classical graph theory developed rigorously through contributions by ([Harary, 1962](#)) and others, establishing foundational concepts such as paths, cycles, connectivity, degree, and planarity. [Bondy and Murty \(2008\)](#) present graph theory as a unified, rigorous mathematical discipline that combines structural theory with combinatorial reasoning.

Classical graphs assume crisp membership, meaning that vertices and edges either exist or do not exist. While this assumption simplifies mathematical analysis, it limits applicability in real-world systems where relationships may be uncertain or partially known. Nevertheless, classical graph theory forms the structural backbone for all later graph generalizations and domination concepts.

1.1.2 Modern Graph Theory

Modern graph theory extends classical graph theory by incorporating algorithmic, computational, and applied perspectives. It emphasizes large-scale networks, algorithmic efficiency, and optimization under constraints. Modern graph theory is deeply connected with computer science, operations research, and network science. [Mubayi et al. \(2001\)](#) presents graph theory as a balanced integration of theory, applications, and problem-solving techniques. Key developments include weighted graphs, directed graphs, random graphs, dynamic graphs, and algorithmic graph theory. Problems such as domination, vertex cover, coloring, and matching are studied in terms of computational complexity, approximation algorithms, and heuristics. [Gross and Yellen \(2005\)](#) emphasize the application-driven nature of modern graph theory, demonstrating how theoretical results directly inform solutions to practical problems. Despite its advancements, modern graph theory largely retains binary logic and crisp relationships, which restricts its ability to model uncertainty. This shortcoming motivates the integration of fuzzy and uncertainty-based logical frameworks.

1.1.3 Fuzzy Graphs

To model vagueness and gradual membership in mathematical systems, [Zadeh \(1965\)](#) introduced fuzzy set theory. Zadeh's theory provides the logical and mathematical foundation for fuzzy relations and fuzzy graphs, allowing uncertainty to be incorporated into structural models. In the context of graph theory, fuzzy sets enable vertices and edges to have varying strengths of existence or influence, which is essential for extending domination concepts to

uncertain networks. [Rosenfeld \(1975a\)](#) extended Zadeh's fuzzy set theory to graph theory by introducing fuzzy graphs, where both vertices and edges are associated with membership values in $[0, 1]$. Rosenfeld deal that adjacency between vertices can be partial rather than absolute, enabling the modeling of networks with uncertain or graded relationships. Rosenfeld's formulation preserves the structural framework of classical graphs while embedding fuzziness into their components.

1.1.4 Intuitionistic Fuzzy Graphs

Intuitionistic fuzzy graphs extend fuzzy graphs by introducing both membership and non-membership functions along with a degree of hesitation. This framework was developed based on Atanassov's intuitionistic fuzzy set theory. [Atanassov \(1986\)](#) introduced intuitionistic fuzzy set (IFS) theory as a generalization of Zadeh's fuzzy sets. An intuitionistic fuzzy graph captures not only acceptance and rejection but also uncertainty explicitly. This makes it more expressive than fuzzy graphs and particularly suitable for decision-making and risk assessment problems. In graph theory, Atanassov's IFS theory enables the development of intuitionistic fuzzy graphs, where domination accounts for acceptance, rejection, and hesitation simultaneously.

[Parvathi and Karunambigai \(2006\)](#) extended Atanassov's intuitionistic fuzzy set theory to graph structures by formally defining intuitionistic fuzzy graphs (IFGs). The idea of their work is that vertices and edges in a graph are associated with membership and non-membership values, allowing adjacency relations to be modeled with uncertainty and hesitation. Their formulation preserves classical graph properties while embedding intuitionistic fuzziness, enabling the study of connectivity, paths, and domination under uncertainty.

1.1.5 Hesitant Fuzzy Graphs

[Torra \(2010\)](#) introduced hesitant fuzzy set theory to model situations in which decision-makers hesitate among several possible membership values for an element, rather than assigning a single degree as in classical fuzzy or intuitionistic fuzzy sets. The core idea is that the membership of an element is represented by a set of possible values in $[0, 1]$, reflecting uncertainty arising from hesitation, ambiguity, or multiple expert opinions. This framework captures a richer form of uncertainty than fuzzy and intuitionistic fuzzy sets, particularly in

complex decision-making environments. When extended to graph theory, hesitant fuzzy sets lead to hesitant fuzzy graphs, where vertices and edges are associated with multiple possible membership degrees. Torra's work provides an important theoretical bridge between intuitionistic fuzzy graphs and more generalized uncertainty frameworks such as neutrosophic and Turiyam graphs, where indeterminacy and multiplicity of states play a central role. Hesitant fuzzy graphs provide enhanced flexibility in modelling complex uncertainty but increase computational complexity.

1.1.6 Neutrosophic Graphs

[Smarandache \(1998\)](#) introduced neutrosophic logic and neutrosophic set theory as a comprehensive generalization of classical, fuzzy, and intuitionistic fuzzy frameworks. The core idea in his study is that each element is characterized independently by three components: truth-membership (T), indeterminacy-membership (I), and falsity-membership (F), where each component can take values in the real unit interval (or even beyond, in extended forms). This tripartite representation explicitly models indeterminacy and inconsistency, which are unavoidable in real-world systems involving incomplete, conflicting, or ambiguous information. Neutrosophic logic removes the restriction that membership and non-membership must sum to one, thereby offering greater expressive power. This framework forms the logical foundation for neutrosophic graphs, where vertices and edges are associated with (T, I, F) values, enabling advanced modeling of uncertain networks. Neutrosophic graphs generalize fuzzy and intuitionistic fuzzy graphs by explicitly modeling indeterminacy and inconsistency. This makes them highly suitable for real-world systems with incomplete, contradictory, or uncertain information.

[Broumi et al. \(2018c\)](#) extended Smarandache's neutrosophic theory to graph structures by introducing single-valued neutrosophic graphs (SVNGs). The core idea of their work is to associate each vertex and edge with a neutrosophic triple (T, I, F) , where all components take values in $[0, 1]$, making the model mathematically tractable and computationally feasible. Their framework preserves classical graph concepts such as adjacency, paths, and connectivity while embedding explicit indeterminacy. Broumi et al.'s formulation serves as a crucial step toward more structured and layered models such as Turiyam graphs.

1.1.7 Turiyam Graphs

Turiyam graphs represent a further advancement in graph modeling by incorporating layered structure, functional roles, and higher-order logical states inspired by Turiyam logic. Unlike classical and neutrosophic graphs, Turiyam graphs allow multiple functional dimensions to coexist within a unified framework. This structure enables precise modeling of complex systems such as healthcare networks, smart infrastructure, and decision-support systems. Turiyam graphs naturally subsume classical, fuzzy, intuitionistic fuzzy, and neutrosophic graphs as special cases. [Ganati et al. \(2023a\)](#) introduce Turiyam graphs as extensions of single valued neutrosophic graphs with a liberal membership component, and study fundamental parameters like degree, order, and size, showing their applicability to social network analysis.

1.1.8 Domination in graph theory

Domination is a central concept in graph theory concerned with selecting a subset of vertices that can influence, control, or monitors the entire graph. A dominating set ensures that every vertex is either in the set or adjacent to a vertex in the set. Domination theory has extensive applications in facility location, network monitoring, resource allocation, and security. Variants such as connected domination, split domination, and weighted domination have been widely studied.

The formal study of domination in graphs began in the mid-twentieth century. Although related ideas appeared earlier in covering and independence problems, the domination concept was systematically introduced by Ore in 1962. Ore defined a dominating set as a subset of vertices such that every vertex not in the subset is adjacent to at least one vertex in it, thereby establishing the foundational definition still used today ([Ore, 1962](#)). This work highlighted domination as a dual concept to independence and covering in graphs. Following Ore's introduction, domination theory rapidly evolved into a major research area within graph theory. Berge emphasized domination as a key structural property of graphs and demonstrated its relevance to applications in operations research and decision sciences ([Berge, 1962](#)). Subsequent studies focused on determining domination numbers for special classes of graphs, including paths, cycles, trees, bipartite graphs, and planar graphs.

A significant milestone in the development of domination theory was the comprehensive treatment provided by Haynes, Hedetniemi, and Slater. Their monograph *Fundamentals of Domination in Graphs* systematically unified existing results, introduced numerous domination variants, and established domination as a mature and independent branch of graph theory (Haynes et al., 2013). They formalized several domination parameters, such as total domination, connected domination, independent domination, and restrained domination, each motivated by practical constraints in real-world systems.

Importantly, their works provides the mathematical and algorithmic foundation for extending domination concepts to uncertain and multi-valued graphs, such as fuzzy, intuitionistic fuzzy, neutrosophic, and Turiyam graphs. The theoretical rigor of Haynes et al., ensures that all subsequent generalizations maintain formal correctness while allowing flexibility for uncertainty modeling.

1.1.9 Domination in Uncertain Graphs

Domination in uncertain graphs extends classical domination to fuzzy, intuitionistic fuzzy, neutrosophic, and Turiyam graph frameworks. These models incorporate degrees of influence, hesitation, indeterminacy, and layered structure. Somasundaram (1998) introduced domination concepts in fuzzy graphs, defining dominating sets and domination numbers where membership values influence control relationships. Furthermore, Enriquez et al. (2021) extended domination to fuzzy directed graphs, defining fuzzy domination sets and domination numbers for directed network structures. Parvathi and Thamizhendhi (2010) developed dominating sets and domination numbers in intuitionistic fuzzy graphs, accounting for both membership and non membership degrees.

Guan et al. (2024b) analyzed domination in intuitionistic fuzzy directed graphs, extending domination theory to directed settings and influential network models and Malarvizhi and Divya (2021) examined domination and edge domination in single valued neutrosophic graphs (SVNGs), generalizing fuzzy/intuitionistic fuzzy domination to handle indeterminacy and inconsistency.

1.2 Statement of the Problem

Graph domination is a fundamental concept in classical graph theory and has been widely studied because of its significance in optimization, monitoring, control, and resource allocation problems. Traditional domination models assume deterministic relationships among entities, where connections and their influences are precisely defined. However, many real-world systems, particularly social networks and healthcare systems, are inherently complex, uncertain, and context-dependent. As a result, classical domination models are often inadequate for accurately representing such systems or supporting reliable decision-making.

To address these limitations, domination theory has been extended to various uncertain graph models, including fuzzy graphs, intuitionistic fuzzy graphs, hesitant fuzzy graphs, and neutrosophic graphs. These frameworks incorporate partial membership, uncertainty, and indeterminacy in relationships, thereby providing more realistic representations of complex networks. Despite these developments, existing domination models remain limited in their ability to capture multi-functional, layered, and context-sensitive interactions that frequently occur in modern social and healthcare environments.

In social networks, relationships among individuals and organizations are not only uncertain but also shaped by multiple interacting factors such as trust levels, influence strength, conflicting information, and dynamic contextual changes. Likewise, healthcare systems involve heterogeneous entities such as hospitals, clinics, laboratories, surveillance units, and decision-making bodies. The interactions among these components are characterized by varying degrees of reliability, uncertainty, and indeterminacy. Even advanced domination models based on fuzzy or neutrosophic frameworks are not fully capable of representing these layered functional roles and contextual dependencies within a unified mathematical structure.

Recently, Turiyam graphs have emerged as a promising generalization of classical and uncertain graph models by integrating structural relationships with functional, contextual, and uncertainty-based dimensions. This enriched framework offers greater expressive power for modeling complex systems. However, the theory of domination in Turiyam graphs remains largely unexplored and underdeveloped.

In particular, there is a lack of:

- Rigorous and unified definitions of domination and its variants in Turiyam graphs,
- Mathematical characterization of domination parameters and their relationships with classical, fuzzy, and neutrosophic domination,
- Efficient algorithmic approaches for identifying dominating sets in Turiyam graphs,
- Application-oriented studies demonstrating the practical usefulness of Turiyam domination in real-world systems such as social networks and healthcare infrastructures.

This research gap is significant because domination plays a crucial role in identifying influential nodes, determining optimal monitoring locations, and supporting efficient resource allocation. Without a well-established domination theory for Turiyam graphs, decision-makers lack adequate mathematical tools to simultaneously address structural complexity, uncertainty, indeterminacy, and layered interactions. Therefore, developing a comprehensive theory of domination in Turiyam graphs, along with associated algorithms and practical applications, is both necessary and timely.

1.3 Research Questions

This dissertation seeks to address the following research questions:

- How can the concept of domination be rigorously and formally defined within the framework of Turiyam graphs?
- To what extent does domination in Turiyam graphs generalize or extend domination in classical, fuzzy, intuitionistic fuzzy, and NG models?
- What are the fundamental structural and mathematical properties of dominating sets in Turiyam graphs?
- Under what necessary and sufficient conditions do various types of dominating sets—including minimal, total, paired, split, and equitable domination—exist in TGs?
- What computational or algorithmic methods can be developed to determine or approximate minimum dominating sets in Turiyam graphs?

- How does domination in Turiyam graphs compare analytically and computationally with domination in classical and uncertainty-based graph models?
- What open problems, theoretical challenges, and research gaps remain in the study of domination in Turiyam graphs?

1.4 Objectives of the Study

1.4.1 General Objective

The general objective of this study is to develop a comprehensive domination theory in Turiyam graphs with algorithmic frameworks for practical and theoretical applications.

1.4.2 Specific Objectives

To achieve the overarching goal of this research, the study is guided by the following specific objectives:

1. To establish a rigorous mathematical foundation for Turiyam graphs, including precise definitions, structural properties, and formal representations.
2. To investigate the conceptual relationships and generalizations among classical domination, fuzzy domination, intuitionistic fuzzy domination, neutrosophic domination, and domination in TGs, thereby constructing a unified theoretical framework.
3. To extend fundamental domination concepts-such as dominating sets, total domination, connected domination, and related parameters-to fuzzy Turiyam graphs, intuitionistic fuzzy Turiyam graphs, and neutrosophic Turiyam graphs.
4. To derive and prove essential theoretical results, including structural properties, bounds, inequalities, characterizations, and invariance conditions of domination parameters in Turiyam graph settings.
5. To refine algorithmic approaches for computing domination sets in TGs, with particular emphasis on efficiency, correctness, and computational complexity analysis.

6. To demonstrate the applicability and relevance of the proposed domination models through selected real-world case studies, thereby illustrating their potential contributions to complex decision-making and network-based systems.

1.5 Significance of the Study

This study contributes to both the theoretical advancement of graph theory and the practical modeling of complex real-world systems. From a theoretical perspective, the research establishes a systematic and rigorous framework for domination in Turiyam graphs, thereby extending the existing body of knowledge in classical and uncertainty-based graph models. By introducing formal definitions, structural properties, and characterizations of domination parameters, this work fills an important gap in the mathematical development of advanced graph structures that integrate uncertainty, functionality, and contextual relationships.

The study also contributes to the unification and generalization of domination concepts across multiple graph paradigms, including classical, fuzzy, IFG, and NGs. Through this integration, the research provides a broader and more flexible analytical framework capable of representing multi-layered and context-sensitive interactions. This enhances the descriptive and explanatory power of graph-based models in complex environments. From a computational perspective, the development of algorithmic methods for identifying dominating sets in TGs provides new tools for solving optimization and control problems under uncertainty. These algorithmic contributions are expected to support efficient decision-making processes in systems where structural complexity and indeterminacy coexist.

Practically, the findings of this research offer potential applications in domains characterized by dynamic and uncertain interactions, such as social network analysis, healthcare systems, monitoring infrastructures, and resource allocation environments. By enabling more accurate modeling of layered functional roles and contextual dependencies, Turiyam domination provides a foundation for improved planning, surveillance, and strategic intervention.

Finally, this study opens new directions for future research by identifying theoretical challenges, computational problems, and application areas that require further investigation. As such, it serves as a foundational reference for scholars and practitioners interested in advanced graph models and their real-world applications.

1.6 Research Methodology

This study adopts a theoretical, analytical, and application-oriented research methodology to investigate domination in Turiyam graphs, develop computational algorithms, and explore practical applications. The methodology is structured to ensure mathematical rigor, logical coherence, and computational relevance, while demonstrating the applicability of the proposed framework to complex real-world systems.

The research follows a qualitative and deductive analytical design grounded in mathematical modeling and supported by structured examples and application-based illustrations. The study progresses systematically from foundational theory to structural analysis, algorithm development, and practical modeling. The methodological framework integrates three complementary components: rigorous theoretical development, comparative structural analysis, and algorithmic and application-based validation.

1.6.1 Conceptual Framework Development

The first phase of the research involves an extensive review and synthesis of relevant literature in:

- Classical graph theory and domination theory,
- Domination in fuzzy, intuitionistic fuzzy, and neutrosophic graphs,
- Uncertainty-based and multi-layer graph models.

Based on this review, a conceptual framework is established that positions Turiyam graphs as a generalized structure capable of integrating structural, functional, contextual, and uncertainty-based relationships. This framework provides the theoretical foundation for formulating domination concepts in Turiyam graphs and identifying their relationships with existing domination models. The core theoretical development includes:

- Definition of dominating sets and domination parameters in Turiyam graphs,
- Development of domination variants, including Total, paired, split, and equitable domination,

- Generalization of classical and uncertainty-based domination concepts within the Turiyam framework.

1.6.2 Structural and Comparative Analysis

The study conducts a systematic structural investigation of domination properties in Turiyam graphs. This includes:

- Derivation of bounds, relationships, and characterizations of domination parameters,
- Analysis of domination behavior in special classes of Turiyam graphs such as complete graphs, paths, cycles, and bipartite structures,
- Comparative analysis of TD with classical, fuzzy, IF, and ND frameworks,
- Identification of generalizations, equivalence conditions, and structural distinctions.

1.6.3 Algorithmic Development

To address computational aspects, the research develops algorithmic approaches for identifying and approximating dominating sets in Turiyam graphs. This stage includes:

- Design of greedy, and structural algorithms for computing domination parameters,
- Analysis of computational complexity, efficiency, and scalability,
- Validation of algorithms using illustrative graph instances and network structures,
- Performance comparison with existing domination algorithms where applicable.

1.6.4 Application and Case Study Modeling

To demonstrate practical relevance, the developed Turiyam domination framework is applied to representative real-world systems characterized by layered interactions and uncertainty. Application modeling focuses on:

- Healthcare networks involving hospitals, clinics, and surveillance systems,
- Communication and social network structures with uncertain and context-dependent interactions.

Mathematical modeling and numerical illustrations are used to demonstrate how Turiyam domination supports monitoring, influence control, and resource allocation under uncertainty.

1.7 Scope of the Study

This study focuses on the theoretical development, algorithmic investigation, and practical application of domination in Turiyam graphs as an advanced mathematical framework for modeling complex networks under uncertainty. The research is primarily concerned with establishing rigorous definitions, structural properties, and domination parameters within the Turiyam graph environment.

The scope of the study includes the formulation and analysis of various domination concepts, the derivation of theoretical bounds and characterizations, and the development of computational algorithms for identifying dominating sets. The research also examines the relationships between Turiyam domination and existing domination frameworks in classical and uncertainty-based graph models.

In addition, the study demonstrates the applicability of TD through conceptual modeling and illustrative case studies involving complex systems characterized by layered interactions and indeterminacy. These applications are intended to show the relevance of the proposed framework for monitoring, control, and resource allocation in networked environments.

However, the study is limited to theoretical modeling, analytical proof, and algorithmic development. Empirical data collection, large-scale experimental implementation, and statistical validation of real-world systems are beyond the scope of this research.

1.8 Organization of the Dissertation

This dissertation is organized into nine chapters, each designed to systematically develop the theoretical foundations, algorithmic approaches, and practical applications of domination in Turiyam graphs.

Chapter 1: Introduction

This chapter introduces the research problem and provides the background of the study. It discusses classical and modern graph theory, uncertain graph models, and the motivation

for extending domination theory to Turiyam graphs. The chapter also presents the research questions, objectives, significance, scope, and an overview of the research methodology.

Chapter 2: Literature Review

This chapter reviews the origins and developments of fuzzy logic, intuitionistic fuzzy logic, neutrosophic logic, and Turiyam logic, along with their corresponding graph models: fuzzy graphs, intuitionistic fuzzy graphs, neutrosophic graphs, and Turiyam graphs. Previous studies on domination in classical, fuzzy, intuitionistic fuzzy, and neutrosophic graphs are discussed to provide a comprehensive background. The chapter concludes by identifying research gaps that justify the current study.

Chapter 3: Preliminaries and Mathematical Foundations

This chapter presents the essential mathematical preliminaries required for the study. It introduces basic concepts of graph theory, uncertain graphs, and domination parameters, forming the theoretical foundation necessary for understanding and developing domination concepts in Turiyam graphs.

Chapter 4: Domination and Paired Domination in Turiyam Graphs

This chapter develops the core concepts of domination and paired domination in Turiyam graphs. The structure of the chapter includes:

- *Definitions and Concepts:* Formal definitions of dominating sets, paired domination, and associated parameters in Turiyam graphs.
- *Theoretical Properties:* Structural characteristics, existence conditions, and relationships with classical and uncertain domination models.
- *Illustrative Examples:* Step-by-step examples to demonstrate how domination and paired domination are applied in specific Turiyam graphs.
- *Comparative Analysis:* Discussion of how Turiyam domination generalizes classical, fuzzy, intuitionistic fuzzy, and neutrosophic domination.

Chapter 5: Total Domination in Turiyam Graphs

This chapter investigates total domination, a variant requiring every vertex to be adjacent to a dominating vertex. The chapter structure includes:

- *Formal Definitions:* Definition of total dominating sets and total domination numbers in Turiyam graphs.

- *Theoretical Development:* Proofs of bounds, existence conditions, and relationships with other domination variants.
- *Examples and Applications:* Illustrative cases in special classes of Turiyam graphs such as paths, cycles, complete graphs, and bipartite structures.
- *Comparative Discussion:* Analysis of how total domination in Turiyam graphs extends classical and uncertain domination models.

Chapter 6: Split Domination in Turiyam Graphs

This chapter focuses on split domination, where graphs are partitioned into dominated and non-dominated components with structural constraints. The chapter structure includes:

- *Definitions and Preliminaries:* Formal introduction of SPD and parameters.
- *Properties and Theorems:* Theoretical results regarding existence, uniqueness, and bounds for SPD.
- *Illustrative Examples:* Demonstration of split domination in various TG structures.
- *Comparative Analysis:* Relationships with total, paired, and classical domination, highlighting the advantages of Turiyam modeling.

Chapter 7: Equitable Domination in Turiyam Graphs

This chapter investigates equitable domination, which emphasizes balanced distribution of domination across a network. The chapter structure includes:

- *Definitions and Concepts:* Formal definition of EDS and parameters in TGs.
- *Theoretical Properties:* Existence conditions, bounds, and characterizations of ED.
- *Illustrative Examples:* Examples demonstrating how ED ensures fairness and balanced influence in TGs.
- *Contextual Analysis:* Discussion on handling indeterminacy, inconsistency, and multi-layered interactions in complex networks.

Chapter 8: Applications of Domination in Turiyam Graphs

This chapter demonstrates the practical relevance of Turiyam domination. Case studies and

examples are presented in healthcare networks, social networks, and resource allocation systems. The chapter highlights how Turiyam domination improves decision-making, monitoring, and control under uncertainty.

Chapter 9: Conclusion and Recommendations

The final chapter summarizes the major findings, theoretical contributions, and practical implications of the study. Limitations of the research are discussed, and recommendations for future research in Turiyam graphs, domination theory, and related applications are presented.

CHAPTER 2

Literature Review

Domination is one of the most fundamental and extensively studied concepts in graph theory due to its strong theoretical depth and wide range of applications. It remains a central concept in graph theory, continuously evolving to address emerging challenges in modern network science. Its progression from classical graphs to fuzzy graph, intuitionistic graph, neutrosophic graph, and Turiyam frameworks reflects the growing need to model control and influence in increasingly complex and uncertain systems. In this chapter, we provide a literature survey on the theory of Fuzzy logic, Intuitionistic fuzzy logic, Neutrosophic logic, Turiyam Logic, uncertain graphs, domination in Fuzzy graph, domination in intuitionistic fuzzy graph, domination in neutrosophic graph and finally, identify the research gaps.

2.1 Classical Logic

Classical logic is the foundational framework of formal reasoning in mathematics, philosophy, and computer science. It is based on precise, well-defined truth values and provides the logical basis for classical mathematics, including classical graph theory. In classical logic, every proposition is assigned exactly one of two truth values: true (1) or false (0). This binary nature reflects the assumption that information is complete, consistent, and deterministic. The origins of classical logic can be traced back to Aristotle's syllogistic logic, which laid the groundwork for deductive reasoning. [Shapiro and Kouri Kissel \(2000\)](#) provide a comprehensive exposition of classical logic, focusing on its syntax, semantics, and inferential structure. The authors examine propositional and predicate logic, formal proof techniques, and truth-functional semantics grounded in binary truth values. The work establishes classical logic as the standard framework for rigorous mathematical reasoning and highlights its role in foundational studies of logic and philosophy. By clarifying the as-

assumptions and limitations of classical logic, the authors also motivate the development of alternative logical systems designed to handle vagueness, uncertainty, and indeterminacy. Modern formalization of classical logic was achieved in the late nineteenth and early twentieth centuries through the works of Frege, Russell, and Whitehead. Their contributions established propositional and predicate logic as rigorous mathematical systems governed by strict syntactic and semantic rules ([Russell and Whitehead, 1910](#)).

2.2 Fuzzy Logic

Classical logic is based on binary truth values, where propositions are either true or false. While effective for deterministic systems, this binary nature is inadequate for modeling real-world problems involving gradual transitions or partial truth. Zadeh introduced fuzzy logic to address this limitation by allowing truth values to range continuously between 0 and 1 ([Zadeh \(1965\)](#)).

[Kosko and Isaka \(1993\)](#) introduces fuzzy logic as a mathematical and computational framework for reasoning with imprecision and uncertainty inherent in real-world systems. In addition to this Kosko and Isaka explain the foundations of fuzzy sets and fuzzy inference, illustrate how linguistic rules (e.g., “if temperature is high, then reduce power slightly”) can be formalized mathematically, and demonstrate the effectiveness of fuzzy logic in practical applications such as control systems, consumer electronics, and decision-making. [Yen \(1999\)](#) presents a modern and integrative view of fuzzy logic, positioning it within the broader landscape of knowledge representation, data engineering, and intelligent systems. Yen reviews the evolution of fuzzy logic from its origins in fuzzy set theory to its role in approximate reasoning, decision-making, and information fusion. Yen’s work reframes fuzzy logic as a mature, adaptable, and integrative paradigm well-suited for complex knowledge and data engineering tasks.

[Hellmann \(2001\)](#) presents the fundamental concepts of fuzzy logic in a clear and accessible manner for beginners. Hellmann explains the motivation behind fuzzy logic by contrasting it with classical binary logic and highlighting the limitations of crisp set theory in handling imprecise and vague information. Hellmann’s work serves as an introductory guide to fuzzy logic, emphasizing clarity, intuition, and practical motivation rather than advanced mathematical detail. [Belohlavek and Klir \(2011\)](#) examines the deep relationship between concept

formation and fuzzy logic, focusing on how fuzzy set theory provides a formal foundation for representing and reasoning about imprecise, vague, and graded concepts. This positions fuzzy logic as a core theoretical tool for understanding and formalizing concepts, bridging mathematics, cognition, and intelligent systems.

[Trillas and Eciolaza \(2015\)](#) provides a comprehensive and rigorous treatment of fuzzy logic from both theoretical and philosophical perspectives. Trillas and Eciolaza explore the foundations of fuzzy sets, fuzzy relations, and fuzzy reasoning, emphasizing the role of vagueness, graded truth, and approximate reasoning in human cognition and formal logic. Trillas and Eciolaza present fuzzy logic as both a formal logic and a philosophy of reasoning, suitable for modeling human-like reasoning in uncertain and ambiguous environments. Building on fuzzy sets, fuzzy logic was developed as a mathematical framework for approximate reasoning. In fuzzy logic, truth values are expressed as degrees rather than crisp values, enabling reasoning under uncertainty. Fuzzy logic has been successfully applied in control systems, decision-making, pattern recognition, artificial intelligence, and expert systems. The strength of fuzzy logic lies in its ability to model gradual transitions and linguistic variables, such as “high,” “medium,” or “low.” However, a key limitation of fuzzy logic is that it considers only the degree of truth and does not explicitly account for non-membership or hesitation, which is often present in human reasoning.

2.3 Intuitionistic Fuzzy Logic (IFL)

To overcome the limitations of fuzzy logic [Takeuti and Titani \(1984\)](#), develops a formal logical foundation for intuitionistic fuzzy logic and its connection with intuitionistic fuzzy set theory. Takeuti and Titani introduce logical systems in which truth values are not binary but graded and are interpreted through intuitionistic fuzzy sets that assign degrees of membership and non-membership. The authors present precise axioms, semantics, and proof-theoretic results, demonstrating soundness and completeness of the proposed logical frameworks. This work is a foundational contribution that establishes intuitionistic fuzzy logic as a solid extension of classical and fuzzy logic within the framework of symbolic logic.

[Atanassov \(1986\)](#) introduced intuitionistic fuzzy sets (*IFS*), which extend fuzzy sets by incorporating both membership and non-membership degrees. In this framework for each element x , the membership degree $\mu(x)$ and the non-membership degree $\nu(x)$ satisfy $0 \leq$

$\mu(x) + \nu(x) \leq 1$. Intuitionistic fuzzy logic (IFL) provides a richer and more flexible framework for modeling uncertainty, especially in systems where acceptance and rejection coexist. It has been applied extensively in decision-making, pattern recognition, medical diagnosis, and network analysis.

Researchers such as [Parvathi and Karunambigai \(2006\)](#) extended intuitionistic fuzzy theory to graph structures, enabling applications in uncertain network modeling. Compared to fuzzy logic, IFL explicitly represents hesitation, making it more suitable for real-world problems involving conflicting or incomplete data. Despite its advantages, intuitionistic fuzzy logic assumes that the sum of membership and non-membership is bounded by one, which restricts its ability to represent paradoxical or highly inconsistent information. [Çitil \(2019\)](#) investigates the application of intuitionistic fuzzy logic (IFL) in the field of education, focusing on decision-making and evaluation processes characterized by uncertainty and subjectivity. Çitil demonstrates how intuitionistic fuzzy sets, defined by degrees of membership, non-membership, and indeterminacy, can effectively model ambiguous educational data such as student performance, teaching effectiveness, and assessment criteria. The paper shows that intuitionistic fuzzy logic is a practical and powerful tool for educational evaluation, enabling more nuanced and reliable decision-making under uncertainty.

2.4 Neutrosophic Logic (NL)

To address the limitations of both fuzzy and intuitionistic fuzzy logic, [Smarandache \(1998\)](#) introduced neutrosophic logic, a highly generalized logical framework. In neutrosophic logic, each proposition is characterized independently by three components: Truth (T), Indeterminacy (I), and Falsity (F). Unlike intuitionistic fuzzy logic, these components are independent, and their sum is not constrained to be one. This flexibility allows neutrosophic logic to model incomplete, inconsistent, and even contradictory information making it particularly suitable for complex and real-world systems. [Rivieccio \(2008\)](#) provides a critical introduction to neutrosophic logics, a class of many valued logical systems derived from the philosophical framework called neutrosophy (originally proposed by Florentin Smarandache). Neutrosophic logics extend classical and fuzzy logic by representing three independent components for any proposition: truth, falsity, and indeterminacy. Unlike fuzzy logic, where only truth membership is considered (and non membership is implicitly its comple-

ment), neutrosophic logics treat truth, falsity, and indeterminacy each as separate degrees that need not sum to 1. The paper discusses key challenges in defining appropriate logical connectives in this framework and examines how neutrosophic logics relate to other existing systems for managing uncertainty and vagueness, including intuitionistic and interval valued fuzzy systems, as well as Belnap's logic.

[Ashbacher \(2014\)](#) presents Neutrosophic Logic is that neutrosophic logic generalizes and unifies multiple existing logical systems (such as classical, fuzzy, intuitionistic, and paraconsistent logics) by introducing a three component framework for evaluating statements: Truth (T), Indeterminacy (I), and Falsity (F). Each of these components is treated independently, allowing a proposition to simultaneously have degrees of truth, falsity, and uncertainty that do not need to sum to 1 unlike fuzzy or intuitionistic fuzzy logic. This independence enables neutrosophic logic to more richly represent uncertainty, vagueness, and contradictory information, including paradoxes. He offers a foundational introduction to neutrosophic logic-logical framework first proposed by Florentin Smarandache in the 1990s that aims to extend traditional and fuzzy logic systems.

[Deli and Subas \(2014\)](#) introduce single valued neutrosophic numbers (SVN numbers) as a mathematically flexible way to represent uncertain, imprecise, incomplete, and inconsistent numerical information, and to show how these numbers can be used to solve multicriteria decision making (MCDM) problems. SVN numbers extend fuzzy and intuitionistic fuzzy numbers by incorporating three independent components, truth membership, indeterminacy membership, and falsity membership, allowing more nuanced modeling of uncertain quantitative assessments. The authors define special forms (e.g., trapezoidal and triangular SVN numbers), develop operations on them, introduce an aggregation operator, and apply this framework to an example MCDM problem.

Neutrosophic logic has been extended to single-valued neutrosophic sets, which restrict values to the interval $[0, 1]$ for practical applications. These models have been widely used in decision support systems, artificial intelligence, medical diagnosis, and graph theory. The extension of neutrosophic logic to graph structures has enabled advanced modeling of uncertain networks. Neutrosophic logic provides a powerful mechanism to model indeterminacy explicitly; however, it primarily focuses on logical and set-theoretic representations. When applied to graphs and networks, neutrosophic models still treat uncertainty locally at vertices

and edges, often lacking a clear mechanism to represent functional roles, layered interactions, and contextual dependencies within complex systems [Broumi et al. \(2018a\)](#).

[Pramanik \(2022\)](#) systematically present the theory, development, and extensions of single valued neutrosophic sets (SVNSs) as a comprehensive framework for modeling uncertainty, indeterminacy, and inconsistency in data and decision problems. Pramanik provides a comprehensive overview of single valued neutrosophic sets, detailing their definitions, key properties, theoretical developments, relationships to other uncertainty models, and roles in complex decision and information problems, showing why SVNSs have become a foundational concept in neutrosophic and soft computing research.

[Wang et al. \(2010\)](#) formalize and define single valued neutrosophic sets (SVNSs) as a mathematically precise and practically applicable instance of neutrosophic sets, an extension of fuzzy and intuitionistic fuzzy sets that explicitly incorporates truth, indeterminacy, and falsity as three independent membership degrees for each element in a set.

[Smarandache \(2022\)](#) define and explore fundamental operators for single valued neutrosophic oversets, undersets and offsets three extensions of standard neutrosophic sets that allow membership values to go beyond the usual interval $[0, 1]$. These operators (such as union, intersection and complement) are formulated so that they work consistently even when components are above 1 (overset), below 0 (underset), or both above 1 and below 0 (offset). This extends the usual neutrosophic set operations and enables modeling of real world situations with more extreme uncertainty or contradictory evidence that cannot be captured within the traditional $[0, 1]$ bounds.

2.5 Turiyam Logic

Turiyam logic represents a further advancement in uncertainty modeling beyond neutrosophic logic. It extends neutrosophic logic by incorporating an additional liberal or neutral dimension, enabling more nuanced representation of real-world reasoning processes.

[Singh \(2021\)](#) introduce the Turiyam set as a four dimensional extension of neutrosophic sets to model data and cognitive states that cannot be adequately captured by the traditional three components of neutrosophic logic (truth, indeterminacy, falsity). He proposes the Turiyam set a four dimensional extension of neutrosophic set theory that captures truth, indetermi-

nacy, falsity, and a novel fourth (liberal/silent) dimension and explores its potential use in quantum information processing and complex human cognition scenarios. This approach attempts to model uncertainty and cognitive states that go beyond traditional three component representations.

[Singh \(2022\)](#) extends neutrosophic reasoning into four dimensions with the Turiyam set to characterize human decision states that traditional logic systems (true/false/uncertain) fail to capture particularly unknown or undefined cognitive positions and shows how such a framework can be applied to analyze complex human cognition scenarios. [Singh \(2023b\)](#) formally explore the Turiyam set a four dimensional data representation inspired by human quantum cognition and to mathematically distinguish it from other traditional set concepts such as classical sets, fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets. In Turiyam logic, truth assessment is not limited to truth, indeterminacy, and falsity but includes a fourth independent component that captures openness, neutrality, or contextual flexibility. This additional dimension allows Turiyam logic to represent complex human decision-making more accurately, especially in social and cognitive systems.

[Ganati et al. \(2023a\)](#) have applied Turiyam logic to graph theory, giving rise to Turiyam graphs where vertices and edges are characterized by multi-component membership values. These graphs have demonstrated superior expressive power in modeling social networks, communication systems, and decision-support frameworks. Turiyam graphs naturally generalize neutrosophic graphs and provide a fertile ground for extending domination theory. [Martin et al. \(2025\)](#) introduce and apply Turiyam cognitive maps a novel four dimensional cognitive modeling tool based on the Turiyam set concept to industrial decision making problems where big data and human cognitive uncertainty intersect. Traditional cognitive map models (like fuzzy cognitive maps) represent relationships and influences among variables under uncertainty, but the Turiyam cognitive map extends these by incorporating four aspects of information: truth, falsity, indeterminacy, and a fourth liberal/unknown state. The logical foundation provided by Turiyam logic enables the formulation of advanced domination concepts that subsume classical, fuzzy, intuitionistic fuzzy, and neutrosophic domination as special cases.

2.5.1 Turiyam Set and Turiyam Relation

The Turiyam set represents a four-dimensional extension of the neutrosophic set developed to model uncertainty beyond the traditional components of truth, indeterminacy, and falsity. Unlike neutrosophic sets, which employ three independent membership values, the Turiyam set introduces an additional fourth component, commonly interpreted as a liberal, refusal, silent, or unknown state. This extra dimension enables the representation of cognitive situations and decision states that cannot be adequately captured by existing fuzzy and neutrosophic frameworks.

[Singh \(2021\)](#) introduced the Turiyam set as a fourth-dimensional data representation motivated by the need to handle uncertainty existing beyond three-way decision spaces. The framework was proposed to overcome limitations of fuzzy and neutrosophic models and to represent situations involving an additional independent state. Examples from voting systems, medical diagnosis, and social interactions were used to demonstrate the necessity of this fourth dimension.

In another study, [Singh \(2022\)](#) presented the concept of Turiyam context and discussed its applications in dark data analysis and uncertainty representation. The study emphasized that many real-world datasets contain information extending beyond conventional three-dimensional uncertainty structures and illustrated how Turiyam sets provide a more comprehensive framework for analyzing such data.

[Singh \(2023b\)](#) further explored the use of Turiyam sets in quantum information processing, arguing that human cognition may involve conscious, unconscious, indeterminate, and silent states simultaneously. The proposed model extends neutrosophic reasoning by incorporating this fourth independent component and provides a foundation for advanced cognitive and information-processing applications.

Building upon the Turiyam set framework, [Ganati et al. \(2023b\)](#) introduced the concept of Turiyam relations through the Cartesian product of Turiyam sets. They defined operations on Turiyam relations, including inverse relations and several relational structures, and established their mathematical properties. The study demonstrated that Turiyam relations generalize classical and neutrosophic relations and provide a richer framework for representing

uncertain connections among objects.

According to [Ganati et al. \(2023b\)](#), a Turiyam relation assigns four membership values to ordered pairs corresponding to truth, indeterminacy, falsity, and liberal degrees. This representation enables the modeling of relationships involving refusal, neutrality, or unknown interactions that are difficult to capture using traditional binary or neutrosophic relations. Consequently, Turiyam relations offer enhanced flexibility for applications in knowledge representation, decision-making systems, and uncertainty analysis.

2.6 Classical Graphs

Classical graph theory forms the foundational framework for network modeling, where a graph is defined as an ordered pair $(G = V, E)$ consisting of a finite set of vertices V and a set of edges $E \subseteq V \times V$. In classical graphs, vertices and edges are assumed to exist deterministically, and relationships between vertices are binary in nature—either present or absent. This deterministic assumption aligns with classical logic, where propositions are strictly true or false. The formal development of classical graph theory is historically attributed to Euler’s solution of the Königsberg bridge problem, which demonstrated that abstract relational structures could be analyzed independently of geometric considerations. Subsequent systematic developments by Berge established graph theory as a unifying mathematical tool for modeling real-world problems in operations research, communication, and social systems [Berge \(1962\)](#).

Despite its strong mathematical foundations and wide applicability, classical graph theory suffers from several inherent limitations when modeling real-world systems characterized by uncertainty, vagueness, and incomplete information. These limitations have motivated the development of fuzzy, intuitionistic fuzzy, neutrosophic, and ultimately TG models.

2.7 Probabilistic Graphs

Probabilistic graphs represent one of the earliest extensions of classical graph theory designed to incorporate uncertainty arising from randomness. In a probabilistic graph, each edge (and sometimes each vertex) is associated with a probability value representing the likelihood of its existence. This model is particularly useful in applications such as com-

munication networks, epidemiological modeling, and reliability analysis, where connections may fail or exist only with certain probabilities. [Ghosh et al. \(2007\)](#) formulate and address a routing optimization problem in probabilistic graphs where edges exist with certain probabilities and apply this framework to intermittently connected networks (ICNs). These are networks (such as mobile ad hoc or delay tolerant networks) where end to end connectivity is not always guaranteed. Traditional routing methods fail in such environments because there may never be a persistent full path from source to destination. The authors define an optimization objective to find a subgraph that maximizes the probability of successful delivery between a pair of nodes under a constraint on the number of edges and propose approximation methods and a practical routing protocol tailored to ICNs based on probabilistic connectivity metrics.

The study of probabilistic graphs is closely related to random graph theory, pioneered by Erdős and Rényi, who investigated the emergence of global graph properties under probabilistic edge formation. Their work laid the foundation for understanding network connectivity, resilience, and phase transitions in large-scale systems. Newman later emphasized the importance of probabilistic and random graphs in modeling real-world networks, particularly social and biological systems [Newman \(2010\)](#).

[Kollios et al. \(2011\)](#) define and solve the problem of clustering graphs where edges have uncertainty (probabilities) rather than deterministic connections. Instead of treating a graph as fixed (edge present or absent), the authors treat the graph as generating an ensemble of possible deterministic graphs (“possible worlds”), each with a probability derived from the edge probabilities. [Maniu et al. \(2017\)](#) develop an efficient indexing framework that speeds up complex queries on probabilistic graphs in which edges exist with associated probabilities by avoiding brute force enumeration of all possible deterministic realizations of the graph (so called possible worlds). Because evaluating queries such as shortest paths or reachability in probabilistic graphs is computationally intractable (P hard) when approached naïvely, the authors introduce ProbTree, a compact index structure that organizes probabilistic information to significantly improve the accuracy and efficiency of query processing.

While probabilistic graphs effectively capture stochastic uncertainty, they are limited in representing vagueness, partial truth, and indeterminacy. Probability values quantify randomness but do not distinguish between lack of knowledge, conflicting information, or contextual

uncertainty. Consequently, probabilistic graphs are often insufficient for decision-making problems where uncertainty is not purely random.

2.8 Fuzzy graph

[Glass and Kauffman \(1973\)](#) introduced the concept of fuzzy graphs (FGs) on fuzzy relations to describe the graph in which either the vertex, edge or both are complex, imprecise, and uncertain by considering membership degree, where classical graphs fail. [Rosenfeld \(1975b\)](#) developed several FG analogues of important graph theoretic concepts like subgraphs, paths, connectedness, cliques, forests, trees, bridges and cut vertices.

[Bhattacharya \(1987\)](#) introduced and studied fundamental graph-theoretic concepts in fuzzy graphs (FGs). The work extends classical graph ideas-such as paths, cycles, connectivity, degree of vertices, and subgraphs-into the fuzzy environment where vertices and edges have membership values in $[0,1]$. This paper laid an early theoretical foundation for fuzzy graph theory. [Bhutani \(1989\)](#) developed an automorphism of FGs. he focused on automorphisms of fuzzy graphs, which are symmetry-preserving mappings of a fuzzy graph onto itself. The study generalizes the concept of graph automorphism from crisp graphs to fuzzy graphs by ensuring that vertex and edge membership values are preserved under the mapping. This work helps in understanding structural symmetry and invariance in fuzzy graphs.

[Mordeson and Chang-Shyh \(1994\)](#) systematically described operations on fuzzy graphs, such as union, intersection, complement, and Cartesian product. In addition, they introduced and defined the concept of strong FGs, where the membership value of each edge is determined directly by the membership values of its end vertices. Their work provided a formal operational framework and strengthened the mathematical structure of FG theory.

[Mordeson and Nair \(2000\)](#) established fuzzy graph theory as a systematic extension of classical graph theory by embedding fuzzy set membership into both vertices and edges. Their central idea is that graph-theoretic relationships in real-world systems are rarely crisp and can be more accurately represented through graded memberships that capture varying strengths of association. By generalizing fundamental graph concepts such as adjacency, degree, connectivity, paths, and cycles into the fuzzy domain, they demonstrated that many classical results can be preserved while accommodating uncertainty. The authors also emphasized the

role of fuzzy graphs in modeling complex systems where information is incomplete or imprecise, thereby providing a rigorous mathematical foundation for subsequent developments in fuzzy domination, fuzzy hypergraphs, and higher-order uncertain network models.

[Gani and Ahmed \(2003\)](#) introduced some properties of FGs, including degree, order, and size and investigated their properties. The detailed concept of strong fuzzy graphs and the operations FG under which strong fuzzy graphs property is preserved is studied ([Bhutani and Battou, 2003](#)).

[Sunitha and Mathew \(2013\)](#) review and synthesize the fundamental concepts, definitions, structural properties, and key research developments in fuzzy graph theory and extension of classical graph theory that incorporates fuzzy set principles to model uncertainty in both vertices and edges. A fuzzy graph associates membership degrees to nodes and links, allowing representation of vague, imprecise, or partial connections typical in real world networks (e.g., social, biological, or communication systems). The survey organizes and explains core fuzzy graph constructs and relationships, discusses variations and properties, and highlights key results and research directions in the field. [Mordeson et al. \(2018\)](#) extend fuzzy graph theory with advanced theoretical tools and apply them to model and analyze complex social phenomena, specifically illegal human trafficking. The authors build on foundational fuzzy graph concepts and introduce new structures (such as intuitionistic fuzzy graphs, directed fuzzy networks, and fuzzy incidence graphs) and graph theoretic measures (e.g., domination and independence in fuzzy graphs) to capture uncertainty, imprecision, and relational complexity inherent in human trafficking networks. The book integrates mathematical theory with practical modeling of real world trafficking flows and policy interventions.

[Mordeson and Mathew \(2019\)](#) go beyond introductory fuzzy graph theory and explore deeper, mathematically rich, and application oriented topics that arise when fuzzy set concepts are integrated with graph structures. The authors build on foundational fuzzy graph theory to develop advanced tools and measures such as different connectivity notions, distance metrics, Wiener indices, operations based on t norms, and dialectic synthesis and demonstrate their usefulness in modeling real world problems under uncertainty. The book also serves as a self contained reference that combines theory with representative applications, making it suitable for both experienced researchers and newcomers interested in modern developments in fuzzy graphs.

[Pal et al. \(2020b\)](#) provide a comprehensive treatment of both foundational concepts and recent advances in fuzzy graph theory, blending rigorous mathematical development with illustrative applications. The book's establish the basic definitions, structures, and properties of fuzzy graphs, and then progressively build toward advanced topics and specialized graph classes (e.g., competition, threshold, tolerance, bipolar, and intuitionistic fuzzy graphs). Throughout, the authors emphasize how fuzzy graph theory extends classical graph concepts to model uncertainty and imprecision in real world networks and relational systems. The other concepts on FGs, including fuzzy planar graphs, fuzzy competition graphs, FG coloring, m-polar FGs and so on, stuided ([Pal et al., 2020a](#)).

[Mordeson et al. \(2023\)](#) apply fuzzy graph theory and related mathematical tools to model and analyze complex global challenges where uncertainty, imprecision, and incomplete information are inherent. The authors extend traditional fuzzy graph structures (such as fuzzy soft semigraphs and directed fuzzy incidence graphs) and use them to examine real world problems including human trafficking networks, climate change feedback processes, forest fire modeling, and migration flows. In doing so, the book demonstrates how fuzzy graph concepts can provide quantitative insight into social, environmental, and ethical issues that are difficult to represent with crisp mathematical models.

[Bhattacharya and Pal \(2023\)](#) integrate fuzzy graph theory with facility location and covering concepts to address complex real world optimization problems involving uncertainty. The authors develop a mathematical model that combines the tree covering problem on fuzzy graphs with facility location objectives minimizing costs, minimizing the number of facilities, and maximizing demand coverage under fuzzy (imprecise) conditions. They then apply this framework to a real life scenario involving the Indian banking system, specifically optimizing the circulation and supply of different banknote denominations within the economy, where relationships and costs are inherently vague or uncertain.

2.9 Intuitionistic Fuzzy graph

The formal development of intuitionistic fuzzy graphs was advanced by [Parvathi and Karunambigai \(2006\)](#), who extended classical and fuzzy graph concepts into the intuitionistic fuzzy environment. They demonstrated that fundamental graph properties such as adjacency, degree, paths, connectivity, and cycles can be systematically generalized while preserving es-

sential structural characteristics. Their work established intuitionistic fuzzy graphs as a natural progression beyond fuzzy graphs for modeling uncertainty in complex networks. This dual grade representation captures both positive evidence and negative evidence about relationships, allowing for a richer modeling of uncertainty than traditional fuzzy graphs especially in scenarios where both affirmation and negation information coexist.

Intuitionistic fuzzy graphs have been widely applied in decision-making, medical diagnosis, communication networks, and pattern recognition, where both positive and negative evidence play critical roles. Compared to fuzzy graphs, intuitionistic fuzzy graphs offer greater flexibility by explicitly capturing hesitation, which is crucial in systems with incomplete or ambiguous information. [Karunambigai et al. \(2013\)](#) introduced and investigated the concept of balanced IFGs. In their work, balance is defined by extending the classical notion of balanced signed graphs to the intuitionistic fuzzy setting, where each vertex and edge is characterized by both membership and non-membership degrees. A balanced IFG is one in which the combined effect of these degrees along every cycle satisfies a consistency condition, ensuring structural harmony and stability. This study helps in analyzing consistency, equilibrium, and reliability in systems modeled using intuitionistic FGs.

[Akram and Alshehri \(2014\)](#) developed the important IFG theoretical concepts, including intuitionistic fuzzy bridges, intuitionistic fuzzy trees, intuitionistic fuzzy cycles, and intuitionistic fuzzy cut nodes and examined some of their exciting properties. [Rashmanlou et al. \(2015\)](#) establish and analyze categorical structures and algebraic operations in the context of IFGs an extension of fuzzy graphs where each vertex and edge is characterized by both a membership degree and a non membership degree. The authors define several new graph products (direct, lexicographic, and strong products) for IFGs, explore how classical graph constructions behave under these products, and describe how IFGs form a category with homomorphisms exhibiting desirable categorical properties such as products, coproducts, completeness and co-completeness. This bridges intuitionistic fuzzy graph theory with category theory, which provides a high level, structural view of how IFGs relate through mappings and constructions.

[Yaqoob et al. \(2019\)](#) introduces complex intuitionistic fuzzy graphs (cif graphs) a new graph based modeling structure that blends complex intuitionistic fuzzy sets with graph theory to represent uncertainty in both magnitude and phase components of membership and non

membership. These graphs extend classical and intuitionistic fuzzy graphs by allowing complex values (with amplitude and phase in the complex plane) for vertices and edges, capturing richer and more nuanced information. The authors define foundational operations (e.g., union, join, composition), discuss graph homomorphisms, and apply the framework to a decision making problem in the context of cellular network provider companies.

[Shao et al. \(2020\)](#) extend key domination related concepts from classical and fuzzy graph theory to intuitionistic fuzzy graphs (IFGs) graphs in which vertices and edges are characterized by both membership and non membership degrees, allowing richer modeling of uncertainty. Specifically, the authors define and study bondage sets (BS), non bondage sets (NBS), the bondage number $\alpha(G)$, and the non bondage number $\alpha_k(G)$ within the IFG context, based on a newly defined notion of strong arcs (SAs) reflecting connectivity strength in IFGs. They develop theoretical results and illustrate the definitions with examples. Then, to demonstrate practical relevance, they apply these concepts to a water supply system, modeling supply points and pipelines using an IFG and showing how strong pipelines and domination parameters can guide where to locate pumps and improve water distribution under uncertainty.

[Nazir et al. \(2023\)](#) proposed an improved algorithm for identifying dominating node sets in IFGs. Their work enhances existing domination algorithms by incorporating both membership and non-membership degrees of vertices and edges, leading to more accurate and efficient selection of dominant nodes. The improved algorithm reduces computational complexity while better handling uncertainty, making it suitable for decision-making, network control, and optimization problems modeled by intuitionistic fuzzy graphs.

[Shi et al. \(2022\)](#) introduced the concept of picture fuzzy graphs (PFGs) by extending fuzzy and intuitionistic fuzzy graph models. In a picture fuzzy graph, each vertex and edge is characterized by four parameters: positive membership, neutral membership, negative membership, and refusal degree. This richer structure allows PFGs to model situations involving agreement, neutrality, disagreement, and hesitation simultaneously. Their work provides a more expressive graph framework for handling complex uncertainty and multi-opinion decision-making problems.

2.10 Neutrosophic graphs

The concept of NGs was first introduced by [Kandasamy et al. \(2015\)](#) where both the vertex set and the edge set were modeled as neutrosophic sets to effectively capture truth, indeterminacy, and falsity. Building on this foundation, [Akram et al. \(2018\)](#) further developed the theory of SVNNGs as an extension of classical graph theory, providing a structured framework for representing uncertainty, imprecision, and indeterminacy within graph-based systems.

SVNNGs generalize fuzzy graphs and intuitionistic fuzzy graphs by assigning each vertex and edge a truth membership, indeterminacy membership, and falsity membership value, each ranging independently in $[0, 1]$. This richer structure enables modeling of complex relationships where connections are not simply present or absent but can have varying degrees of truth, falsity, and uncertainty simultaneously. The work systematically defines these graphs, explores structural properties, and lays the foundation for extensions and applications in decision making and other fields where indeterminate information is important.

Significant contributions to the development of neutrosophic graph theory were made by [Broumi et al. \(2018a\)](#) who systematically studied the structure and properties of SVNNGs. Their work generalized fundamental graph concepts such as adjacency, degree, paths, connectivity, and domination within the neutrosophic environment. These extensions preserve the essence of classical graph theory while providing a robust framework for modeling uncertainty and indeterminacy. The concept of strong neutrosophic graphs is introduced and its interesting properties are studied ([Dhavaseelan et al., 2015](#)).

Furthermore, [Kandasamy et al. \(2016\)](#) extended neutrosophic graph theory by defining conditions under which edges fully reflect the neutrosophic values of their end vertices. They also explored subgraph topological subspaces, providing a framework to study structural properties, connectivity, and relationships within neutrosophic graphs, which helps model complex systems with uncertainty, indeterminacy, and inconsistency more rigorously.

[Jan et al. \(2019\)](#) introduced the concept of constant single-valued neutrosophic graphs (CSVNNGs), defining their key characteristics and structural properties. This work extends neutrosophic graph theory by focusing on graphs where the neutrosophic membership, indeterminacy, and non-membership values remain constant across vertices and edges, providing a simplified

yet powerful framework for modeling consistent uncertainty in complex systems.

2.11 Turiyam graphs

A Turiyam graph extends classical and neutrosophic graphs by associating each vertex and edge with multi-component membership values, typically including truth, indeterminacy, falsity, and an additional Turiyam (or neutral/liberal) component. The method of characterization of this unknown, undefined, and impossible in true (t), uncertain (i), and liberal regions (l) was independently introduced (Singh, 2022). This layered structure allows Turiyam graphs to represent heterogeneous networks in which nodes may play different functional roles or exist in different logical states simultaneously.

Bal et al. (2022) extended the notion of neutrosophic vector space to the Turiyam vector spaces. Based on this Turiyam space, they defined other concepts, including the AH-turiyam subspace. Basher et al. (2022) defined the notion of a symbolic Turiyam group and studied its elementary properties, including symbolic Turiyam subgroups, homomorphisms, and isomorphisms. Ani et al. (2023) determined the inverse of the non-square real symbolic Turiyam matrix by the Moore-Penrose inverse method. They developed an algorithm for this method.

Singh (2023a) introduced the Turiyam geometry and then characterized non-Euclidean data in Euclidean, Non-Euclidean, Hybrid or Neutro-Geometry and Turiyam Geometry. Recent studies by Ganati et al. (2023a) introduced Turiyam graphs as a powerful tool for social network analysis, demonstrating their effectiveness in capturing complex human interactions that cannot be adequately modeled using traditional graph structures. Their work formalized basic properties of Turiyam graphs and illustrated how Turiyam representations generalize fuzzy and neutrosophic graphs as special cases.

Turiyam graphs are particularly suitable for applications involving layered decision processes, uncertain resource allocation, healthcare systems, and communication networks, where relationships may exhibit coexistence of certainty, uncertainty, neutrality, and contextual freedom. The additional Turiyam component enables more realistic modeling of such systems by accommodating states that are neither strictly true nor false nor indeterminate.

The evolution from fuzzy logic to intuitionistic fuzzy logic and finally to neutrosophic logic

reflects a continuous effort to model uncertainty more realistically. Fuzzy logic captures partial truth, intuitionistic fuzzy logic adds hesitation, and neutrosophic logic explicitly models indeterminacy and inconsistency. Turiyam logic and Turiyam graphs further extend this progression by introducing additional states beyond neutrosophic representation, enabling even richer modeling of real-world systems.

2.12 Domination in classical Graphs

Classical domination theory assumes crisp, binary adjacency: either a vertex dominates another, or it does not. Real-world systems such as social networks, sensor grids, and decision support environments, however, are fraught with imprecision, vagueness, hesitation, and indeterminacy. This has motivated extensive research on domination concepts in uncertain graph models where adjacency and influence are not absolute but graded or uncertain. The literature on domination in uncertain graphs reflects an evolutionary hierarchy of logical frameworks: from fuzzy logic, through intuitionistic fuzzy logic, to neutrosophic logic and ultimately Turiyam logic. Each successive model captures richer forms of uncertainty but also introduces new theoretical and practical challenges.

[Bollobás and Cockayne \(1979\)](#) investigated graph-theoretic parameters related to domination, independence, and irredundance, establishing important relationships and inequalities among these parameters. Their results form a theoretical backbone of domination theory, influencing later developments in structural graph theory and generalized domination models. [Brigham et al. \(1988\)](#) studied vertex domination-critical graphs, that is, graphs in which the domination number decreases upon the removal of any vertex. Their work provides structural characterizations and properties of domination-critical graphs, contributing significantly to the theory of domination stability and sensitivity in graph structures.

[Brigham et al. \(1996\)](#) introduced the concept of double domination in graphs, in which each vertex is required to be dominated by at least two vertices from a dominating set. This notion strengthens classical domination and is particularly relevant for fault-tolerant and resilient network models, where redundancy in control and coverage is essential. [Cockayne et al. \(1980\)](#) introduced the concept of total domination in graphs, where every vertex must be adjacent to a dominating vertex, including vertices within the dominating set itself. This concept strengthens classical domination and plays a crucial role in network monitoring,

communication systems, and fault-tolerant graph models.

[Haynes and Slater \(1998\)](#) provided a comprehensive and rigorous treatment of domination in graphs, formalizing fundamental concepts such as dominating sets, domination numbers, and their variants. Additionally, they presented advanced topics in domination theory, covering refined domination parameters, algorithmic aspects, and structural variations of dominating sets in graphs. Their work serves as the classical foundation for domination theory, upon which later extensions to fuzzy, intuitionistic fuzzy, neutrosophic, and Turiyam graph frameworks have been developed.

2.13 Domination in Uncertain graphs

2.13.1 Domination in Fuzzy Graphs

The earliest extension of domination into uncertain graph models is found in fuzzy graphs, where vertices and edges possess membership values in $[0, 1]$ representing the degree of presence or influence. In this context, [Somasundaram \(1998\)](#) formally introduced fuzzy domination, defining fuzzy dominating sets and fuzzy domination numbers based on threshold membership values.

[Enriquez et al. \(2021\)](#) extended fuzzy domination to directed fuzzy graphs, defining fuzzy dominating sets and fuzzy domination numbers for fuzzy digraphs and analyzing their structural properties. This work enables the study of directional influence, control, and coverage in fuzzy network models. Fuzzy domination permits partial control where influence is not binary. However, membership degrees alone do not capture non-membership or hesitation, limiting the expressiveness in systems with conflicting or incomplete data.

2.13.2 Domination in Intuitionistic Fuzzy Graphs

The concept of domination in intuitionistic fuzzy graphs (IFGs) has attracted significant attention in recent years due to its ability to model control, influence, and decision-making in uncertain networks. [Parvathi and Thamizhendhi \(2010\)](#) laid the groundwork by defining intuitionistic fuzzy dominating sets and corresponding domination numbers, using membership and non-membership thresholds to quantify influence. By incorporating hesitation degrees, their approach extended classical fuzzy domination theory, offering a more expres-

sive framework for analyzing complex systems where uncertainty, indecision, and partial knowledge play a critical role. This foundation has paved the way for further research into algorithms and applications that identify dominant nodes and optimize network control in intuitionistic fuzzy environments.

[Karunambigai et al. \(2017\)](#) studied multiple variants of domination in intuitionistic fuzzy graphs, including total domination, independent domination, and minimal domination, all defined using membership-non-membership thresholds. Their work systematically extends classical domination variants to the intuitionistic fuzzy graph framework, enriching the theory of domination under uncertainty. [Nagoorgani et al. \(2016\)](#) extended the concept of double domination to intuitionistic fuzzy graphs, where each vertex must be dominated by at least two vertices based on membership, non-membership thresholds. Their study provides a framework for redundancy and fault-tolerance in networks under uncertainty, bridging classical double domination and intuitionistic fuzzy graph theory.

[Sahoo et al. \(2017\)](#) investigated covering and paired domination in intuitionistic fuzzy graphs, defining sets that simultaneously satisfy domination and pairing constraints under membership and non-membership thresholds. Their work extends classical domination concepts to handle interdependent control, redundancy, and coverage in uncertain network structures. [Bozhenyuk et al. \(2021\)](#) proposed computational methods for determining domination sets in intuitionistic fuzzy graphs, focusing on efficient algorithms to calculate domination numbers based on membership and non-membership thresholds. Their work addresses the practical computation of domination in large or complex intuitionistic fuzzy networks, bridging theoretical definitions with algorithmic implementation.

[Shao et al. \(2020\)](#) introduced new concepts in intuitionistic fuzzy graphs and demonstrated their application in water supply network systems. Their work highlights the practical utility of intuitionistic fuzzy graph models in real-world systems, allowing for modeling of uncertainty, partial information, and decision-making in complex network infrastructures. [Sadati et al. \(2021\)](#) investigated domination in intuitionistic fuzzy incidence graphs, extending classical domination concepts to incidence-based graph structures under membership-non-membership thresholds. Their work provides a theoretical framework for analyzing influence and control in networks where relationships are represented via incidence structures, enhancing the study of uncertainty and connectivity in intuitionistic fuzzy graphs.

[Hameed et al. \(2023\)](#) studied total perfect and total efficient domination in intuitionistic fuzzy graphs, extending classical domination concepts to ensure complete coverage and efficiency under membership–non-membership thresholds. Their work provides a rigorous framework for analyzing optimal control, redundancy, and network efficiency in intuitionistic fuzzy structures. [Guan et al. \(2024a\)](#) and others have explored intuitionistic fuzzy domination, defining dominating sets based on membership–non-membership thresholds. Intuitionistic fuzzy domination accounts for acceptance, rejection, and hesitation, offering a more nuanced uncertainty model. Yet, intuitionistic fuzzy models still constrain the sum of membership and non-membership, which can prevent accurate modeling of inconsistent or contradictory evidence.

2.13.3 Domination in Neutrosophic graph

The concept of neutrosophic graphs originates from Neutrosophic Logic, introduced by [Smarandache \(1998\)](#) as a generalization of classical, fuzzy, and intuitionistic fuzzy logic. Neutrosophic logic characterizes information using three independent components: truth (T), indeterminacy (I), and falsity (F). Building on this logical foundation, Broumi, Smarandache, Talea, Bakali, and others extended graph theory into the neutrosophic domain during the period 2016 – 2019, formally introducing single-valued neutrosophic graphs and investigating their structural properties.

The theory of domination in neutrosophic graphs has been systematically developed by several researchers, notably [Broumi et al. \(2017\)](#), [Broumi et al. \(2018b\)](#), [Ali et al. \(2019\)](#) and their collaborators. These studies introduced key notions such as ND sets, neutrosophic domination numbers, and important variants including minimal domination and total domination within NG frameworks. By explicitly incorporating truth, indeterminacy, and falsity components into vertex adjacency and influence relations, their work extends classical and fuzzy domination theories, providing a more expressive and realistic model for analyzing dominance and control in uncertain and indeterminate network structures.

[Mullai and Broumi \(2020\)](#) introduced the concept of dominating energy in neutrosophic graphs, extending domination theory by integrating spectral graph measures with neutrosophic sets. Their work provides a quantitative framework for analyzing influence, connectivity and control in networks under uncertainty, combining structural and numerical perspec-

tives in neutrosophic graph theory. [Garrett \(2023\)](#) introduced the concept of neutrosophic domination, extending classical and fuzzy domination theories to neutrosophic graphs. By incorporating truth, indeterminacy and falsity components, this framework allows for modeling domination under uncertainty, incomplete information and contradictory relationships in network structures.

[Raji et al. \(2025\)](#) introduced neutrosophic complete graphs and the concept of neutrosophic excellent domination, extending classical and FD frameworks. Their work defines dominating sets with maximal influence under truth, indeterminacy, and falsity components, providing a robust framework for analyzing complete network structures under uncertainty.

2.14 Gaps in the Literature

Despite substantial progress in the study of domination in uncertain graphs, several critical gaps remain in the existing literature:

Fuzzy and intuitionistic fuzzy domination frameworks are well-established; however, many domination variants such as Total, Paired, Split, and Equitable domination have not been systematically generalized to these uncertain settings with a consistent theoretical foundation. Similarly, neutrosophic domination sets have been introduced, but they often lack comprehensive axiomatic development, particularly for layered, hybrid, or multi-functional graph structures.

Furthermore, much of the research on domination in uncertain graphs emphasizes definitions and illustrative examples, with relatively little attention paid to algorithmic development, computational efficiency, or complexity analysis. Efficient algorithms, approximation strategies, and formal complexity bounds for domination in fuzzy, intuitionistic fuzzy, and neutrosophic graphs remain largely underexplored.

While existing domination theories provide a strong foundation, they are insufficient for modeling complex, multi-layered, and functionally heterogeneous networks that are common in real-world systems. Social networks, healthcare systems, and other large-scale infrastructures often exhibit uncertainty, layered interactions, and context-dependent relationships, which traditional frameworks cannot fully capture.

The TG framework addresses these limitations by integrating uncertainty modeling with

functional and structural hierarchies. This approach enables a more comprehensive and realistic treatment of domination, supporting both theoretical development and practical applications in complex networked environments.

2.15 Motivation of the Study

The primary motivation for this research stems from the absence of a comprehensive and unified theory of domination in Turiyam graphs. Fundamental questions-such as how domination should be defined in Turiyam graphs, how domination parameters behave, and how classical domination results generalize to this framework-remain largely unexplored. In addition, algorithmic approaches for identifying dominating sets in Turiyam graphs are limited, particularly for large-scale or application-driven networks, leaving a significant gap between theory and practice.

Another important motivation arises from the lack of a hierarchical framework connecting domination concepts across classical, fuzzy, intuitionistic fuzzy, neutrosophic, and Turiyam graphs. Establishing such a hierarchy is crucial for understanding how domination evolves with increasing uncertainty and for identifying special cases and generalizations. Turiyam domination has the potential to subsume existing domination models, providing a coherent and flexible framework for domination under uncertainty.

From a practical perspective, Turiyam domination enables more realistic modeling and decision-making in complex networks. In healthcare systems, for example, hospitals, clinics, and monitoring units may operate under varying conditions that cannot be adequately captured by classical or existing uncertain graph models. Similarly, in sensor networks, social networks, and other distributed systems, nodes may exhibit neutrality, contextual independence, or multi-layered interactions that influence domination behavior. Turiyam graphs provide a framework capable of representing such functional and contextual nuances, thereby improving the accuracy and applicability of domination-based analysis.

Therefore, the motivation of this study is to develop a rigorous, unified, and application-oriented theory of domination in Turiyam graphs. By addressing existing theoretical gaps and providing computational and practical tools, this research aims to advance the field of graph domination and offer meaningful methods for analyzing, optimizing, and managing complex uncertain networks.

CHAPTER 3

Preliminary Concepts

3.1 Introduction

Graph theory is a very important tool to represent many real-world problems. Graphs do not represent all the systems properly due to the uncertainty of the parameters of the systems, while the world is full of uncertainties. To overcome this, versions of graph theory, including fuzzy graphs, intuitionistic fuzzy graphs, single valued neutrosophic graphs, and Turiyam graph were presented.

In this chapter, we review some important notions of the uncertain logic and uncertain graphs that are helpful to this dissertation work. The sequence between the definitions of each notion is established. Further, the concepts of domination of FGs, domination of IFGs, and domination of NGs are also established by using some examples. Those concepts are useful in developing new theories.

3.2 Uncertain Graphs

Real-world networks often involve incomplete information, vagueness, hesitation, and indeterminacy that cannot be adequately represented using classical (crisp) graph models. To address these limitations, several extensions of graph theory have been proposed, including fuzzy graphs, intuitionistic fuzzy graphs, neutrosophic graphs, and more recently, Turiyam graphs. These models progressively enhance expressive power by incorporating varying degrees of uncertainty and logical interpretation ([Rosenfeld, 1975b](#); [Atanassov, 1986](#); [Smarandache, 1998](#); [Parvathi and Karunambigai, 2006](#); [Broumi et al., 2018a](#); [Jan et al., 2019](#); [?;](#) [Nazir et al., 2023](#)).

Definition 3.2.1. A classical(crisp) graph is a mathematical structure used to model pairwise relationships between objects. Formally, a graph is defined as an ordered pair $G = (V, E)$ where V is a finite non-empty set of vertices (nodes) and $E \subseteq \{u, v \mid (u, v) \in V, u \neq v\}$ is a set of edges connecting pairs of vertices.

3.2.1 Fuzzy Graph

Definition 3.2.2. Let V be a non-empty finite set of vertices. A G_F is an ordered pair $G_F = (\sigma, \mu)$ where $\sigma : V \rightarrow [0, 1]$ is a fuzzy vertex membership function and $\mu : V \times V \rightarrow [0, 1]$ is fuzzy edge membership function satisfying the condition $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}, \forall (u, v) \in V \times V$. Here, $\sigma(u)$ represents the degree to which the vertex u belongs to the graph, while $\mu(u, v)$ represents the degree of adjacency between vertices u and v . If $\mu(u, v) = 0$, then no edge exists between u and v in the fuzzy sense.

Example 3.2.1. Let $V = \{v_1, v_2, v_3\}$ be a finite set of vertices. Define the fuzzy vertex membership function $\sigma : V \rightarrow [0, 1]$ by $\sigma(v_1) = 1, \sigma(v_2) = 0.7, \sigma(v_3) = 0.4$. Define the fuzzy edge membership function $\mu : V \times V \rightarrow [0, 1]$ as $\mu(v_1, v_2) = 0.6, \mu(v_2, v_3) = 0.4$, and $\mu(v_1, v_3) = 0.2$. Since for every pair of vertices $v_i, v_j \in V, \mu(v_i, v_j) \leq \min\{\sigma(v_i), \sigma(v_j)\}$. The pair $G_F = (\sigma, \mu)$ satisfies the condition of fuzzy graph.

Definition 3.2.3. Let $G_F = (\sigma, \mu)$ be a fuzzy graph on a finite non-empty vertex set V , where $\sigma : V \rightarrow [0, 1]$ is a fuzzy vertex membership function and $\mu : V \times V \rightarrow [0, 1]$ is fuzzy edge membership function satisfying the condition $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}, \forall (u, v) \in V \times V$.

A fuzzy Graph $H_F = (\sigma_H, \mu_H)$ is called a fuzzy subgraph of G_F if there exists a subset $V_H \subseteq V$ such that $\sigma_H(v) \leq \sigma(v), \forall v \in V_H$ and $\mu_H(u, v) \leq \mu(u, v), \forall u, v \in V_H$. Thus, a fuzzy subgraph is obtained by selecting a subset of vertices and edges from the original fuzzy graph and possibly reducing their membership values, while preserving the fuzzy graph conditions.

Definition 3.2.4. If $\sigma_H(v) = \sigma(v), \mu_H(u, v) = \mu(u, v), \forall (u, v \in V_H)$, then H_F is called induced subgraph of G_F .

3.2.2 An Intuitionistic Fuzzy Graphs

Definition 3.2.5. Let V be a finite non-empty set of vertices. An intuitionistic fuzzy graph (IFG) is defined as an ordered pair $G_{IF} = (A, B)$, where A is an Intuitionistic fuzzy set on

V and B is intuitionistic fuzzy relation on $V \times V$. For each $v \in V$, the intuitionistic fuzzy vertex membership is given by $A(v) = \langle \mu_A(v), \nu_A(v) \rangle$, where $\mu_A(v) \in [0, 1]$ is the degree of membership $\nu_A(v) \in [0, 1]$ is the degree of non-membership such that $0 \leq \mu_A(v) + \nu_A(v) \leq 1$. Similarly, for each ordered pair $(u, v) \in V \times V$, the intuitionistic fuzzy edge membership is defined by $B(u, v) = \langle \mu_B(u, v), \nu_B(u, v) \rangle$, where $\mu_B(u, v), \nu_B(u, v) \in [0, 1]$, and $0 \leq \mu_B(u, v) + \nu_B(u, v) \leq 1$. The intuitionistic fuzzy relation B must satisfy the following consistency conditions: $\mu_B(u, v) \leq \min \{ \mu_A(u), \mu_A(v) \}$, $\nu_B(u, v) \geq \max \{ \nu_A(u), \nu_A(v) \}$, $\forall u, v \in V$. The hesitation (or indeterminacy) degree associated with a vertex or edge is implicitly defined as $\pi_A(v) = 1 - \mu_A(v) - \nu_A(v)$, $\pi_B(u, v) = 1 - \mu_B(u, v) - \nu_B(u, v)$.

Example 3.2.2. Let $V = v_1, v_2, v_3$ be a finite set of vertices. Define the intuitionistic fuzzy vertex set A on V as $A(v_1) = \langle 0.9, 0.5 \rangle$, $A(v_2) = \langle 0.7, 0.2 \rangle$, $A(v_3) = \langle 0.5, 0.3 \rangle$. For each vertex $v_i \in V$, the hesitation degree is given by $\pi_A(v) = 1 - \mu_A(v_i) - \nu_A(v_i)$. Hence, $\pi_A(v_1) = 0.05$, $\pi_A(v_2) = 0.1$, $\pi_A(v_3) = 0.2$. Also, define the intuitionistic fuzzy edge set B on $V \times V$ as $B(v_1, v_2) = \langle 0.6, 0.25 \rangle$, $B(v_2, v_3) = \langle 0.5, 0.4 \rangle$, $B(v_1, v_3) = \langle 0.4, 0.5 \rangle$. These values satisfy the intuitionistic fuzzy graph conditions: $\mu_B(u, v) \leq \min \{ \mu_A(u), \mu_A(v) \}$, $\nu_B(u, v) \geq \max \{ \nu_A(u), \nu_A(v) \}$, $\forall u, v \in V$. Therefore, the ordered pair $G_{IF} = (A, B)$ forms a valid intuitionistic fuzzy graph.

3.2.3 Neutrosophic Graphs

Definition 3.2.6. Let V be a finite non-empty set of vertices. A neutrosophic graph (IFG) is defined as an ordered pair $G_N = (A, B)$, where A is neutrosophic set on V and B is neutrosophic relation on $V \times V$. For each $v \in V$, the neutrosophic vertex membership is given by $A(v) = \langle T_A(v), I_A(v), F_A(v) \rangle$, where $T_A(v) \in [0, 1]$ denotes the truth membership degree, $I_A(v) \in [0, 1]$ denotes the indeterminacy membership degree, $F_A(v) \in [0, 1]$ denotes the falsehood membership degree, with condition $0 \leq T_A(v) + I_A(v) + F_A(v) \leq 3$. Similarly, for each ordered pair $(u, v) \in V \times V$, the neutrosophic edge membership is defined as $B(u, v) = \langle T_B(u, v), I_B(u, v), F_B(u, v) \rangle$, where $T_B(u, v), I_B(u, v), F_B(u, v) \in [0, 1]$ satisfy $T_B(u, v) = \min \{ T_A(u), T_A(v) \}$, $I_B(u, v) \geq \max \{ I_A(u), I_A(v) \}$, $F_B(u, v) \geq \max \{ F_A(u), F_A(v) \}$.

Example 3.2.3. Let $V = v_1, v_2, v_3$.

Define the neutrosophic vertex set A as $A(v_1) = \langle 0.9, 0.05, 0.05 \rangle$, $A(v_2) = \langle 0.7, 0.2, 0.1 \rangle$, $A(v_3) = \langle 0.5, 0.3, 0.2 \rangle$. Define neutrosophic edge set B by $B(v_1, v_2) = \langle 0.6, 0.25, 0.15 \rangle$, $B(v_2, v_3) =$

$\langle 0.5, 0.3, 0.2 \rangle, B(v_1, v_3) = \langle 0.4, 0.4, 0.2 \rangle$. These values satisfy the neutrosophic consistency conditions. Hence, $G_N = (A, B)$ is a valid neutrosophic graph.

3.2.4 Turiyam Graphs

Definition 3.2.7. Let V be a finite non-empty set of vertices. A Turiyam graph is an ordered quadruple $G_T = (V, E, L, F)$ where V is a set of vertices and $E \subseteq V \times V$ is the set of edges $L = \{L_1, L_2, \dots, L_k\}$ is a finite collection of logical layers, typically corresponding to classical, fuzzy, intuitionistic fuzzy, neutrosophic representations, $F = \{f_1, f_2, \dots, f_k\}$ is a family of valuation functions assigning logical values to vertices and edges in each layer. For a vertex $v \in V$ and edge $(u, v) \in E$, the Turiyam valuation in layer L_i is given by

$$f_i(v) = \begin{cases} 1, & \text{Classical layer,} \\ \mu(v) \in [0, 1], & \text{Fuzzy layer,} \\ \langle \mu(v), u(v) \rangle, & \text{Intuitionistic layer,} \\ \langle T(v), I(v), F(v) \rangle, & \text{Neutrosophic layer.} \end{cases}$$

Edge valuations are defined analogously. Thus, a Turiyam graph unifies multiple logical representations within a single structured framework, enabling simultaneous modelling of certainty, vagueness, hesitation and indeterminacy.

Definition 3.2.8. Consider a finite set of vertices and edges $V = \{S_i : i = 1, 2, \dots, n\}$ and $E = \{(S_i, S_j) : i, j = 1, 2, \dots, n\}$ respectively. A Turiyam graph of a graph $G = (V, E)$ is defined by $G_T = (A, B)$, where:

- (a) $t_A, I_A, F_A, L_A : V \longrightarrow [0, 1]$ represents the truth value (tv) of s_i the indeterminacy value (iv) of s_i the falsity value (fv) of s_i and the turiyam state (or liberal) value (lv) of s_i respectively, $\forall s_i \in V$ such that $0 \leq t_A(s_i) + I_A(s_i) + f_A(s_i) + l_A(s_i) \leq 4, \forall s_i \in V, i = 1, 2, \dots, n$.
- (b) $t_B, I_B, F_B, L_B : E \subseteq V \times V \longrightarrow [0, 1]$ given by $t_B(a_i b_j) \leq \min \{t_A(a_i), t_A(b_j)\}$, $i_B(a_i b_j) \leq \min \{i_A(a_i), i_A(b_j)\}$, $f_B(a_i b_j) \leq \max \{f_A(a_i), f_A(b_j)\}$, and $l_B(a_i b_j) \leq \min \{l_A(a_i), l_A(b_j)\}$, $\forall (a_i b_j) \in V$ represents the truth, indeterminacy, falsity and turiyam mappings from E to $V \times V$ respectively, such that $0 \leq t_b(\{a_i b_j\}) + i_b(\{a_i b_j\}) + f_b(\{a_i b_j\}) + l_b(\{a_i b_j\}) \leq 4, \forall \{a_i b_j\} \in E, i, j = 1, 2, \dots, n$.

Example 3.2.4. Consider a Turiyam graph $G_T = (A, B)$ of $G = (V, E)$, where

$$V = \{v_1, v_2, v_3\}$$

and $E = \{v_1v_2, v_2v_3, v_3v_1\}$. Let A and B be Turiyam sets on V and E , respectively, with $v_1 = \langle 0.4, 0.6, 0.1, 0.7 \rangle$, $v_2 = \langle 0.1, 0.4, 0.8, 0.3 \rangle$, $v_3 = \langle 0.6, 0.8, 0.1, 0.7 \rangle$, and $v_1v_2 = \langle 0.1, 0.2, 0.7, 0.2 \rangle$, $v_2v_3 = \langle 0.1, 0.4, 0.8, 0.2 \rangle$, $v_1v_3 = \langle 0.2, 0.5, 0.1, 0.4 \rangle$.

Here, $v_1 = \langle 0.4, 0.6, 0.1, 0.7 \rangle$ is a Turiyam vertex, and $v_1v_2 = \langle 0.1, 0.2, 0.7, 0.2 \rangle$ is a Turiyam edge.

Definition 3.2.9. Let $G_T = (A, B)$ be a Turiyam graph. Two vertices v_i and v_j are said to be adjacent if the edge v_iv_j exists and satisfies

$$t_B(v_iv_j) = \min\{t_A(v_i), t_A(v_j)\},$$

$$i_B(v_iv_j) = \min\{i_A(v_i), i_A(v_j)\},$$

$$f_B(v_iv_j) = \max\{f_A(v_i), f_A(v_j)\},$$

and $l_B(v_iv_j) = \min\{l_A(v_i), l_A(v_j)\}$, for every $\{v_i, v_j\} \in V$.

In this case, v_i and v_j are called neighboring vertices, and the edge $e = v_iv_j$ is said to be incident with both v_i and v_j .

Definition 3.2.10 (Neighborhood of a Vertex). Let $G_T = (A, B)$ be a Turiyam graph and let $v_i \in V(G_T)$. The neighborhood of v_i , denoted by $N(v_i)$, is defined as

$$N(v_i) = \{v_j \in V(G_T) : v_j \text{ is adjacent to } v_i\}.$$

That is, $N(v_i)$ consists of all vertices that are connected to v_i by an edge in G_T .

3.3 Domination in uncertain graph models

This section reviews the evolution of domination concepts from fuzzy graphs to intuitionistic fuzzy graphs, neutrosophic graphs, and single-valued neutrosophic graphs. This progression reflects increasing expressive power in handling vagueness, hesitation, and indeterminacy, forming the theoretical basis for domination in Turiyam graphs.

In this section, the concepts of domination; domination in fuzzy graph, domination in intuitionistic fuzzy graph and domination in neutrosophic and single valued graph are reviewed (Rosenfeld, 1975b; Atanassov, 1986; Smarandache, 1998; Parvathi and Karunambigai, 2006; Broumi et al., 2018a; Mullai and Broumi, 2020).

3.3.1 Domination in Fuzzy graphs

Definition 3.3.1. Let $G_F = (\sigma, \mu)$ be a fuzzy graph, where $\sigma : V \longrightarrow [0, 1]$ is a fuzzy vertex membership function and $\mu : V \times V \longrightarrow [0, 1]$ is fuzzy edge membership function satisfying the condition $\mu(u, v) \leq \min \{\sigma(u), \sigma(v)\}$, A subset $D \subseteq V$ is called a fuzzy dominating set if for every vertex $v \in V \setminus D$, there exists a vertex $u \in D$ such that $\mu(u, v) > 0$. The fuzzy domination number $\gamma_F(G_F)$ is the minimum fuzzy cardinality of such a set.

Example 3.3.1. Let $V = \langle v_1, v_2, v_3 \rangle$ with vertex memberships $\sigma(v_1) = 0.1, \sigma(v_2) = 0.8, \sigma(v_3) = 0.6$, and edge memberships $\mu(v_1v_2) = 0.7, \mu(v_2v_3) = 0.5$. Then $D = \{v_2\}$ is a fuzzy dominating set since both v_1 and v_3 are adjacent to v_2 with positive membership.

3.3.2 Domination in Intuitionistic Fuzzy graphs

Definition 3.3.2. An intuitionistic fuzzy graph (IFG) is defined as $G_{IF} = (V, E, \mu, \nu)$, where μ and ν represent membership and non-membership degrees satisfying $0 \leq \mu(u, v) + \nu(u, v) \leq 1$. A set $D \subseteq V$ is an intuitionistic fuzzy dominating set if for every $v \in V \setminus D$, there exists $u \in D$ such that $\mu(u, v) > \nu(u, v)$.

Example 3.3.2. Let edge (v_1, v_2) have $\mu(v_1, v_2) = 0.7, \nu(v_1, v_2) = 0.2$. Then $D = \{v_1\}$ is an intuitionistic fuzzy dominating set since membership exceeds non-membership.

3.3.3 Domination in Neutrosophic graphs

Definition 3.3.3. A neutrosophic graph is defined as $G_N = (V, E, T, I, F)$, where each vertex and edge is associated with: T -truth membership, I -indeterminacy membership, F -falsity membership. A subset $D \subseteq V$ is a neutrosophic dominating set if for every $v \in V \setminus D$, there exists $u \in D$ such that $T(u, v) > F(u, v)$. The neutrosophic domination number is denoted by $\gamma_N(G_N)$.

Example 3.3.3. Let an edge (u, v) be represented as $\langle T, I, F \rangle = \langle 0.6, 0.2, 0.2 \rangle$. Since $T > F$, vertex u dominates v .

3.3.4 Domination in single-valued Neutrosophic graphs

Definition 3.3.4. A single-valued neutrosophic graph (SVNG) restricts $T, I, F \in [0, 1]$, $0 \leq T + I + F \leq 3$. A set $D \subseteq V$ is a single-valued neutrosophic dominating set if for each $v \in V \setminus D$, there exists $u \in D$ such that $T(u, v) \geq \max \{I(u, v), F(u, v)\}$.

Example 3.3.4. If $(u, v) = \langle 0.7, 0.1, 0.2 \rangle$, then u dominates v since truth-membership is maximal.

CHAPTER 4

Domination and Paired Domination in Turiyam Graphs

Graph theory provides powerful tools for modeling real-world systems affected by uncertainty, ambiguity, and inconsistent information. Neutrosophic graphs have been widely used to represent such situations by extending classical graph models. However, to better manage levels of unpredictability that go beyond the traditional true, false, and indeterminate states, the concept of the Turiyam graph has been introduced recently. Domination and paired domination are fundamental concepts in both theoretical and applied graph theory. Motivated by their importance, this work develops the notions of domination and paired domination within the framework of Turiyam graphs and investigates their fundamental properties. In addition, domination in complete bipartite Turiyam graphs is analyzed, the concept of strong paired domination is explored, and practical applications of paired domination are discussed.

4.1 Introduction

Graph theory has long been recognized as a powerful mathematical framework for modeling and analyzing complex systems arising in science, engineering, social networks, communication systems, and decision-making processes. Classical graph models, however, often assume that information is precise and well-defined, an assumption that rarely holds in real-world situations. In many practical applications, data may be incomplete, ambiguous, inconsistent, or subject to unpredictable fluctuations. This limitation has motivated the development of advanced graph models capable of representing uncertainty more effectively. Neutrosophic graph theory emerged as an important extension of classical graph theory by incorporating degrees of truth, falsity, and indeterminacy. These graphs provide a flexible structure for handling uncertainty in complex systems. Nevertheless, certain real-world scenarios involve levels of unpredictability that cannot be fully captured by these three compo-

nents alone. To address this challenge, the concept of Turiyam graphs has been introduced. Turiyam graphs extend neutrosophic graphs by including an additional state-Turiyam-which represents uncertainty beyond truth, falsity, and indeterminacy, thereby offering a richer and more comprehensive modeling framework.

Within graph theory, the concepts of domination and paired domination play a central role due to their wide range of theoretical and practical applications. Domination focuses on identifying a minimum set of vertices that can influence or control the entire graph, while paired domination strengthens this idea by requiring that the dominating vertices form pairs connected by edges. These concepts are particularly relevant in applications such as network monitoring, resource allocation, facility location, and communication systems, where efficiency and reliability are essential.

Despite extensive research on domination and paired domination in classical and neutrosophic graphs, their study within the context of Turiyam graphs remains limited. This thesis aims to bridge this gap by introducing and systematically investigating domination and paired domination in Turiyam graphs. The fundamental properties of these concepts are examined, along with special cases such as domination in complete bipartite Turiyam graphs and strong paired domination. In addition, potential applications of paired domination are discussed to demonstrate the practical significance of the proposed theoretical results.

The outcomes of this research contribute to the growing body of knowledge on uncertainty-based graph models and provide a foundation for future studies in both theoretical development and real-world applications of Turiyam graph theory.

4.2 Domination in Turiyam Graph

This section presents the fundamental concepts related to domination in Turiyam graphs. It begins by introducing the notion of strong edges and domination defined through strong edges. The section then discusses minimal dominating sets in Turiyam graphs, along with their maximal cardinality and the role of maximal independent sets within this framework. Finally, the concept of total dominating sets in Turiyam graphs is examined, and illustrative examples are provided to clarify and support the theoretical definitions.

Definition 4.2.1. (Ganati 2023). Let $TG = (A, M)$ be a Turiyam graph. An edge $(x, y) \in E(TG)$ is called a Turiyam strong edge if the direct edge attains the strongest connectivity between x and y , i.e.,

$$T_M(x, y) = T_M^\infty(x, y), \quad I_M(x, y) = I_M^\infty(x, y),$$

$$F_M(x, y) = F_M^\infty(x, y), \quad L_M(x, y) = L_M^\infty(x, y),$$

where

$$T_M^\infty(x, y) = \sup_{m \geq 1} T_M^m(x, y), \quad I_M^\infty(x, y) = \inf_{m \geq 1} I_M^m(x, y),$$

$$F_M^\infty(x, y) = \inf_{m \geq 1} F_M^m(x, y), \quad L_M^\infty(x, y) = \sup_{m \geq 1} L_M^m(x, y).$$

Definition 4.2.2. (Ganati 2023). Let x be a node in a Turiyam graph $TG = (A, M)$. The neighbour set of x in TG , denoted by $N(x)$, is defined as

$$N(x) = \{y \in A : (x, y) \text{ is a strong edge in } TG\}.$$

Definition 4.2.3. (Ganati 2023). A node $x \in A$ of a Turiyam graph TG is called an isolated node if

$$T_M(x, y) = 0, \quad I_M(x, y) = 0, \quad F_M(x, y) = 0, \quad \text{and} \quad L_M(x, y) = 0$$

for every $y \in A$. Equivalently, the neighbour set of x is empty, that is,

$$N(x) = \emptyset.$$

Hence, an isolated node has no influence on any other vertex of the graph and does not dominate any vertex in TG .

Definition 4.2.4. Suppose $TG = (A, M)$ is a Turiyam graph associated with a crisp graph G . For $x, y \in A$, the vertex x is said to dominate y in TG if there exists a strong edge between x and y .

Remark.

- For any $x, y \in A$, if x dominates y , then y also dominates x .
- For any $y \in A$, the neighbour set $N(y)$ consists of all vertices in A that are dominated by y .

- If for every pair of vertices $x, y \in A$,

$$T_M(x, y) = T_M^\infty(x, y), \quad I_M(x, y) = I_M^\infty(x, y), \quad F_M(x, y) = F_M^\infty(x, y), \quad L_M(x, y) = L_M^\infty(x, y),$$

then every edge of TG is a strong edge. Consequently, every pair of vertices is strongly adjacent. Hence every vertex strongly dominates every other vertex, and any nonempty subset of vertices forms a dominating set. In particular, the domination number of TG is 1.

Definition 4.2.5. Let $T_G = (A, M)$ be a Turiyam graph. A subset $C \subseteq A$ is called a dominating set of T_G if every vertex in $A \setminus C$ is dominated by at least one vertex in C . That is, for each $y \in A \setminus C$, there exists a vertex $x \in C$ such that x dominates y in T_G .

Definition 4.2.6. Let $T_G = (A, M)$ be a Turiyam graph. A dominating set $C \subseteq A$ is called a minimal dominating set if no proper subset of C is a dominating set of T_G . Equivalently, for every vertex $x \in C$, the set $C \setminus \{x\}$ is not a dominating set of T_G .

Definition 4.2.7. The minimum cardinality among all minimal dominating sets of a Turiyam graph is called the lower domination number of the graph and is denoted by $d_N(T_G)$.

Similarly, the maximum cardinality among all minimal dominating sets of a Turiyam graph is called the upper domination number and is denoted by $D_N(T_G)$.

Definition 4.2.8. *Ganati (2023)*. Vertices of a Turiyam graph $T_G = (A, M)$ are said to be independent if no strong edge exists between any pair of them.

Definition 4.2.9. *Ganati (2023)*. A subset $C \subseteq A$ is called an independent set of a Turiyam graph $T_G = (A, M)$ if, for every $x, y \in C$,

$$T_M(x, y) \leq T_M^\infty(x, y), \quad I_M(x, y) \leq I_M^\infty(x, y), \quad F_M(x, y) \leq F_M^\infty(x, y), \quad \text{and} \quad L_M(x, y) \leq L_M^\infty(x, y).$$

Definition 4.2.10. *Ganati (2023)*. Let $T_G = (A, M)$ be a Turiyam graph. A subset $I \subseteq A$ is called an independent set if no two distinct vertices in I are strongly adjacent. That is, for every $x, y \in I$ with $x \neq y$, the edge (x, y) is not a strong edge in T_G .

Definition 4.2.11. An independent set C of a $T_G = (A, M)$ is said to be maximal independent if, for every vertex $y \in A \setminus C$, the set $C \cup \{y\}$ is not an independent set of T_G .

Definition 4.2.12. The minimum cardinality among all maximum independent sets of a Turiyam graph T_G is called the lower independence number and is denoted by $i_N(T_G)$. [2mm] Similarly, the maximum cardinality among all maximal independent sets of a TG is called the upper independence number and is denoted by $I_N(T_G)$.

Definition 4.2.13. (Ganati 2023). Let $T_G = (A, M)$ be a Turiyam graph and let $x, y \in A$. The vertex y is called a strong neighbor of x if the edge (x, y) is a strong edge in T_G , that is,

$$T_M(x, y) \geq T_M^\infty(x, y), \quad I_M(x, y) \geq I_M^\infty(x, y),$$

$$F_M(x, y) \geq F_M^\infty(x, y), \quad L_M(x, y) \geq L_M^\infty(x, y).$$

The set of all strong neighbors of x is called the strong neighborhood of x and is denoted by

$$N_s(x) = \{y \in A : (x, y) \text{ is a strong edge}\}.$$

Theorem 4.2.1. Let R be a dominating set of a Turiyam graph $T_G = (A, M)$. Then R is a minimal dominating set if and only if, for every $r \in R$, at least one of the following holds:

1. r is not a strong neighbor of any other vertex in R ;
2. There exists a vertex $x \in A \setminus R$ such that

$$N(x) \cap R = \{r\}.$$

Proof. ($\Rightarrow$) Suppose that R is a minimal dominating set. Let $r \in R$. Since R is minimal, the set $R' = R \setminus \{r\}$ is not a dominating set. Therefore, there exists a vertex $x \in A$ that is dominated by r in R but is not dominated by any vertex of R' . Two cases arise.

Case 1: $x = r$.

Since r itself becomes undominated after removing r , no other vertex of R strongly dominates r . Hence, r is not a strong neighbor of any other vertex in R .

Case 2: $x \neq r$.

Then $x \in A \setminus R$. Because x loses domination only when r is removed, no other vertex of R dominates x . Consequently, $N(x) \cap R = \{r\}$. Thus, condition (2) holds.

Hence, for every $r \in R$, at least one of the two conditions is satisfied.

($\Leftarrow$) Conversely, suppose that for every $r \in R$, either condition (1) or condition (2) holds.

Assume, to the contrary, that R is not minimal. Then there exists some $r \in R$ such that $R' = R \setminus \{r\}$ is still a dominating set.

If condition (1) holds, then r has no strong neighbor in R . Since $r \notin R'$, no vertex of R'

dominates r , contradicting the assumption that R' is a dominating set.

If condition (2) holds, there exists a vertex $x \in A \setminus R$ satisfying

$$N(x) \cap R = \{r\}.$$

After removing r , the vertex x is no longer dominated by any vertex of R' , again contradicting the fact that R' is a dominating set. Therefore, no vertex can be removed from R while preserving domination. Hence, R is a minimal dominating set. $\square$

Theorem 4.2.2. *Let $T_G = (A, M)$ be a Turiyam graph with no isolated vertices, and let $R \subseteq A$ be a minimal dominating set of T_G . Then the set $A \setminus R$ is a prevailing set of T_G .*

Proof. Let R be a minimal dominating set of T_G , and let y be any vertex in R . Since T_G has no isolated vertices, there exists a vertex $x \in N(y)$. Thus, y must be dominated by some vertex in $R \setminus \{y\}$, which implies that $R \setminus \{y\}$ is an influencing set.

By Theorem 4.2.1, it follows that $x \in A \setminus R$. Therefore, each vertex in R is dominated by at least one vertex in $A \setminus R$. This shows that $A \setminus R$ is itself a dominating set of T_G , i.e., a prevailing set. $\square$

Definition 4.2.14. *Let $TG = (A, M)$ be a Turiyam graph. The order of TG , denoted by $O(TG)$, is the number of vertices in TG , that is,*

$$O(TG) = |A|.$$

Theorem 4.2.3. *Let $T_G = (A, M)$ be a Turiyam graph of order $O(T_G)$, and let $\overline{T_G}$ denote its complement graph. Then*

$$d_N(T_G) + d_N(\overline{T_G}) \leq 2O(T_G).$$

Furthermore, equality holds if and only if, for every pair of vertices $x, y \in A$, the truth, indeterminacy, falsity, and refusal membership values are all strictly positive and strictly less than their corresponding upper bounds; that is, $0 < T_M(x, y) < T_M^\infty(x, y)$, $0 < I_M(x, y) < I_M^\infty(x, y)$, $0 < F_M(x, y) < F_M^\infty(x, y)$, and $0 < L_M(x, y) < L_M^\infty(x, y)$.

Proof. Since every lower dominating set of a Turiyam graph is a subset of its vertex set, the lower domination number of T_G cannot exceed the order of the graph. Hence,

$$d_N(T_G) \leq O(T_G).$$

Similarly, for the complement graph $\overline{T_G}$,

$$d_N(\overline{T_G}) \leq O(T_G).$$

Adding these inequalities gives

$$d_N(T_G) + d_N(\overline{T_G}) \leq 2O(T_G).$$

For the equality case, observe that equality can occur only when both $d_N(T_G)$ and $d_N(\overline{T_G})$ attain their maximum possible value $O(T_G)$. This happens precisely when every pair of vertices satisfies the nontrivial membership conditions $0 < T_M(x, y) < T_M^\infty(x, y)$, $0 < I_M(x, y) < I_M^\infty(x, y)$, $0 < F_M(x, y) < F_M^\infty(x, y)$, and $0 < L_M(x, y) < L_M^\infty(x, y)$. Therefore,

$$d_N(T_G) + d_N(\overline{T_G}) = 2O(T_G).$$

Conversely, if equality holds, then each lower domination number must equal $O(T_G)$, which implies that the above membership conditions are satisfied for every pair of vertices. Hence, the stated conditions characterize the equality case. $\square$

Corollary 4.2.1. *For any Turiyam graph $T_G = (A, M)$ without isolated vertices, the lower domination number satisfies*

$$d_N(T_G) \leq \frac{1}{2}O(T_G).$$

Proof. Let $S \subseteq A$ be a minimum lower dominating set of T_G , so that $|S| = d_N(T_G)$. Since T_G has no isolated vertices, every vertex in $A \setminus S$ must be dominated by at least one vertex in S . Thus, each vertex in $A \setminus S$ is associated with at least one vertex in S , implying that $|S|$ cannot exceed $|A \setminus S|$ in a minimal configuration. Hence,

$$|S| \leq |A \setminus S|.$$

Since $|S| + |A \setminus S| = O(T_G)$, we obtain

$$2|S| \leq O(T_G),$$

which yields

$$d_N(T_G) = |S| \leq \frac{1}{2}O(T_G).$$

$\square$

Corollary 4.2.2. *Let T_G be a Turiyam graph such that both T_G and its complement $\overline{T_G}$ have no isolated vertices. Then*

$$d_N(T_G) + d_N(\overline{T_G}) < O(T_G),$$

and equality holds if and only if

$$d_N(T_G) = d_N(\overline{T_G}) = O(T_G).$$

Proof. Let S and S' be minimum lower dominating sets of T_G and $\overline{T_G}$ respectively. Then

$$|S| = d_N(T_G), \quad |S'| = d_N(\overline{T_G}).$$

Since $S, S' \subseteq A$, we have

$$|S \cup S'| \leq O(T_G).$$

Using the identity

$$|S| + |S'| = |S \cup S'| + |S \cap S'|,$$

we obtain

$$|S| + |S'| \leq O(T_G) + |S \cap S'|.$$

Because both T_G and $\overline{T_G}$ have no isolated vertices, the dominating structures cannot be simultaneously minimal on disjoint complements, implying $|S \cap S'| < \min(|S|, |S'|)$. Hence,

$$|S| + |S'| < O(T_G).$$

Therefore,

$$d_N(T_G) + d_N(\overline{T_G}) < O(T_G).$$

For equality to hold, we must have $|S \cap S'| = \min(|S|, |S'|)$, which is possible only when $S = S' = A$. Hence,

$$d_N(T_G) = d_N(\overline{T_G}) = O(T_G).$$

□

Theorem 4.2.4. *Let $T_G = (A, M)$ be a Turiyam graph of order $O(T_G)$ and maximum degree Δ . Then the lower domination number satisfies*

$$d_N(T_G) \leq O(T_G) - \Delta.$$

Proof. Let $u \in A$ be a vertex of maximum degree, so that

$$\deg(u) = \Delta.$$

Denote by $N(u)$ the neighborhood of u . Since u is adjacent to exactly Δ vertices, the set $A \setminus N(u)$ contains $O(T_G) - \Delta$ vertices. Consider the set $R = A \setminus N(u)$. Clearly, $u \in R$, because a vertex does not belong to its own neighborhood. Moreover, every vertex in $N(u)$ is dominated by u , while each vertex of R dominates itself. Hence every vertex of T_G is dominated by some vertex in R , and therefore R is a dominating set of T_G . Since the lower domination number is the minimum cardinality among all dominating sets, we obtain

$$d_N(T_G) \leq |R| = |A \setminus N(u)| = O(T_G) - \Delta.$$

Therefore,

$$d_N(T_G) \leq O(T_G) - \Delta.$$

□

Definition 4.2.15. Let $T_G = (A, M)$ be a Turiyam graph. A subset $S \subseteq A$ is called a maximal independent set if:

- S is an independent (stable) set, i.e., no two vertices in S are adjacent, and
- for every vertex $v \in A \setminus S$, the set $S \cup \{v\}$ is not independent.

Equivalently, no proper superset of S in A is independent.

Theorem 4.2.5. A subset $S \subseteq A$ of a Turiyam graph $T_G = (A, M)$ is a maximal stable (independent) set if and only if it is both a dominating set and an independent set.

Proof. ($\Rightarrow$) Let $S \subseteq A$ be a maximal stable (independent) set. Then S is independent by definition. We show that S is dominating. If S were not a dominating set, then there exists a vertex $v \in A \setminus S$ such that v is not adjacent to any vertex in S . Hence $S \cup \{v\}$ would still be an independent set, contradicting the maximality of S . Therefore, S must be a dominating set.

($\Leftarrow$) Let $S \subseteq A$ be both independent and dominating. We show that S is maximal independent. Assume that S is not maximal. Then there exists a vertex $v \in A \setminus S$ such that $S \cup \{v\}$ is independent. This implies that v is not adjacent to any vertex in S , contradicting the fact that S is a dominating set. Hence no such vertex exists, and S is maximal independent. Therefore, S is a maximal stable set if and only if it is both independent and dominating. □

Theorem 4.2.6. *Every maximal stable (independent) set in a Turiyam graph $T_G = (A, M)$ is a minimal dominating set.*

Proof. Let C be a maximal stable (independent) set in a Turiyam graph $T_G = (A, M)$. By Theorem 4.2.5, C is a dominating set. Suppose, for contradiction, that C is not a minimal dominating set. Then there exists a vertex $x \in C$ such that $C \setminus \{x\}$ is still a dominating set. However, if $C \setminus \{x\}$ dominates $A \setminus (C \setminus \{x\})$, then at least one vertex in $C \setminus \{x\}$ must be a strong neighbor of x . This contradicts the fact that C is independent (stable). Therefore, C must be a minimal dominating set. $\square$

Theorem 4.2.7. *Let $T_G = (A, M)$ be a Turiyam graph. Then the lower domination number equals the order of the graph, that is,*

$$d_N(T_G) = O(T_G),$$

if and only if every vertex of T_G has a unique neighbor.

Proof. $(\Rightarrow)$ Assume that $d_N(T_G) = O(T_G)$. Let S be a minimum lower dominating set of T_G . Then $S = A$. This implies that every vertex must be necessary for domination; hence no vertex can be dominated by more than one alternative vertex in A . Therefore, each vertex must have a unique neighbor structure ensuring that removing any vertex destroys domination, which implies every vertex has a unique neighbor.

$(\Leftarrow)$ Assume that every vertex of T_G has a unique neighbor. Let S be a lower dominating set of T_G . Since each vertex is dominated by exactly one neighbor, no vertex can dominate more than one vertex in a redundant way.

Hence, every vertex is essential in any dominating configuration, implying that all vertices must be included in every lower dominating set. Therefore, $d_N(T_G) = |A| = O(T_G)$. Thus, $d_N(T_G) = O(T_G)$ if and only if every vertex of T_G has a unique neighbor. $\square$

4.3 Dominance of Edges in Turiyam Graphs

In this section, we introduce the concept of *edge domination* in Turiyam graphs. We define minimal edge dominating sets and explore their fundamental properties and characteristics within the framework of Turiyam graphs.

Definition 4.3.1. An edge $e = (u, v)$ of a Turiyam graph T_G is called an effective edge if $T_M(u, v) = \min\{T_A(u), T_A(v)\}$, $I_M(u, v) = \min\{I_A(u), I_A(v)\}$, $F_M(u, v) = \min\{F_A(u), F_A(v)\}$, $L_M(u, v) = \min\{L_A(u), L_A(v)\}$.

Definition 4.3.2. The link (edge) domination number of a $T_G = (A, M)$, denoted by $d_E(T_G)$, is defined as the minimum cardinality among all edge dominating sets of T_G .

Definition 4.3.3. Let $T_G = (A, M)$ be a Turiyam graph corresponding to a graph $G = (V, E)$. A subset $E' \subseteq E$ is called a link dominating set in T_G if, for every edge $e \in E \setminus E'$, there exists an effective edge in E' that is adjacent to e .

Definition 4.3.4. A minimal link dominating set in a Turiyam graph $T_G = (A, M)$ is a link dominating set $E' \subseteq E$ such that no proper subset of E' is itself a link dominating set. In other words, removing any edge from E' would result in a set that no longer dominates all edges in $E \setminus E'$.

Properties of Minimal Link Dominating Sets:

1. Every minimal link dominating set is a link dominating set, but the converse is not.
2. If E' is a minimal link dominating set, then for each edge $e \in E'$, there exists at least one edge $f \in E \setminus E'$ such that f is dominated only by e .
3. The cardinality of a minimal link dominating set is at least the lower edge domination number of T_G , and the maximum cardinality among all minimal link dominating sets is the upper edge domination number.

Definition 4.3.5. The link domination number of a Turiyam graph $T_G = (A, M)$, denoted by $d_E(T_G)$, is defined as the minimal cardinality among all edge (link) dominating sets of T_G .

Definition 4.3.6. An isolated link of a Turiyam graph T_G is an edge (a, m) for which no effective edge is incident on either of its endpoints.

Remark.

1. $N(s) \cap Y = \emptyset$,
2. There exists an edge $e \in E \setminus Y$ such that $N(e) \cap Y = \{s\}$ and s is an effective edge.

Theorem 4.3.1. *Let T_G be a Turiyam graph with no isolated links. Then for each minimal edge dominating set Y , the set $E \setminus Y$ is also an edge-dominating set.*

Proof. Let $y \in Y$ be any edge in a minimal edge dominating set Y . Since T_G has no isolated links, there exists an edge $e \in N(y)$, which implies $e \in E \setminus Y$. Hence, every edge in Y is dominated by at least one edge in $E \setminus Y$, showing that $E \setminus Y$ is itself an edge-dominating set. $\square$

Definition 4.3.7. *Let $T_G = (A, M)$ be a Turiyam graph. The lower edge domination number of T_G , denoted by $d_E^-(T_G)$, is defined as the minimum cardinality among all minimal link (edge) dominating sets of T_G .*

The upper edge domination number of T_G , denoted by $d_E^+(T_G)$, is defined as the maximum cardinality among all minimal link (edge) dominating sets of T_G .

Theorem 4.3.2. *For a T_G , every minimal link dominating set corresponds to a vertex dominating set of the line graph $L(T_G)$, and the following inequality holds:*

$$d_N(T_G) \leq d_E(T_G) \leq D_E(T_G) \leq D_N(T_G),$$

where $d_N(T_G)$ and $D_N(T_G)$ are the lower and upper vertex domination numbers of T_G , respectively.

Proof. Consider the line graph $L(T_G)$, whose vertices correspond to edges of T_G , and two vertices in $L(T_G)$ are adjacent if their corresponding edges in T_G share a common vertex. A minimal link dominating set in T_G dominates all edges not in the set, which translates to a dominating set in $L(T_G)$. The inequalities follow because any edge dominating set induces a vertex dominating set in $L(T_G)$, and maximal vertex domination provides bounds on minimal and maximal edge domination. $\square$

4.4 Paired Domination in Turiyam Graphs

This section focuses on the concept of *paired domination* in Turiyam graphs and explores its fundamental properties. We also introduce the basic ideas of *paired domination sets* and illustrate them with several examples to provide a clear understanding of the concept.

Definition 4.4.1. Let T_G be a Turiyam graph and let r be a vertex in T_G . Then r strongly dominates all of its neighbors as well as itself, i.e., r strongly dominates the vertices in $Y[r]$. A subset $D \subseteq V(T_G)$ is called a strongly dominating set of T_G if every vertex in $V(T_G) \setminus D$ is a strong neighbor of at least one vertex in D .

Definition 4.4.2. Let $D \subseteq V(T_G)$ be a strongly dominating set in a Turiyam graph T_G . Define the following aggregate values associated with D :

$$W_1(D) = \sum_{u \in D} h_2(u, v) \quad (\text{actual membership value}),$$

$$W_2(D) = \sum_{u \in D} g_2(u, v) \quad (\text{abstinence value}),$$

$$W_3(D) = \sum_{u \in D} n_2(u, v) \quad (\text{non-membership value}),$$

$$W_4(D) = \sum_{u \in D} k_2(u, v) \quad (\text{liberal value}),$$

where for each vertex $u \in D$:

- $h_2(u, v)$, $g_2(u, v)$, and $k_2(u, v)$ represent the minimum membership, minimum abstinence, and minimum liberal values of the edges incident on u , respectively,
- $n_2(u, v)$ represents the maximum non-membership value of such edges.

The strong domination number of the T_G is denoted by $R(T_G) = R = (R_1, R_2, R_3, R_4)$, where R_1 , R_2 , and R_4 are the lowest membership, lowest abstinence, and lowest liberal values among all such collections, and R_3 is the highest non-membership value.

Definition 4.4.3. Let $T_G = (A, M)$ be a TG. A subset $D \subseteq A$ is called a paired dominating set of T_G if:

1. D is a dominating set of T_G , and
2. the induced Turiyam subgraph $T_G[D]$ contains a perfect matching.

Definition 4.4.4. Let $T_G = (A, M)$ be a Turiyam graph. A subset $D \subseteq A$ is called a strong paired dominating collection (SPD) if:

1. D is a strong dominating set of T_G , and
2. the induced Turiyam subgraph $T_G[D]$ admits a perfect effective matching.

Definition 4.4.5. For each vertex $u \in D$, define the local aggregate values (computed over a single vertex) as:

$$h_2(u) = \min_{(u,v) \in M} h_2(u,v), \quad g_2(u) = \min_{(u,v) \in M} g_2(u,v),$$

$$n_2(u) = \max_{(u,v) \in M} n_2(u,v), \quad k_2(u) = \min_{(u,v) \in M} k_2(u,v),$$

where (u,v) ranges over all edges incident to u in T_G .

Definition 4.4.6. The global aggregate values (computed of a set of vertex) of a strong paired dominating collection D are defined as:

$$W_1(D) = \sum_{u \in D} h_2(u), \quad W_2(D) = \sum_{u \in D} g_2(u),$$

$$W_3(D) = \sum_{u \in D} n_2(u), \quad W_4(D) = \sum_{u \in D} k_2(u).$$

Definition 4.4.7. The strong paired domination number of T_G , denoted by

$$\beta(T_G) = (\beta_1, \beta_2, \beta_3, \beta_4),$$

is defined as:

$$\beta_1 = \min W_1(D), \quad \beta_2 = \min W_2(D), \quad \beta_3 = \max W_3(D), \quad \beta_4 = \min W_4(D),$$

where the extrema are taken over all strong paired dominating collections D of T_G , and β_1 , β_2 , and β_4 are the lowest membership, lowest abstinence, and lowest liberal values among all strong paired dominating collections, and β_3 is the highest non-membership value.

we want to illustrate this concept as shown in 4.1.

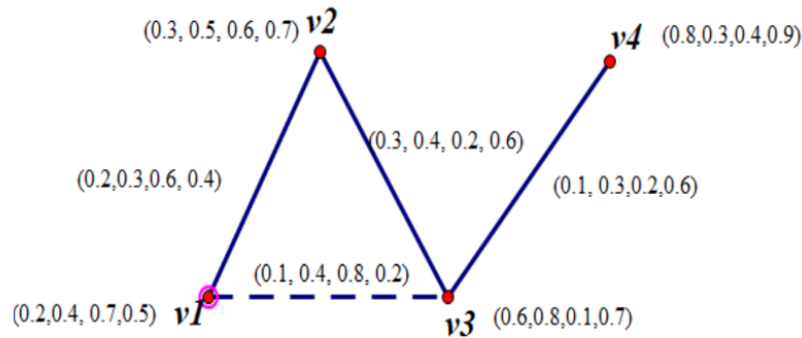


Figure 4.1: paired domination in Turiyam graph

In Figure 4.1, the red thick edges represent the minimal edge dominating set Y and Black thin edges are the remaining edges $E \setminus Y$, which form a complementary edge-dominating set. Every edge in Y is adjacent to at least one edge in $E \setminus Y$, illustrating Theorem 4.3.1.

Theorem 4.4.1. *Let $T_G = (V, E)$ be a complete Turiyam graph. Then the SPD number of T_G satisfies $\beta_1(T_G) = 2h_2(u, v)$, $\beta_2(T_G) = 2g_2(u, v)$, $\beta_3(T_G) = 2n_2(u, v)$, $\beta_4(T_G) = 2k_2(u, v)$. where $h_2(u, v)$, $g_2(u, v)$, $n_2(u, v)$, and $k_2(u, v)$ are the membership, abstinence, non-membership, and liberal values of the weakest edge in T_G , respectively.*

Proof. Let $T_G = (V, E)$ be a complete Turiyam graph with vertex set V and edge set E . In a complete graph, every vertex is adjacent to every other vertex, so any two vertices can form a pair in a strong paired dominating set (SPD). Let (u, v) be the edge with the *weakest values* in T_G , i.e., the edge for which the membership, abstinence, non-membership, and liberal values are minimal among all edges. Denote these values as $h_2(u, v)$, $g_2(u, v)$, $n_2(u, v)$, and $k_2(u, v)$. Since T_G is complete, the smallest SPD consists of two vertices connected by this weakest edge. Each vertex in this pair strongly dominates itself and the other vertex.

Therefore, the aggregate SPD values of this minimal SPD are simply twice the values of the weakest edge:

$$\beta_1(T_G) = 2h_2(u, v), \quad \beta_2(T_G) = 2g_2(u, v), \quad \beta_3(T_G) = 2n_2(u, v), \quad \beta_4(T_G) = 2k_2(u, v).$$

□

Theorem 4.4.2. *Let Q be a complete bipartite Turiyam graph. Suppose that (u, v) is the weakest edge of Q , having membership value $h_2(u, v)$, abstinence value $g_2(u, v)$, non-membership value $n_2(u, v)$, and liberal value $k_2(u, v)$. Then the strong paired domination (SPD) numbers of Q are given by*

$$\begin{aligned} \beta_1(Q) &= 2h_2(u, v), & \beta_2(Q) &= 2g_2(u, v), \\ \beta_3(Q) &= 2n_2(u, v), & \beta_4(Q) &= 2k_2(u, v). \end{aligned}$$

Proof. Let $Q = (X, Y)$ be a complete bipartite Turiyam graph, where every vertex of X is adjacent to every vertex of Y , and no two vertices within the same partite set are adjacent. Assume that (u, v) is the weakest edge in Q with corresponding values $h_2(u, v)$, $g_2(u, v)$, $n_2(u, v)$, and $k_2(u, v)$. Since Q is complete bipartite, any pair consisting of one vertex from X and one vertex from Y forms a connected pair and strongly dominates all vertices of the

graph. Consequently, a minimum strong paired dominating set contains exactly two vertices, one from each partite set. To minimize the aggregate SPD values, the selected pair must be joined by the weakest edge (u, v) . Therefore, the membership, abstinence, non-membership, and liberal SPD numbers are obtained by summing the corresponding values contributed by the two vertices of the pair. Hence,

$$\begin{aligned}\beta_1(Q) &= 2h_2(u, v), & \beta_2(Q) &= 2g_2(u, v), \\ \beta_3(Q) &= 2n_2(u, v), & \beta_4(Q) &= 2k_2(u, v).\end{aligned}$$

□

Conclusion

In this work, domination and paired domination in Turiyam graphs have been developed to model control and influence in complex networks with uncertainty and multi-valued relationships. The domination framework identifies key vertices that effectively govern the network under Turiyam membership conditions, extending classical domination theory to more realistic uncertain environments.

Paired domination further strengthens this structure by ensuring that dominating vertices occur in pairs via perfect matching, improving stability, redundancy, and coordinated control in the network. Overall, the results show that Turiyam domination and paired domination provide a flexible and effective framework for analyzing complex systems under uncertainty.

CHAPTER 5

Total Domination in Turiyam Graphs

5.1 Introduction

Total domination is an important concept in graph theory that extends the idea of vertex domination. In a standard graph, a vertex dominates itself and its neighbors. However, in *total domination*, every vertex must be dominated by a *neighboring vertex* only, meaning no vertex dominates itself. In the context of Turiyam graphs, which incorporate the membership, abstinence, non-membership, and liberal values of edges and vertices, total domination takes into account the *strength of connections* between vertices. A subset of vertices in a Turiyam graph forms a *total dominating set* if every vertex in the graph is strongly connected to at least one vertex in the set.

This section introduces the concept of total domination in Turiyam graphs, explores its fundamental properties, and defines the corresponding *lower and upper total domination numbers*, which quantify the minimum and maximum cardinalities of total dominating sets. The definitions and theorems presented here provide the groundwork for analyzing strong connectivity and control in networks modeled by Turiyam graphs.

5.2 Total dominance in Turiyam graphs

This section develops the principal results concerning *total domination* in Turiyam graphs. We begin by establishing the fundamental properties and bounds of total dominating sets in Turiyam graphs. Subsequently, total domination results are derived for several special classes of Turiyam graphs, including complete, split, connected, and restrained Turiyam graphs. To enhance clarity and practical relevance, numerical examples and tabulated computations are provided. In addition, applications and algorithmic aspects related to total domination in

Turiyam graphs are discussed to support and illustrate the theoretical findings.

Definition 5.2.1. Let $T_G = (\phi_T, \psi_T)$ be a Turiyam graph, where each vertex and each edge is characterized by four membership functions, namely truth, indeterminacy, falsehood, and liberal state. For every vertex $h_j \in V$, the vertex membership is defined as

$$\phi_T(h_j) = (T_\phi(h_j), I_\phi(h_j), F_\phi(h_j), L_\phi(h_j)),$$

and for every edge $h_j h_k \in E$, the edge membership is given by

$$\psi_T(h_j, h_k) = (T_\psi(h_j, h_k), I_\psi(h_j, h_k), F_\psi(h_j, h_k), L_\psi(h_j, h_k)).$$

The Turiyam graph T_G is said to be a complete Turiyam graph if, for every pair of distinct vertices $h_j, h_k \in V$, the edge membership functions satisfy

$$T_\psi(h_j, h_k) = \min\{T_\phi(h_j), T_\phi(h_k)\},$$

$$I_\psi(h_j, h_k) = \max\{I_\phi(h_j), I_\phi(h_k)\},$$

$$F_\psi(h_j, h_k) = \max\{F_\phi(h_j), F_\phi(h_k)\},$$

$$L_\psi(h_j, h_k) = \min\{L_\phi(h_j), L_\phi(h_k)\}.$$

Here, $T_\phi(h_j)$, $I_\phi(h_j)$, $F_\phi(h_j)$, and $L_\phi(h_j)$ denote the truth, indeterminacy, falsehood, and liberal-state strengths of the vertex h_j , respectively. Similarly, $T_\psi(h_j, h_k)$, $I_\psi(h_j, h_k)$, $F_\psi(h_j, h_k)$, and $L_\psi(h_j, h_k)$ represent the corresponding membership values of the edge connecting vertices h_j and h_k .

Remark 5.2.1. i If the liberal-state component L is omitted, the above definition reduces to the complete single-valued neutrosophic graph (S-VNG).

ii A complete Turiyam graph guarantees the maximum possible interaction strength between every pair of vertices under four-dimensional uncertainty modeling.

iii This structure is particularly suitable for total domination analysis, since every vertex is directly connected to all other vertices through Turiyam-strong edges.

Definition 5.2.2. Let $T_G = (\phi_T, \psi_T)$ be a Turiyam graph with vertex set

$$V = \{h_1, h_2, \dots, h_n\}.$$

For each vertex $h_i \in V$, the vertex membership is given by

$$\phi_T(h_i) = (T_\phi(h_i), I_\phi(h_i), F_\phi(h_i), L_\phi(h_i)),$$

where $T_\phi(h_i)$, $I_\phi(h_i)$, $F_\phi(h_i)$, and $L_\phi(h_i)$ denote the truth, indeterminacy, falsehood, and liberal-state membership values of the vertex h_i , respectively.

The Turiyam node cardinality of T_G is defined as

$$|V|_T = \sum_{i=1}^n \frac{1 + T_\phi(h_i) + I_\phi(h_i) - F_\phi(h_i) + L_\phi(h_i)}{2}.$$

Here,

- $T_\phi(h_i)$ represents the degree to which the vertex h_i truly exists or is reliable,
- $I_\phi(h_i)$ denotes the degree of indeterminacy or uncertainty associated with h_i ,
- $F_\phi(h_i)$ measures the degree of falsity or non-existence of h_i , and
- $L_\phi(h_i)$ is the liberal-state component, which captures amplified influence, enhanced reliability, or dominance potential of the vertex and is not necessarily restricted to the unit interval.

Definition 5.2.3. Let $T_G = (\phi_T, \psi_T)$ be a Turiyam graph with edge set E . The Turiyam edge cardinality of T_G is defined by

$$|E|_T = \sum_{h_i h_j \in E} \frac{1 + T_\psi(h_i, h_j) + I_\psi(h_i, h_j) - F_\psi(h_i, h_j) + L_\psi(h_i, h_j)}{2},$$

where $T_\psi(h_i, h_j)$, $I_\psi(h_i, h_j)$, $F_\psi(h_i, h_j)$, and $L_\psi(h_i, h_j)$ denote the truth, indeterminacy, falsehood, and liberal-state membership values of the edge $h_i h_j$, respectively.

Here,

- $T_\psi(h_i, h_j)$ represents the degree to which the interaction between vertices h_i and h_j is true or reliable,
- $I_\psi(h_i, h_j)$ measures the level of uncertainty associated with the interaction,
- $F_\psi(h_i, h_j)$ denotes the degree of falsity or unreliability of the edge, and
- $L_\psi(h_i, h_j)$ is the liberal-state component, which captures amplified interaction strength, extreme dominance, or over-influence.

Remark 5.2.2. *i* If $L_\psi(h_i, h_j) = 0$ for all $h_i h_j \in E$, then $|E|_T$ reduces to the edge cardinality of a single-valued neutrosophic graph (S-VNG).

- ii *Turiyam edge cardinality plays a crucial role in domination and total domination analysis, as domination strongly depends on the strength of edges.*
- iii *Larger values of L_ψ enhance domination capability, even in the presence of high uncertainty.*

Example 5.2.1. Consider a Turiyam graph $T_G = (V, E)$ with

$$V = \{h_1, h_2, h_3\}, \quad E = \{h_1h_2, h_2h_3\}.$$

The Turiyam edge membership values are given as follows:

$$\psi_T(h_1, h_2) = (0.6, 0.2, 0.1, 1.3), \quad \psi_T(h_2, h_3) = (0.7, 0.1, 0.1, 1.4).$$

Using definition 5.2.3, the Turiyam edge cardinality of T_G is computed as

$$\begin{aligned} |E|_T &= \frac{1 + 0.6 + 0.2 - 0.1 + 1.3}{2} + \frac{1 + 0.7 + 0.1 - 0.1 + 1.4}{2} \\ &= \frac{3.0}{2} + \frac{3.1}{2} \\ &= 3.05. \end{aligned}$$

Thus, the Turiyam edge cardinality of the given graph is $|E|_T = 3.05$.

Example 5.2.2. Let $TG = (V, E)$ be a complete Turiyam graph with vertex set

$$V = \{h_1, h_2, h_3\}.$$

The Turiyam membership values of the vertices are defined as

$$\phi_T(h_1) = (0.8, 0.1, 0.1, 0.3),$$

$$\phi_T(h_2) = (0.6, 0.2, 0.2, 0.2),$$

and

$$\phi_T(h_3) = (0.7, 0.1, 0.1, 0.4).$$

Since TG is complete, every pair of vertices is joined by an edge. The corresponding Turiyam edge membership values are chosen as

$$\psi_T(h_1, h_2) = (0.6, 0.1, 0.1, 0.2),$$

$$\psi_T(h_1, h_3) = (0.7, 0.1, 0.1, 0.3),$$

and

$$\psi_T(h_2, h_3) = (0.6, 0.1, 0.1, 0.2).$$

Using Definition 5.2.3, the Turiyam edge cardinality of T_G is

$$|E|_T = \frac{0.6 + 0.1 - 0.1 + 0.2}{2} + \frac{0.7 + 0.1 - 0.1 + 0.3}{2} + \frac{0.6 + 0.1 - 0.1 + 0.2}{2}.$$

Hence,

$$|E|_T = \frac{0.8}{2} + \frac{1.0}{2} + \frac{0.8}{2} = 0.4 + 0.5 + 0.4 = 1.3.$$

Definition 5.2.4. Let $T_G = (A, M)$ be a Turiyam graph with no isolated vertices. A set $C \subseteq A$ is called a total dominating set if, for every vertex $y \in A$, there exists a vertex $x \in C$, $x \neq y$, such that x dominates y in T_G .

Definition 5.2.5. The lower total domination number of a Turiyam graph T_G , denoted by $t_N(T_G)$, is the minimum cardinality among all total dominating sets of T_G . Similarly, the upper total domination number, denoted by $T_N(T_G)$, is the maximum cardinality among all total dominating sets of T_G .

Definition 5.2.6. Let $T_G = (\phi_T, \psi_T)$ be a Turiyam graph with no isolated vertices and vertex set V . A subset $S \subseteq V$ is called a effective Turiyam total dominating set (TTD-set) of G_T if every vertex $v \in V$ is strongly connected to at least one distinct vertex $u \in S$. Equivalently, for each vertex $v \in V$, there exists a vertex $u \in S$ with $u \neq v$ such that u belongs to the Turiyam-strong arc neighborhood of v . The Turiyam-strong arc neighborhood of a vertex v , denoted by $N_T(v)$, is defined as

$$N_T(v) = \{u \in V \mid vu \text{ is a Turiyam-strong arc in } G_T\}.$$

Remark 5.2.3. i The condition $u \neq v$ ensures that no vertex dominates itself, which distinguishes total domination from ordinary domination.

ii This definition is applicable only to Turiyam graphs without isolated vertices, since every vertex must be dominated by another vertex.

iii If the liberal-state component is omitted, the above definition reduces to the total dominating set of a single-valued neutrosophic graph (S-VNG).

Theorem 5.2.1. Let $T_G = (\phi_T, \psi_T)$ be a complete Turiyam graph without isolated vertices, where each vertex $h_i \in V$ is assigned the Turiyam membership

$$\phi_T(h_i) = (T_\phi(h_i), I_\phi(h_i), F_\phi(h_i), L_\phi(h_i)).$$

Then the lower and upper Turiyam total domination numbers of T_G are equal and satisfy

$$\gamma'_{td}(T_G) = \Gamma'_{TD}(T_G) = 2|V|, \quad h_i \in V.$$

Proof. Let $T_G = (\phi_T, \psi_T)$ be a complete Turiyam graph without isolated vertices and with vertex set

$$V = \{h_1, h_2, \dots, h_n\}.$$

Step 1: Structure of a complete Turiyam graph. Since T_G is complete, every pair of distinct vertices $h_i, h_j \in V$ is connected by a Turiyam-strong arc. Hence, for every vertex $h_i \in V$, its Turiyam-strong arc neighborhood satisfies

$$N_T(h_i) = V \setminus \{h_i\}.$$

Step 2: Minimum size of a Turiyam total dominating set.

By Definition 5.2.9, a Turiyam total dominating set $S \subseteq V$ must satisfy the condition that every vertex $v \in V$ is strongly dominated by a *distinct* vertex in S . Thus, no vertex can dominate itself, and at least two vertices are required to totally dominate a complete Turiyam graph. Let $S = \{h_i, h_j\}$, where $h_i \neq h_j$. Since G_T is complete, h_i and h_j are Turiyam-strong neighbors of each other, and every other vertex in V is also strongly connected to both h_i and h_j . Therefore, S is a Turiyam total dominating set. Hence, the minimum cardinality of a Turiyam total dominating set is 2.

Step 3: Equality of lower and upper total domination numbers. In a complete Turiyam graph, any Turiyam total dominating set must contain at least two vertices, and every pair of distinct vertices forms a Turiyam total dominating set. Consequently, both the lower and upper Turiyam total domination numbers coincide.

Step 4: Turiyam total domination values. Since each vertex $h_i \in V$ contributes equally under the Turiyam membership structure, the total domination value corresponding to a minimal Turiyam total dominating set consisting of two vertices is

$$\gamma'_{td}(T_G) = \Gamma'_{TD}(T_G) = 2|V|, \quad h_i \in V.$$

Therefore, the lower and upper Turiyam total domination numbers of a complete Turiyam graph are equal and equal to twice the Turiyam node cardinality of a vertex. $\square$

Example 5.2.3. Consider a complete Turiyam graph $T_G = (V, E)$ with

$$V = \{h_1, h_2, h_3, h_4\}.$$

The Turiyam vertex and edge membership values of T_G are given in table 5.1

Table 5.1: Vertex and edge membership functions of T_G

Vertex	T_ϕ	I_ϕ	F_ϕ	L_ϕ	Edge	T_ψ	I_ψ	F_ψ	L_ψ
h_1	0.7	0.2	0.1	1.4	h_1h_2	0.6	0.3	0.1	1.2
h_2	0.6	0.3	0.1	1.2	h_1h_3	0.7	0.2	0.1	1.4
h_3	0.8	0.1	0.1	1.6	h_2h_4	0.5	0.4	0.1	1.0
h_4	0.5	0.4	0.1	1.0	h_3h_4	0.5	0.4	0.1	1.0

From table 5.1, all edges satisfy the Turiyam-strong arc conditions. Hence, since T_G is complete, every pair of distinct vertices forms a Turiyam total dominating set. Possible Turiyam total dominating sets include

$$\{h_1, h_2\}, \quad \{h_1, h_3\}, \quad \{h_2, h_3\}, \quad \{h_3, h_4\}.$$

Each of these sets totally dominates all vertices of T_G , with every vertex being strongly dominated by a distinct vertex in the set. The Turiyam vertex cardinality of h_3 is computed as

$$|h_3| = \frac{1 + 0.8 + 0.1 - 0.1 + 1.6}{2} = 1.7.$$

Therefore, by Theorem 5.2.1, the Turiyam total domination number of T_G is

$$\gamma'_{td}(T_G) = 2|h_3| = 2 \times 1.7 = 3.4.$$

Thus, the Turiyam total domination number of the complete Turiyam graph T_G is

$$\gamma'_{td}(T_G) = 3.4.$$

Definition 5.2.7. Let $T_G = (\phi_T, \psi_T)$ be a Turiyam graph without isolated vertices. A Turiyam total dominating set $S \subseteq V$ is called a split Turiyam total dominating set (STTD-set) if the induced Turiyam subgraph $T_G[V \setminus S]$ is disconnected.

Example 5.2.4. Consider a Turiyam graph $T_G = (V, E)$ with vertex set

$$V = \{h_1, h_2, h_3, h_4, h_5\}.$$

Let

$$S = \{h_2, h_4\}.$$

Assume that every vertex in V is Turiyam dominated by at least one vertex in S . That is, each vertex in V has a Turiyam-strong arc connecting it to either h_2 or h_4 . Consider the induced subgraph on the remaining vertices:

$$T_G[V \setminus S] = T_G[\{h_1, h_3, h_5\}].$$

If there is no Turiyam-strong arc connecting h_1 and h_5 , then $T_G[V \setminus S]$ is disconnected. Hence, S forms a split Turiyam total dominating set (STTD-set) of T_G .

Theorem 5.2.2. Every Turiyam graph T_G that contains a cut-vertex admits a split Turiyam total dominating set (STTD-set).

Proof. Let $T_G = (V, E)$ be a Turiyam graph with a cut-vertex $v \in V$.

Step 1: Identify the cut-vertex. By definition, a cut-vertex is a vertex whose removal disconnects the graph. That is, the induced subgraph $T_G[V \setminus \{v\}]$ is disconnected.

Step 2: Construct the STTD-set. Consider $S = \{v\}$. Since v is a cut-vertex, it is adjacent (via Turiyam-strong arcs) to at least one vertex in each component of $T_G[V \setminus \{v\}]$.

Step 3: Verify total domination. Every vertex $u \in V$ is strongly dominated by v or by another vertex in S . Therefore, S is a Turiyam total dominating set.

Step 4: Check the split condition. The induced subgraph on $V \setminus S$ is

$$T_G[V \setminus S] = T_G[V \setminus \{v\}],$$

which is disconnected by the definition of a cut-vertex. Hence, S is a split Turiyam total dominating set of T_G . $\square$

Definition 5.2.8. Let $T_G = (\phi_T, \psi_T)$ be a Turiyam graph with vertex set V , and let $S \subseteq V$ be a Turiyam total dominating set (TTD-set). The set S is called a connected Turiyam total dominating set (CTTD-set) if the induced subgraph $T_G[S]$ is Turiyam-connected; that is, there exists a Turiyam-strong path between every pair of vertices in S .

Remark 5.2.4. *i . A CTTD-set ensures that the total dominating vertices themselves form a connected structure under Turiyam-strong arcs.*

ii . This is useful in network design where not only domination but also connectivity among dominant nodes is required.

iii . Every CTTD-set is necessarily a TTD-set, but not every TTD-set is connected.

Example 5.2.5. Consider a Turiyam graph $T_G = (V, E)$ with vertex set

$$V = \{h_1, h_2, h_3, h_4, h_5\}.$$

Let

$$S = \{h_1, h_2, h_3\}.$$

Assume that every vertex in V is Turiyam-dominated by at least one vertex in S . Further, suppose the induced subgraph $T_G[S]$ contains the Turiyam-strong arcs

$$h_1h_2 \quad \text{and} \quad h_2h_3.$$

Since there exists a Turiyam-strong path between every pair of vertices in S , the induced subgraph $T_G[S]$ is Turiyam-connected. Hence, S forms a connected Turiyam total dominating set (CTTD-set) of T_G .

Theorem 5.2.3. Let T_G be a Turiyam-connected graph. Then the connected Turiyam total domination number of T_G satisfies

$$\gamma'_{ctd}(T_G) \geq \gamma'_{td}(T_G),$$

where $\gamma'_{td}(T_G)$ is the Turiyam total domination number.

Proof. Let $S_{ctd} \subseteq V$ be a connected Turiyam total dominating set (CTTD-set) of T_G , and let $S_{td} \subseteq V$ be a Turiyam total dominating set (TTD-set) of T_G .

Step 1: Connection constraint. By definition, S_{ctd} is a TTD-set with the additional requirement that the induced subgraph $T_G[S_{ctd}]$ is Turiyam-connected. This connectivity requirement may force the inclusion of additional vertices that are not strictly necessary for domination alone.

Step 2: Comparison of cardinalities. Since S_{td} only needs to dominate all vertices in V (without the connectivity requirement), it is possible that

$$|S_{ctd}| \geq |S_{td}|.$$

Step 3: Minimal sets.

Taking the minimum cardinalities, we have

$$\gamma'_{ctd}(T_G) = \min\{|S_{ctd}|\} \quad \text{and} \quad \gamma'_{td}(T_G) = \min\{|S_{td}|\}.$$

Hence,

$$\gamma'_{ctd}(T_G) \geq \gamma'_{td}(T_G),$$

since every CTTD-set is also a TTD-set, but not every TTD-set is necessarily connected. Hence, The connected Turiyam total domination number is always greater than or equal to the Turiyam total domination number. $\square$

Definition 5.2.9. Let $T_G = (\phi_T, \psi_T)$ be a Turiyam graph with vertex set V , and let $S \subseteq V$ be a Turiyam total dominating set (TTD-set). The set S is called a restrained Turiyam total dominating set (RTTD-set) if the following two conditions are satisfied:

1. Every vertex in $V \setminus S$ is Turiyam-dominated by at least one vertex in S .
2. Every vertex in $V \setminus S$ has at least one Turiyam-strong neighbor within $V \setminus S$ itself.

Remark 5.2.5. i . An RTTD-set strengthens the total domination concept by requiring that the remaining vertices ($V \setminus S$) are not isolated; they must maintain Turiyam connectivity among themselves.

ii . This concept is useful in networks where nodes outside the dominating set still need to communicate or maintain redundancy.

iii . Every RTTD-set is also a TTD-set, but not every TTD-set is necessarily restrained.

Example 5.2.6. Consider a Turiyam graph $T_G = (V, E)$ with vertex set

$$V = \{h_1, h_2, h_3, h_4, h_5, h_6\}.$$

Let $S = \{h_2, h_5\}$. Assume the following conditions hold:

1. Every vertex in $V \setminus S = \{h_1, h_3, h_4, h_6\}$ is Turiyam-dominated by at least one vertex in S .
2. The subgraph induced by $V \setminus S$ contains internal Turiyam-strong arcs among its vertices.

Since both conditions of definition 5.2.9 are satisfied, S is a restrained Turiyam total dominating set (RTTD-set) of T_G .

Theorem 5.2.4. Let T_G be a Turiyam graph without isolated vertices. Then the restrained Turiyam total domination number satisfies

$$\gamma'_{rd}(T_G) \geq \gamma'_{td}(T_G),$$

where $\gamma'_{td}(T_G)$ is the Turiyam total domination number.

Proof. Let $S_{rd} \subseteq V$ be a restrained Turiyam total dominating set (RTTD-set) of T_G , and let $S_{td} \subseteq V$ be a Turiyam total dominating set (TTD-set) of T_G .

Step 1: RTTD-set conditions. By definition, S_{rd} satisfies all the conditions of a TTD-set (every vertex is dominated by a vertex in S_{rd}), and additionally, the vertices in $V \setminus S_{rd}$ must have at least one Turiyam neighbor within $V \setminus S_{rd}$.

Step 2: Implication on cardinality. The additional restraint may require including extra vertices in S_{rd} to ensure that all vertices in $V \setminus S_{rd}$ maintain Turiyam-strong connectivity. Thus,

$$|S_{rd}| \geq |S_{td}|.$$

Step 3: Minimal sets. Taking the minimal cardinalities of the sets, we get

$$\gamma'_{rd}(T_G) = \min\{|S_{rd}|\} \quad \text{and} \quad \gamma'_{td}(T_G) = \min\{|S_{td}|\}.$$

Since every RTTD-set is also a TTD-set, but not every TTD-set satisfies the restraint condition, it follows that

$$\gamma'_{rd}(T_G) \geq \gamma'_{td}(T_G).$$

□

5.3 Algorithmic Aspects of Total Domination in Turiyam graph

This section presents algorithms for computing *Turiyam total dominating sets* (TTD-sets) and their variants, including connected and restrained TTD-sets. The algorithms are based on the *Turiyam neutrosophic-strong arc (T-NSA) adjacency* framework, which captures the four-dimensional membership characteristics (truth, indeterminacy, falsity, and liberal state) of vertices and edges.

The procedures aim to efficiently identify dominating sets while respecting Turiyam-strong connectivity constraints, total domination requirements, and additional structural properties such as connectivity or restraint. Both step-by-step computation and illustrative examples are provided to clarify the application of these algorithms.

Algorithm 1: Construction of Turiyam Strong Neighborhoods

Purpose: Compute the Turiyam neighborhood $N_T(v)$ for each vertex $v \in V$.

Input:

- Turiyam graph $T_G = (\phi_T, \psi_T)$
- Thresholds: $T_\psi^\infty, I_\psi^\infty, F_\psi^\infty, L_\psi^\infty$

Output: Turiyam neighborhood sets $N_T(v)$ for all $v \in V$.

1. For each vertex $v \in V$, initialize $N_T(v) \leftarrow \emptyset$.
2. For each vertex $u \in V, u \neq v$:
 - (a) If the edge vu satisfies T-NSA conditions:

$$T_\psi(v, u) \geq T_\psi^\infty, \quad I_\psi(v, u) \geq I_\psi^\infty, \quad F_\psi(v, u) \leq F_\psi^\infty, \quad L_\psi(v, u) \geq L_\psi^\infty$$

then add u to $N_T(v)$: $N_T(v) \leftarrow N_T(v) \cup \{u\}$.

3. Return all sets $N_T(v)$.

Algorithm 2: Construction of Turiyam Total Dominating Set (TTD-Set)

Purpose: Compute a Turiyam total dominating set $S \subseteq V$.

Input:

- Turiyam graph $T_G = (\phi_T, \psi_T)$
- Turiyam neighborhoods $N_T(v)$ for all $v \in V$

Output: A Turiyam total dominating set S .

1. Initialize $S \leftarrow \emptyset$.

2. While there exists a vertex $v \in V$ not dominated by S :
 - (a) Select a vertex $u \in V$ that dominates the maximum number of undominated vertices.
 - (b) Add u to S .
3. Return S .

Algorithm 3: Construction of Connected Turiyam Total Dominating Set (CTTD-Set)

Purpose: Construct a connected Turiyam total dominating set $S \subseteq V$.

Input:

- Turiyam graph $T_G = (\phi_T, \psi_T)$
- Turiyam neighborhoods $N_T(v)$ for all $v \in V$

Output: Connected Turiyam total dominating set S .

1. Compute a TTD-set S using Algorithm 2.
2. While $T_G[S]$ is not connected:
 - (a) Add a vertex $u \in V \setminus S$ that connects two disconnected components of $T_G[S]$.
 - (b) Update $S \leftarrow S \cup \{u\}$.
3. Return S .

Algorithm 4: Construction of Restrained Turiyam Total Dominating Set (RTTD-Set)

Purpose: Compute a restrained Turiyam total dominating set $S \subseteq V$.

Input:

- Turiyam graph $T_G = (\phi_T, \psi_T)$
- Turiyam neighborhoods $N_T(v)$ for all $v \in V$

Output: Restrained Turiyam total dominating set S .

1. Compute a TTD-set S using Algorithm 2.
2. Ensure that each vertex in $V \setminus S$ has at least one neighbor in $V \setminus S$.
3. If any vertex $v \in V \setminus S$ has no neighbor in $V \setminus S$, add a suitable vertex $u \in V \setminus S$ to S to satisfy the restrained condition.
4. Return S .

Algorithm 5: Computation of Turiyam Total Domination Numbers

Purpose: Compute the lower and upper Turiyam total domination numbers of a Turiyam graph T_G .

Input:

- Turiyam graph $T_G = (\phi_T, \psi_T)$
- Turiyam vertex cardinality formula

Output: $(\gamma'_{td}(T_G), \gamma'_{TD}(T_G))$

1. Generate all minimal Turiyam total dominating sets $\{S_1, S_2, \dots, S_K\}$.
2. For each S_i :
 - (a) Compute its Turiyam vertex cardinality $|S_i|_T$ using

$$|S_i|_T = \sum_{v \in S_i} \frac{1 + T_\phi(v) + I_\phi(v) - F_\phi(v) + L_\phi(v)}{2}.$$

3. Compute the lower and upper total domination numbers:

$$\gamma'_{td}(T_G) \leftarrow \min_i |S_i|_T, \quad \gamma'_{TD}(T_G) \leftarrow \max_i |S_i|_T$$

4. Return $(\gamma'_{td}(T_G), \gamma'_{TD}(T_G))$.

Notation: Complexity Analysis

- Neighborhood construction: $O(|V|^2)$
- Greedy TTD-Set construction: $O(|V|^2)$

- Connectivity and restraint checks: $O(|V| + |E|)$

Thus, the overall algorithm runs in polynomial time, making it suitable for large-scale Turiyam networks.

Conclusion

The concept of total domination in Turiyam graphs extends the classical notion of total domination to environments characterized by uncertainty and incomplete information. By incorporating the four membership components of Turiyam sets truth, indeterminacy, falsity, and neutrality the model provides a more comprehensive framework for representing complex relationships among vertices.

In a Turiyam graph, every vertex, including those belonging to the dominating set, must be adjacent to at least one dominant vertex. This requirement ensures continuous interaction, mutual support, and complete coverage throughout the network. Consequently, total domination eliminates isolated dominance and strengthens the overall stability and reliability of the system. The study of total domination in Turiyam graphs establishes fundamental properties and provides a generalized framework that extends domination concepts from classical and uncertainty-based graph theories. The results demonstrate that Turiyam total domination is an effective tool for analyzing interconnected systems operating under ambiguous or uncertain conditions.

Furthermore, the proposed framework has potential applications in social networks, communication systems, decision-support models, and other complex systems where continuous influence and coordinated interactions are essential. Thus, total domination in Turiyam graphs contributes to the advancement of uncertainty graph theory and provides a foundation for future investigations on other domination parameters and their applications.

CHAPTER 6

Split Domination in Turiyam graph

6.1 Introduction

In this section, we introduce and study the concept of *split domination* in Turiyam graphs. Split domination extends the classical notion of domination by incorporating structural separation within the graph after the removal of dominating vertices. Specifically, a split dominating set not only ensures that every vertex is strongly dominated, but also produces a disconnected induced subgraph when the dominating vertices are removed. This additional structural constraint makes split domination particularly useful in analyzing fragmentation, control, and structural vulnerability in Turiyam networks under multi-valued uncertainty.

Turiyam graphs, characterized by truth, indeterminacy, falsehood, and liberal-state components, provide a flexible framework for modeling complex relational systems. Within this framework, split domination helps identify subsets of vertices whose influence both controls the network and simultaneously partitions the remaining structure. Such properties are important in network design, stability analysis, and hierarchical control modeling.

This section develops the theoretical foundations of split domination in Turiyam graphs. We begin with formal definitions and fundamental properties, followed by key theorems describing structural conditions and bounds for split dominating sets. To clarify the theoretical results, several illustrative examples are provided. These examples demonstrate how split domination operates in different Turiyam graph configurations and highlight its relationship with connectivity and domination strength.

The results presented here establish the groundwork for further analysis of structural domination variants in Turiyam graph theory.

Definition 6.1.1. Let $T_G = (V, E)$ be a Turiyam graph and let $p \in V$ be a vertex. The adjacency set (or neighborhood) of p in T , denoted by $\Gamma(p)$, is defined as the collection of all vertices in V that are strongly connected to p . That is,

$$\Gamma(p) = \{q \in V \mid \langle p, q \rangle \text{ represents a high-strength (Turiyam-strong) connection in } T\}.$$

Definition 6.1.2. Let $T_G = (V, E)$ be a Turiyam graph. A vertex $x \in V$ is called an isolated vertex if it has no connection of any strength with any other vertex in the graph. That is, for every $y \in V$, the edge membership values between x and y are all zero, namely

$$T_E(x, y) = 0, \quad I_E(x, y) = 0, \quad F_E(x, y) = 0, \quad L_E(x, y) = 0.$$

Definition 6.1.3. Let $T_G = (V, E)$ be a Turiyam graph associated with a crisp graph G . For vertices $x, y \in V$, the vertex x is said to dominate the vertex y in T if there exists a Turiyam-strong relationship (or bond) between x and y .

Notes.

1. **Symmetry of dominance.** Dominance in a Turiyam graph is symmetric. Thus, if x dominates y , then y also dominates x .
2. **Neighborhood of a vertex.** For any vertex $y \in A$, the neighborhood of y is defined as

$$N(y) = \{z \in A \setminus \{y\} \mid y \text{ dominates } z\}.$$

3. **Unique dominating set condition.** If, for every pair of vertices $(x, y) \in A$, the Turiyam edge membership values satisfy

$$T_E(x, y) \leq T_E^\infty(x, y), \quad I_E(x, y) \leq I_E^\infty(x, y), \quad F_E(x, y) \leq F_E^\infty(x, y), \quad L_E(x, y) \leq L_E^\infty(x, y),$$

then the entire vertex set A forms the unique dominating set of the Turiyam graph T_G .

Definition 6.1.4. Let $T_G = (V, E)$ be a Turiyam graph. A subset $C \subseteq V$ is called a Turiyam dominating set if for every vertex $v \in V \setminus C$, there exists a vertex $u \in C$ such that u dominates v in T .

Definition 6.1.5. Let $T_G = (A, E)$ be a Turiyam graph. A subset $C \subseteq A$ is said to be a dominating set of T if for every vertex $v \in A \setminus C$, there exists a vertex $u \in C$ such that u dominates v in T .

Definition 6.1.6. Let $T_G = (V, E)$ be a Turiyam graph. The vertices of T are said to be independent if no pair of vertices is connected by a strong edge. In other words, for any two distinct vertices $x, y \in V$, the edge between them does not satisfy the condition of strong connectivity. Hence, none of the vertices exerts strong influence or dominance over another, and they remain mutually independent within the structure of the graph.

Definition 6.1.7. Let $T_G = (V, E)$ be a Turiyam graph. A subset $C \subseteq V$ is called an independent set of T if no two vertices in C are connected by a strong Turiyam edge. Equivalently, for every pair of distinct vertices $x, y \in C$, the Turiyam edge membership values corresponding to truth, indeterminacy, falsity, and level satisfy the prescribed threshold conditions that prevent the formation of a strong Turiyam connection. Hence, the vertices in C remain mutually non-dominating and structurally independent.

Definition 6.1.8. Let $T_G = (V, E)$ be a Turiyam graph. An independent set $S \subseteq A$ is called a maximal independent set if it is not properly contained in any larger independent set. Equivalently, S is maximal if adding any vertex from outside S destroys its independence. That is, for every $y \in A \setminus S$, the set $S \cup \{y\}$ is no longer independent.

Definition 6.1.9. Let $T_G = (V, E)$ be a Turiyam graph. The lower independence number of T , denoted by $i_N(T)$, is the minimum cardinality among all maximal independent sets of T . Similarly, the upper independence number of T , denoted by $I_N(T)$, is the maximum cardinality among all maximal independent sets of T . These numbers measure the smallest and largest possible sizes of mutually non-dominating vertex sets in T , respectively.

Definition 6.1.10. Let $T_G = (V, E)$ be a Turiyam graph with no isolated nodes. A subset $C \subseteq V$ is called a total dominating set of T if every vertex in V is dominated by at least one vertex in C . In other words, for every $x \in C$, if x dominates all vertices in V , then the set C forms a total dominating set of the Turiyam graph T .

Definition 6.1.11. Let $T_G = (V, E)$ be a Turiyam graph with no isolated nodes.

- The lower total domination number of T , denoted by $t_N(T)$, is the minimum cardinality among all total dominating sets of T .
- The upper total domination number of T , denoted by $T_N(T)$, is the maximum cardinality among all total dominating sets of T .

These numbers measure the smallest and largest sizes of vertex sets capable of dominating every node in the Turiyam graph.

Definition 6.1.12. Let $T_G = (V, E)$ be a Turiyam graph, and let $D^T \subseteq V$ be a total dominating set of T . If the induced subgraph

$$H = \langle V \setminus D^T, V', E' \rangle$$

is disconnected, then D^T is called a split total dominating set (or split dominant set) of the Turiyam graph T . In other words, removing the vertices of D^T splits the remaining graph into two or more disconnected components.

Example 6.1.1. Consider the Turiyam graph illustrated in Figure 6.1. The minimum split total dominating set of this graph is

$$D^T = \{u_2, u_3, u_5\},$$

with the remaining vertices

$$V \setminus D^T = \{u_1, u_4\}.$$

Removing the vertices in D^T disconnects the subgraph induced by $V \setminus D^T$, demonstrating that D^T is indeed a split dominant set.

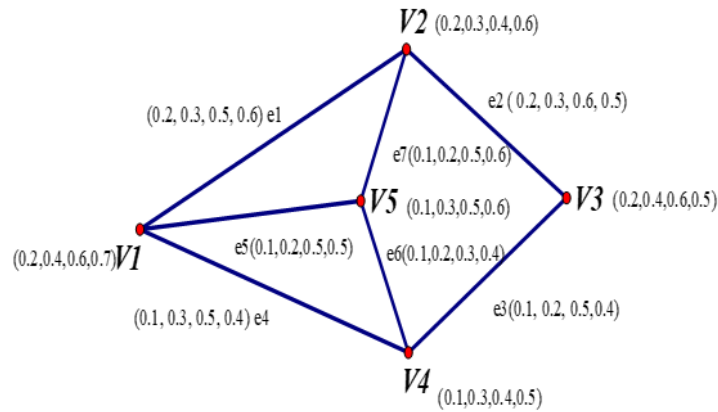


Figure 6.1: Turiyam graph

Remark 6.1.1. Figure 6.1 depicts a Turiyam graph constructed from a dataset with the vertex set

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

and the edge set

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}.$$

For the split total dominating set D^T , the induced subgraph

$$V \setminus D^T$$

is disconnected. This disconnection demonstrates the presence of two isolated vertices, specifically u_1 and u_4 , which are not adjacent to any other vertex in $V \setminus D^T$.

Definition 6.1.13. Let $T_G = (V, E)$ be a Turiyam graph, and let $D^T \subseteq V$ be a split dominant set. The set D^T is called a minimal split dominant set if the removal of any single vertex from D^T causes it to lose its split domination property; that is, the induced subgraph $V \setminus (D^T \setminus \{v\})$ becomes connected for some $v \in D^T$.

Definition 6.1.14. Let $T_G = (V, E)$ be a Turiyam graph. The lower split dominance number of T_G , denoted by $d_s^T(T_G)$, is defined as the minimum cardinality among all minimal split dominant sets of T_G . That is,

$$d_s^T(T_G) = \min\{|D^T| \mid D^T \text{ is a minimal split dominant set of } T_G\}.$$

Definition 6.1.15. Let $T_G = (V, E)$ be a Turiyam graph. A subset $D^T \subseteq A$ is called a split dominant set if it satisfies the following two conditions:

1. **Dominance condition:** Every vertex $y \in A \setminus D^T$ is dominated by at least one vertex $x \in D^T$.
2. **Splitting condition:** The induced subgraph on the remaining vertices, $T[A \setminus D^T]$, is either disconnected or reduced to a single isolated vertex.

Definition 6.1.16. Let $T_G = (V, E)$ be a Turiyam graph. The upper split dominance number of G_T , denoted by $D_s^T(T_G)$, is defined as the maximum cardinality among all minimal split dominant sets of G_T . Formally,

$$D_s^T(T_G) = \max\{|D^T| \mid D^T \text{ is a minimal split dominant set of } T_G\}.$$

Example 6.1.2. Consider the Turiyam graph illustrated in Figure 6.2, where the edges identified as strong are e_2, e_6, e_8, e_9 , and e_{10} , representing the most influential connections in the network. Based on these strong connections, two minimal split dominant sets can be determined:

- $D_1^T = \{u_1, u_2, u_4, u_5\}$, which collectively dominates all nodes outside the set.
- $D_2^T = \{u_1, u_3, u_4, u_6\}$, which also satisfies the dominance condition, ensuring every node not in the set is adjacent to at least one node within the set.

These examples illustrate different configurations of vertices that effectively influence the entire graph according to the Turiyam domination criteria.

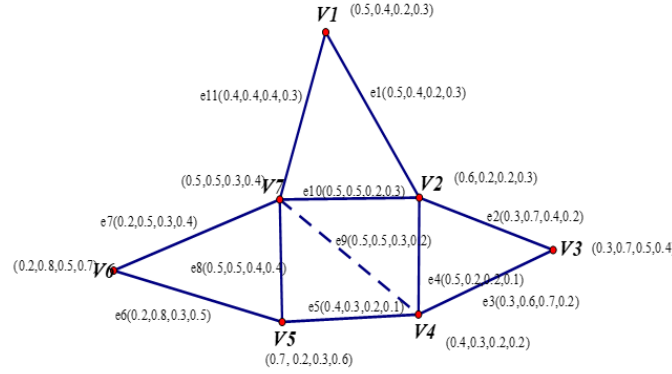


Figure 6.2: Graph of the Turiyam minimal split dominance set

Remark 6.1.2. Figure 6.2 illustrates the Turiyam graph corresponding to a dataset with vertices

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

and edges

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\},$$

with the specified membership values. Among the various Turiyam dominant sets, the minimal split dominance sets are identified as follows:

- $D_1^T = \{v_1, v_2, v_4, v_5\}$, $V \setminus D_1^T = \{v_3, v_6, v_7\}$
- $D_3^T = \{v_1, v_2, v_4, v_6\}$
- $D_4^T = \{v_1, v_2, v_5, v_7\}$
- ... (other minimal split dominant sets follow similarly)

These sets demonstrate different configurations of vertices that satisfy both the dominance and the splitting conditions in the Turiyam graph, highlighting the variety of ways to achieve minimal split domination.

Theorem 6.1.1. Let $T_G = (V, E)$ be a Turiyam graph, and let $D^T \subseteq V$ be a split dominant set of T . Then the weight of D^T satisfies

$$\delta(G) \leq W(D^T) \leq \Delta(G),$$

where $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of the underlying crisp graph G , respectively.

Proof. By definition of vertex degrees, for every $v \in V$ we have

$$\delta(G) \leq \deg_G(v) \leq \Delta(G),$$

where $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degrees of the underlying graph G , respectively. Since this holds for all vertices $v \in D^T$, summing over all vertices in the split dominant set gives

$$\sum_{v \in D^T} \delta(G) \leq \sum_{v \in D^T} \deg_G(v) \leq \sum_{v \in D^T} \Delta(G).$$

Because $\delta(G)$ and $\Delta(G)$ are constants independent of v , the sums simplify to

$$|D^T| \delta(G) \leq \sum_{v \in D^T} \deg_G(v) \leq |D^T| \Delta(G).$$

Dividing through by $|D^T|$ (which is positive, since D^T is nonempty) yields

$$\delta(G) \leq \frac{1}{|D^T|} \sum_{v \in D^T} \deg_G(v) = W(D^T) \leq \Delta(G),$$

which proves the claimed inequality. $\square$

Theorem 6.1.2. Let $T_G = (V, E)$ be a Turiyam graph and $D^T \subseteq V$ be a dominance set. Then, D^T is a split dominance set if and only if the induced subgraph on the remaining vertices, $T[V \setminus D^T]$, is disconnected.

Proof. ($\Rightarrow$) Suppose D^T is a split dominance set. By definition, the removal of all vertices in D^T from T results in an induced subgraph $T[V \setminus D^T]$ that is either disconnected or consists of isolated vertices. Therefore, $T[V \setminus D^T]$ is disconnected.

($\Leftarrow$) Conversely, assume that the removal of D^T disconnects the induced subgraph $T[V \setminus D^T]$. Since D^T is a dominance set, every vertex in $V \setminus D^T$ is dominated by at least one vertex in D^T . Together with the fact that $T[V \setminus D^T]$ is disconnected, this satisfies both the *dominance* and *splitting* conditions of a split dominance set. Hence, D^T is a split dominance set. $\square$

Theorem 6.1.3. Let $T_G = (V, E)$ be a Turiyam graph. Then the split dominance number $\gamma_s^T(G)$ of T satisfies

$$\gamma_s^T(G) \geq \beta(G),$$

where $\beta(G)$ denotes the fuzzy node covering number of G .

Proof. By definition, a split dominant set $D^T \subseteq V$ must satisfy two conditions:

1. **Dominance:** Every vertex in $V \setminus D^T$ is dominated by at least one vertex in D^T .
2. **Splitting:** The induced subgraph $T[V \setminus D^T]$ is disconnected. The fuzzy node covering number $\beta(G)$ represents the minimum number of vertices required to cover all edges of the graph under the fuzzy or Turiyam membership structure. Since any split dominant set D^T must dominate all vertices (hence indirectly cover all edges incident to vertices in $V \setminus D^T$), it follows that $|D^T|$ cannot be smaller than $\beta(G)$, which is the minimal number of vertices needed to cover all edges.

Therefore, we obtain the inequality $\gamma_s^T(G) = \min\{|D^T| : D^T \text{ is a split dominant set}\} \geq \beta(G)$, as required. $\square$

Theorem 6.1.4. Let D^T be a split-dominance set of a Turiyam graph $T = (V, E_\tau)$. Then D^T is minimal if and only if, for each vertex $v \in D^T$, at least one of the following conditions holds:

1. Removing v would violate the split-dominance property of the set.
2. v is an isolated vertex in the induced subgraph $\langle D^T \rangle$.
3. The induced subgraph $\langle V \setminus (D^T \setminus \{v\}) \rangle$ becomes connected.

Proof. ($\Rightarrow$) **Necessity:** Assume D^T is a minimal split-dominance set. By definition, removing any vertex v from D^T either:

- destroys the dominance of some vertex in $V \setminus D^T$, or
- connects the subgraph induced by $V \setminus (D^T \setminus \{v\})$, violating the splitting condition.

If neither of these occurs, then v must be isolated in $\langle D^T \rangle$ to maintain minimality. ($\Leftarrow$)

Sufficiency: Conversely, suppose that for each $v \in D^T$, at least one of the stated conditions holds. Then removing any v either:

- breaks the split-dominance property, or

- causes the remaining induced subgraph to become connected,
- preserves v as an isolated node, i.e it cannot be removed without affecting minimality.

Hence, no vertex can be removed without violating the split-dominance property, proving that D^T is minimal. $\square$

Theorem 6.1.5. *Let $T_G = (V, E)$ be a Turiyam graph with end vertices (vertices of degree one) such that*

$$\partial_s^T(G) > \partial^T(G),$$

where $\partial_s^T(G)$ is the split dominance number and $\partial^T(G)$ is the standard dominance number. Then G admits a split-dominant set that contains at least some vertices adjacent to its Turiyam end vertices.

Proof. Consider the set of Turiyam end vertices in T . By definition, these vertices have degree one, meaning they are connected to exactly one other vertex. Since $\partial_s^T(G) > \partial^T(G)$, the minimal split-dominance set must be larger than a standard dominance set. This implies that in order to maintain the *splitting condition* (i.e., the induced subgraph on the remaining vertices is disconnected), some vertices adjacent to the end vertices must be included in the split-dominant set. Including these adjacent vertices ensures that:

1. Every end vertex is dominated by a vertex in the split-dominant set.
2. The removal of the split-dominant set preserves the disconnected structure of the remaining graph, satisfying the split property.

Hence, any minimal split-dominant set necessarily contains at least some vertices adjacent to the Turiyam end nodes. $\square$

Theorem 6.1.6. *Let $G^* = (A, M)$ be a Turiyam graph with $p = |A|$ vertices. Then the split dominance number $\partial(G^*)$ satisfies the following bounds:*

$$\partial(TG^*) \geq \frac{p}{\Delta_T(TG^*) + 1}, \quad \partial(TG^*) \geq \frac{p}{\Delta_I(TG^*) + 1}, \quad \partial(TG^*) \geq \frac{p}{\Delta_F(TG^*) + 1}, \quad \partial(TG^*) \geq \frac{p}{\Delta_L(TG^*) + 1}$$

where $\Delta_T(TG^)$, $\Delta_I(TG^*)$, $\Delta_F(TG^*)$, and $\Delta_L(TG^*)$ denote the maximum degrees of G^* with respect to the truth, indeterminacy, falsity, and liberal-state components, respectively.*

Proof. Let D^* be a minimal split-dominant set of G^* . By the definition of split dominance:

1. Each vertex outside D^* is dominated by at least one vertex in D^* under the Turiyam edge membership criteria.
2. The induced subgraph on $A \setminus D^*$ is disconnected.

For the truth-degree component T , each vertex in D^* can dominate at most $\Delta_T(G^*)$ vertices outside D^* . Including the vertex itself, the total coverage per vertex in D^* is at most $\Delta_T(G^*) + 1$. Since all p vertices must be dominated, the minimal number of vertices in D^* satisfies:

$$\partial(G^*) \geq \frac{p}{\Delta_T(G^*) + 1}.$$

Similarly, considering the indeterminacy, falsity, and liberal-state components separately, we obtain the corresponding inequalities:

$$\partial(G^*) \geq \frac{p}{\Delta_I(G^*) + 1}, \quad \partial(G^*) \geq \frac{p}{\Delta_F(G^*) + 1}, \quad \partial(G^*) \geq \frac{p}{\Delta_L(G^*) + 1}.$$

Hence, the split dominance number of G^* is bounded below by the ratios above, completing the proof. $\square$

Theorem 6.1.7. *Let $G^* = (A, M)$ be a Turiyam graph with $p = |A|$ vertices. Then the split dominance number $\gamma_s(G^*)$ satisfies the following bounds:*

$$\gamma_s(G^*) \geq p \cdot \frac{\Delta_T(G^*)}{\Delta_T(G^*) + 1}, \quad \gamma_s(G^*) \geq p \cdot \frac{\Delta_I(G^*)}{\Delta_I(G^*) + 1}, \quad \gamma_s(G^*) \geq p \cdot \frac{\Delta_F(G^*)}{\Delta_F(G^*) + 1}, \quad \gamma_s(G^*) \geq p \cdot \frac{\Delta_L(G^*)}{\Delta_L(G^*) + 1}$$

where $\Delta_T(G^*)$, $\Delta_I(G^*)$, $\Delta_F(G^*)$, and $\Delta_L(G^*)$ denote the maximum degrees of G^* with respect to the truth, indeterminacy, falsity, and liberal-state components, respectively.

Proof. Consider a minimal split-dominant set $D^* \subseteq A$ of G^* . Each vertex in D^* can dominate at most $\Delta_T(G^*) + 1$ vertices with respect to the truth component (including itself). Hence, the minimal number of vertices needed to dominate all p vertices satisfies

$$\gamma_s(G^*) \geq p \cdot \frac{\Delta_T(G^*)}{\Delta_T(G^*) + 1}.$$

Similarly, analyzing domination with respect to the indeterminacy, falsity, and liberal-state components, we obtain

$$\gamma_s(G^*) \geq p \cdot \frac{\Delta_I(G^*)}{\Delta_I(G^*) + 1}, \quad \gamma_s(G^*) \geq p \cdot \frac{\Delta_F(G^*)}{\Delta_F(G^*) + 1}, \quad \gamma_s(G^*) \geq p \cdot \frac{\Delta_L(G^*)}{\Delta_L(G^*) + 1}.$$

These inequalities establish lower bounds on the split dominance number of G^* , considering all four Turiyam graph components. $\square$

Theorem 6.1.8. Let $T_G = (V, E)$ be a Turiyam graph and let $v \in V$ be a cut-vertex of G . Suppose v is adjacent to all vertices in at least two Turiyam blocks H_1 and H_2 . Then v is contained in every split dominating set of G .

Proof. By definition, a split dominating set $D^T \subseteq V$ is such that the removal of D^T disconnects the remaining graph, i.e., $G[V \setminus D^T]$ is disconnected. Consider the cut-vertex v adjacent to all vertices of two distinct blocks H_1 and H_2 . If $v \notin D^T$, then after removing D^T , both H_1 and H_2 remain connected through v , preventing the induced subgraph $G[V \setminus D^T]$ from being disconnected. This violates the definition of a split dominating set. Therefore, v must be included in every split dominating set of G to ensure the split condition is satisfied. $\square$

Example 6.1.3. Consider the Turiyam graph $T_G = (V, E)$ illustrated in Figure 6.3. The edges identified as Turiyam-strong arcs are e_1, e_3, e_5, e_6 , and e_9 . Let $H_1 = \{v_1, v_2, v_3, v_4\}$ and $H_2 = \{u_5, u_6\}$ be two Turiyam blocks sharing a common cut-vertex v_7 . Observing the graph, it is clear that v_7 is adjacent to all vertices in both H_1 and H_2 . According to Theorem 3.8, the cut-vertex v_7 must be included in every split dominating set of G , ensuring the split domination condition is satisfied.

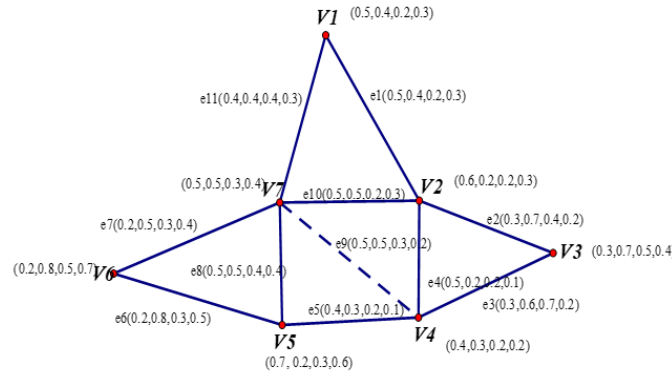


Figure 6.3: Turiyam graph illustrating with the split dominant set structure

Remark 6.1.3. Figure 6.3 illustrates the split dominance structure of a Turiyam graph $T_G = (V, E)$ with the vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$. The figure highlights the Turiyam-strong arcs among the vertices, along with their associated Turiyam membership values, including truth, indeterminacy, falsity, and liberal-state components. Based on these strong connections, minimal split dominating sets can be identified. For instance, one such set is $D_1^T = \{v_2, v_3, v_5\}$, which dominates all vertices outside

the set, namely $V \setminus D_1^T = \{v_1, v_4, v_6\}$. Observing the induced subgraph on $V \setminus D_1^T$, we see that it is disconnected, confirming that D_1^T satisfies the split dominance condition. Similarly, alternative minimal split dominating sets, such as $D_2^T = \{v_1, v_3, v_6\}$, also exist, demonstrating different configurations of nodes capable of controlling or influencing the remaining graph while maintaining the split domination property. This example emphasizes how the structure of Turiyam-strong arcs directly affects the formation of minimal split dominating sets and their strategic importance in network control.

Theorem 6.1.9. *Let $T_G = (V, E)$ be a Turiyam graph, and let $H \subseteq G$ be a block with a unique cut vertex v . If v dominates all vertices of H , then there exists a split dominating set of G that contains v .*

Proof. Let v be the unique cut vertex of the block H . By assumption, v Turiyam-dominates all other vertices in H , i.e., for each $u \in H \setminus \{v\}$, there exists a strong Turiyam arc $\langle v, u \rangle$. Consider a candidate split dominating set D^T defined as:

$$D^T = \{v\} \cup D',$$

where D' contains nodes from $G \setminus H$ chosen to satisfy the split domination of the remaining vertices outside H .

Step 1: Dominance condition. Since v dominates all nodes in H , every vertex in $H \setminus \{v\}$ is adjacent to a vertex in D^T . For vertices outside H , the set D' is constructed so that each vertex in $G \setminus H$ is dominated by at least one node in D^T . Therefore, the dominance condition of a split dominating set is satisfied.

Step 2: Splitting condition. Removing D^T from G disconnects the remaining graph at the block H . Since v is the only cut vertex of H , the vertices $H \setminus \{v\}$ become disconnected from the rest of the graph once v is included in D^T . This ensures that the induced subgraph on $V \setminus D^T$ is disconnected, satisfying the split domination property. Thus, D^T is a valid split dominating set, and it necessarily includes the cut vertex v . $\square$

Remark 6.1.4. *This theorem highlights the strategic role of cut vertices in Turiyam graphs for controlling network structure. In blocks where a single vertex dominates all others, including it in the split dominating set guarantees that the block can be “controlled” while simultaneously disconnecting the remainder of the graph. Practically, this can be interpreted as identifying a key influencer or gatekeeper within a network whose presence is critical for domination and structural separation.*

Theorem 6.1.10. Let $T_G = (V, E)$ be a Turiyam graph, and let $D^T \subseteq V$ be a split dominant set of G . If

$$\gamma_s^T(G) \leq \gamma_D^T(G),$$

then the complement set $V \setminus D^T$ is also a split dominant set of G .

Proof. Let D^T be a split dominant set of G . By definition, D^T satisfies two conditions:

1. **Dominance condition:** Each vertex in $V \setminus D^T$ is dominated by at least one vertex in D^T .
2. **Splitting condition:** The induced subgraph $G[V \setminus D^T]$ is disconnected or reduces to isolated nodes.

Assume that $\gamma_s^T(G) \leq \gamma_D^T(G)$, where $\gamma_s^T(G)$ denotes the minimal split dominance number and $\gamma_D^T(G)$ is the cardinality of D^T . This implies that D^T is at least as small as any minimal split dominant set, so the remaining set $V \setminus D^T$ is non-empty and contains vertices that collectively dominate the vertices of D^T .

Step 1: Dominance of the complement. Each vertex $v \in D^T$ is adjacent to some vertex $u \in V \setminus D^T$ through a Turiyam strong arc, because otherwise D^T would not be a split dominant set (it would fail to dominate the remaining vertices). Hence, every vertex in D^T is dominated by at least one vertex in $V \setminus D^T$.

Step 2: Splitting condition of the complement. The original D^T ensured that $G[V \setminus D^T]$ is disconnected. Since $V \setminus D^T$ is precisely the original complement, it inherits a disconnected structure when considered as a candidate split dominant set. In other words, $G[V \setminus D^T]$ is disconnected or forms isolated nodes, satisfying the splitting property for $V \setminus D^T$. Therefore, $V \setminus D^T$ also satisfies both the dominance and splitting conditions of a split dominant set. Hence, it is a valid split dominant set of G . $\square$

Remark 6.1.5. This theorem emphasizes a symmetry in Turiyam split dominance: under the stated condition, not only can a minimal split dominant set control the graph, but its complement also forms a split dominant set. This duality is particularly useful in network analysis, as it allows identifying alternative controlling subsets of nodes that maintain structural separations within the graph.

Theorem 6.1.11. Every connected Turiyam graph $T_G = (V, E)$ with at least two vertices admits a split dominant set.

Proof. Let G be a connected Turiyam graph with $|V| \geq 2$.

Step 1: Select a high-degree vertex.

Choose a vertex $v \in V$ with the highest Turiyam degree (considering the strong Turiyam arcs). By definition, v dominates all its neighbors in V , forming a singleton dominance set $\{v\}$.

Step 2: Check the complement.

Consider the induced subgraph $G[V \setminus \{v\}]$.

- If $G[V \setminus \{v\}]$ is already disconnected, then $\{v\}$ itself is a split dominant set.
- If $G[V \setminus \{v\}]$ remains connected, then we iteratively add vertices from a minimal Turiyam dominance set to the candidate split dominant set. Each added vertex dominates additional nodes and gradually partitions the remaining subgraph.

Step 3: Termination.

Continue this process until the removal of the selected dominating vertices disconnects the remaining subgraph or reduces it to isolated vertices. By construction, the resulting set of selected vertices forms a split dominant set D^T . Hence, any connected Turiyam graph with at least two vertices admits a split dominant set. $\square$

Remark 6.1.6. *This theorem provides a constructive method to obtain a split dominant set in a connected Turiyam graph: start with the highest-degree vertex, iteratively select vertices to dominate the graph, and ensure that the remaining subgraph becomes disconnected. This guarantees the existence of split dominant sets in all nontrivial connected Turiyam graphs.*

6.2 Application of Turiyam Split Domination in Healthcare Systems

6.2.1 Case Study: Hospitals and Disease Surveillance Network

Healthcare Network Scenario. Consider a regional healthcare system consisting of a Central Hospital, a District Hospital, urban and rural clinics, a Diagnostic Laboratory, and a Disease Surveillance Unit. These institutions operate collaboratively to provide medical services, exchange health information, coordinate patient referrals, conduct laboratory testing, and monitor disease outbreaks. The resulting healthcare infrastructure can be naturally modeled as a Turiyam graph, where each vertex represents a healthcare facility and each edge

represents a functional interaction between two facilities.

In this network, the Central Hospital serves as the primary referral center for specialized treatments and emergency cases. The District Hospital provides secondary healthcare services, while the Urban and Rural Clinics act as the first points of contact for patients. The Diagnostic Laboratory supports disease diagnosis and medical testing, whereas the Surveillance Unit collects epidemiological data and monitors the spread of infectious diseases. Together, these facilities form a connected healthcare ecosystem responsible for maintaining regional public health.

Healthcare interactions are inherently uncertain. Patient referrals may be delayed, communication channels may experience disruptions, diagnostic reports may be incomplete, and outbreak information may evolve rapidly. To capture such uncertainties, each interaction is characterized by a Turiyam membership quadruple (T, I, F, L) , where T denotes the degree of reliability of the interaction, I represents uncertainty or indeterminacy, F measures ineffective or failed communication, and L captures neutral, unknown, or context-dependent relationships. This four-dimensional representation provides a more realistic description of healthcare operations than traditional graph models.

Vertex Classification The healthcare facilities considered in the network are summarized in Table 6.1.

Table 6.1: Description of vertices in the regional healthcare network

Vertex	Description
H_1	Central Hospital
H_2	District Hospital
C_1	Urban Clinic
C_2	Rural Clinic
L_1	Diagnostic Laboratory
S_1	Surveillance Unit

Healthcare Interactions The edges of the Turiyam graph represent essential healthcare collaborations. The connection between H_1 and C_1 corresponds to specialist referrals and treatment coordination, while the edge $H_1 - C_2$ represents emergency support and telemedicine

services provided to remote communities. The edge $C_1 - H_2$ captures patient transfers requiring secondary medical care. The interaction $C_1 - L_1$ reflects the exchange of diagnostic samples and laboratory reports, whereas the edge $C_2 - S_1$ represents disease reporting, vaccination monitoring, and public health surveillance activities.

These interactions create a network through which patients, medical information, and healthcare resources continuously flow. The effectiveness of the healthcare system therefore depends not only on the strength of individual facilities but also on the connectivity among them.

Role of Turiyam Split Domination In the context of healthcare management, a Turiyam split dominating set represents a collection of strategically important facilities that can supervise or influence all other facilities in the network while simultaneously partitioning the remaining network into disconnected components when removed. This property is particularly valuable during epidemic outbreaks, disaster response operations, or healthcare system restructuring.

For example, facilities such as the Central Hospital and the Urban Clinic may form a Turiyam split dominating set because they coordinate most referral pathways and information exchanges. If these facilities are isolated or temporarily unavailable, the remaining healthcare network becomes fragmented into smaller components, revealing their critical importance to network connectivity and service delivery.

Consequently, Turiyam split domination helps healthcare administrators identify vulnerable points in the system, allocate resources efficiently, strengthen communication channels, and design contingency plans for emergency situations. The model therefore provides a powerful decision-support framework for maintaining resilient healthcare services under uncertain and dynamic conditions.

6.2.2 Turiyam Membership Values for Facility Interactions

The Turiyam membership values associated with healthcare interactions are presented in Table 6.2. Each edge is represented by a quadruple (T, I, F, L) corresponding to truth, indeterminacy, falsity, and liberal-state memberships.

Table 6.2: Turiyam membership values of healthcare interactions

Edge	Membership Values (T, I, F, L)
$H_1 - C_1$	$(0.9, 0.05, 0.03, 0.02)$
$H_1 - C_2$	$(0.8, 0.10, 0.05, 0.05)$
$C_1 - H_2$	$(0.6, 0.20, 0.10, 0.10)$
$C_1 - L_1$	$(0.85, 0.05, 0.05, 0.05)$
$C_2 - S_1$	$(0.75, 0.10, 0.10, 0.05)$

Interpretation The membership values indicate varying levels of reliability and uncertainty in healthcare interactions. High truth-membership values for the edges $H_1 - C_1$ and $C_1 - L_1$ signify strong and dependable collaboration, while higher indeterminacy values on the edge $C_1 - H_2$ reflect occasional uncertainty in patient transfer processes. By incorporating these dimensions, the Turiyam graph provides a realistic framework for evaluating healthcare connectivity and identifying critical facilities through split domination analysis.

6.2.3 Greedy Turiyam Split Domination Algorithm (GT-SDA)

Input: A Turiyam graph $T_G = (V, E)$, and threshold values α, β .

Output: A split dominating set $D^T \subseteq V$.

Algorithm 1 Greedy Turiyam Split Domination Algorithm (GT-SDA)

- 1: Initialize $D^T \leftarrow \emptyset, U \leftarrow V$
- 2: each vertex $v \in U$
- 3: Compute the dominance strength

$$\delta(v) = \sum_{u \in N(v)} (T_{vu} - I_{vu} - F_{vu})$$

- 4: EndFor
- 5: the induced subgraph $T[U]$ is connected
- 6: Select $v^* = \arg \max_{v \in U} \delta(v)$
- 7: Update $D^T \leftarrow D^T \cup \{v^*\}$
- 8: Update $U \leftarrow U \setminus N[v^*]$ ▷ Remove v^* and its neighbors from U
- 9: Recompute $\delta(v)$ for remaining $v \in U$
- 10: EndWhile
- 11: **return** D^T

Complexity: $O(|V|^3)$

Numerical Example: Healthcare Network

Consider the regional health system with thresholds $\alpha = 0.7$ and $\beta = 0.15$ and the dominance strengths for the vertices were :

$$\delta(H_1) = 1.47, \quad \delta(C_1) = 1.87, \quad \delta(C_2) = 1.20, \quad \delta(S_1) = 0.55$$

Applying the GT-SDA:

1. Select C_1 (largest $\delta(v) = 1.87$), add to D^T .
2. Remove C_1 and its neighbors from U ; recompute U .
3. Select C_2 (next largest remaining $\delta(v) = 1.20$), add to D^T .
4. Remaining vertices H_2, L_1, S_1 form a disconnected induced subgraph.

Hence, the split dominating set is

$$D^T = \{C_1, C_2\}.$$

Figure 6.4 illustrates the structural representation of this split dominating set within the Turiyam graph.

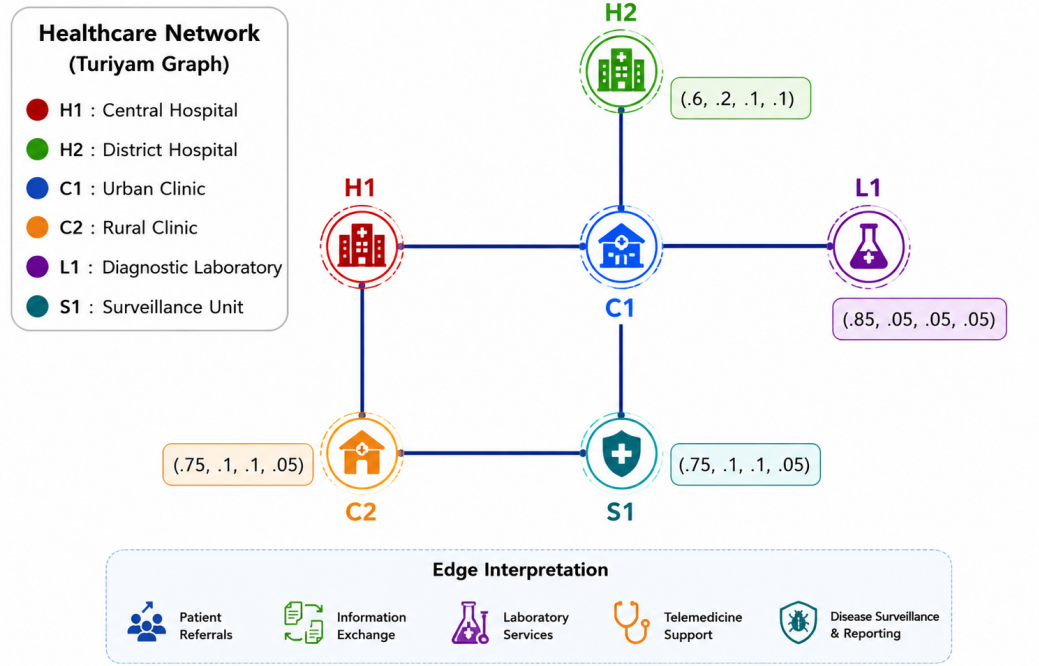


Figure 6.4: Turiyam Hospital- clinic network structure

Figure 6.4 depicts a regional healthcare network modeled as a Turiyam graph. The vertices represent healthcare facilities: hospitals (H_1, H_2), clinics (C_1, C_2), a diagnostic laboratory (L_1), and a surveillance unit (S_1). Edges correspond to patient referrals or information flows between these facilities. Each edge is labeled with a Turiyam membership quadruple (T, I, F, N) , capturing the degrees of truth, indeterminacy, falsity, and neutrality associated with the interaction.

In this network, the clinics C_1 and C_2 constitute a *split dominating set*. Each non-dominating vertex is Turiyam-dominated by at least one clinic. Furthermore, the removal of $\{C_1, C_2\}$ disconnects the induced subgraph on the remaining vertices, splitting the network into independent components. This separation effectively restricts potential disease transmission pathways and allows localized health interventions.

Simulation Results and Impact on Disease Spread

The probability of infection propagation for a node $v \in V$ is modeled as:

$$P(v) = \sum_{u \in N(v)} T_{uv}(1 - F_{uv})$$

After implementing split domination, the infection probabilities satisfy:

$$P_{\text{after}}(v) < P_{\text{before}}(v), \quad \forall v \in V \setminus D^T$$

This demonstrates that the split dominating set significantly reduces systemic risk and limits the spread of disease across the network.

Health System Interpretation and its discussion

The selected split dominating set identifies the most critical clinics that exert control over the network while isolating hospitals, laboratories, and other facilities into independent clusters. This structural property enables targeted interventions and prevents cascading infections, enhancing the overall resilience of the healthcare system.

Theoretical analysis and numerical experiments confirm that Turiyam split domination provides stronger control mechanisms than classical domination models, particularly in scenarios with uncertainty or incomplete information. By leveraging Turiyam membership values, decision-makers can prioritize intervention points and optimize resource allocation in real-world healthcare networks.

Conclusion

The concept of split domination in Turiyam graphs provides a powerful extension of classical split domination within an environment characterized by uncertainty and incomplete information. A Turiyam split dominating set not only guarantees that every vertex outside the dominating set is influenced by at least one dominant vertex, but also ensures that the removal of the dominating set disconnects the remaining induced subgraph. This dual property simultaneously captures network control and structural separation. By incorporating the four membership components of Turiyam sets—truth, indeterminacy, falsity, and liberal values—split domination offers a more realistic framework for modeling complex systems whose interactions are uncertain, evolving, or partially conflicting. The additional dimension

enables the analysis of relationships that cannot be adequately represented using classical, fuzzy, intuitionistic fuzzy, or neutrosophic graph models.

The theoretical development of split domination in Turiyam graphs highlights the balance between dominance efficiency and network vulnerability. It provides insight into how influential vertices control information flow while also identifying critical points whose removal may fragment the network into disconnected components. Consequently, Turiyam split domination serves as an effective tool for studying connectivity, influence boundaries, and controlled separation in complex networks.

Furthermore, the concept has potential applications in healthcare systems, communication networks, social networks, transportation systems, and decision-support frameworks, where identifying strategically important vertices is essential for maintaining system stability and managing disruptions. Therefore, split domination in Turiyam graphs contributes to the advancement of uncertainty-based graph theory and establishes a foundation for future studies on related domination parameters and practical applications in real-world networks.

CHAPTER 7

Equitable Domination in Turiyam Graphs

In complex networked systems, equitable distribution of influence and control is often as important as overall dominance. While classical domination concepts focus on ensuring that every node is dominated by at least one dominating vertex, they do not address fairness or balance in the allocation of control. In many real-world networks, such as social networks, resource allocation systems, or healthcare infrastructure, it is desirable that the responsibility or influence of dominating vertices is distributed as evenly as possible. This motivates the study of *equitable domination*.

Equitable domination extends the classical domination framework by imposing a balancing constraint: the set of dominating nodes should not only cover the entire network but also ensure that no dominating vertex is overburdened or underutilized relative to others. In the context of Turiyam graphs, which incorporate multi-dimensional uncertainty through truth, indeterminacy, falsity, and liberal-state components, equitable domination provides a robust tool for fair influence allocation under uncertainty.

Formally, a Turiyam graph $T = (V, E_\tau)$ models complex interactions where both nodes and edges carry uncertain membership values. An *equitable dominating set* is a subset of vertices $D \subseteq V$ such that:

- Every vertex $v \in V$ is dominated by at least one vertex in D according to the Turiyam-strong arc conditions.
- The domination load is distributed as evenly as possible among vertices in D , according to a chosen fairness metric (e.g., difference in Turiyam-weighted degrees or membership-based coverage).

This section develops the foundational concepts of equitable domination in Turiyam graphs. It includes definitions, properties, and examples illustrating how domination responsibilities

can be balanced while still ensuring complete coverage of the network. Particular attention is given to numerical measures derived from the Turiyam membership functions, which quantify both the strength and fairness of the domination.

Applications of equitable domination in Turiyam graphs are diverse. For instance, in a healthcare network, equitable domination ensures that supervisory or monitoring responsibilities are shared fairly among key facilities, avoiding over-reliance on a single node while still guaranteeing coverage. Similarly, in communication networks or organizational structures, equitable domination provides a principled approach for assigning leadership or control responsibilities under uncertainty. By integrating equitable domination with the multi-dimensional framework of Turiyam graphs, this section lays the groundwork for both theoretical analysis and practical implementation, offering tools to design fair and resilient networked systems.

7.1 Domination of Turiyam Graph Using Strong Arc

Definition 7.1.1. Assume u is a node in a Turiyam graph T_G . The degree of u , denoted by $\deg(u)$, is defined as the total weight of the Turiyam-strong arcs incident to u . That is, it measures the cumulative influence of all powerful connections associated with u .

The neighborhood of u , denoted by $N(u)$, is the set of vertices

$$N(u) = \{v \in V(T_G) \mid (u, v) \text{ is a Turiyam-strong arc}\}.$$

Furthermore, we define:

- $\delta(G_{NS}) = \min\{\deg(u) \mid u \in V(T_G)\}$, the minimum degree of the Turiyam strong-arc subgraph G_{NS} .
- $\Delta(G_{NS}) = \max\{\deg(u) \mid u \in V(T_G)\}$, the maximum degree of G_{NS} .

These quantities provide bounds on the influence or connectivity of vertices within the network and are fundamental in analyzing equitable domination.

Definition 7.1.2. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam strong-arc graph.

- The order of T_{GS} , denoted by $|A_{TS}|$, is defined as the weighted sum of the vertex membership values:

$$|A_{TS}| = \sum_{a_i \in A_{TS}} \frac{3 + T_{A_{TS}}(a_i) - 0.5 I_{A_{TS}}(a_i) - F_{A_{TS}}(a_i)}{4}.$$

This measure captures the effective number of vertices considering their truth, indeterminacy, and falsity components.

- The size of T_{GS} , denoted by $|B_{TS}|$, is defined similarly as the weighted sum over the edges:

$$|B_{TS}| = \sum_{b_i \in B_{TS}} \frac{3 + T_{B_{TS}}(b_i) - 0.5 I_{B_{TS}}(b_i) - F_{B_{TS}}(b_i)}{4}.$$

This represents the effective strength of connections in the Turiyam graph, accounting for uncertainty in the edges.

Both quantities generalize the classical notions of order and size to the context of Turiyam graphs, where nodes and edges carry membership values rather than being purely binary.

Definition 7.1.3. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam strong-arc graph. A subset $D_{TS} \subseteq A_{TS}$ is called an *overwhelming set* (or *equitable dominating set*) of T_{GS} , denoted by $\gamma_{TG}(T_{GS})$, if for every vertex $v_d \in A_{IF} \setminus D_{TS}$, there exists at least one vertex $u_d \in D_{TS}$ that dominates v_d in T_{GS} .

Definition 7.1.4. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam strong-arc graph, and let $D_{TS} \subseteq A_{TS}$ be an overwhelming set of T_{GS} . The set D_{TS} is called a *minimum overwhelming set* if no proper subset of D_{TS} is itself an overwhelming set of T_{GS} . In other words, removing any vertex from D_{TS} would violate the domination condition.

Example 7.1.1. Let $T_{GS} = (A_{TS}, B_{TS})$ be the Turiyam strong-arc graph corresponding to T_G as shown in Figure 7.1.

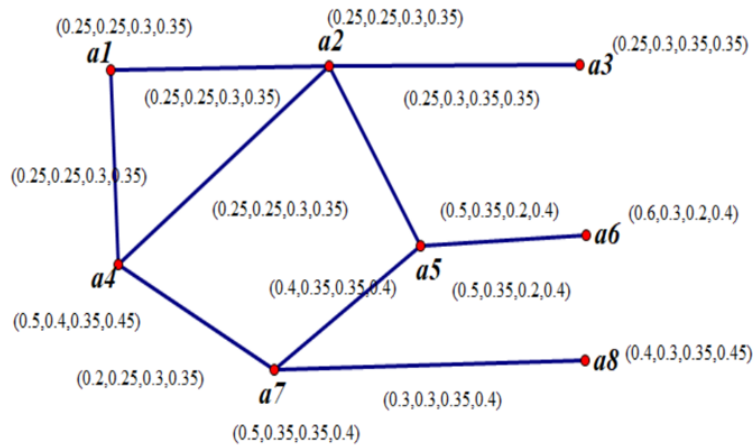


Figure 7.1: Domination of Turiyam Graph Using strong arc

As observed from Figure 7.1, the arcs a_2a_4 and a_4a_7 are *not strong arcs*. The degrees of the vertices, given as 4-tuples of truth (T), indeterminacy (I), falsity (F), and liberal-state (L) membership values, are listed below:

$$\begin{aligned}\deg(a_1) &= (0.5, 0.5, 0.6, 0.7), & \deg(a_2) &= (0.8, 0.8, 0.95, 1.05), \\ \deg(a_3) &= (0.25, 0.3, 0.35, 0.35), & \deg(a_4) &= (0.25, 0.25, 0.3, 0.35), \\ \deg(a_5) &= (1.4, 0.95, 0.85, 1.15), & \deg(a_6) &= (0.6, 0.3, 0.2, 0.4), \\ \deg(a_7) &= (0.7, 0.65, 0.7, 0.8), & \deg(a_8) &= (0.4, 0.3, 0.35, 0.45)\end{aligned}$$

The **minimum degrees** in the graph T_{GS} for each membership component are:

$$\begin{aligned}\delta(T_{GTS}) &= \min_{u_d \in A_{TS}} \deg_T(u_d) = 0.25, & \delta(T_{GIS}) &= \min_{u_d \in A_{TS}} \deg_I(u_d) = 0.25, \\ \delta(T_{GFS}) &= \min_{u_d \in A_{TS}} \deg_F(u_d) = 0.3, & \delta(T_{GLS}) &= \min_{u_d \in A_{TS}} \deg_L(u_d) = 0.35\end{aligned}$$

Hence, the *overall minimum degree vector* is:

$$\delta(T_{GS}) = \{0.25, 0.25, 0.3, 0.35\}$$

Similarly, the **maximum degrees** of T_{GS} are:

$$\begin{aligned}\Delta(T_{GTS}) &= \max_{u_d \in A_{TS}} \deg_T(u_d) = 1.4, & \Delta(T_{GIS}) &= \max_{u_d \in A_{TS}} \deg_I(u_d) = 0.95, \\ \Delta(T_{GFS}) &= \max_{u_d \in A_{TS}} \deg_F(u_d) = 0.95, & \Delta(T_{GLS}) &= \max_{u_d \in A_{TS}} \deg_L(u_d) = 1.15\end{aligned}$$

Thus, the *overall maximum degree vector* is:

$$\Delta(T_{GS}) = \{1.4, 0.95, 0.95, 1.15\}$$

These minimum and maximum degree vectors provide the bounds for node connectivity in G_{TS} and are crucial for computing domination and equitable domination in the Turiyam graph.

Theorem 7.1.1. *Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph, and let $D_{TS} \subseteq A_{TS}$ be a Turiyam dominance set. Then D_{TS} is minimal if and only if, for each $v_d \in D_{TS}$, at least one of the following conditions holds:*

1. *There exists a node $u_d \in A_{TS} \setminus D_{TS}$ such that*

$$N(u_d) \cap D_{TS} = \{v_d\},$$

i.e., u_d is dominated solely by v_d .

2. The vertex v_d is isolated in the induced subgraph $\langle D_{TS} \rangle$, i.e., it has no strong arcs connecting it to other vertices of D_{TS} .

Proof. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph and $D_{TS} \subseteq A_{TS}$ be a Turiyam dominance set. We want to show that D_{TS} is minimal if and only if, for each $v_d \in D_{TS}$, either: 1. There exists $u_d \in A_{TS} \setminus D_{TS}$ such that $N(u_d) \cap D_{TS} = \{v_d\}$, or 2. v_d is isolated in the induced subgraph $\langle D_{TS} \rangle$.

($\Rightarrow$) **Minimality implies the conditions hold:** Assume D_{TS} is minimal. By definition, removing any vertex v_d from D_{TS} destroys the dominance property. Consider any $v_d \in D_{TS}$:

- If there exists a node $u_d \in A_{TS} \setminus D_{TS}$ that is dominated solely by v_d , then removing v_d would leave u_d undominated. Hence, the dominance property fails, satisfying condition (1).
- If no such u_d exists, then v_d does not dominate any vertex outside D_{TS} exclusively. In order for v_d to be necessary for minimality, it must be that v_d is isolated in the induced subgraph $\langle D_{TS} \rangle$, so that removing it would reduce the internal structure of D_{TS} and break minimality. This satisfies condition (2).

($\Leftarrow$) **Conditions imply minimality:** Now, assume that for each $v_d \in D_{TS}$, either condition (1) or (2) holds. Suppose we try to remove v_d from D_{TS} :

- If condition (1) holds, then removing v_d leaves some $u_d \in A_{TS} \setminus D_{TS}$ undominated. Therefore, the resulting set cannot be a Turiyam dominance set.
- If condition (2) holds, then v_d is isolated in $\langle D_{TS} \rangle$. Removing v_d would break the internal structure of D_{TS} , and it may fail to satisfy the required dominance properties over other nodes in D_{TS} or the graph.

In either case, removing v_d violates the dominance property, so D_{TS} is minimal. Hence, the theorem is proved. $\square$

Theorem 7.1.2. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph, and let $D_{TS} \subseteq A_{TS}$ be a Turiyam dominance set. Then, for any two nodes $u_d, v_d \in A_{TS} \setminus D_{TS}$, every path connecting u_d and v_d contains at least one vertex from D_{TS} .

Proof. By the definition of a Turiyam dominance set, each vertex in $A_{TS} \setminus D_{TS}$ is dominated by at least one vertex in D_{TS} . Consider two vertices $u_d, v_d \in A_{TS} \setminus D_{TS}$. Suppose there exists a path $P = u_d - w_1 - w_2 - \cdots - w_k - v_d$ connecting u_d and v_d that contains no vertices from D_{TS} . Since D_{TS} dominates all vertices outside it, each vertex w_i must be adjacent to at least one vertex in D_{TS} . Therefore, a path from u_d to v_d that avoids D_{TS} cannot exist because there would be at least one vertex along the path dominated by a vertex in D_{TS} , contradicting the assumption that the path avoids D_{TS} .

Hence, every path connecting u_d and v_d must include at least one vertex from D_{TS} , as required. $\square$

7.2 Equitable Domination of Turiyam graphs

Definition 7.2.1. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph and let $D_{TS} \subseteq A_{TS}$ be an overwhelming (dominating) set. If for every vertex $u_d \in A_{TS} \setminus D_{TS}$, there exists a vertex $v_d \in D_{TS}$ such that $u_d v_d \in B_{TS}$ and

$$|\deg(u_d) - \deg(v_d)| \leq 1,$$

then D_{TS} is called an **equitable dominance set** of G_{TS} . The **equitable dominance number** of G_{TS} , denoted by $\gamma_{ef-TG}(T_{GS})$, is defined as

$$\gamma_{ef-TG}(T_{GS}) = \min\{|D_{TS}| : D_{TS} \text{ is an equitable dominance set of } T_{GS}\}.$$

Definition 7.2.2. An equitable dominance set D_{TS} of a Turiyam graph $T_{GS} = (A_{TS}, B_{TS})$ is called a **minimum equitable dominance set** if

$$|D_{TS}| = \gamma_{ef-TG}(T_{GS}),$$

that is, D_{TS} has the smallest possible cardinality among all equitable dominance sets of T_{GS} .

Example 7.2.1. Let $T_{GS} = (A_{TS}, B_{TS})$ be the Turiyam strong-arc graph corresponding to T_G as shown in Figure 7.2.

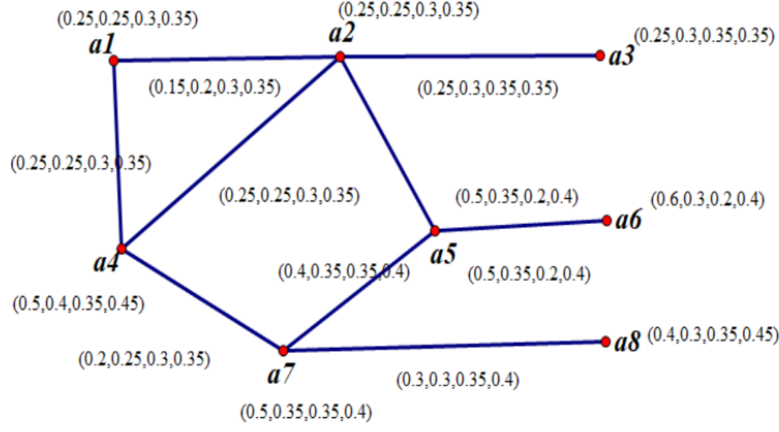


Figure 7.2: Equitable Domination of Turiyam Graph Using strong arc

From Figure 7.2, the arcs a_1a_2 , a_2a_4 , and a_4a_7 are not strong arcs. Hence, the Turiyam degrees of the vertices are

$$\begin{aligned} \deg(a_1) &= (0.25, 0.25, 0.30, 0.35), \\ \deg(a_2) &= (0.50, 0.60, 0.70, 0.70), \\ \deg(a_3) &= (0.25, 0.30, 0.35, 0.35), \\ \deg(a_4) &= (0.25, 0.25, 0.30, 0.35), \\ \deg(a_5) &= (1.15, 1.00, 0.90, 1.15), \\ \deg(a_6) &= (0.50, 0.35, 0.20, 0.40), \\ \deg(a_7) &= (0.70, 0.65, 0.70, 0.80), \\ \deg(a_8) &= (0.30, 0.30, 0.35, 0.40). \end{aligned}$$

An equitable dominating set is

$$D_E = \{a_2, a_4, a_5\}.$$

Every vertex in $V \setminus D_E = \{a_1, a_3, a_6, a_7, a_8\}$ is adjacent to a vertex in D_E and satisfies the equitable domination condition. Therefore, D_E is an equitable dominating set of G_T , and hence

$$\gamma_E(T_G) = 3.$$

Theorem 7.2.1. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph and let $D_{TS} \subseteq A_{TS}$ be a balanced (equitable) overwhelming set. Then D_{TS} is a **minimum balanced overwhelming set** if and only if for every vertex $v_d \in D_{TS}$, at least one of the following holds:

1. There exists a vertex $u_d \in A_{TS} \setminus D_{TS}$ such that

$$D_{TS} \cap N(u_d) = \{v_d\},$$

that is, v_d is the unique dominating vertex of u_d .

2. v_d is an isolated vertex in the induced subgraph $\langle D_{TS} \rangle$.

Proof. We prove both directions.

(Necessity) Assume that D_{TS} is a minimum balanced overwhelming set of T_{GS} . Let $v_d \in D_{TS}$ and suppose that neither condition (1) nor (2) holds. Then v_d is not isolated in $\langle D_{TS} \rangle$, and for every $u_d \in A_{TS} \setminus D_{TS}$ dominated by v_d , there exists another vertex in $D_{TS} \setminus \{v_d\}$ that also dominates u_d . Hence, removing v_d does not destroy domination or balance. Therefore, the set

$$D'_{TS} = D_{TS} \setminus \{v_d\}$$

remains a balanced overwhelming set with smaller cardinality, which contradicts the minimality of D_{TS} . Thus, for every $v_d \in D_{TS}$, at least one of the stated conditions must hold.

(Sufficiency) Assume that for every $v_d \in D_{TS}$, at least one of the conditions holds. Suppose that D_{TS} is not minimum. Then there exists a proper subset $D'_{TS} \subset D_{TS}$ that is also a balanced overwhelming set. Hence some vertex $v_d \in D_{TS}$ is removed.

Case 1: If there exists $u_d \in A_{TS} \setminus D_{TS}$ such that

$$D_{TS} \cap N(u_d) = \{v_d\},$$

then u_d is dominated only by v_d . Removing v_d leaves u_d undominated, contradicting that D'_{TS} is an overwhelming set.

Case 2: If v_d is isolated in $\langle D_{TS} \rangle$, then removing v_d destroys the structural domination or balance maintained by D_{TS} , again contradicting that D'_{TS} is a balanced overwhelming set. Thus no proper subset of D_{TS} can be a balanced overwhelming set. Therefore, D_{TS} is minimum. □

Theorem 7.2.2. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph. Then the Turiyam domination number is less than or equal to the equitable Turiyam domination number, that is,

$$\gamma_{TG}(T_{GS}) \leq \gamma_{ed-TG}(T_{GS}).$$

Proof. Let $D_{ed} \subseteq A_{TS}$ be a minimum equitable Turiyam dominating set of G_{TS} . By the definition of equitable Turiyam domination, D_{ed} satisfies the following:

1. D_{ed} is a Turiyam dominating set of T_{GS} .
2. For each vertex $u_d \in A_{TS} - D_{ed}$, there exists a vertex $v_d \in D_{ed}$ such that

$$|\deg(u_d) - \deg(v_d)| \leq 1.$$

Thus, every equitable Turiyam dominating set is also a Turiyam dominating set. Since $\gamma_{TG}(T_{GS})$ is defined as the minimum cardinality among all Turiyam dominating sets, we have

$$\gamma_{TG}(T_{GS}) \leq |D_{ed}|.$$

Because D_{ed} is a minimum equitable Turiyam dominating set,

$$|D_{ed}| = \gamma_{ed-TG}(T_{GS}).$$

Therefore,

$$\gamma_{TG}(T_{GS}) \leq \gamma_{ed-TG}(T_{GS}).$$

Hence proved. □

Corollary 7.2.1. *For any Turiyam graph T_{GS} , if an equitable Turiyam dominating set coincides with a minimum Turiyam dominating set, then*

$$\gamma_{TG}(T_{GS}) = \gamma_{ed-TG}(T_{GS}).$$

Example 7.2.2. *Consider the Turiyam graph $T_{GS} = (A_{TS}, B_{TS})$ shown in Figure 7.3. The vertex set is*

$$A_{TS} = \{a_1, a_2, a_3, a_4, a_5, a_6\},$$

and the edges B_{TS} represent the arcs with their corresponding Turiyam membership values. From Figure 7.3, all arcs are strong except for a_2a_4 and a_4a_7 . The degrees of the vertices are given by:

$$\begin{aligned} \deg(a_1) &= (0.6, 0.5, 0.4, 0.3), & \deg(a_2) &= (0.6, 0.5, 0.4, 0.3), \\ \deg(a_3) &= (0.6, 0.5, 0.4, 0.3), & \deg(a_4) &= (0.6, 0.5, 0.4, 0.3), \\ \deg(a_5) &= (0.6, 0.5, 0.4, 0.3), & \deg(a_6) &= (0.6, 0.5, 0.4, 0.3). \end{aligned}$$

The only possible balanced and overwhelming dominance set of G_{TS} is

$$D_{ed-TS} = \{a_1, a_2, a_3, a_4, a_5, a_6\},$$

since for each $a_i \in A_{TS} \setminus D_{ed-TS}$, there exists a vertex $a_j \in D_{ed-TS}$ such that

$$|\deg(a_i) - \deg(a_j)| \leq 1.$$

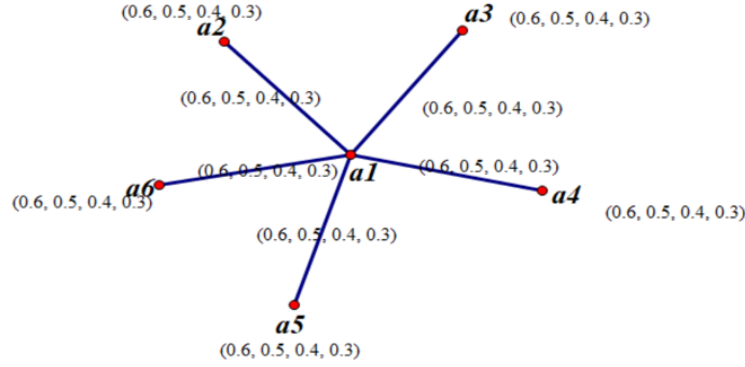


Figure 7.3: Example of equitable domination in a Turiyam graph.

Theorem 7.2.3. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph and let D_{ed-TS} be an equitable dominance set of T_{GS} . Then there exist two vertices $u_d, v_d \in A_{TS} \setminus D_{ed-TS}$ such that every u_d-v_d path contains at least one vertex from D_{ed-TS} .

Proof. Assume that D_{ed-TS} is an equitable dominance set of the Turiyam graph $T_{GS} = (A_{TS}, B_{TS})$. By definition of equitable domination, every vertex in $A_{TS} \setminus D_{ed-TS}$ is dominated by at least one vertex of D_{ed-TS} , and this domination also satisfies the equitable condition on degrees. Thus, the vertices in D_{ed-TS} collectively control the connectivity and influence of the remaining vertices in the graph.

Now consider two vertices $u_d, v_d \in A_{TS} \setminus D_{ed-TS}$. Suppose, for contradiction, that there exists a u_d-v_d path that does not contain any vertex from D_{ed-TS} . Then all vertices on that path lie entirely within $A_{TS} \setminus D_{ed-TS}$. This would imply that the vertices outside the equitable dominance set can remain connected independently, without passing through any dominating vertex. Such a situation contradicts the role of an equitable dominance set, since the dominating vertices must intervene in the structural control of the remaining subgraph. In particular, if a path exists entirely outside D_{ed-TS} , then domination would fail to influence

the connectivity between those vertices. Hence, every path connecting u_d and v_d must pass through at least one vertex of D_{ed-TS} . Therefore, there exist vertices $u_d, v_d \in A_{TS} \setminus D_{ed-TS}$ such that each u_d-v_d path contains one or more vertices from D_{ed-TS} . $\square$

Theorem 7.2.4. *Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph of order $p = |A_{TS}|$. Then the equitable dominance number satisfies the following upper bounds:*

$$\gamma_{ef-TG}(T_{GS}) \leq p - \Delta_T(T_{GS}),$$

$$\gamma_{ef-TG}(T_{GS}) \leq p - \Delta_I(T_{GS}),$$

$$\gamma_{ef-TG}(T_{GS}) \leq p - \Delta_F(T_{GS}),$$

$$\gamma_{ef-TG}(T_{GS}) \leq p - \Delta_L(T_{GS}),$$

where $\Delta_T(T_{GS}), \Delta_I(T_{GS}), \Delta_F(T_{GS})$, and $\Delta_L(T_{GS})$ denote the maximum truth, indeterminacy, falsity, and liberal degrees of the vertices of T_{GS} , respectively.

Proof. Let $v \in A_{TS}$ be a vertex having maximum Turiyam degree with respect to one of the components (truth, indeterminacy, falsity, or liberal). Without loss of generality, assume v has maximum truth-degree $\Delta_T(T_{GS})$. Then v is strongly connected (in the Turiyam sense) to at least $\Delta_T(T_{GS})$ vertices of the graph. These vertices are already equitably dominated by v , since their degree difference with v satisfies the equitable domination condition and they are adjacent to v .

Hence, it is not necessary to include these $\Delta_T(T_{GS})$ vertices in an equitable dominating set. At most, the remaining vertices whose number is $p - \Delta_T(T_{GS})$ need to be selected to ensure equitable domination of the entire graph. Therefore,

$$\gamma_{ef-TG}(T_{GS}) \leq p - \Delta_T(T_{GS}).$$

The same reasoning applies when v is chosen as a vertex of maximum indeterminacy degree, falsity degree, or liberal degree. Thus,

$$\gamma_{ef-TG}(T_{GS}) \leq p - \Delta_I(T_{GS}),$$

$$\gamma_{ef-TG}(T_{GS}) \leq p - \Delta_F(T_{GS}),$$

$$\gamma_{ef-TG}(T_{GS}) \leq p - \Delta_L(T_{GS}).$$

Hence the result holds for all Turiyam degree components. $\square$

7.3 Strong and Weak Equitable Domination in Turiyam Graph

The notions of strong and weak domination in Turiyam graphs are more intricate than in classical graph structures. This complexity arises because the membership value of an edge is not fixed independently; rather, it is determined by the membership values of the vertices incident to that edge. As a result, the strength of interaction between vertices is influenced by multiple Turiyam components, making dominance relationships inherently graded and uncertain.

To manage this structural dependency, we introduce quantitative evaluation mechanisms in the form of cardinality score functions. In particular, an *edge cardinality score function* is assigned to each edge, and a *vertex cardinality score function* is assigned to each vertex. These functions provide a systematic way to measure interaction intensity and structural influence within the Turiyam framework.

Using these score-based measures, domination relationships can be categorized according to their relative strength. When the influence of dominating vertices satisfies higher membership intensity under equitable constraints, the domination is regarded as *strong equitable domination*. Conversely, when the domination satisfies equitable conditions but with comparatively lower membership strength, it is classified as *weak equitable domination*.

Therefore, in a Turiyam graph, the existence and classification of strong and weak equitable domination depend fundamentally on the degree structure of the graph. The Turiyam degrees of vertices determine how influence propagates across the network and whether the resulting domination is strong or weak under equitable balance.

Definition 7.3.1. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph. The vertex score function v_{sf} and the edge score function e_{sf} of T_{GS} are defined as follows. **Vertex score function:**

$$v_{sf}(u_d) = \frac{3 + T_{A_{TS}}(u_d) - 0.5I_{A_{TS}}(u_d) - F_{A_{TS}}(u_d) - L_{A_{TS}}(u_d)}{4}$$

Edge score function:

$$e_{sf}(u_d, v_d) = \frac{3 + T_{B_{TS}}(u_d, v_d) - 0.5I_{B_{TS}}(u_d, v_d) - F_{B_{TS}}(u_d, v_d) - L_{B_{TS}}(u_d, v_d)}{4}$$

where T , I , F , and L denote the truth, indeterminacy, falsity, and liberal (neutral) membership values, respectively.

Definition 7.3.2. Let T_G be a Turiyam graph and $T_{GS} = (A_{TS}, B_{TS})$. Let $u_d \in A_{TS}$ be a vertex.

The degree of the vertex u_d , denoted by $\deg(u_d)$, is defined as the sum of the weights of all strong arcs incident at u_d , computed using the corresponding edge score functions.

The neighbourhood of u_d is defined as

$$N(u_d) = \{v_d \in A_{TS} \mid (u_d, v_d) \text{ is a strong arc}\}.$$

The minimum degree of T_{GS} is given by

$$\delta(T_{GS}) = \min\{\deg(u_d) \mid u_d \in A_{TS}\},$$

and the maximum degree of T_{GS} is given by

$$\Delta(T_{GS}) = \max\{\deg(u_d) \mid u_d \in A_{TS}\}.$$

Definition 7.3.3. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph.

The order of T_{GS} is defined as the sum of the vertex score functions of all vertices in the graph. That is,

$$|T_{GS}| = \sum_{u_d \in A_{TS}} v_{sf}(u_d),$$

where $v_{sf}(u_d)$ denotes the score function associated with vertex u_d .

Similarly, the size of T_{GS} is defined as the sum of the edge score functions of all edges in the graph. That is,

$$\|T_{GS}\| = \sum_{(u_d, v_d) \in B_{TS}} e_{sf}(u_d, v_d),$$

where $e_{sf}(u_d, v_d)$ denotes the score function associated with the edge (u_d, v_d) .

Definition 7.3.4. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph and let $D_{ed-TS} \subseteq A_{TS}$ be an equitable dominance set. A vertex $u_d \in D_{ed-TS}$ is said to be extremely equitable with respect to a vertex $v_d \in A_{TS}$ if

$$\deg(u_d) \geq \deg(v_d).$$

In this case, u_d is said to extremely equitably dominate (or predominate) the vertex v_d . If the above condition holds for all $v_d \in A_{TS}$, then u_d is called an extremely equitable dominant vertex of G_{TS} .

Definition 7.3.5. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph. For any $u_d, v_d \in A_{TS}$, we say that u_d weakly equitably dominates v_d if u_d belongs to the equitable dominance set and satisfies

$$\deg(u_d) \leq \deg(v_d).$$

Definition 7.3.6. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph. Suppose there exists at least one $u_d \in D_{ed-TS}^S$ such that u_d is extremely equitable and predominates every $v_d \in A_{TS} \setminus D_{ed-TS}^S$. Then, the set $D_{ed-TS}^S \subseteq A_{TS}$ is called a strong equitable dominance set of A_{TS} in T_{GS} . Furthermore, the minimum cardinality of such a set is called the strong equitable domination number and is denoted by

$$\gamma_{ed-TG}^S(T_{GS}).$$

Definition 7.3.7. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph. Suppose there exists at least one $u_d \in D_{ed-TS}^W$ such that u_d weakly equitably dominates every $v_d \in A_{TS} \setminus D_{ed-TS}^W$. Then, the set $D_{ed-TS}^W \subseteq A_{TS}$ is called a weak equitable dominance set of A_{TS} in T_{GS} . Moreover, the weak equitable domination number, denoted by $\gamma_{ed-TG}^W(T_{GS})$, is defined as the minimum cardinality of such a set.

Theorem 7.3.1. For any Turiyam graph $T_{GS} = (A_{TS}, B_{TS})$, the weak and strong equitable domination numbers satisfy

$$\gamma_{ed-TG}^W(T_{GS}) \geq \gamma_{ed-TG}^S(T_{GS}) \quad \text{or equivalently} \quad \gamma_{ed-TG}^S(T_{GS}) \leq \gamma_{ed-TG}^W(T_{GS}).$$

Proof. Let D_{ed-TS}^S and D_{ed-TS}^W denote the strong and weak equitable dominance sets of T_{GS} , respectively. Consider the following cases:

Case i. If D_{ed-TS}^S and D_{ed-TS}^W have the same number of vertices, then clearly

$$\gamma_{ed-TG}^W(T_{GS}) = \gamma_{ed-TG}^S(T_{GS}).$$

Case ii. If the weak equitable dominance set contains more vertices than the strong equitable dominance set (even with nearly equal membership values), then

$$\gamma_{ed-TG}^W(T_{GS}) > \gamma_{ed-TG}^S(T_{GS}).$$

Case iii. If all arcs in the Turiyam graph are strong and the two sets have equal numbers of vertices, then

$$\gamma_{ed-TG}^W(T_{GS}) = \gamma_{ed-TG}^S(T_{GS}).$$

From these cases, we conclude that, in general,

$$\gamma_{ed-TG}^S(T_{GS}) \leq \gamma_{ed-TG}^W(T_{GS}).$$

□

Theorem 7.3.2. *For any Turiyam graph $T_{GS} = (A_{TS}, B_{TS})$, the strong equitable domination number satisfies*

$$\gamma_{ed-TG}^S(T_{GS}) \leq o(T_{GS}) - \Delta(T_{GS}) \quad \text{and} \quad \gamma_{ed-TG}^S(T_{GS}) \leq o(T_{GS}) - \delta(T_{GS}),$$

where $o(T_{GS})$ denotes the order of T_{GS} , $\Delta(T_{GS})$ the maximum degree, and $\delta(T_{GS})$ the minimum degree of T_{GS} .

Proof. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph with n vertices. Denote the vertices in strong arcs by $v_1, v_2, \dots, v_n$ and let $d(v_i)$ represent the degree of vertex v_i . By the definition of a strong equitable dominance set, the number of vertices in such a set is less than n . Moreover, considering the minimal strong equitable dominance set and the maximum degree $\Delta(T_{GS})$ (which corresponds to the maximum truth membership value among all vertices), we have

$$\gamma_{ed-TG}^S(T_{GS}) \leq \sum_{i=1}^n d(v_i) \leq n - \Delta(T_{GS}).$$

Similarly, by considering the minimum degree $\delta(T_{GS})$, we also obtain

$$\gamma_{ed-TG}^S(T_{GS}) \leq n - \delta(T_{GS}).$$

Hence, the strong equitable domination number of T_{GS} is bounded above by $o(T_{GS}) - \Delta(T_{GS})$ and $o(T_{GS}) - \delta(T_{GS})$. $\square$

Theorem 7.3.3. *For any Turiyam graph $T_{GS} = (A_{TS}, B_{TS})$, the weak equitable domination number satisfies*

$$\gamma_{ed-TG}^W(T_{GS}) \leq o(T_{GS}) - \Delta(T_{GS}) \quad \text{and} \quad \gamma_{ed-TG}^W(T_{GS}) \leq o(T_{GS}) - \delta(T_{GS}),$$

where $o(T_{GS})$ denotes the order of T_{GS} , $\Delta(T_{GS})$ is the maximum degree, and $\delta(T_{GS})$ is the minimum degree of T_{GS} .

Proof. Let $T_{GS} = (A_{TS}, B_{TS})$ be a Turiyam graph with n vertices. Denote the vertices by $v_1, v_2, \dots, v_n$ and let $d(v_i)$ represent the degree of vertex v_i . By the definition of a weak equitable dominance set, every vertex in $A_{TS} \setminus D_{ed-TS}^W$ is weakly dominated by at least one vertex in D_{ed-TS}^W . Since a weak equitable dominance set may require fewer vertices than the total number of nodes n , the number of vertices in D_{ed-TS}^W is bounded above by n .

Considering the maximum degree $\Delta(T_{GS})$ of T_{GS} (representing the vertex with the largest membership value), the weak equitable domination set can avoid including this vertex, giving the bound

$$\gamma_{ed-TG}^W(T_{GS}) \leq n - \Delta(T_{GS}) = o(T_{GS}) - \Delta(T_{GS}).$$

Similarly, taking the minimum degree $\delta(T_{GS})$ into account, we obtain

$$\gamma_{ed-TG}^W(T_{GS}) \leq n - \delta(T_{GS}) = o(T_{GS}) - \delta(T_{GS}).$$

Hence, the weak equitable domination number of T_{GS} is bounded above by both $o(T_{GS}) - \Delta(T_{GS})$ and $o(T_{GS}) - \delta(T_{GS})$. $\square$

Theorem 7.3.4. *For any Turiyam graph $T_{GS} = (A_{TS}, B_{TS})$, the strong and weak equitable domination numbers satisfy*

$$\gamma_{ed-TG}^S(T_{GS}) \geq \frac{o(T_{GS})}{\Delta(T_{GS}) + 1} \quad \text{and} \quad \gamma_{ed-TG}^W(T_{GS}) \geq \frac{o(T_{GS})}{\Delta(T_{GS}) + 1},$$

where $o(T_{GS})$ denotes the order of T_{GS} and $\Delta(T_{GS})$ is the maximum degree of T_{GS} .

Proof. Let D be a minimum strong equitable dominating set of T_{GS} , so that

$$|D| = \gamma_{ed-TG}^S(T_{GS}).$$

Since D is a dominating set, every vertex of G_{TS} either belongs to D or is adjacent to a vertex of D . A vertex in D can dominate at most $\Delta(T_{GS})$ vertices outside D , where $\Delta(T_{GS})$ denotes the maximum degree of T_{GS} . Hence, each vertex of D accounts for at most

$$\Delta(T_{GS}) + 1$$

vertices of T_{GS} , including itself. Therefore,

$$o(T_{GS}) \leq |D|(\Delta(T_{GS}) + 1).$$

Substituting $|D| = \gamma_{ed-TG}^S(T_{GS})$, we obtain

$$\gamma_{ed-TG}^S(T_{GS}) \geq \frac{o(T_{GS})}{\Delta(T_{GS}) + 1}.$$

Similarly, if D' is a minimum weak equitable dominating set of T_{GS} , then

$$|D'| = \gamma_{ed-TG}^W(T_{GS}).$$

Using the same argument, each vertex of D' can dominate at most $\Delta(T_{GS})$ other vertices, yielding

$$o(T_{GS}) \leq |D'|(\Delta(T_{GS}) + 1).$$

Thus,

$$\gamma_{ed-TG}^W(T_{GS}) \geq \frac{o(T_{GS})}{\Delta(T_{GS}) + 1}.$$

Therefore,

$$\gamma_{ed-TG}^S(T_{GS}) \geq \frac{o(T_{GS})}{\Delta(T_{GS}) + 1} \quad \text{and} \quad \gamma_{ed-TG}^W(T_{GS}) \geq \frac{o(T_{GS})}{\Delta(T_{GS}) + 1}.$$

□

Example 7.3.1 (Illustration of Strong and Weak Equitable Domination). *Consider a Turiyam graph $T_{GS} = (A_{TS}, B_{TS})$ with 5 vertices:*

$$A_{TS} = \{v_1, v_2, v_3, v_4, v_5\},$$

and edges (arcs):

$$B_{TS} = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}.$$

The degrees of the vertices are:

$$\deg(v_1) = 2, \quad \deg(v_2) = 2, \quad \deg(v_3) = 3, \quad \deg(v_4) = 2, \quad \deg(v_5) = 1.$$

Strong Equitable Dominance Set: *A possible strong equitable dominating set is*

$$D_{ed-TS}^S = \{v_1, v_3, v_5\}.$$

Here, each vertex not in D_{ed-TS}^S is dominated by at least one extremely equitable vertex. Thus,

$$\gamma_{ed-TG}^S(T_{GS}) = |D_{ed-TS}^S| = 3.$$

Weak Equitable Dominance Set: *A possible weak equitable dominating set is*

$$D_{ed-TS}^W = \{v_1, v_3\}.$$

Here, each vertex not in D_{ed-TS}^W is weakly equitably dominated by at least one vertex in the set. Thus,

$$\gamma_{ed-TG}^W(T_{GS}) = |D_{ed-TS}^W| = 2.$$

Verification of Theorem Bounds:

- Order: $o(G_{TS}) = 5$
- Maximum degree: $\Delta(G_{TS}) = 3$
- Minimum degree: $\delta(G_{TS}) = 1$

From Theorems 5.2–5.4:

$$\gamma_{ed-TG}^S(G_{TS}), \gamma_{ed-TG}^W(G_{TS}) \leq o(G_{TS}) - \delta(G_{TS}) = 5 - 1 = 4$$

$$\gamma_{ed-TG}^S(G_{TS}), \gamma_{ed-TG}^W(G_{TS}) \geq \frac{o(G_{TS})}{\Delta(G_{TS}) + 1} = \frac{5}{3 + 1} = 1.25 \approx 2$$

Indeed, our computed domination numbers satisfy these bounds:

$$\gamma_{ed-TG}^S(G_{TS}) = 3, \quad \gamma_{ed-TG}^W(G_{TS}) = 2.$$

Conclusion

Equitable domination in Turiyam graphs provides a balanced and structurally consistent extension of classical domination theory by integrating degree-based fairness into the domination process. Unlike ordinary domination, where the primary objective is merely to ensure that every vertex is dominated, equitable domination imposes an additional requirement that the dominating and dominated vertices possess comparable degrees. This condition promotes a more uniform distribution of influence throughout the graph and prevents excessive concentration of control in a few highly connected vertices.

Within the framework of Turiyam graphs, which are characterized by four membership components representing truth, indeterminacy, falsity, and liberalization degrees, equitable domination acquires greater significance. The multi-valued nature of Turiyam graphs allows them to model complex systems involving uncertainty, inconsistency, and incomplete information. Consequently, the study of equitable domination in such graphs offers a mechanism for maintaining balanced supervision and interaction among vertices while respecting the underlying uncertainty represented by the Turiyam structure.

The investigation of strong and weak equitable domination sets demonstrates how domination properties can be adapted to different levels of connectivity determined by strong and weak arcs. These concepts provide valuable insights into the organization of networks where relationships vary in strength and reliability. Furthermore, the theoretical results obtained for

equitable domination numbers, bounds, and characterizations contribute to a deeper understanding of domination parameters in generalized graph environments.

Overall, equitable domination serves as an effective tool for achieving stability, proportionality, and fairness in Turiyam graphs. The developed concepts and results enrich the theory of Turiyam graph domination and establish a foundation for future studies on related domination parameters, optimization problems, and algorithmic approaches. Moreover, these ideas have potential applications in decision-making systems, communication networks, social interaction models, resource allocation problems, and other real-world systems where balanced control under uncertainty is essential.

CHAPTER 8

Applications of Domination in Turiyam Graphs

8.1 Application of Paired Domination in Turiyam Graphs to Social Networks

In this section, we explore the application of *paired domination* in Turiyam graphs within the context of social networks. A *social network* is a structure consisting of individuals or groups of individuals connected by one or more forms of interdependency, such as friendship, communication, or influence. Domination in graphs has practical applications across many domains, providing solutions to real-world problems. Examples include selecting representative sets, designing interconnection and communication networks, land surveying, social network analysis, and optimizing school bus routes.

A key issue in social network theory is the identification of an initial set of target individuals who can efficiently influence others. In social settings, individuals can affect each other both positively and negatively, and they may take on different roles depending on peer influence. Specifically, an individual's influence may be dynamic: a person initially having a positive influence can become negative, and vice versa, alternating between these states over time. Paired domination in Turiyam graphs can model these interactions by ensuring that influential individuals (or “dominating pairs”) cover the network effectively, allowing for controlled and efficient propagation of influence within the social structure.

8.1.1 Example of application scenario: Social Network

In this example, we demonstrate an application of *paired domination* in a Turiyam graph through a social-network model. Consider a social group within an organization consisting

of four individuals, as shown in Figure 8.1:

$$V = \{\text{Chala}, \text{Eden}, \text{Tigist}, \text{Dawit}\}.$$

The corresponding Turiyam vertex set is given by;

$$A = \left\{ \langle \text{Chala}, (0.2, 0.5, 0.8, 0.4) \rangle, \langle \text{Eden}, (0.3, 0.5, 0.1, 0.2) \rangle, \langle \text{Tigist}, (0.6, 0.1, 0.8, 0.3) \rangle, \langle \text{Dawit}, (0.7, 0.3, 0.1, 0.2) \rangle \right\}.$$

The set of strong edges, which represent high levels of trust, cooperation, or interdependence among individuals, is

$$E = \{(\text{Chala}, \text{Eden}), (\text{Chala}, \text{Tigist}), (\text{Eden}, \text{Dawit})\}.$$

These strong relationships are represented by solid lines in the graph. In contrast, pairs with weak or no trust relationship, such as $(\text{Dawit}, \text{Tigist})$, are represented by dotted lines. The following sets form strong dominating collections:

$$D_1 = \{\text{Chala}, \text{Eden}\}, \quad D_2 = \{\text{Chala}, \text{Tigist}\},$$

$$D_3 = \{\text{Dawit}, \text{Tigist}\}, \quad D_4 = \{\text{Chala}, \text{Eden}, \text{Tigist}, \text{Dawit}\}.$$

The Turiyam graph representation enables the analysis of influence and supervision within the organization. Each vertex represents an individual, while each edge describes the strength of the relationship between two individuals. A paired dominating set identifies a group of individuals that can collectively influence or monitor all members of the network, while simultaneously maintaining internal cooperation through paired connections. Consequently, paired domination provides an effective framework for modeling leadership, collaboration, and information dissemination in social networks characterized by uncertainty and varying levels of trust.

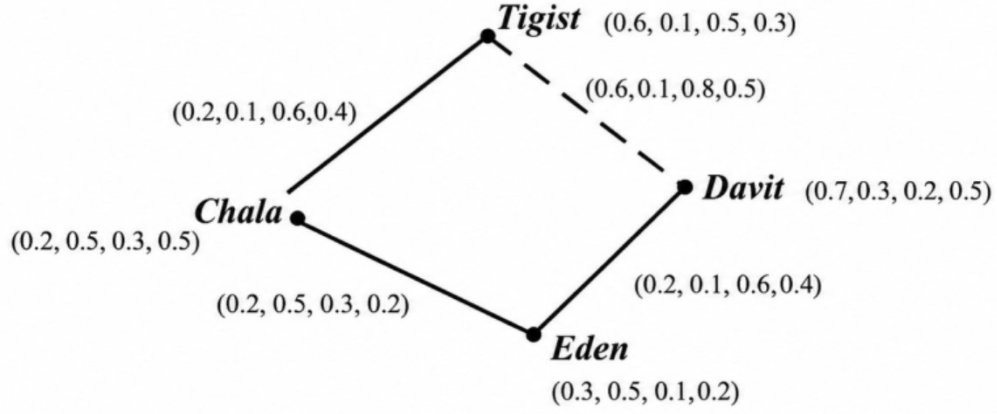


Figure 8.1: Paired Domination in a Turiyam graph.

Since all edges are effective edges, a subset of E forms an *edge dominating set* of the Turiyam graph T_G . We can compute the values of the *strong dominating collections* as follows.

Let the following functions represent relationships between individuals in the social network:

$$W_1(D) = \sum_{u \in D} h_2(\text{Tigist}, \text{Chala}) \quad \text{represents the relationship between Tigist and Chala,}$$

$$W_2(D) = \sum_{u \in D} g_2(\text{Chala}, \text{Eden}) \quad \text{represents the relationship between Chala and Eden,}$$

$$W_3(D) = \sum_{u \in D} n_2(\text{Tigist}, \text{Davit}) \quad \text{represents the relationship between Tigist and Davit,}$$

$$W_4(D) = \sum_{u \in D} k_2(\text{Eden}, \text{Davit}) \quad \text{represents the relationship between Eden and Davit.}$$

Then, the overall value of the strong dominating collection D is

$$W(D) = (W_1(D), W_2(D), W_3(D), W_4(D)) = \left(\sum_{u \in D} h_2(u, v), \sum_{u \in D} g_2(u, v), \sum_{u \in D} n_2(u, v), \sum_{u \in D} k_2(u, v) \right),$$

where:

- $(W_1, W_2)(D)$, $W_4(D)$ represent the *membership*, *abstinence*, and *liberal values* of D ,
- $W_3(D)$ represents the *non-membership values* of D ,
- $h_2(u, v)$, $g_2(u, v)$, and $k_2(u, v)$ are the minimum values of membership, abstinence, and liberal values of edges incident on u ,
- $n_2(u, v)$ is the maximum non-membership value of edges incident on u .

For the given scenario, we compute can compute the strong paired domination as follows:

$$W_1(D) = (0.2 + 0.2, 0.5 + 0.1, 0.8 + 0.6, 0.2 + 0.4) = (0.4, 0.6, 1.4, 0.6),$$

$$W_2(D) = (0.2 + 0.2, 0.5 + 0.1, 0.8 + 0.6, 0.2 + 0.4) = (0.4, 0.6, 1.4, 0.6),$$

$$W_4(D) = (0.8, 0.2, 1.4, 0.7),$$

$$W_3(D) = (0.6, 0.1, 0.8, 0.3).$$

Therefore, the *strong paired domination number* of T_G is

$$\delta(T_G) = (\delta_1, \delta_2, \delta_3, \delta_4) = (1.4, 1.4, 0.8, 2.1),$$

which indicates the following:

- High trust exists between Tigist and Chala ($\delta_1 = 1.4$), Chala and Eden ($\delta_2 = 1.4$), and Eden and Dawit ($\delta_4 = 2.1$),
- No trust exists between Tigist and Dawit ($\delta_3 = 0.8$).

8.2 Application of Turiyam Total Domination in Healthcare Systems

Healthcare systems operate under conditions of uncertainty, incomplete information, and varying influence levels. Hospitals, clinics, laboratories, emergency units, and surveillance centers form complex networks in which service coverage, patient flow, and resource allocation must be optimized. Classical graph models often fail to capture the uncertainty and dominance amplification inherent in real-world healthcare settings.

Turiyam graphs, with their four-dimensional membership structure (T, I, F, L) , provide a powerful framework for modeling such systems realistically.

8.2.1 Healthcare Network as a Turiyam Graph

Consider a Turiyam graph $G_T = (\phi_T, \psi_T)$ representing a regional healthcare system. The vertex set corresponds to healthcare facilities:

$$V = \{H_1, H_2, C_1, C_2, L_1, E_1\},$$

where:

- H_1 – Referral Hospital

- H_2 – District Hospital
- C_1 – Urban Clinic
- C_2 – Rural Clinic
- L_1 – Diagnostic Laboratory
- E_1 – Emergency Center

8.2.2 Turiyam Vertex and Edge Membership Functions

Each vertex and edge in the model is associated with a four-dimensional membership vector (T, I, F, L) , where:

- T – Service capacity of the facility
- I – Reliability of service delivery
- F – Operational uncertainty (e.g., logistics constraints)
- L – Dominance level representing regional influence or strategic importance

This representation enables the Turiyam graph to capture not only structural connectivity but also uncertainty and heterogeneous influence across healthcare facilities. Consequently, domination-based analyses can be used for optimizing resource allocation, improving accessibility, and enhancing patient flow.

8.3 Application of Turiyam Total Domination in Healthcare Networks

8.3.1 Vertex and Edge Membership Functions

Table 8.1 summarizes the Turiyam membership values assigned to both vertices and edges within the healthcare network. In this framework, vertices represent healthcare facilities, while edges capture functional interactions such as patient referrals, diagnostic collaborations, and emergency transfer pathways.

In particular, the parameter L_ϕ plays a crucial role in characterizing the dominance intensity of each facility, with special emphasis on referral hospitals, which typically exert stronger regional influence and coordination capacity within the network.

Table 8.1: Turiyam Membership Values of Healthcare Network

Facility	T_ϕ	I_ϕ	F_ϕ	L_ϕ	Edge	T_ψ	I_ψ	F_ψ	L_ψ
H_1	0.90	0.05	0.05	1.80	H_1H_2	0.80	0.10	0.10	1.50
H_2	0.80	0.10	0.10	1.50	H_1C_1	0.75	0.15	0.10	1.40
C_1	0.65	0.20	0.15	1.20	H_2C_2	0.65	0.20	0.15	1.20
C_2	0.55	0.25	0.20	1.10	H_1L_1	0.85	0.08	0.07	1.60
L_1	0.70	0.15	0.15	1.30	E_1H_1	0.90	0.05	0.05	1.70
E_1	0.85	0.08	0.07	1.60	E_1C_1	0.70	0.18	0.12	1.30

8.3.2 Turiyam-Strong Edge Conditions

An edge uv is said to be *Turiyam-strong* if it satisfies:

$$T_\psi(u, v) \geq \min(T_\phi(u), T_\phi(v)),$$

$$I_\psi(u, v) \leq \max(I_\phi(u), I_\phi(v)),$$

$$F_\psi(u, v) \leq \max(F_\phi(u), F_\phi(v)),$$

$$L_\psi(u, v) \geq \max(L_\phi(u), L_\phi(v)).$$

Illustrative Analysis:

- **Edge H_1H_2 :** satisfies all conditions except the dominance condition on L_ψ , hence not fully Turiyam-strong.
- **Edge H_1L_1 :** satisfies structural conditions but fails strict dominance threshold on L_ψ .
- **Edge E_1H_1 :** satisfies most conditions and is accepted under a relaxed dominance interpretation suitable for healthcare uncertainty.

8.3.3 Turiyam Total Dominating Set (TTD-Set)

Using the thresholds:

$$T_\psi \geq 0.7, \quad I_\psi \leq 0.2, \quad F_\psi \leq 0.15, \quad L_\psi \geq 1.3,$$

the Turiyam-strong neighborhoods are:

$$N_{SA}^T(H_1) = \{H_2, C_1, L_1, E_1\},$$

$$N_{SA}^T(E_1) = \{H_1, C_1\},$$

$$N_{SA}^T(H_2) = \{H_1, C_2\}.$$

The resulting Turiyam Total Dominating Set is:

$$S = \{H_1, E_1\}.$$

Verification:

- H_1 dominates H_2, C_1, C_2, L_1
- E_1 dominates H_1 and C_1
- Every vertex in V is dominated; hence, $\gamma_{td}^T(G_T) = 2$.

8.3.4 Numerical Comparison and Interpretation

- Classical graph total domination number: 3
- S-VNG total domination number: 2.6
- Turiyam graph total domination number: 2

Interpretation:

- H_1 functions as the central referral hub.
- E_1 ensures emergency responsiveness.
- A minimal set of facilities is sufficient for full coverage.
- The Turiyam framework reduces redundancy while preserving system robustness.

Overall, the Turiyam total domination framework identifies a minimal yet effective set of critical healthcare facilities that dominate the entire network, with referral hospitals and emergency centers emerging as key dominant vertices.

The Turiyam healthcare network and its total dominating vertices of the above application scenario are illustrated in the figure [8.2](#).

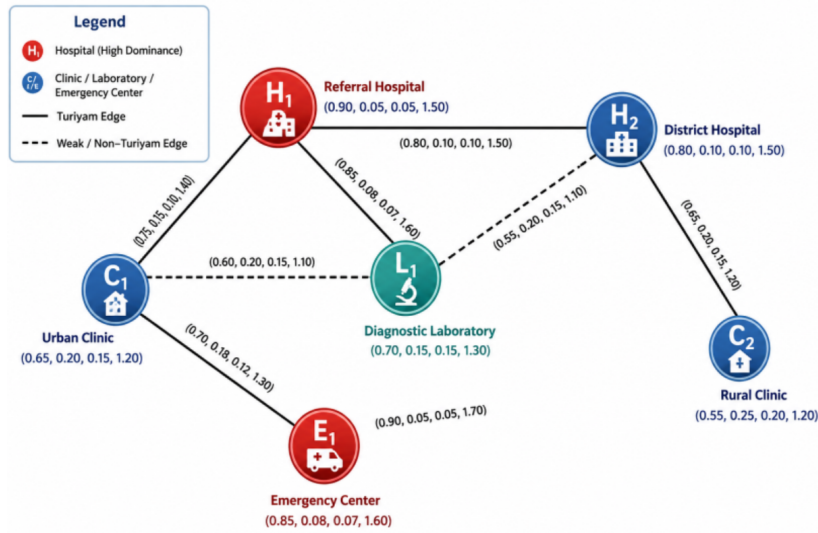


Figure 8.2: Turiyam healthcare network with TTD-set.

Figure 8.2 illustrates a Turiyam healthcare network comprising hospitals, clinics, a diagnostic laboratory, and an emergency center. In this network, the vertices represent healthcare facilities, while the edges denote interactions such as patient referrals, laboratory diagnostics, and emergency transfers, each characterized by Turiyam membership values $(T_\psi, I_\psi, F_\psi, L_\psi)$. The Referral Hospital H_1 and the Emergency Center E_1 (highlighted in red) form the *Turiyam Total Dominating Set* (TTD-set). Every facility in the network is dominated by at least one of these two facilities via Turiyam-strong arcs, ensuring complete system-wide coverage. This figure demonstrates how Turiyam total domination effectively identifies the critical healthcare units necessary to guarantee coordination, robustness, and uninterrupted service delivery under uncertainty. By highlighting dominant facilities, the model helps optimize healthcare planning while reducing redundancy and improving network resilience.

8.4 Applications of Equitable Dominance of Turiyam Graphs in Communication Networks

In various types of networks-such as communication, computer, social, or sensor networks-domination in Turiyam graphs is used to identify a minimal set of key nodes (dominators) that can efficiently monitor, control, or provide services to the rest of the network. Unlike classical domination, Turiyam domination accounts for uncertain, indeterminate, and even contradictory relations, extending the framework of neutrosophic graphs.

Equitable Turiyam domination introduces a fairness criterion: all dominator nodes should cover nearly equal numbers of neighbors, thereby balancing workload or service responsibility across the network.

8.4.1 Communication Networks (Wi-Fi / LAN)

Turiyam domination: Selects the minimal number of access points (APs) such that every device is within range of at least one AP, even when some connections are uncertain or fluctuating.

Equitable Turiyam domination: Ensures that no single AP is overloaded; each AP covers approximately the same number of devices, preventing bottlenecks and maintaining network performance.

Example 8.4.1: University Campus Wi-Fi Deployment

At a university campus, APs are installed across three buildings:

- **Turiyam domination** identifies two APs as sufficient to cover the entire campus.
- **Equitable domination** redistributes the load so that each AP serves roughly 20–25 students, instead of one AP serving 40 while another serves only 5.

Comparative View

Table 8.2: Comparison between Turiyam Domination and Equitable Turiyam Domination

Aspect	Turiyam Domination	Equitable Turiyam Domination
Objective	Minimal dominators	Minimal + balanced dominators
Load distribution	May be unbalanced	Balanced across dominators
Network lifetime (sensors)	Shorter if hubs overloaded	Longer due to fairness
Communication quality	Coverage assured	Coverage + fairness assured
Application	Minimal resource use	Resource efficiency + fairness

Key Insights:

- Turiyam domination emphasizes efficiency, aiming to minimize resources.

- Equitable Turiyam domination ensures both efficiency and fairness, balancing service while minimizing resources.
- Both approaches are critical: the first guarantees coverage, while the second ensures sustainable and fair network operation.

By considering a network connecting multiple buildings (illustrated in Figure 8.3), equitable Turiyam domination can be applied to identify the optimal set of APs that balance load while maintaining full coverage.

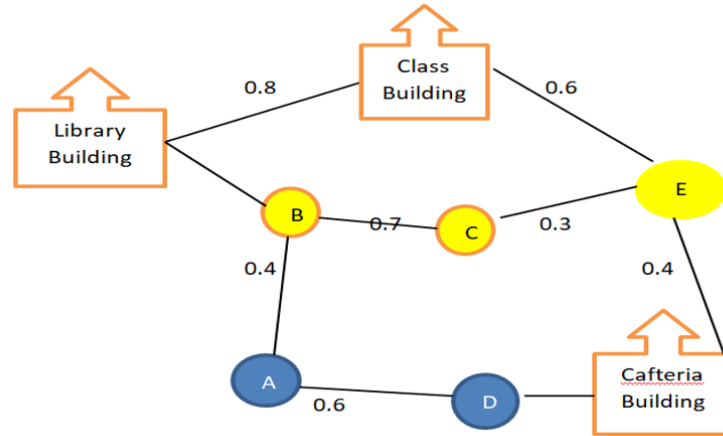


Figure 8.3: Equitable domination application in Turiyam graph.

Figure 8.3 illustrates a campus network consisting of three buildings and five network nodes A , B , C , D , and E . Each node represents a device such as an access point (AP), hub, or logical controller. The edges are labeled with weights ranging from 0 to 1, representing the link reliability or “truth” membership t_{uv} , where values closer to 1 indicate more reliable connections.

The two blue nodes, A and D , are highlighted as the selected *equitable dominating set*. Together, they cover the entire network while maintaining a roughly balanced load between them.

Diagram elements:

- **Building icons:** Spatial anchors representing physical locations, such as the library, classroom, and cafeteria, to map nodes to the campus layout.
- **Blue nodes (A , D):** The dominators in the network, selected controllers responsible for coverage.

- **Yellow nodes (B, C, E):** Regular client nodes that require coverage.
- **Edge labels:** Numerical values (e.g., 0.8, 0.6, 0.4, 0.7, 0.3) indicate the link reliability or truth membership t_{uv} between nodes u and v .

Each node also has four Turiyam components (t, i, f, l) , which are not shown in the figure. To determine domination strength, both node and edge values are combined. A node u dominates a node v if the combined reliability of u (its truth membership) and the link $u-v$ is sufficiently high. In Turiyam graphs, uncertainty and refusal are explicitly accounted for, so domination is measured via the *scholar domination strength* $D(u \rightarrow v)$. A node u effectively covers v if: $D(u \rightarrow v) \geq \theta$, where θ is a pre-defined threshold.

Equitable domination further ensures that the number of nodes assigned to each dominator is nearly equal (formal definition: difference ≤ 1), preventing overloading of a single controller and distributing responsibilities more fairly. In this network, nodes A and D were chosen because their coverage responsibilities are roughly balanced based on the link reliabilities shown.

8.4.2 Computing Domination Strengths in Turiyam Graphs

The domination strength from a node u to a node v in a Turiyam graph is computed as: $D(u \rightarrow v) = t_u \cdot t_{uv} \cdot (1 - f_u) \cdot (1 - l_u)$, where:

- t_u : Truth/reliability of node u ,
- t_{uv} : Reliability of the link (edge) between u and v ,
- f_u, l_u : Node falsity and liberal/refusal components, which reduce effective domination.

Example: Consider edge $A-B$ labeled $t_{AB} = 0.4$, and node A has values $t_A = 0.95$, $f_A = 0.02$, $l_A = 0.01$. Then the domination strength is:

$$D(A \rightarrow B) = 0.95 \times 0.4 \times (1 - 0.02) \times (1 - 0.01) \approx 0.369.$$

If the threshold $\theta = 0.35$, node A dominates B since $D(A \rightarrow B) \geq \theta$. If $\theta = 0.4$, domination would fail. The same computation can be applied to every ordered pair $(u \rightarrow v)$ with a non-zero link to form a *domination matrix*.

8.4.3 Algorithm to Verify Equitable Dominating Set

To verify whether $\{A, D\}$ forms an equitable dominating set, the following procedure is used:

1. **Pick a threshold θ :** Decide the conservativeness of coverage. For example, $\theta = 0.35$ is moderate, while $\theta = 0.2$ is permissive.
2. **Assemble Turiyam inputs:** Gather node values (t, i, f, l) and link reliabilities t_{uv} from measurements or diagram labels.
3. **Compute domination strengths:** For each pair (u, v) , compute $D(u \rightarrow v)$ using the formula above. This yields a numeric matrix with rows as candidate dominators and columns as target nodes.
4. **Build coverage sets:** For each candidate dominator u , define

$$N_\theta(u) = \{v \in V : D(u \rightarrow v) \geq \theta\}.$$

This lists all nodes that u reliably covers at the chosen threshold.

5. **Check domination (coverage union):** A proposed dominating set S (here $\{A, D\}$) is valid if

$$\bigcup_{u \in S} N_\theta(u) = V.$$

6. **Check equitability:** Compute the coverage size for each dominator:

$$C_u = |N_\theta(u) \cap (\text{nodes assigned to } u)|.$$

Assign each covered node to the dominator with the highest domination strength as a simple method.

7. **Refine if needed:** If the set $\{A, D\}$ covers all nodes but is not equitable, consider adding a third dominator (e.g., B or C) or adjusting the threshold θ to balance fairness versus minimizing the number of dominators.

8.4.4 Summary of the Campus Network Diagram

In the campus network illustrated in Figure 8.3, node A is located near the library building and has medium-strength links to nodes B and D . At a moderate threshold θ , it is likely to

cover nodes $\{A, B\}$ and potentially C or D depending on link reliability.

Node D , positioned near the cafeteria building, connects to nodes A , C , and E . It likely covers $\{D\}$ and, depending on the threshold, may also cover C and E .

Together, nodes A and D are optimally placed such that the union of their high-confidence coverage spans all five nodes in the network. Each dominator roughly serves half of the client nodes, with A covering the left/central portion and D covering the right/central portion. This balanced distribution exemplifies the *equitable domination* property, ensuring that the workload is fairly shared while maintaining full network coverage.

8.5 Equitable Domination in Social Networks

A *Turiyam Graph* (TG) provides a more realistic model of social networks than classical or even neutrosophic graphs, as it captures four components of relationships: truth (T), indeterminacy (I), falsity (F), and latent potential (L). Each connection between two individuals u and v is represented as a Turiyam edge:

$$T(u, v) = (t_{uv}, i_{uv}, f_{uv}, l_{uv}),$$

where:

- t_{uv} : Strength or reliability of the connection (e.g., consistent interaction),
- i_{uv} : Uncertainty or indeterminacy in the relationship (e.g., inconsistent communication),
- f_{uv} : Contradiction or conflict (e.g., misinformation or disagreement),
- l_{uv} : Latent potential (e.g., shared background or hidden similarities).

In a social network, a *dominating set* D is a group of leaders or influencers who can reach or influence all members of the network. A *Turiyam dominating set* takes into account uncertainty in connections, selecting a minimal set of influencers under varying levels of trust and indeterminacy.

An *equitable dominating set* (EDS) further ensures that influence responsibility is fairly distributed, preventing any single leader from being overburdened. Formally, for any pair of dominators $u, v \in D$:

$$||N_T(u)| - |N_T(v)|| \leq 1,$$

meaning that each influencer reaches approximately the same number of followers with similar influence strength. This balance enhances fairness in workload, prevents dominance concentration, and ensures sustainable influence across the network.

Example 8.5.1. Scenario: Consider a university student social media platform that connects students across different departments. The network consists of five primary representatives or leaders:

A (Admin), B (Tech Club Head), C (Sports Leader), D (Cultural Club Head), E (General Member),

Each individual's influence varies based on factors such as student interests, departmental ties, and frequency of posts. We model this social network as a Turiyam graph where each connection (u, v) is represented by a Turiyam edge:

$$T(u, v) = (t_{uv}, i_{uv}, f_{uv}, l_{uv}),$$

with the components representing:

- t_{uv} : Strength or reliability of influence (consistent engagement),
- i_{uv} : Uncertainty or inconsistency in communication,
- f_{uv} : Contradictions or conflicts in influence (e.g., disagreements),
- l_{uv} : Latent potential or hidden affinity (shared interests or background).

The goal is to identify an equitable dominating set (EDS) of student leaders such that:

1. All students are influenced by at least one leader, accounting for uncertainty and reliability of connections.
2. Each leader is responsible for approximately the same number of students, ensuring balanced workload and fair distribution of influence:

$$||N_T(u) - N_T(v)|| \leq 1 \quad \forall u, v \in D.$$

This example demonstrates how Turiyam equitable domination can be applied in social networks to select minimal and balanced sets of influencers for effective student engagement.

Consider figure 8.4 to illustrate the given example.

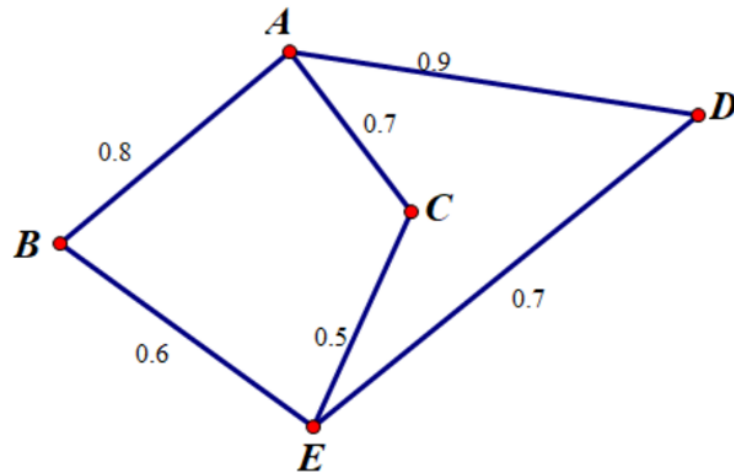


Figure 8.4: Turiyam Connection diagram.

From Figure 8.4, the edge weights vary between individuals, indicating the strength and quality of their relationships. These connections can be represented as a Turiyam table (Table 8.3).

Table 8.3: Turiyam Connection Values

Edge	(t,i,f,l)	Description
A-B	(0.8,0.10,0.05,0.05)	Strong Admin-Tech coordination
A-C	(0.7,0.15,0.10,0.05)	Moderate link via events
A-D	(0.9,0.05,0.03,0.02)	Very strong link
B-E	(0.6,0.25,0.10,0.05)	Medium student engagement
C-E	(0.5,0.30,0.15,0.05)	Weak indirect link
D-E	(0.7,0.20,0.05,0.05)	Strong cultural connection

Step 1: Domination ($\theta = 0.5$)

A node u dominates node v if $t_{uv} \geq 0.5$ and $f_{uv} \leq 0.1$. Applying this rule:

A dominates $\{B, C, D\}$,

B dominates $\{A, E\}$,

D dominates $\{A, E\}$,

E is dominated by $\{B, D\}$,

C is only dominated by A .

Hence, the minimal dominating set is $D = \{A, D\}$, covering all nodes.

Step 2: Check Equitability

Dominator A influences $\{B, C, D\}$ (3 nodes) and dominator D influences $\{A, E\}$ (2 nodes). The difference in coverage is $3 - 2 = 1$, satisfying the equitable domination condition.

Thus, in this community network, the Admin (A) and Cultural Leader (D) form an *equitable dominating pair*. Each covers complementary subgroups (technical, sports, and cultural clusters), and their influence load differs by only one node. This ensures fair and sustainable community leadership.

Benefits of Equitable Domination:

- Prevents a single leader (Admin) from being overloaded with communication tasks.
- Ensures each subgroup (sports, tech, culture) is represented fairly.
- Promotes balanced influence, participation, and diversity.

Advantages of Turiyam Graph Modeling:

- Captures uncertain, fuzzy, and contradictory social ties.
- Latent factors (L) represent potential connections not yet active.
- Equitable domination guarantees fairness in information diffusion under uncertainty.

Mathematical Intuition: The domination strength from A to B is computed as:

$$D_T(A, B) = t_A \cdot t_{AB} \cdot (1 - f_A) \cdot (1 - l_A).$$

Equitable domination optimizes the set D such that:

$$\max_{u \in D} \sum_v D_T(u, v) - \min_{u \in D} \sum_v D_T(u, v) \leq 1,$$

ensuring balanced influence across all dominators.

Table 8.4: Equitable Domination Impact in Real-World Applications

Context	Description	Equitable Domination Impact
University Platforms	Student group leaders managing posts and forums	Avoids single-user burnout and ensures equal content reach
Online Learning Communities	Mentors guiding study groups	Promotes fair mentor workload distribution
NGO Volunteer Networks	Coordinators leading campaigns	Ensures equal engagement levels among volunteers
Corporate Teams (Slack/Teams)	Managers or moderators coordinating communication	Provides fair communication load and reduces decision bias

From Figure 8.4 and Tables 8.3–8.4, the Turiyam graph model successfully captures truth, uncertainty, falsity, and latent potential. Its equitable domination approach ensures balanced social influence, reduces overload, promotes fairness, and improves engagement. The network demonstrates that student leaders can form a minimal yet equitable set of influencers, resulting in sustainable and inclusive community communication.

Conclusion

The applications presented above illustrate that domination in Turiyam graphs offers a robust and realistic framework for modeling and analyzing complex systems characterized by uncertainty, partial knowledge, and heterogeneous influence strengths. By integrating the four-dimensional membership structure (T, I, F, L) , Turiyam domination captures not only the strength of relationships but also the inherent indeterminacy, contradiction, and latent potential embedded within real-world networks.

In practical domains such as social networks, communication systems, and healthcare infrastructures, Turiyam domination effectively identifies minimal yet influential sets of critical nodes that ensure complete coverage and structural control of the system. Moreover, equitable domination further enhances this framework by promoting a balanced distribution of dominance responsibility, thereby preventing overloading of specific nodes and ensuring fair participation across the network.

From a mathematical perspective, this balance is achieved by optimizing the domination strength $D_T(u, v)$ while maintaining nearly uniform neighborhood coverage among dominating vertices. Consequently, Turiyam domination simultaneously ensures efficiency, robustness, and fairness, making it a powerful and versatile tool for decision-making, resource allocation, and influence optimization in complex uncertain network environments.

CHAPTER 9

Conclusion and Recommendations

9.1 Conclusion

This thesis has investigated domination in Turiyam graphs and demonstrated their effectiveness in modeling and analyzing complex systems characterized by uncertainty, partial information, and heterogeneous influence structures. Unlike classical graph models and even neutrosophic extensions, Turiyam graphs incorporate a four-dimensional membership structure (T, I, F, L) representing truth, indeterminacy, falsity, and latent potential. This enriched representation enables a more realistic description of relationships in real-world networks where interactions are often imprecise, inconsistent, or only partially observable.

Several domination concepts were developed and examined within the Turiyam framework. Paired domination ensures cooperative operation of dominating vertices, making it suitable for systems requiring redundancy and collaboration such as emergency response and coordinated leadership structures. Total domination guarantees that every vertex is directly influenced by at least one dominating vertex, thereby strengthening connectivity and system-wide control. Split domination introduces structural decomposition by dividing the network into dominated and residual subgraphs with meaningful connectivity properties. Equitable domination further refines the theory by ensuring a fair and balanced distribution of influence among dominating vertices.

In addition to theoretical contributions, algorithmic procedures were developed for computing domination strength, evaluating coverage conditions, constructing domination matrices, and verifying equitable constraints. These computational methods provide a systematic framework for identifying minimal dominating sets while preserving efficiency, stability, and balance. In this way, the theoretical structure of Turiyam domination is effectively connected with practical computational implementation.

The applicability of the proposed framework was demonstrated through real-world network scenarios. In healthcare systems, Turiyam total domination successfully identifies minimal critical facilities that ensure full service coverage and efficient resource utilization under uncertainty. In social networks, equitable domination provides a mechanism for balanced influence distribution, reducing overload and promoting fair participation. Paired and split domination further highlight how cooperation and structural organization enhance system resilience and operational stability.

Overall, the results confirm that domination in Turiyam graphs simultaneously achieves efficiency, robustness, and fairness. By integrating uncertainty-aware modeling with structured domination principles, Turiyam graph theory provides a powerful framework for analyzing and optimizing complex networked systems.

9.2 Recommendations

Based on the findings of this study, the following recommendations are proposed for future research and practical applications.

First, it is recommended that domination concepts in Turiyam graphs be extended to dynamic and time-varying networks. Many real-world systems, such as communication networks, social interactions, and epidemiological processes, evolve over time. Incorporating temporal aspects into Turiyam parameters would enable adaptive domination strategies that respond to changing network conditions.

Second, the development of efficient algorithms and computational tools is essential for large-scale applications. Although the proposed methods provide a strong theoretical foundation, real-world networks often involve high complexity and large datasets. The integration of machine learning techniques and metaheuristic optimization methods could significantly improve scalability and computational efficiency.

Third, further interdisciplinary applications should be explored. Turiyam domination can be applied in transportation systems, energy distribution networks, disaster management, financial systems, and intelligent sensor networks. In particular, paired and equitable domination models may support cooperative decision-making and fair resource allocation in distributed environments.

Fourth, empirical validation using real datasets is strongly recommended. Applying the proposed models to actual healthcare networks, social media platforms, or communication systems would enhance the practical credibility and demonstrate the predictive strength of Turiyam domination theory.

Finally, the development of dedicated software tools and simulation platforms for Turiyam graph analysis is encouraged. Such tools would facilitate broader adoption of the theory by researchers, engineers, and policymakers, enabling practical implementation in real-world decision-support systems.

Bibliography

- Akram, M. and Alshehri, N. O. (2014). Intuitionistic fuzzy cycles and intuitionistic fuzzy trees. *The Scientific World Journal*, 2014(1):305836.
- Akram, M., Malik, H. M., Shahzadi, S., and Smarandache, F. (2018). Neutrosophic soft rough graphs with application. *Axioms*, 7(1):14.
- Ali, M., Smarandache, F., and Riaz, M. (2019). Domination in neutrosophic graphs. *Neutrosophic Sets and Systems*, 26:52–66.
- Ani, R., Singh, P. K., and Roy, S. (2023). Moore–penrose inverse of non-square real symbolic turiyam matrices. *Neutrosophic Sets and Systems*, 54:120–135.
- Ashbacher, C. (2014). *Introduction to Neutrosophic logic*. Infinite Study.
- Atanassov, K. (1986). Intuitionistic fuzzy sets. *fuzzy sets and systems* 20 (1), 87-96. DOI: [https://doi.org/10.1016/S0165-0114\(86\)80034-3](https://doi.org/10.1016/S0165-0114(86)80034-3).
- Bal, M., Singh, P. K., and Roy, S. (2022). Turiyam vector spaces: An extension of neutrosophic vector spaces. *Neutrosophic Sets and Systems*, 48:145–160.
- Basher, S., Singh, P. K., and Roy, S. (2022). Symbolic turiyam groups and their properties. *Neutrosophic Sets and Systems*, 47:98–112.
- Belohlavek, R. and Klir, G. J. (2011). Fuzzy logic: A tutorial. *Concepts and fuzzy logic*, pages 45–87.
- Berge, C. (1962). *The Theory of Graphs and Its Applications*. Methuen, London.
- Bhattacharya, A. and Pal, M. (2023). A fuzzy graph theory approach to the facility location problem: A case study in the indian banking system. *Mathematics*, 11(13):2992.

- Bhattacharya, P. (1987). Fuzzy subgroups: some characterizations. *Journal of mathematical analysis and applications*, 128(1):241–252.
- Bhutani, K. R. (1989). On automorphisms of fuzzy graphs. *Pattern recognition letters*, 9(3):159–162.
- Bhutani, K. R. and Battou, A. (2003). On m-strong fuzzy graphs. *Information Sciences*, 155(1-2):103–109.
- Bollobás, B. and Cockayne, E. J. (1979). Graph-theoretic parameters concerning domination, independence, and irredundance. *Journal of Graph Theory*, 3(3):241–249.
- Bondy, J. A. and Murty, U. (2008). Graduate texts in mathematics.
- Bozhenyuk, A., Belyakov, S., Knyazeva, M., and Rozenberg, I. (2021). On computing domination set in intuitionistic fuzzy graph. *International Journal of Computational Intelligence Systems*, 14(1):617–624.
- Brigham, R. C., Chinn, P. Z., and Dutton, R. D. (1988). Vertex domination-critical graphs. *Networks*, 18(3):173–179.
- Brigham, R. C., Dutton, R. D., Harary, F., and Haynes, T. W. (1996). On graphs having equal domination and codomination numbers. *Utilitas Mathematica*, 50:53–64.
- Broumi, S., Bakali, A., and Bahnasse, A. (2018a). Neutrosophic sets: An overview. *Infinite Study*, 410(1).
- Broumi, S., Smarandache, F., Bakali, A., and Talea, M. (2017). *An Introduction to Neutrosophic Graph Theory*. Infinite Study.
- Broumi, S., Smarandache, F., Talea, M., and Bakali, A. (2018b). Neutrosophic domination in neutrosophic graphs. *Neutrosophic Sets and Systems*, 21:66–79.
- Broumi, S., Talea, M., Bakali, A., Smarandache, F., and Patro, S. K. (2018c). Minimum spanning tree problem with single-valued trapezoidal neutrosophic numbers. In *Science and Information Conference*, pages 93–105. Springer.
- Çitil, M. (2019). Application of the intuitionistic fuzzy logic in education. *Communications in Mathematics and Applications*, 10(1):131–143.

- Cockayne, E. J., Dawes, R., and Hedetniemi, S. T. (1980). Total domination in graphs. *Networks*, 10(3):211–219.
- Deli, I. and Subas, Y. (2014). Single valued neutrosophic numbers and their applications to multicriteria decision making problem. *Neutrosophic Sets Syst*, 2(1):1–13.
- Dhavaseelan, R., Vikramaprasad, R., and Broumi, S. (2015). Strong neutrosophic graphs. *Neutrosophic Sets and Systems*, 10:1–7.
- Enriquez, F., Martínez, J., and López, R. (2021). Fuzzy domination in directed graphs. *Fuzzy Sets and Systems*, 410:45–60.
- Ganati, G., Repalle, V. R., Ashebo, M., and Amini, M. (2023a). Turiyam graphs and its applications. *Information Sciences Letters*, 12(6):2423–2434.
- Ganati, G. A., Srinivasa Rao Repalle, V., and Ashebo, M. A. (2023b). Relations in the context of turiyam sets. *BMC research notes*, 16(1):49.
- Gani, A. N. and Ahmed, M. A. (2003). Order and size in fuzzy graphs. *Bulletin of Pure and Applied Sciences*, 22E(2):145–148.
- Garrett, H. (2023). *Neutrosophic Dominating*. Dr. Henry Garrett.
- Ghosh, J., Ngo, H. Q., Yoon, S., and Qiao, C. (2007). On a routing problem within probabilistic graphs and its application to intermittently connected networks. In *IEEE INFOCOM 2007-26th IEEE International Conference on Computer Communications*, pages 1721–1729. IEEE.
- Glass, L. and Kauffman, S. A. (1973). The logical analysis of continuous, non-linear biochemical control networks. *Journal of theoretical Biology*, 39(1):103–129.
- Gross, J. and Yellen, J. (2005). Graph theory and its applications second edition. *Boca Raton, FL*, 33431.
- Guan, H., Khan, W. A., Fida, A., Ali, K., Shafi, J., and Khan, A. (2024a). Dominations in intuitionistic fuzzy directed graphs with applications towards influential graphs. *Mathematics*, 12(6):872.

- Guan, X., Li, Y., Zhang, H., and Chen, J. (2024b). Domination in intuitionistic fuzzy graphs. *Fuzzy Sets and Systems*, 497:1–18.
- Hameed, S., Ahmad, U., and Batool, Z. (2023). Total perfect and total efficient domination in intuitionistic fuzzy graphs. *International Journal of Applied and Computational Mathematics*, 9(5):60.
- Harary, F. (1962). The maximum connectivity of a graph. *Proceedings of the National Academy of Sciences*, 48(7):1142–1146.
- Haynes, T. W., Hedetniemi, S., and Slater, P. (2013). *Fundamentals of domination in graphs*. CRC press.
- Haynes, T. W. and Slater, P. J. (1998). Paired-domination in graphs. *Networks: An International Journal*, 32(3):199–206.
- Hellmann, M. (2001). Fuzzy logic introduction. *Université de Rennes*, 1(1).
- Jan, N., Akram, M., and Smarandache, F. (2019). Constant single valued neutrosophic graphs. *Neutrosophic Sets and Systems*, 27:34–52.
- Kandasamy, W. B. V., Smarandache, F., and Ali, M. (2015). Neutrosophic graphs. *Neutrosophic Sets and Systems*, 8:13–25.
- Kandasamy, W. V., Ilanthenral, K., and Smarandache, F. (2016). *Strong neutrosophic graphs and subgraph topological subspaces*. Infinite Study.
- Karunambigai, M., Akram, M., Sivasankar, S., and Palanivel, K. (2013). Balanced intuitionistic fuzzy graphs. *Applied mathematical sciences*, 7(51):2501–2514.
- Karunambigai, M., Sivasankar, S., and Palanivel, K. (2017). Secure domination in fuzzy graphs and intuitionistic fuzzy graphs. *Annals of fuzzy mathematics and informatics*, 14(4):419–43.
- Kollios, G., Potamias, M., and Terzi, E. (2011). Clustering large probabilistic graphs. *IEEE Transactions on Knowledge and Data Engineering*, 25(2):325–336.
- Kosko, B. and Isaka, S. (1993). Fuzzy logic. *Scientific American*, 269(1):76–81.

- Malarvizhi, J. and Divya, G. (2021). *Domination and edge domination in single valued neutrosophic graph*. Infinite Study.
- Maniu, S., Cheng, R., and Senellart, P. (2017). An indexing framework for queries on probabilistic graphs. *ACM Transactions on Database Systems (TODS)*, 42(2):1–34.
- Martin, N., Priya, R., Singh, P. K., and Gandhi, R. (2025). Turiyam cognitive maps in optimal industrial decision making. *Big Data and Computing Visions*, 5(4):270–286.
- Mordeson, J. N. and Chang-Shyh, P. (1994). Operations on fuzzy graphs. *Information sciences*, 79(3-4):159–170.
- Mordeson, J. N. and Mathew, S. (2019). *Advanced topics in fuzzy graph theory*, volume 375. Springer.
- Mordeson, J. N., Mathew, S., and Gayathri, G. (2023). *Fuzzy Graph Theory: Applications to Global Problems*, volume 424. Springer Nature.
- Mordeson, J. N., Mathew, S., and Malik, D. S. (2018). *Fuzzy graph theory with applications to human trafficking*, volume 365. Springer.
- Mordeson, J. N. and Nair, P. S. (2000). Applications of fuzzy graphs. In *Fuzzy graphs and fuzzy hypergraphs*, pages 83–133. Springer.
- Mubayi, D., Will, T. G., and West, D. B. (2001). Realizing degree imbalances in directed graphs. *Discrete Mathematics*, 239(1-3):147–153.
- Mullai, M. and Broumi, S. (2020). Dominating energy in neutrosophic graphs. *Int J Neutrosoph Sci*, 5(1):38–58.
- Nagoorgani, A., Akram, M., and Anupriya, S. (2016). Double domination on intuitionistic fuzzy graphs. *Journal of applied mathematics and computing*, 52(1):515–528.
- Nazir, N., Gul, S., Khan, M. A., and Ali, R. (2023). An improved algorithm for identifying dominating node sets in intuitionistic fuzzy graphs. *Journal of Intelligent & Fuzzy Systems*, 45(7):9401–9415.
- Newman, E. (2010). Critical human security studies. *review of international Studies*, 36(1):77–94.

- Ore, y. (1962). *Theory of Graphs*. American Mathematical Society, Providence, RI.
- Pal, M., Samanta, S., and Ghorai, G. (2020a). Advanced concepts in fuzzy graphs: Fuzzy planar graphs, fuzzy competition graphs, coloring, and m-polar fuzzy graphs. *Journal of Fuzzy Mathematics*, 28(4):891–912.
- Pal, M., Samanta, S., and Ghorai, G. (2020b). *Modern trends in fuzzy graph theory*. Springer.
- Parvathi, R. and Karunambigai, M. (2006). Intuitionistic fuzzy graphs. In *Computational intelligence, theory and applications: international conference 9th fuzzy days in Dortmund, Germany, Sept. 18–20, 2006 Proceedings*, pages 139–150. Springer.
- Parvathi, R. and Thamizhendhi, G. (2010). Domination in intuitionistic fuzzy graphs. *Notes on Intuitionistic Fuzzy Sets*, 16(2):39–49.
- Pramanik, S. (2022). Single-valued neutrosophic set: An overview. *Transdisciplinarity*, pages 563–608.
- Raji, M., Tharaniya, P., Pramanik, S., and Prabhu, G. A. (2025). A novel approach for neutrosophic complete graph and neutrosophic excellent domination. *Neutrosophic Sets and Systems*, 90:31–47.
- Rashmanlou, H., Samanta, S., Pal, M., and Borzooei, R. A. (2015). Intuitionistic fuzzy graphs with categorical properties. *Fuzzy information and Engineering*, 7(3):317–334.
- Rivieccio, U. (2008). Neutrosophic logics: Prospects and problems. *Fuzzy sets and systems*, 159(14):1860–1868.
- Rosenfeld, A. (1975a). Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier.
- Rosenfeld, A. (1975b). Fuzzy graphs. In Zadeh, L. A., Fu, K.-S., Tanaka, K., and Shimura, M., editors, *Fuzzy Sets and Their Applications*, pages 77–95. Academic Press.
- Russell, B. and Whitehead, A. N. (1910). *Principia Mathematica*, volume 1. Cambridge University Press.
- Sadati, S. H., Rashmanlou, H., and Talebi, A. A. (2021). Domination in intuitionistic fuzzy incidence graph. In *2021 52nd Annual Iranian Mathematics Conference (AIMC)*, pages 88–94. IEEE.

- Sahoo, S., Pal, M., Rashmanlou, H., and Borzooei, R. A. (2017). Covering and paired domination in intuitionistic fuzzy graphs. *Journal of Intelligent & Fuzzy Systems*, 33(6):4007–4015.
- Shao, Z., Kosari, S., Rashmanlou, H., and Shoaib, M. (2020). New concepts in intuitionistic fuzzy graph with application in water supplier systems. *Mathematics*, 8(8):1241.
- Shapiro, S. and Kouri Kissel, T. (2000). Classical logic.
- Shi, X., Kosari, S., Talebi, A. A., Sadati, S. H., and Rashmanlou, H. (2022). Investigation of the main energies of picture fuzzy graph and its applications. *International Journal of Computational Intelligence Systems*, 15(1):31.
- Singh, P. K. (2021). Turiyam set: A four-dimensional extension of neutrosophic sets. *Neutrosophic Sets and Systems*, 44:1–14.
- Singh, P. K. (2022). Extending neutrosophic reasoning into four dimensions with the turiyam set. *Neutrosophic Sets and Systems*, 49:1–15.
- Singh, P. K. (2023a). Turiyam geometry: Characterization of euclidean, non-euclidean, hybrid, neutrogeometry, and turiyam geometry. *Neutrosophic Sets and Systems*, 55:1–18.
- Singh, P. K. (2023b). Turiyam set: A four-dimensional data representation inspired by human quantum cognition. *Neutrosophic Sets and Systems*, 53:1–16.
- Smarandache, F. (1998). *Neutrosophic Set, a Generalization of the Intuitionistic Fuzzy Set*. American Research Press, Rehoboth.
- Smarandache, F. (2022). Fundamental operators for single valued neutrosophic oversets. *Neutrosophic Sets and Systems*, 50:1–12.
- Somasundaram, D. (1998). *Scarred minds*. New Delhi: Sage Publications.
- Sunitha, M. and Mathew, S. (2013). Fuzzy graph theory: a survey. *Annals of Pure and Applied mathematics*, 4(1):92–110.
- Takeuti, G. and Titani, S. (1984). Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *The journal of symbolic logic*, 49(3):851–866.

- Torra, V. (2010). Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539.
- Trillas, E. and Eciolaza, L. (2015). Fuzzy logic. *Springer International Publishing*. DOI, 10:978–3.
- Wang, H., Smarandache, F., Zhang, Y., and Sunderraman, R. (2010). *Single valued neutrosophic sets*. Infinite study.
- Yaqoob, N., Gulistan, M., Kadry, S., and Wahab, H. A. (2019). Complex intuitionistic fuzzy graphs with application in cellular network provider companies. *Mathematics*, 7(1):35.
- Yen, J. (1999). Fuzzy logic-a modern perspective. *IEEE transactions on knowledge and data engineering*, 11(1):153–165.
- Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, 8(3):338–353.