



**WOLLEGA UNIVERSITY  
SCHOOL OF GRADUATE STUDIES**

**Boundary Layer Flow of Nanofluids past Some  
Geometrical Shapes Embedded in Porous Media**

**PhD Dissertation**

**Ayele Tulu Yadete**

**April 21, 2021  
Nekemte, Ethiopia**



**WOLLEGA UNIVERSITY  
SCHOOL OF GRADUATE STUDIES**

**Boundary Layer Flow of Nanofluids past Some  
Geometrical Shapes Embedded in Porous Media**

**A Dissertation Submitted to School of Graduate Studies in  
Partial Fulfillment of the Requirements for the Degree of  
Doctor of Philosophy in Fluid Dynamics**

**Ayele Tulu Yadete**

**Advisor: Wubshet Ibrahim (PhD, Assoc. Professor)**

**April 21, 2021  
Nekemte, Ethiopia**



**WOLLEGAUNIVERSITY**  
**SCHOOL OF GRADUATE STUDIES**  
**P.O. Box: 395, Nekemte, Ethiopia.**

---

## APPROVAL SHEET FOR SUBMITTING FINAL DISSERTATION

As members of the Board of Examining of the Final PhD dissertation open defense, we certify that we have read and evaluated the thesis prepared by Ayele Tulu under the title "Boundary Layer Flow of Nanofluids past Some Geometrical Shape Embedded in Porous Media" in Fluid Dynamics.

|                   |           |       |
|-------------------|-----------|-------|
| _____             | _____     | _____ |
| Chairperson       | Signature | Date  |
| _____             | _____     | _____ |
| Internal Examiner | Signature | Date  |
| _____             | _____     | _____ |
| External Examiner | Signature | Date  |

### **Final Approval and Acceptance** **Dissertation Approved By:**

|                 |           |       |
|-----------------|-----------|-------|
| _____           | _____     | _____ |
| Department PGC  | Signature | Date  |
| _____           | _____     | _____ |
| Dean of College | Signature | Date  |

### **Certification of the Final Dissertation**

I hereby certify that all the correction and recommendation suggested by the board of examiners are incorporated into the final dissertation entitled "Boundary Layer Flow of Nanofluids past Some Geometrical Shape Embedded in Porous Media" by Ayele Tulu

|             |           |       |
|-------------|-----------|-------|
| _____       | _____     | _____ |
| Dean of SGS | Signature | Date  |

# Table of Contents

|                                                  |          |
|--------------------------------------------------|----------|
| Table of Contents                                | iii      |
| List of Tables                                   | vii      |
| List of Figures                                  | ix       |
| Acronyms                                         | xiv      |
| Nomenclatures                                    | xv       |
| Greek symbols                                    | xvii     |
| Subscripts                                       | xviii    |
| Statement of the Author                          | xix      |
| Acknowledgement                                  | xx       |
| Biographical Sketch of the Researcher            | xxi      |
| Abstract                                         | xxii     |
| <b>1 General Introduction</b>                    | <b>1</b> |
| 1.1 Background of the Study . . . . .            | 1        |
| 1.2 Statement of the Problem . . . . .           | 9        |
| 1.3 Objective of the Study . . . . .             | 10       |
| 1.3.1 General Objective of the Study . . . . .   | 10       |
| 1.3.2 Specific Objectives of the Study . . . . . | 10       |

|          |                                                   |           |
|----------|---------------------------------------------------|-----------|
| 1.4      | Scope of the Study . . . . .                      | 11        |
| 1.5      | Significance of the Study . . . . .               | 11        |
| 1.6      | Dissertation Outline . . . . .                    | 12        |
| <b>2</b> | <b>Literature Review</b>                          | <b>15</b> |
| 2.1      | Fluid Dynamics . . . . .                          | 15        |
| 2.1.1    | Description of Fluid . . . . .                    | 16        |
| 2.1.2    | Fluid Transport . . . . .                         | 16        |
| 2.1.3    | Approaches to Study Fluid Dynamics . . . . .      | 17        |
| 2.2      | Mode of Heat and Mass Transfer . . . . .          | 18        |
| 2.2.1    | Conduction/Mass Diffusion . . . . .               | 19        |
| 2.2.2    | Convection . . . . .                              | 19        |
| 2.2.3    | Radiation . . . . .                               | 20        |
| 2.3      | Boundary Layers Flow . . . . .                    | 21        |
| 2.3.1    | Velocity Boundary Layer . . . . .                 | 22        |
| 2.3.2    | Thermal Boundary Layer . . . . .                  | 23        |
| 2.3.3    | Concentration Boundary Layer . . . . .            | 24        |
| 2.4      | Governing Equations of Fluid Dynamics . . . . .   | 24        |
| 2.4.1    | Continuity Equation . . . . .                     | 25        |
| 2.4.2    | Momentum Equations . . . . .                      | 26        |
| 2.4.3    | Internal Energy Equation . . . . .                | 28        |
| 2.4.4    | Initial and Boundary Conditions . . . . .         | 30        |
| 2.5      | Nanofluids . . . . .                              | 31        |
| 2.5.1    | Effective Thermophysical Properties of Nanofluids | 33        |
| 2.5.2    | Hybrid Nanofluids . . . . .                       | 35        |
| 2.5.3    | Carbon Nanotubes . . . . .                        | 37        |
| 2.6      | Non-Newtonian Fluids . . . . .                    | 38        |
| 2.7      | Some Properties Affecting Fluid Flow . . . . .    | 39        |
| 2.7.1    | Magnetohydrodynamics (MHD) . . . . .              | 39        |
| 2.7.2    | Porous Media . . . . .                            | 40        |
| 2.7.3    | Viscous Dissipation . . . . .                     | 41        |
| 2.7.4    | Cattaneo-Christov Heat Flux Model . . . . .       | 42        |
| 2.7.5    | Slip Boundary Conditions . . . . .                | 44        |
| <b>3</b> | <b>Methodology of the Study</b>                   | <b>45</b> |
| 3.1      | Approaches Used in the Study . . . . .            | 45        |
| 3.2      | Spectral Collocation Method . . . . .             | 47        |
| 3.2.1    | Collocation Points . . . . .                      | 49        |

|          |                                                                                                                                    |            |
|----------|------------------------------------------------------------------------------------------------------------------------------------|------------|
| 3.2.2    | Differentiation Matrix . . . . .                                                                                                   | 50         |
| 3.2.3    | Spectral Quasilinearization Method (SQLM) . . .                                                                                    | 52         |
| 3.2.4    | Spectral Relaxation Method (SRM) . . . . .                                                                                         | 56         |
| <b>4</b> | <b>MHD Boundary Layer Flow of Nanofluid Past a Wedge in a Porous Media with Viscous Effects</b>                                    | <b>60</b>  |
| 4.1      | Introduction . . . . .                                                                                                             | 60         |
| 4.2      | Mathematical Formulation . . . . .                                                                                                 | 63         |
| 4.3      | Numerical Method . . . . .                                                                                                         | 68         |
| 4.4      | Results and Discussion . . . . .                                                                                                   | 71         |
| 4.5      | Conclusion . . . . .                                                                                                               | 77         |
| <b>5</b> | <b>MHD Slip Flow of a Casson Nanofluid over a Stretching Cylinder Embedded in a Porous Medium with Cattaneo-Christov Heat Flux</b> | <b>82</b>  |
| 5.1      | Introduction . . . . .                                                                                                             | 82         |
| 5.2      | Mathematical Formulation . . . . .                                                                                                 | 85         |
| 5.3      | Method of Solutions . . . . .                                                                                                      | 90         |
| 5.4      | Results and Discussion . . . . .                                                                                                   | 94         |
| 5.5      | Conclusion . . . . .                                                                                                               | 102        |
| <b>6</b> | <b>Unsteady Mixed Convection flow of Nanofluid Past a Moving Wedge Embedded in a Porous Media</b>                                  | <b>107</b> |
| 6.1      | Introduction . . . . .                                                                                                             | 107        |
| 6.2      | Mathematical Formulation . . . . .                                                                                                 | 111        |
| 6.3      | Numerical Method . . . . .                                                                                                         | 117        |
| 6.4      | Results and Discussion . . . . .                                                                                                   | 120        |
| 6.5      | Conclusion . . . . .                                                                                                               | 129        |
| <b>7</b> | <b>MHD Slip Flow of CNTs-Ethylene glycol Nanofluid due to a Stretchable Rotating Disk with Cattaneo-Christov Heat Flux Model</b>   | <b>131</b> |
| 7.1      | Introduction . . . . .                                                                                                             | 131        |
| 7.2      | Mathematical Description of Problem . . . . .                                                                                      | 134        |
| 7.3      | Numerical Method of Solutions . . . . .                                                                                            | 140        |
| 7.4      | Results and Discussion . . . . .                                                                                                   | 141        |
| 7.5      | Conclusion . . . . .                                                                                                               | 148        |

|           |                                                                                                                                                       |            |
|-----------|-------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| <b>8</b>  | <b>Natural Convection Flow of CNTs-Magnetite/Water Hybrid Nanofluids Due to Stretching Surface with Second Order Slip Flow and Variable Viscosity</b> | <b>153</b> |
| 8.1       | Introduction . . . . .                                                                                                                                | 153        |
| 8.2       | Mathematical Description of the Problem . . . . .                                                                                                     | 157        |
| 8.3       | Method of Solutions . . . . .                                                                                                                         | 163        |
| 8.4       | Results and Discussion . . . . .                                                                                                                      | 165        |
| 8.5       | Conclusion . . . . .                                                                                                                                  | 171        |
| <b>9</b>  | <b>Mixed Convection Flow of Hybrid Nanofluids due to a Spinning Cone with Variable Viscosity and Thermal Conductivity</b>                             | <b>177</b> |
| 9.1       | Introduction . . . . .                                                                                                                                | 177        |
| 9.2       | Mathematical Description of the Problem . . . . .                                                                                                     | 182        |
| 9.3       | Method of Solutions . . . . .                                                                                                                         | 188        |
| 9.4       | Results and Discussion . . . . .                                                                                                                      | 190        |
| 9.5       | Conclusion . . . . .                                                                                                                                  | 198        |
| <b>10</b> | <b>Conclusion and Recommendation</b>                                                                                                                  | <b>203</b> |
| 10.1      | General Conclusion . . . . .                                                                                                                          | 203        |
| 10.2      | Recommendation for the Future Work . . . . .                                                                                                          | 204        |
|           | <b>Bibliography</b>                                                                                                                                   | <b>206</b> |
|           | <b>Appendix</b>                                                                                                                                       | <b>222</b> |

# List of Tables

|     |                                                                                                           |     |
|-----|-----------------------------------------------------------------------------------------------------------|-----|
| 4.1 | Comparison of the SQLM results of $-f''(0)$ and $-\theta'(0)$<br>for some $m$ . . . . .                   | 72  |
| 4.2 | Computations of $-f''(0)$ , $-\theta'(0)$ and $-\phi'(0)$ for various<br>parameters . . . . .             | 81  |
| 5.1 | SRM Solutions Convergence . . . . .                                                                       | 94  |
| 5.2 | Comparison of $Nu_x$ and $Sh_x$ for $N_b$ and $N_t$ . . . . .                                             | 95  |
| 5.3 | Values of $-f''(0)$ , $-\theta'(0)$ and $-\phi'(0)$ for some parameters                                   | 105 |
| 5.4 | Values of $-\theta'(0)$ and $-\phi'(0)$ for some parameters . . . .                                       | 106 |
| 6.1 | Thermo-physical properties of base fluid and nanoparticles                                                | 115 |
| 6.2 | Comparison of the SQLM results of $-\theta'(0)$ when $\phi_s = 0$<br>for some $Pr$ . . . . .              | 121 |
| 6.3 | Values of $-f''(0)$ , $-\theta'(0)$ and $-\phi'(0)$ for $A$ and $\phi_s$ . . . .                          | 129 |
| 6.4 | Values of $-f''(0)$ , $-\theta'(0)$ and $-\phi'(0)$ for $\lambda, Bi_1, Bi_2$ . . .                       | 130 |
| 7.1 | Thermo-physical properties of SWCNTs, MWCNTs and<br>ethylene glycol at temperature $20^\circ C$ . . . . . | 137 |
| 7.2 | SQLM Convergence for $SWCNTs - C_2H_6O_2$ nanofluid.                                                      | 142 |

|     |                                                                                                                                 |     |
|-----|---------------------------------------------------------------------------------------------------------------------------------|-----|
| 7.3 | SQLM Convergence for $MWCNTs - C_2H_6O_2$ nanofluid                                                                             | 143 |
| 7.4 | Values of $\sqrt{f''(0)^2 + h'(0)^2}$ and $-\theta'(0)$ for $SWCNTs - C_2H_6O_2$ and $MWCNTs - C_2H_6O_2$ nanofluid. . . . .    | 144 |
| 8.1 | Thermo-physical properties of base fluid, CNTs, and $Fe_3O_4$                                                                   | 160 |
| 8.2 | Thermo-physical Model of Mono Nanofluid . . . . .                                                                               | 161 |
| 8.3 | Thermo-physical Model of Hybrid Nanofluid . . . . .                                                                             | 162 |
| 8.4 | SQLM Convergence for $f''(0)$ and $-\theta'(0)$ of $CNTs - Fe_3O_4$ with $H_2O$ hybrid nanofluid . . . . .                      | 165 |
| 8.5 | Comparison of the present result of $f''(0)$ with $\delta_{v_1}$ . . . . .                                                      | 166 |
| 8.6 | Comparison of the present result of $-\theta'(0)$ with $Pr$ . . . . .                                                           | 166 |
| 8.7 | Thermal conductivity ( $k_{nf}$ ) of $CNTs - H_2O$ nanofluid and $k_{hnf}$ of $(CNTs - Fe_3O_4)/H_2O$ hybrid nanofluid. . . . . | 167 |
| 8.8 | Values of $f''(0)$ and $-\theta'(0)$ of $(MWCNTs - Fe_3O_4)/H_2O$ hybrid nanofluid with $\phi_{s_1} = 0.01, A = 0$ . . . . .    | 176 |
| 9.1 | Thermophysical properties of Engine oil, MWCNTs, and $Al_2O_3$ . . . . .                                                        | 185 |
| 9.2 | Thermophysical Properties Model of Nanofluid . . . . .                                                                          | 186 |
| 9.3 | Thermophysical Properties Model of Hybrid Nanofluid . . . . .                                                                   | 186 |
| 9.4 | SRM Convergence analysis. . . . .                                                                                               | 191 |
| 9.5 | Comparison of skin friction coefficients and local Nusselt number with previous work for $Pr$ and $\lambda$ . . . . .           | 192 |

# List of Figures

|     |                                                                                                                   |    |
|-----|-------------------------------------------------------------------------------------------------------------------|----|
| 2.1 | The development of the velocity boundary layer for flow over a flat plate and the different flow regimes. . . . . | 22 |
| 2.2 | Thermal boundary layer development on an isothermal flat plate. . . . .                                           | 23 |
| 2.3 | Stress components on three faces of fluid element. . . . .                                                        | 27 |
| 2.4 | Carbon nanotubes structures. . . . .                                                                              | 37 |
| 3.1 | Chebyshev collocation points. . . . .                                                                             | 50 |
| 3.2 | Chebyshev differentiation matrix D. . . . .                                                                       | 52 |
| 4.1 | Physical model of flow configuration. . . . .                                                                     | 64 |
| 4.2 | Velocity profiles for various values of $\beta$ . . . . .                                                         | 73 |
| 4.3 | Velocity profiles for various values of $\kappa$ . . . . .                                                        | 74 |
| 4.4 | Velocity profiles for various values of $M$ . . . . .                                                             | 75 |
| 4.5 | Temperature profiles for various values of $M$ . . . . .                                                          | 75 |
| 4.6 | Concentration profiles for various values of $M$ . . . . .                                                        | 76 |
| 4.7 | Concentration profiles for various values of $Le$ . . . . .                                                       | 76 |
| 4.8 | Temperature profiles for various values of $Pr$ . . . . .                                                         | 77 |
| 4.9 | Concentration profiles for various values of $Pr$ . . . . .                                                       | 78 |

|      |                                                                   |     |
|------|-------------------------------------------------------------------|-----|
| 4.10 | Temperature profiles for various values of $Ec$ . . . . .         | 78  |
| 4.11 | Concentration profiles for various values of $Ec$ . . . . .       | 79  |
| 4.12 | Temperature profiles for various values of $Nb$ . . . . .         | 79  |
| 4.13 | Concentration profiles for various values of $Nb$ . . . . .       | 79  |
| 4.14 | Temperature profiles for various values of $Nt$ . . . . .         | 80  |
| 4.15 | Concentration profiles for various values of $Nt$ . . . . .       | 80  |
| 5.1  | Physical model and coordinate system . . . . .                    | 86  |
| 5.2  | Velocity profiles for various values of $\gamma$ . . . . .        | 96  |
| 5.3  | Velocity profiles for various values of $Ha$ . . . . .            | 96  |
| 5.4  | Velocity profiles for various values of $\delta_v$ . . . . .      | 97  |
| 5.5  | Temperature profiles for various values of $\gamma$ . . . . .     | 98  |
| 5.6  | Temperature profiles for various values of $Ha$ . . . . .         | 99  |
| 5.7  | Temperature profiles for various values of $Ec$ . . . . .         | 100 |
| 5.8  | Temperature profiles for various values of $\delta_t$ . . . . .   | 101 |
| 5.9  | Temperature profiles for various values of $Nb$ . . . . .         | 102 |
| 5.10 | Temperature profiles for various values of $Nt$ . . . . .         | 102 |
| 5.11 | Temperature profiles for various values of $\epsilon$ . . . . .   | 103 |
| 5.12 | Temperature profiles for various values of $\alpha_t$ . . . . .   | 103 |
| 5.13 | Concentration profiles for various values of $Le$ . . . . .       | 104 |
| 5.14 | Concentration profiles for various values of $\delta_c$ . . . . . | 104 |
| 5.15 | Concentration profiles for various values of $\alpha_c$ . . . . . | 105 |
| 6.1  | Physical model and coordinate system . . . . .                    | 111 |
| 6.2  | Velocity profiles for various values of $\phi_s$ . . . . .        | 122 |

|      |                                                                    |     |
|------|--------------------------------------------------------------------|-----|
| 6.3  | Concentration profiles for various values of $\phi_s$ . . . . .    | 122 |
| 6.4  | Velocity profiles for various values of $A$ . . . . .              | 123 |
| 6.5  | Temperature profiles for various values of $A$ . . . . .           | 123 |
| 6.6  | Concentration profiles for various values of $A$ . . . . .         | 124 |
| 6.7  | Temperature profiles for various values of $Nr$ . . . . .          | 124 |
| 6.8  | Temperature profiles for various values of $Ec$ . . . . .          | 125 |
| 6.9  | Temperature profiles for various values of $Df$ . . . . .          | 125 |
| 6.10 | Concentration profiles for various values of $Yc$ . . . . .        | 125 |
| 6.11 | Concentration profiles for various values of $Sc$ . . . . .        | 126 |
| 6.12 | Concentration profiles for various values of $Sr$ . . . . .        | 127 |
| 6.13 | Velocity profiles for various values of $\lambda$ . . . . .        | 127 |
| 6.14 | Temperature profiles for various values of $\lambda$ . . . . .     | 128 |
| 6.15 | Temperature profiles for various values of $Bi_1$ . . . . .        | 128 |
| 6.16 | Concentration profiles for various values of $Bi_2$ . . . . .      | 129 |
| 7.1  | Physical model of the flow problem. . . . .                        | 135 |
| 7.2  | Radial Velocity profiles for various values of $\phi_s$ . . . . .  | 143 |
| 7.3  | Temperature profiles for various values of $\phi_s$ . . . . .      | 145 |
| 7.4  | Radial Velocity profiles for various values of $A_s$ . . . . .     | 146 |
| 7.5  | Tangential velocity profiles for various values of $A_s$ . . . . . | 147 |
| 7.6  | Temperature profiles for various values of $A_s$ . . . . .         | 148 |
| 7.7  | Tangential velocity profiles for various values of $M$ . . . . .   | 148 |
| 7.8  | Temperature profiles for various values of $M$ . . . . .           | 149 |
| 7.9  | Radial velocity profiles for various values of $\gamma$ . . . . .  | 149 |

|      |                                                                                               |     |
|------|-----------------------------------------------------------------------------------------------|-----|
| 7.10 | Tangential Velocity profiles for various values of $\delta_v$ . . . . .                       | 150 |
| 7.11 | Temperature profiles for various values of $\delta_t$ . . . . .                               | 150 |
| 7.12 | Temperature profiles for various values of $\alpha_t$ . . . . .                               | 151 |
| 7.13 | Skin friction coefficient for various values of $M$ and $\phi_s$ . . . . .                    | 151 |
| 7.14 | Skin friction coefficient for various values of $A$ and $\gamma$ . . . . .                    | 152 |
| 7.15 | Local heat transfer rate for various values of $M$ and $\phi_s$ . . . . .                     | 152 |
| 7.16 | Local heat transfer rate for various values of $Pr$ and $\alpha_t$ . . . . .                  | 152 |
| 8.1  | Physical model of the flow problem. . . . .                                                   | 158 |
| 8.2  | Velocity profiles for various values of $\phi_{s_2}$ . . . . .                                | 167 |
| 8.3  | Temperature profiles for various values of $\phi_{s_2}$ . . . . .                             | 168 |
| 8.4  | Velocity profiles for various values of $\alpha_v$ . . . . .                                  | 169 |
| 8.5  | Temperature profiles for various values of $\alpha_v$ . . . . .                               | 170 |
| 8.6  | Velocity profiles for various values of $Gr_x$ . . . . .                                      | 171 |
| 8.7  | Temperature profiles for various values of $\tau$ . . . . .                                   | 172 |
| 8.8  | Velocity profiles for various values of $\delta_{v_2}$ . . . . .                              | 173 |
| 8.9  | Temperature profiles for various values of $\delta_{v_2}$ . . . . .                           | 173 |
| 8.10 | Temperature profiles for various values of $H$ . . . . .                                      | 174 |
| 8.11 | Temperature for various values of $Pr$ . . . . .                                              | 174 |
| 8.12 | Skin friction coefficient $C_f$ for various values of $A$ and $\phi_{s_1}$ . . . . .          | 174 |
| 8.13 | Skin friction coefficient $C_f$ for various values of $\alpha_v$ and $\delta_{v_2}$ . . . . . | 175 |
| 8.14 | Local heat transfer rate for various values of $A$ and $\phi_{s_1}$ . . . . .                 | 175 |
| 9.1  | Physical model of the problem . . . . .                                                       | 183 |

|      |                                                                                                 |     |
|------|-------------------------------------------------------------------------------------------------|-----|
| 9.2  | Tangential Velocity profiles for various values of $\phi_{s_1}$ and $\phi_{s_2}$ . . . . .      | 193 |
| 9.3  | Temperature profiles for various values of $\phi_{s_1}$ and $\phi_{s_2}$ . . .                  | 194 |
| 9.4  | Tangential Velocity profiles for various values of $\alpha_v$ . . . .                           | 194 |
| 9.5  | Tangential Velocity profiles for various values of $\lambda_m$ . . . .                          | 195 |
| 9.6  | Tangential Velocity profiles for various values of $\kappa$ . . . . .                           | 196 |
| 9.7  | Tangential Velocity profiles for various values of $\epsilon$ . . . . .                         | 197 |
| 9.8  | Temperature profiles for various values of $\epsilon$ . . . . .                                 | 197 |
| 9.9  | Temperature profiles for various values of $\zeta$ . . . . .                                    | 198 |
| 9.10 | Temperature profiles for various values of $Nr$ . . . . .                                       | 198 |
| 9.11 | Temperature profiles for various values of $Ec$ . . . . .                                       | 199 |
| 9.12 | Tangential Skin friction coefficient versus $\zeta$ for various values of $\alpha_v$ . . . . .  | 199 |
| 9.13 | Azimuthal Skin friction coefficient versus $\zeta$ for various values of $\alpha_v$ . . . . .   | 200 |
| 9.14 | Tangential Skin friction coefficient versus $\zeta$ for various values of $\lambda_m$ . . . . . | 201 |
| 9.15 | Azimuthal Skin friction coefficient versus $\zeta$ for various values of $\lambda_m$ . . . . .  | 201 |
| 9.16 | Local heat transfer rate versus $\zeta$ for various values of $\epsilon$ . .                    | 202 |
| 9.17 | Local heat transfer rate versus $\zeta$ for various values of $Nr$ .                            | 202 |
| 9.18 | Local heat transfer rate versus $\zeta$ for various values of $Ec$ .                            | 202 |

# Acronyms

|        |                                    |
|--------|------------------------------------|
| CFD    | Computational Fluid Dynamics       |
| CNTs   | Carbon Nanotubes                   |
| EO     | Engine oil                         |
| FDM    | Finite difference method           |
| FEM    | Finite element method              |
| MHD    | Magnetohydrodynamic                |
| MWCNTs | Multi wall carbon nanotubes        |
| SQLM   | Spectral quasilinearization method |
| SRM    | Spectral relaxation method         |
| SWCNTs | Single wall carbon Nanotubes       |

# Nomenclatures

|              |                                       |
|--------------|---------------------------------------|
| $A$          | Unsteady parameter                    |
| $B$          | Magnetic field strength               |
| $Bi$         | Biot numbers                          |
| $C$          | Concentration of fluids               |
| $C_f$        | Local Skin friction coefficient       |
| $c_p$        | Specific heat capacity of materials   |
| $C_s$        | Concentration susceptibility          |
| $\mathbf{D}$ | Chebyshev Differentiation matrix      |
| $D_B$        | Brownian diffusion coefficient        |
| $D_f$        | Dufour number                         |
| $D_m$        | Mass diffusivity of the concentration |
| $D_T$        | Thermophoresis diffusion coefficient  |
| $E$          | Specific energy of fluid elements     |
| $Ec$         | Eckert number                         |
| $f$          | Dimensionless stream function         |
| $Gr$         | Local Grashof number                  |
| $g$          | Acceleration due to gravity           |
| $h_f$        | Connective heat transfer coefficient  |
| $h_s$        | Connective mass transfer coefficient  |
| $\mathbf{I}$ | Identity matrix                       |
| $K$          | Permeability of porous medium         |
| $k$          | Thermal conductivity of materials     |

|              |                                          |
|--------------|------------------------------------------|
| $L$          | Characteristic length                    |
| $Le$         | Lewis number                             |
| $M$          | Magnetic field parameter                 |
| $m$          | Falkner-Skan power-law parameter         |
| $M_w$        | Surface mass flux                        |
| $N$          | Number of collocation points             |
| $Nb$         | Brownian motion parameter                |
| $Nr$         | Thermal radiation parameter              |
| $Nt$         | Thermophoresis parameter                 |
| $Nu$         | Local Nusselt number                     |
| $p$          | Pressure term of the fluids              |
| $Pr$         | Prandtl number                           |
| $\mathbf{q}$ | Thermal flux vector                      |
| $q_r$        | Radiation heat flux                      |
| $q_w$        | Surface heat flux                        |
| $Re$         | Local Reynolds number                    |
| $Sc$         | Schmidt number                           |
| $Sr$         | Soret number                             |
| $S$          | General source term                      |
| $Sh$         | Local Sherwood number                    |
| $T$          | Temperature of the fluid                 |
| $T_m$        | Mean fluid temperature                   |
| $T_n$        | Chebyshev polynomials of the first kind  |
| $\mathbf{u}$ | Velocity vector of the fluid particles   |
| $u$          | Fluid velocity in the x direction        |
| $U_n$        | Chebyshev polynomials of the second kind |
| $u_\infty$   | Free stream velocity                     |
| $V$          | Volume of the materials                  |
| $v$          | Fluid velocity in the y direction        |
| $w$          | Fluid velocity in the z direction        |

# Greek symbols

|            |                                         |
|------------|-----------------------------------------|
| $\alpha$   | Thermal diffusivity of materials        |
| $\beta$    | Pressure gradient parameter             |
| $\gamma$   | Non-Newtonian Casson fluid parameter    |
| $\delta$   | Boundary condition slip parameter       |
| $\epsilon$ | Variable thermal conductivity parameter |
| $\zeta$    | Dimension of reciprocal time            |
| $\theta$   | Dimensionless temperature               |
| $\kappa$   | Permeability parameter                  |
| $\lambda$  | Moving wedge parameter                  |
| $\mu$      | Dynamic viscosity of fluid              |
| $\nu$      | Kinematic viscosity of fluid            |
| $\xi$      | Dimensionless similarity variable       |
| $\rho$     | Density of materials                    |
| $\sigma$   | Electrical conductivity of materials    |
| $\tau$     | Viscous stress                          |
| $\tau_w$   | Wall shear stress                       |
| $\Phi_v$   | Viscus dissipation function             |
| $\phi$     | Dimensionless concentration             |
| $\phi_s$   | Volume fraction of nanoparticle         |
| $\psi$     | Stream function                         |
| $\omega$   | Curvature parameter                     |
| $\Omega$   | Angular velocity                        |

# Subscripts

|          |                              |
|----------|------------------------------|
| $bf$     | Base fluids                  |
| $hnf$    | Hybrid nanofluids            |
| $nf$     | Nanofluids                   |
| $s$      | Nanoparticles                |
| $s_1$    | The first nanoparticles      |
| $s_2$    | The second nanoparticles     |
| $w$      | Condition at the surface     |
| $0$      | Constant/reference condition |
| $\infty$ | Condition at the free stream |

# Statement of the Author

I, Ayele Tulu Yadete hereby declare and affirm that the dissertation entitled 'Boundary Layer Flow of Nanofluids past Some Geometrical Shapes Embedded in Porous Media' is my own work conducted under the supervision of Dr. Wubshet Ibrahim (Associate Professor). I have followed all the ethical principles of scholar in the problem formulation, computation, result analysis and completion of this dissertation. All scholarly matter that is included in the dissertation has been given recognition through citation. I have adequately cited and referenced all the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and I have not misrepresented, fabricated, or falsified any idea / data / fact / source in my submission. This dissertation is submitted in partial fulfillment of the requirement for a degree from the Post Graduate Studies at Wollega University. I further declare that this dissertation has not been submitted to any other institution anywhere for the award of any academic degree, diploma or certificate.

I understand that any violation of the above will be cause for disciplinary action by the University and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Name: Ayele Tulu

Signature \_\_\_\_\_

Date: April 25, 2021

Department: Mathematics

College: Natural and Computational Sciences

# Acknowledgement

Most of all, I would want to praise the almighty God for what He has done in my life and blessing me in all aspects of my life.

Next, I want to thank my supervisor Dr. Wubshet Ibrahim for his continuous valuable professional guidance, constructive comment and devotion of his precious time with patience and brotherly approach starting from the early planning to the completion of this dissertation. Through his guidance I have learnt the research methods used in the applied mathematics particularly in fluid dynamics. Besides, his countless effort in forming me an independent mathematician is really appreciated.

Also, I want to thank Professor Sandile Motsa for his guidance in writing mathematical program in Matlab for Spectral Collocation methods.

I want to express my sincerely appreciation to the Ambo University for giving me the chance with full sponsor. Moreover, I want to extend my gratitude to all instructors, supportive staffs and the management body of Wollega University particularly, department of Mathematics for all their supports and making suitable learning atmosphere.

My special gratitude particularly goes to my wife Hawi Bayisa for all her supports and being with me in all aspects of my life. My heartfelt gratitude also goes to my lovely daughters and son: Sifan, Moyika, Ebise, and Yobsan for their endless love to me. Finally, I want to thank those who are directly or indirectly supported me in the study.

# Biographical Sketch of the Researcher

Mr. Ayele Tulu Yadete was born in Oromia Regional State, Bale Zone, in 1978 G.C. He attended his primary and secondary educations at Agerfa Primary and Goba Secondary schools, respectively. After completion of his Secondary schools, he joined Bahirdar University and graduated with BEd degree in Mathematics in July, 2001 G.C. After his graduation, he worked at Bako and Holeta Agricultural TVET College as a Mathematics instructor. Then, he joined Haramaya University and graduated with MSc degree in Applied Mathematics in November, 2011 G.C. Since March 2012 G.C he has been working at Ambo University as a Mathematics lecture. Also, he joined the School of Graduate Studies at Wollega University in December, 2017 to pursue his PhD study in Fluid dynamics.

# Abstract

In this dissertation work the boundary layer flow of nanofluids past geometrical shapes such as wage, cylinder, disc, and cone embedded in porous media were studied. The physical problems of fluids flow are governed by highly nonlinear partial differential equations. Using suitable similarity transformations, the governing equations of flow were transformed into nonlinear higher order ordinary differential equations. Then, the computational solutions were obtained using recently developed numerical methods namely, Spectral quasilinearization method (SQLM) and Spectral relaxation method (SRM). The results of the study were compiled in the form of published papers on six different problems of incompressible fluid flow. In the study, nanoparticles of *Ag*, *CNTs*,  $Al_2O_3$ , &  $Fe_3O_4$  were also employed with water, ethylene glycol and Engine oil as the base fluids. Effects of embedded parameters such as magnetic field, viscous dissipation, nanoparticle volume fraction, Non-Newtonian fluid parameter, thermal radiation, pressure gradient, velocity and thermal slip effects, Cattaneo Christove heat flux Model, variable viscosity and thermal conductivity, and convective boundary condition on velocity, temperature and concentration distributions as well as skin friction coefficient, local heat and mass transfer rates were examined. The computational results were presented via embedded parameters with graphs and tables. The results revealed that the skin friction coefficient was improved while the local heat transfer rate was reduced with higher values of nanoparticle volume fraction, unsteadiness, viscous dissipation, Non-Newtonian Casson fluid, pressure gradient, permeability, magnetic field, temperature and dependent viscosity parameters. Also, higher skin friction coefficient and lower local heat transfer rate were reported in the hybrid nanofluids than mono nanofluids. In addition, hybrid nanofluids show better flow distributions with good stability of thermal properties than mono nanofluids. Thus, the obtained results are promising for the application of hybrid nanofluids in the nanotechnology sectors. Moreover, the accuracy of the solutions was checked against the previously published results and an excellent agreement were obtained. Finally, following the obtained results, conclusions and recommendations were indicated.

**Keywords:** Boundary layer flow, Carbon nanotubes, Hybrid nanofluids, Porous media, Skin friction coefficient, Spectral collocation method

# Chapter 1

## General Introduction

### 1.1 Background of the Study

Most of the fluid dynamics problems can be modeled by partial differential equations. These equations are extremely nonlinear and most of the time it is impossible to get their exact solutions. Fluid dynamics problems are often arise in various engineering and science disciplines, for example, in aerodynamics of aircraft and vehicles, power plant, chemical process engineering, marine engineering, environmental engineering, biomedical engineering, and so on. as described in (Versteeg & Malalasekera, 2007). Thus, to study fluid dynamics, it is more convenient to treat fluids as continuum rather than as a group of discrete molecules, and this concept is called the continuum hypothesis (Edward et al., 2005).

Under this hypothesis, the general problem of fluid dynamics can be described in two ways, the Eulerian description and Lagrangian description (Edward et al., 2005). In the Eulerian description, a complete set of fluid property values are related with every point in the spatial grid defined by a particular coordinate system whereas in Lagrangian description, any particle of the fluid is selected and observation is made of its particular motion whatever changes take place in fluid property. This study

considered only the Eulerian description of fluid motion.

Fluid flow situations such as an external flow which take place when an object moves through a fluid or a fluid passes over a surface of a body existing theoretically challenging problems. Not only are these problems nonlinear, but other complicating factors such as boundary conditions, steady or unsteady flow, laminar or turbulent flow, and incompressible or compressible flow are some of them come into play (Edward et al., 2005). Such external flows are present everywhere around us, both in nature and as a result of modern technology. For instance, the development of aircraft, ships and land vehicles could not have happened without a good understanding of external flow.

In view of these, the theory of boundary layer flow in fluid flow over a surface was developed by Ludwig Prandtl in 1904, and is milestones in the development of fluid dynamics (Bergman et al., 2011). Later on, the study of Prandtl's boundary layer theory was extended by various researchers (Falkner & Skan, 1931; Hartree, 1937). Moreover, the velocity boundary layer examined by Khan et al., 2014; Rasheed et al., 2017; Ibrahim & Hindebu, 2019 to mention few of them. In the study of boundary layer theory three different boundary layers flow namely velocity, thermal, and concentration boundary layers were discussed. Now, this thesis discusses the above mentioned three boundary layers.

Boundary layer flow over various shaped bodies like flat plate, wedge, disc, cone, cylinder, and sphere is ensured in many practical applications such as crude oil extraction, geothermal systems, thermal insulation, heat exchangers, space technology, astrophysics, energy storage systems, nuclear power management system, geological science, micro-inverter chips, hospitality and health care systems (Nagendramma et al., 2015; Ramzan et al., 2019). Owing to these several applications, many researchers have been motivated to understand and study the physical features of boundary layer flow of fluid past these shaped bodies. Some of these include the works of Eckert

(1942), Turkeyilmazoglu (2015), Ghalambaz et al. (2015), Malik et al.(2016), and Kuttan et al. (2019).

A porous medium is a region which contains pores that offers resistance to fluid flow. The volume of the fluid to flow across a porous medium is determined by the porosity or permeability of the medium, and the basic law governing the flow of fluids through porous media is Darcy's Law (Makanda et al., 2015). The flow of fluid in heat storage in beds, biological systems, irrigation problems, and the flow of water in aquifers are some examples of fluid flow in porous media (Khalid et al., 2015). Fluid flow through porous media has significant practical application in material sciences and engineering such as Petroleum industries, seepage of water in river beds, filtration and purification of chemical processes and many others (Rajput, 2017). Numerous studies have been conducted on fluid flow in porous media by (Nagendramma & Leelaratnam, 2017; Oyelakin et al., 2017; Prasannakumara et al., 2017; Rajput et al., 2017; Ramesh et al., 2017; Maraj et al., 2018; Tlili et al., 2018) are some of them.

Heat is a kind of energy that can be conveyed from one system to another due to temperature variation. Heat can be transferred in three different modes namely, conduction, convection, and radiation heat transfer (Bergman et al., 2011). In a static fluid, heat is transferred by conduction while it is by convection and radiation in a flowing fluid. Also, mass transfer involves the existence of two states at different mass gradient, and the key driving force for mass transfer is the concentration difference. Likewise, in fluid, mass can be transferred by means of diffusion and convection, but there is no radiation take place in mass transfer (Cengel & Ghajar, 2015). In the presence of bulk fluid motion, heat and mass are transferred through a fluid by convection.

Depending on how the fluid motion is initiated, convection can be classified as natural/free and forced convection (Cengel & Ghajar, 2015). In natural convection, any fluid motion is instigated by density variances due to temperature gradient that

creates buoyant forces which enable the fluid motion. In forced convection, the fluid is forced to flow over a surface or in a pipe by external means such as pump or fan. The blend of forced and free convection flow is known as mixed convection flow. Mixed convection flow arises in several transport processes such as in heat exchangers, solar collectors, nuclear reactors, electronic equipment; and it plays a significant role in various engineering processes (Rosali et al., 2016). Natural convection fluid flow has useful application in thermal insulation, extraction of petroleum, metal processing, performance of lubricants, application of paints and extrusion of plastic sheets (Makanda et al., 2015b). Currently, both natural and mixed convection flows of fluid have been examined by different scholars (Malik et al., 2016; Reddy & Chamkha, 2016; Rosali et al., 2016; Nageeb et al., 2017; Kuttan et al., 2019; Manjunatha et al., 2019) to mention few.

Different types of fluids are usually used as heat transmission in heat transfer applications. The wide applications of heat transfer in different industrial and engineering applications have required the availability of efficient thermal performance techniques. Owing to this, researchers have continuously worked on developing advanced heat transfer fluids that have significantly higher thermal conductivity than usually used fluids such as water, engine oils, ethylene glycol, and etc. (Choi & Eastman, 1995). In the few past decades, nanofluids that are produced by scattering of nanoparticle metallic structures in the working fluid is considered to be one of the novel and efficient technique; and have gained remarkable attention of nowadays researchers (Khanafer & Vafai, 2018).

Common working fluids such as water, engine oils and ethylene glycol have limited thermal performances which restrict their use in modern-day heat transfer applications. Nanofluids comprise of nanoscale particles such as copper, alumina, carbides, nitrides, metal oxides, graphite, and carbon nanotubes which enhances the thermal conductivity of base fluids (Ahmad et al., 2017; Ali et al., 2018). They have extensive

applications in modern systems of heating and cooling, solar cells, generation of new fuels, hybrid-powered engines, cancer therapy, drug delivery and medicine (Aziz et al., 2018). Due to these various applications of nanofluids, many studies associated with the flow of nanofluids have been conducted (Khan et al., 2017; Amirsom et al., 2019; Chamkha et al., 2019; Mahanthesh et al., 2019).

In contrast, the availability of some nanoparticles like Cu, Ag and Au with better thermal conductivity is limited. In addition, if they exist, some unmodified nanoparticles might be causes the toxicity risks (Parveen et al., 2016). Because of these, much effort has been given to develop well-defined and highly stable nanoparticles which improve the thermal conductivity of base fluid. Consequently, the innovative classes of fluids namely, hybrid nanofluids have been prepared using hybrid (two or more) nanoparticles. Some studies spectacle that hybrid nanoparticles have the capability to advance the heat transfer rates more efficiently with low cost as compared to single nanoparticles (Shi et al., 2010; Sundar et al., 2014, Esfe et al, 2018).

Hybrid nanofluids have momentous applications in various engineering and biomedical fields such as in manufacturing, transportation, nuclear safety, military, modern electronic devices and supercomputers, pharmaceutical and drug delivery (Manjunatha et al., 2019). Due to the above mentioned facts, many researchers are involved in hybrid nanofluids exploration (Afrand et al., 2016; Sundar et al, 2017; Asadi et al., 2018; Esfe et al., 2018; Mohebbi et al, 2019, etc.).

Currently, heat transfer of carbon nanofluids (CNTs) has received great attention due to their extensive applications in the fields of nanotechnology and medicine. Nanoparticles of carbon nanotubes were first discovered by Lijima (1991). CNTs have a notable conductivity which supports them to form a network of conductive tubes. Because of their unique morphology, mechanical properties, electronic structural and innovative physicochemical features, CNTs are the most resourceful material of this century (Prajapati et al, 2011; Khalid et al, 2018). Besides, the existence of carbon

chains in CNTs does not convey any hazard to the atmosphere (Alsagri et al., 2019).

Contingent on their number of concentric layers of rolled grapheme sheets, CNTs are usually categorized into single wall carbon nanotubes (SWCNTs) and multi wall carbon nanotubes (MWCNTs) (Alsagri et al., 2019). SWCNTs contain catalyst for their synthesis and encompass a particular graphene sheet encircled around and form a cylinder. MWCNTs can be prepared without catalyst and contain several rolled layers of graphite with complex structure (Khan et al., 2018). CNTs have significant uses in nanotechnology, energy storage, hardware, optics, ceramic, biomedical, thermal defense and in different fields of material sciences and engineering (Hayat et al., 2017). At present, these modern applications of Carbon nanotubes in nanofluids have attracted different researchers (Esfe et al., 2018; Jabbari et al., 2019; Mohebbi et al., 2019; Sreedevi et al., 2019; Palanisamy & Kumar, 2019; Ramzan et al. 2019).

In the last few decades, the study of motion of non-Newtonian fluids has been a popular area of research due to its extensive role in industrial and engineering applications such as in power engineering, food engineering, petroleum production and in the industries of polymer solutions (Malik et al., 2017; Gangadhar et al., 2018). In non-Newtonian fluids, the relationship between the shear stress and rate of strain is very complicated in comparison to the Newtonian fluids. The well-known Navier-Stokes equations of flow may not completely describe the noticeable features of non-Newtonian fluids (Ibrahim & Hindebu, 2019). Hence, to analysis the non-Newtonian features of fluid flow, several rheological models were developed for different fluids according to their nature.

Casson fluid model is one of non-Newtonian fluids, and it was first derived by Casson (1959). It starts to flow while yield stress is less than the shear stress whereas it acts like an elastic solid when shear stress is less than the yield stress. The study of Casson fluid in some flow fields has received attention by many researchers due to

its application in food processing, metallurgy, bio-engineering operations, drilling operations, manufacturing of pharmaceutical products and biological fluids (Pramanik, 2013; Malik et al., 2014; Makanda et al., 2015; Makinde et al., 2017; Nagendramma & Leelaratnam, 2017). Also, the Casson non-Newtonian fluid model was considered in this study.

Furthermore, the effects of thermophysical features such as magnetohydrodynamic (MHD), viscous dissipation, thermal radiation, mode of heat flux, slip conditions, and etc. on fluid flow cannot be ignored as they have a direct impact on heat transfer. MHD is the study of fluid flow in electrically conducting fluids with magnetic properties that affect fluid, and it has wide applications in technological processes such as in MHD power generator designs, plasma studies, petroleum industries, design for cooling of nuclear reactors, construction of heat exchangers, blood flow measurements, electric motors and etc. (Makanda et al., 2015; Rehman et al., 2017). Some studies on MHD boundary layer flow of nanofluid and heat transfer were included in the works of Alama et al. (2016), Alam et al. (2017), Ibrahim (2017), Rasheed et al. (2017), Nasir et al., (2018), Tlili et al. (2018), Alsagri et al. (2019), Mahanthesh et al. (2019), and so on.

Viscous dissipation is the heat energy that is created due to friction between fluid layers, and it affects the temperature distribution by playing a role like an energy source (Pal and Mandal, 2015). Hence, viscous dissipation can affect heat transfer rate; and its effect was considered in fluid flow with heat transfer problems by different studies (Ibrahim & Shanker, 2014; Haile & Shankar, 2015; Majety et al., 2016; Sheri et al., 2016, etc.). The impact of thermal radiation in fluid flow offers a theoretically challenging problem. At high temperatures, radiation meaningfully affect temperature distribution and heat transfer (Makanda et al., 2015; Pal & Mandal, 2015).

The dynamics of heat transfer phenomenon is a focus of modern-day study, because of its wide applications in residential, industrial production, engineering and life sciences (Khan et al., 2015; Shehzad et al., 2016). Accordingly, to regulate the side effects of heat transfer, and to use efficiently researchers have explained the features of heat transfer and proposed different heat flux models. For the first time, Fourier developed the most popular classical heat flux model (Akber et al., 2017). However, this model has a limitation and leads to a parabolic energy equation for the temperature field. Later, to overcome the unrealistic feature of this model, Cattaneo-Christov heat flux model was developed (Christov, 2009). Presently, considerable attention has been given in the study of Cattaneo-Christov heat flux mode (Makinde et al., 2017; Gangadhar et al., 2018; Kundu et al., 2018). As well, both of the above heat flux models were employed in this study.

The governing equations of a fluid flow are described using mathematical statements of the conservation laws of physics such as conservation of mass, Newton's second law, and first law of thermodynamics (Versteeg & Malalasekera, 2007; Edward et al., 2005). These governing equations of fluid flow are a complex set of nonlinear partial differential equations, and it is a challenge to get their exact solutions analytically. Thus, this study focused on finding the numerical solutions of boundary layer problems of incompressible fluid flow.

Most of the boundary layer problems of fluid flow were studied using numerical methods such as Runge-Kutta with shooting scheme, finite difference, and spectral homotopy analysis methods. However, recently, spectral collocation based numerical methods namely, Spectral quasilinearization method and Spectral relaxation method were developed (Motsa, 2013; Motsa et al., 2014; Magagula et al., 2016). Apart from other numerical methods, the spectral collocation method chooses a basis function that is global to the entire computational domain with high degree polynomials or

trigonometric polynomials (Thibeault, 2009). Moreover, the method is easy to implement, adaptable to various problems and provide more accurate approximations (Gottlieb & Hesthaven, 2001; Motsa et al., 2013b). Also, these spectral collocation based methods require less grid points to achieve a high accurate results and efficient compared to methods like finite difference and finite element methods (Motsa et al., 2014). As a result, these spectral collocation based numerical methods were used for this study.

## 1.2 Statement of the Problem

Today, due to vast applications of nanofluids in the nanotechnology and biomedical sectors, extensive theoretical and experimental investigations of nanofluids with respect to stability, preparation technique, particle size, particle shape, and particles type have been done. Besides, boundary layer flow of nanofluids over a stretching/shrinking sheets was widely investigated. However, in view of all the mentioned literature survey and to the best of the researcher knowledge, there is a scarcity of information observed in the nanofluids dynamics, such as the effects of different thermophysical parameters on the boundary layer flow of nanofluids past geometrical shapes (like wedge, cylinder, disc and cone); the characteristics of different nanofluids during flow through porous media; The mode of heat and mass transfer in nanofluids flow with time, type and amount of nanoparticles concentration, type of base fluids, flow geometry, and boundary conditions. Thus, in this study, it is attempted to answer the following research questions.

- How the mathematical model and numerical algorithm are formulated for the problem under study?
- What are the effects of different embedded parameters on the flow and heat transfer of nanofluids past some geometry (wage, cylinder, disc and cone)?

- What are the theoretical steps employed in using Spectral collection method to solve fluid flow problem?
- What are the proper values of nanoparticles to base fluids to be used to obtain stable nanofluids flow with better heat transfer?
- Which nanoparticles and base fluids can give better flow distributions with good stability of thermal properties?

## **1.3 Objective of the Study**

### **1.3.1 General Objective of the Study**

The general objective of this dissertation is to study the boundary layer flow of nanofluid past geometrical bodies such as wedge, cylinder, cone, and disc embedded in porous media computationally, employing SQLM and SRM.

### **1.3.2 Specific Objectives of the Study**

To this end, specific objectives of the study are:

1. To analyze steady two-dimensional MHD boundary layer flow of nanofluids past a wedge in porous media with viscous dissipation effect.
2. To investigate MHD slip flow of a Casson nanofluid flow over linearly stretching cylinder in porous media in the presence of Cattaneo-Christov heat flux model.
3. To examine unsteady mixed convection flow of nanofluids past a moving wedge embedded in porous media with thermal radiation, viscous dissipation, and chemical reaction effects.
4. To inspect MHD slip flow of CNTs with ethylene glycol nanofluids flow and heat transfer due to stretchable rotating disk with Cattaneo-Christov heat flux

model.

5. To scrutinize unsteady natural convection flow of  $CNTs - Fe_3O_4$ /Water hybrid nanofluids caused by stretching surface embedded in a porous medium with second order velocity slip.
6. To examine the effects of variable viscosity and thermal conductivity on  $MWCNTs - Al_2O_3$ /Engine oil hybrid nanofluid flow due to spinning cone in a porous medium.

## 1.4 Scope of the Study

In this study, the problems of both steady and unsteady two- and three-dimensional laminar incompressible boundary layer flow of nanofluids and hybrid nanofluids past geometrical bodies such as wedge, cylinder, cone, and disc embedded in porous media were considered. Nanofluids with both Newtonian and Casson non-Newtonian fluids as base fluids with nanoparticles such as  $Cu$ ,  $Ag$ ,  $CNTs$ ,  $Al_2O_3$ , &  $Fe_3O_4$  were used. Also, both natural and mixed convection heat and mass transfer were examined with both Fourier and non-Fourier heat flux Model. Furthermore, the effects of governing parameters such as magnetic field, viscous dissipation, pressure gradient, thermal radiation, chemical reaction, heat generation/absorption, velocity and thermal slip factors were analyzed. The problems were solved numerically employing the recently developed spectral collocation based numerical methods namely, Spectral quasilinearization method (SQLM) and Spectral relaxation method (SRM).

## 1.5 Significance of the Study

Nowadays, various researches have been conducted regarding the preparation and flow of nanofluids due to their vast applications in different nanotechnology sectors. However, there is scarcity of information observed on the boundary layer flow of

nanofluids past some geometrical shape embedded in porous media. Therefore, the study have the following benefits.

- The findings of the study can be used in different applications of nanotechnology and biomedical sectors.
- The numerical algorithms and their matlab codes developed for the study can help researchers to carry out further study.
- The results of the study can fill the information gap observed in the study area, and it may recommend for further study.
- The material produced from the study can be used as a reference for other similar research.

## 1.6 Dissertation Outline

This dissertation paper is divided into ten chapters. The first chapter is the general introduction which introduces the back ground of the study, that is, the boundary layer flow of nanofluids and some important concepts which are essential in the study. Also, some important topics of the study such as statement of the problem, objective of the study, scope of the study, and significance of the study are specified in the first chapter.

In Chapter 2, some important description of fluid flow such as boundary layer flow, mode of heat and mass transfer, governing equations of fluid flow, basic concepts of nanofluids, and some thermophysical properties affecting fluids flow are highlighted.

In Chapter 3, the basic ideas of Spectral collocation method, and the detailed description of the numerical methods which are used in the dissertation namely, Spectral quasilinearization method (SQLM) and Spectral relaxation method (SRM) are given.

In Chapter 4, the problem of steady two-dimensional MHD boundary layer flow of nanofluids past a wedge in porous media with viscous dissipation effect is studied. Using suitable similarity transformations, the governing equations of flow are transformed into a system of nonlinear ordinary differential equations, then it is solved numerically employing SQLM.

In Chapter 5, the non-Newtonian Casson nanofluid flow over linearly stretching cylinder in non-Darcy porous medium in the presence of Cattaneo-Christov heat flux model is analyzed. Effects of normally applied magnetic field, variable thermal conductivity, viscous dissipation and wall slip factors are taken into account. The governing equations of the flow is numerically solved employing SRM.

In Chapter 6, the problem of unsteady two-dimensional mixed convection flow of nanofluids past a moving wedge embedded in porous media is considered. Nanofluids which are produced from  $Cu$ ,  $Ag$  and  $Al_2O_3$  nanoparticles with  $H_2O$  as a base fluid are used. The effects of governing parameters such as unsteadiness, nanoparticle volume fraction, thermal radiation, viscous dissipation, chemical reaction, and convective boundary condition are examined. Introducing appropriate similarity variables, the governing equations of fluid flow are solved numerically using SQLM.

In Chapter 7, CNTs with ethylene glycol nanofluid flow and heat transfer due to stretchable rotating disk with Cattaneo-Christov heat flux model is investigated. Both SWCNTs and MWCNTs nanoparticles are considered with ethylene glycol as the base fluid. The fundamental flow governing equations are transformed into dimensionless system, and solved numerically using SQLM.

In Chapter 8, unsteady natural convection flow of CNTs-Magnetite/Water hybrid nanofluids caused by stretching surface embedded in a porous medium is studied. Both hybrid nanoparticles of  $SWCNTs - Fe_3O_4$  and  $MWCNTs - Fe_3O_4$  are used with water as base fluid. Effects of hybrid nanoparticles volume fraction, second order velocity slip flow and temperature dependent viscosity are also investigated employing

SQLM.

In Chapter 9, the effects of variable viscosity and thermal conductivity with mixed convection, thermal radiation and viscous dissipation effects on *MWCNTs* – *Al<sub>2</sub>O<sub>3</sub>*/engine oil hybrid nanofluid flow due to vertically inverted spinning cone embedded in a porous medium is analyzed. The numerical solutions for the flow problem are obtained numerically employing the SRM.

In Chapter 10, finally, general conclusion of the study is given. Furthermore, future direction of the research work is also recommended.

The next section includes all the references that are cited in the dissertation. At the end, appendices which consist of the published papers is included.

# Chapter 2

## Literature Review

### 2.1 Fluid Dynamics

Fluid dynamics is the branch of science which deals with fluids in motion. Fluid is a substance that deforms continually, and a particle whose linear dimensions are negligible. Actual fluids are classified as liquids and gases (ideal fluids)(Edward et al., 2005). Liquids are incompressible fluids, that is, their volumes do not change with change in pressure. On the other hand, gases are compressible (perfect fluids), which undergo change in volume whenever the pressure changes. The term hydrodynamics is often used to the science of moving incompressible fluids (Bergman et.al., 2011)

It is more convenient to treats fluids as a continuous substance or continuum, rather than as a group of discrete molecules. Thus, this concept is called the *continuum hypothesis* (Cengel & Ghajar, 2015 ). In using the continuum hypothesis, the fundamental molecular structure of a fluid is suitably overlooked and replaced by a limited set of fluid properties such as velocity, pressure, density etc. defined at each point in the fluid at every instant. Hence, each fluid property is assumed to be a continuous function of position and time; and mathematically, the continuum hypothesis permits the use of differential calculus in the modeling and solution of fluid dynamics problems (Edward et al., 2005). Moreover, for a continuous or ideal fluid

we can describe a fluid particle as the fluid confined within an infinitesimal volume whose size is so small that it may be viewed as a geometrical point.

### 2.1.1 Description of Fluid

Under the continuum hypothesis, the general problem of fluid dynamics can be described in two ways, the *Eulerian description* and *Lagrangian description* (Edward et al., 2005). In the Eulerian description of continuum fluid dynamics, physical space is described by means of a coordinate system that serves as a background for fluid motion. In this method, a complete set of fluid property values are related with every point in the spatial grid defined by a particular coordinate system. The Eulerian approach establishes a field description of a fluid such as velocity, density, pressure, temperature fields, etc.; and each of them has a corresponding functional dependence on position and time (Edward et al., 2005). That is, in Eulerian description, we select any point fixed as space occupied by the fluid and observation is made of whatever changes take place in velocity, density, pressure etc. as the fluid passes through this point. Moreover, if  $P(x, y, z)$  is the point under reference, then  $x, y$ , and  $z$  do not depend upon the time parameter  $t$ , thus,  $\frac{dx}{dt}$ ,  $\frac{dy}{dt}$ , and  $\frac{dz}{dt}$  do not exist. In contrast, in Lagrangian description, observations are made at each point and each instant. That is, any particle of the fluid is selected and observation is made of its particular motion whatever changes take place in velocity, density, pressure, temperature etc. and it is pursued throughout its course (Edward et al., 2005). This study considered the Eulerian description of fluid motion.

### 2.1.2 Fluid Transport

In the continuum Eulerian description, to analyze the transport of mass, momentum, energy, concentration and other properties of fluid motion across a surface, it is useful to know that fluid motion happens in macroscopic and microscopic length scales (Cengel & Ghajar, 2015 ). The macroscopic (bulk) motion of fluid molecules is

modeled by the fluid velocity field; and this observable flow of fluid particles across a surface is called *convective transport*. The transport resulting from the microscopic (molecular-scale) motion of fluid molecules is called *diffusive transport*. Convective transport can occur only in fluid in motion whereas diffusive transport may happen in fluids in motion or fluids at rest however it is exist only in the occurrence of a spatial difference in a fluid property (Edward et al., 2005). In a moving fluid, both modes of fluid transport are promising. However, normally diffusive transport by far less than the convective transport; and it affects the distribution of a transported quantity along its gradients in all directions whereas convective transport influence only in the flow direction (Cengel & Ghajar, 2015 ). Thus, in the study, convective and diffusive transports are very important topics that encounters in every fluid flow, and their models are presented under governing equations of fluid flow.

### **2.1.3 Approaches to Study Fluid Dynamics**

There are three approaches to study fluid dynamics. These are experimental, theoretical and computational approaches (Versteeg & Malalasekera, 2007). Among these, experimental approach is the oldest approach, possibly for the first time employed by Archimedes. Experimental method is a very popular approach, but it is an expensive venture and time taking. Then scholars have seek the theoretical approach that mathematical equations which govern the fluid flow problems were employed and captured the fluid behavior within a closed form solution using formulas that can be readily used. This is possibly the simplest of the approaches. However, its scope is somewhat limited since not every fluid flow renders itself to such an approach. Moreover, the resulting equations may be too complicated to solve easily. Then, the third approach that is Computational methods emanated from the work. In this approach, the complicated governing equations of fluid flow problems are solved by computing them using a computer (Makanda et al., 2015). This approach has the advantage

that a wide variety of fluid flows may be computed with minimum cost. Due to this, nowadays the emerging discipline Computational Fluid Dynamics (CFD) has become a very powerful approach in industry and research (Versteeg & Malalasekera, 2007). Also, the computational approaches is used for this study.

## 2.2 Mode of Heat and Mass Transfer

Heat is a kind of energy that can be conveyed from one system to another due to temperature variation. The transfer of energy by means of heat is permanently from the higher to the lower-temperature medium, and it ends once the two mediums reach equal temperature. Heat can be conveyed in three different modes. These are *conduction*, *convection*, and *radiation* heat transfer (Bergman et.al., 2011). In a still fluid, heat transfer is by conduction and probably through radiation while it is by convection and radiation in a flowing fluid (Makanda et al., 2015). With absence of radiation, depending on the existence of any bulk fluid motion, heat transfer through a fluid is either by conduction or convection. Conduction in a fluid can be regarded as a special case of convection in the absence of any fluid motion since convection can be noticed as joined of conduction and fluid motion. Thus, when we deal with heat transfer through a fluid, we have either conduction or convection, but not both (Cengel & Ghajar, 2015).

Mass transfer involves the existence of two states at different chemical compositions, and the key driving force for mass transfer is the concentration difference. It denotes the flow of a chemical species from a high to a lower concentration states. Similar to heat transfer, mass is transferred by means of *conduction* (known as *mass diffusion*) and *convection*, but there is no radiation take place in mass transfer (Bergman et.al., 2011).

### 2.2.1 Conduction/Mass Diffusion

Conduction is the transfer of energy from the more robust particles of a material to the nearby less robust ones due to interactions among the particles. Heat Conduction can arise in solids, liquids, and gases. In solids, conduction is caused by the amalgamation of shaking of the molecules in a lattice and the energy transport by free electrons (Ibrahim & Hindebu, 2019). In liquids and gases, it is as a result of the collisions and diffusion of the molecules in the course of their random motion. The rate of heat conduction/mass diffusion through a medium is determined by the geometry of the medium, the material of the medium, the thickness of the medium, and the temperature(concentration for mass) difference through the medium (Ibrahim & Hindebu, 2019). Moreover, the rate of heat conduction is proportional to the temperature gradient along normal to the surfaces and is denoted by *Fourier's law of heat conduction*. Also, the rate of mass diffusion of a chemical species in a stationary medium is proportional to the concentration gradient along normal to the surfaces and is stated by *Fick's law of diffusion* (Bergman et.al., 2011).

### 2.2.2 Convection

Convection is the mechanism of heat and mass transfer through a fluid in the presence of bulk fluid motion. Heat convection is the heat transfer mechanism that involves both heat conduction (molecular diffusion) and bulk fluid motion (Rajput et al., 2017). Whereas, mass convection is the mass transfer mechanism between a surface and a moving fluid that involves both mass diffusion and bulk fluid motion. In the limiting case of no bulk fluid motion, heat and mass convection respectively reduce to heat conduction and mass diffusion (Cengel & Ghajar, 2015). Convection is classified as natural/free and forced convection, depending on how the fluid motion is initiated. In forced convection, the fluid is forced to flow over a surface or in a pipe by external means such as a pump or a fan. In natural or free convection, any fluid motion is

caused by density differences due to temperature gradient; and this creates buoyant forces which facilitate fluid flow (Cengel & Ghajar, 2015).

Natural convection fluid flow has practical application such as thermal insulation, extraction of petroleum resources, metal processing, performance of lubricants, application of paints, and extrusion of plastic sheets (Makanda et al., 2015b). Fluid flow caused by free or natural convection depends on a number of factors such as the geometry, orientation, variation of temperature on the surface and thermo-physical properties of the fluid (Makanda et al., 2015b). Mixed convection flow is the combination of forced and free convection flow and plays an important role in many engineering processes. Mixed convection flow is important when the buoyancy force due to the temperature difference between the free stream and the solid surface become large, which in turn significantly affect the flow and the thermal fields (Rosali et al., 2016). This flow arises in many transport processes such as heat exchangers, solar collectors, nuclear reactors, electronic equipment etc. The dimensionless parameter which describes natural convection is called the Grashof number (Rajput et al., 2017). The Grashof number is the ratio of the buoyancy forces to the viscous forces. The effects of both free and mixed convection are investigated in this research.

### **2.2.3 Radiation**

Heat transfer by conduction or convection takes place from a high-temperature medium to a lower-temperature one. It is interesting that radiation heat transfer can occur between two bodies separated by a medium colder than both bodies. For example, solar radiation reaches the surface of the earth after passing through cold air layers at high altitudes (Cengel & Ghajar, 2015). The theoretical foundation of radiation was established by physicist James Clerk Maxwell in 1864. He postulated that accelerated charges or changing electric currents give rise to electric and magnetic fields (Makanda, 2015). These rapidly moving fields are called electromagnetic radiation.

The type of electromagnetic radiation that is relevant to heat transfer is the thermal radiation emitted as a result of energy transitions of molecules, atoms, and electrons of a substance. The rate of thermal radiation emission increases with an increase in temperature (Cengel & Ghajar, 2015).

The consideration of thermal radiation effects in fluid flow gives rise to a theoretically challenging problem. When radiation emission is incident on fluid flow it affects certain fluid properties. To model the movement of thermal radiation, the Rosseland approximation is applied (Mustafa et al., 2015). This approximation assumes that the heat flux is proportional to the temperature change and that heat flows from the solid surface to the fluid (El-Kabeir & El-Sayed, 2012). Furthermore, it is assumed that temperature differences within the flow are small. At high temperatures, radiation effects significantly affect temperature distribution and heat transfer (Pal & Mandal, 2015). The impact of thermal radiation is also considered in some problems of this study.

## **2.3 Boundary Layers Flow**

An external flow happens whenever an object moves through a fluid or a fluid passes by the surface of a structure. Such flows are present universally around us, both in nature and as a result of modern technology. For example, the development of aircraft, ships, and land vehicles such as automobiles could not have happened without a good understanding of external flow. Nowadays, understanding the relationship between an object moving through a fluid or fluid moving over an object or adjacent to a structure has vast practical significance in industry and different technical problems. Owing to these, before a century, the boundary layer theory was developed and is milestones in the development of fluid dynamics (Edward et al., 2005). Thus, the three-boundary layers namely, velocity, thermal, and concentration boundary layers are considered in this study.

### 2.3.1 Velocity Boundary Layer

The theory of boundary layer flow in fluid flow over a surface is attributed mainly to Ludwig Prandtl in 1904. To highlight the theory, it is considered that the fluid to comprise of adjacent layers piled on top of each other. Due to wall viscous shearing effects, the velocity of fluid particles immediately adjacent to the surface becomes zero and the fluid layer next to the surface becomes attached to the surface because of the no-slip condition (Klaus, 2017). Then, the fluid layer slows down the motion of molecules of the next layer, and the layers of the fluid above the surface are moving with different velocities. Hence, the incidence of the plate is moved up to certain normal distance  $\delta_v$ , then after which the free-stream velocity remains basically unchanged (see Fig. 2.1, Source: Cengel & Ghajar, 2015). This thin layer adjacent to the surface of a solid body in which the effects of the viscous shearing forces caused by fluid viscosity are felt is called the *velocity boundary layer* (Bergman et al., 2011).

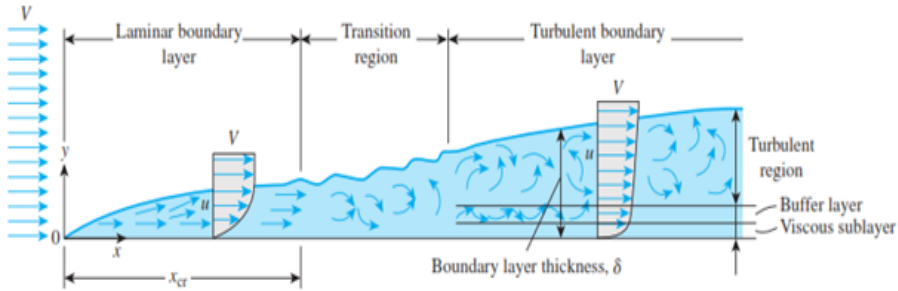


Figure 2.1: The development of the velocity boundary layer for flow over a flat plate and the different flow regimes.

The fluid velocity  $u$  increases as the distance from the wall increase, and the boundary layer thickness  $\delta_v$  is defined as the distance from the surface to the height above the surface at which the free stream velocity  $u = 0.99V$ . Thus, the flow over a plate are divided in to two regions by this proposed line of  $u = 0.99V$ . These

are the *boundary layer region*, wherein the viscous effects and the velocity changes are momentous; and the *irrotational flow region*, wherein the frictional effects are insignificant and the velocity remains unchanged (Cengel & Ghajar, 2015). The thickness of boundary layer is a function of the ratio of inertial forces and viscous forces, that is  $\delta_v(x) \sim \frac{x}{(Re_x)^{1/2}}$ , and at low Reynolds number, viscous forces govern the entire boundary layer and the flow is *laminar*, whereas for high Reynolds number inertia forces dominated the boundary layer and fluid becomes *turbulent* (Klaus, 2017).

### 2.3.2 Thermal Boundary Layer

Like velocity boundary layer, a thermal boundary layer develops when a fluid at a specified temperature flows over a surface at a different temperature. To highlight this concept, the study considered that the flow of a fluid at a uniform temperature of  $T_\infty$  over an isothermal flat plate at temperature  $T_w$ , as shown in Fig. 2.2, (Source: Bergman et al., 2011). The fluid particles in the layer nearby to the surface achieve thermal equilibrium with the plate surface temperature  $T_w$ . In turn, these fluid particles exchange energy with those particles in the adjoining-fluid layers, then temperature gradient develops in the flow field which varies from  $T_w$  to  $T_\infty$  as adequately far from the surface.

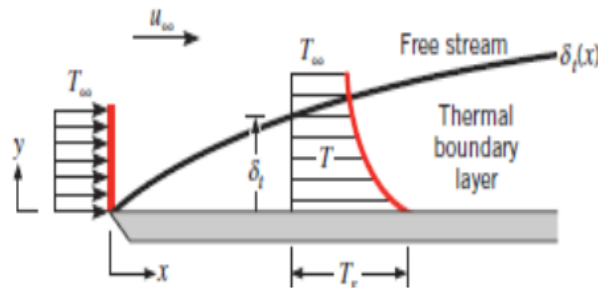


Figure 2.2: Thermal boundary layer development on an isothermal flat plate.

The flow region over the surface in which the temperature gradients exist is important, it is known as the *thermal boundary layer*, and its thickness  $\delta_t$  at any position along the surface is defined as the distance from the surface at which the temperature difference  $T - T_w = 0.99(T_\infty - T_w)$  (Cengel and Ghajar, 2015). Also, at the outer edge of the thermal boundary layer since  $T_w = 0$ , the temperature  $T = 0.99T_\infty$ . Moreover, the thickness of the thermal boundary layer grows in the flow direction, because the impacts of heat transfer are identified at bigger distances from the surface more downstream (Bergman et.al., 2011).

### 2.3.3 Concentration Boundary Layer

Consider the flow of molar concentration of species  $A$  at the isothermal surface be  $C_{Aw}$  and the free stream value be  $C_{A\infty}$ . If the amount of  $C_{Aw}$  varies from  $C_{A\infty}$  transfer of a concentration  $A$  by convection will take place and it is continuously saturated to the free stream value far from the surface. Similarly, in mass convection it is defined that the region of the fluid in which concentration gradients exist as the *concentration boundary layer*; and its thickness  $\delta_c$  for a species  $A$  at a identified position on the surface is usually defined as the value of  $\delta_c = \frac{C_{Aw} - C_A}{C_{Aw} - C_{A\infty}} = 0.99$  (Bergman et al., 2011). Further, the effects of species transfer enter farther into the free stream and the concentration boundary layer grows as distance from the leading edge enhances.

## 2.4 Governing Equations of Fluid Dynamics

In this section the differential form of the governing equations of fluid dynamics were developed and described how the equations were used to get a complete description of fluids flow. The governing equations of Newtonian fluids flow characterize mathematical statements of the *conservation laws of physics*. That is, the mass of a given fluid is conserved, the rate of change of momentum equal to the sum of all the forces applied on a fluid particle (*Newton's second law*), and the rate of change of energy is

equal to the sum of the rate of heat addition to and the rate of work done on a fluid particle (*first law of thermodynamics*) as reported in (Versteeg & Malalasekera, 2007; Edward et al., 2005). The governing equations of fluid dynamics are a complex set of nonlinear partial differential equations, and it is hard to get their exact solutions analytically. Nowadays, however, finding numerical solution of the governing equations is much promised.

### 2.4.1 Continuity Equation

When a region of a fluid contains neither source nor sink, the amount of fluid with in the region is conserved in accordance with the particles of conservation of matter (Cengel & Ghajar, 2015). This means that, the rate of generation of mass with in a given volume is equal to the net flow of mass from the volume, and this is formulated mathematically by means of *equation of continuity*, and it is given in more compact vector notations as (Versteeg and Malalasekera, 2007):

$$\frac{\partial \rho}{\partial t} = \text{div}(\rho \mathbf{u}) \quad (2.4.1)$$

where  $\rho$  is the density (mass per unit volume), and  $\mathbf{u} = \langle u, v, w \rangle$  is the velocity of a fluid particles.

The continuity equation consists of only the fluid density and the fluid velocity, and it applies to all fluids, Newtonian and non-Newtonian, compressible and incompressible, and for the entire range of flow velocities (Edward et al., 2005). Thus, the continuity equation expresses the law of conservation of mass at each point in the fluid, and it is one of the basic governing equation of fluid dynamics (Cengel & Ghajar, 2015). Hence, to examine the motion of fluid flow, the continuity equation must be satisfied at every point in the flow field.

### 2.4.2 Momentum Equations

Newton's second law states that the rate of change of momentum of a fluid particle equal to the vector sum of all the forces applied on the particle. To formulate the equation of momentum, it is considered that the motion of small rectangular parallelepiped of viscous fluid having  $P(x, y, z)$  as center and  $\delta_x$ ,  $\delta_y$ , and  $\delta_z$  as its lengths of edges parallel to the fixed coordinate axes. The mass  $\rho\delta_x\delta_y\delta_z$  of the fluid element remains constant and assume element moves along with the fluid. The rates of change of  $x$ -,  $y$ - and  $z$ -momentum per unit volume of a fluid particle are given by (Versteeg and Malalasekera, 2007).

$$\rho \frac{Du}{Dt}, \quad \rho \frac{Dv}{Dt}, \quad \rho \frac{Dw}{Dt} \quad (2.4.2)$$

where  $\frac{D}{Dt}$  is the total or substantive derivative, that is, the rate of change following a fluid particle per unit mass, and for the fluid property  $\Phi$  is given as.

$$\frac{D\Phi}{Dt} = \frac{\partial\Phi}{\partial t} + \text{div}(\Phi\mathbf{u}) \quad (2.4.3)$$

Basically, there are two types of forces acting on a fluid element. One of them is surface force such as pressure forces, viscous forces, and gravity force, etc.; and the other is body force such as centrifugal force and electromagnetic force and so on (Edward et al., 2005). The surface force is proportional to the surface area and acts on the boundary of the body, and it is common practice to include their contributions as separate terms in the momentum equation. The body force is proportional to the mass of the body on which it acts and their effect is included as source term in the momentum equation. The normal force per unit area is known as the normal stress and is also called *pressure* while the tangential force per unit area is called the *viscous stress* (Versteeg & Malalasekera, 2007).

The state of stress of a fluid element is defined in terms of the pressure which is denoted by  $p$  and the nine viscous stress components which are also denoted by

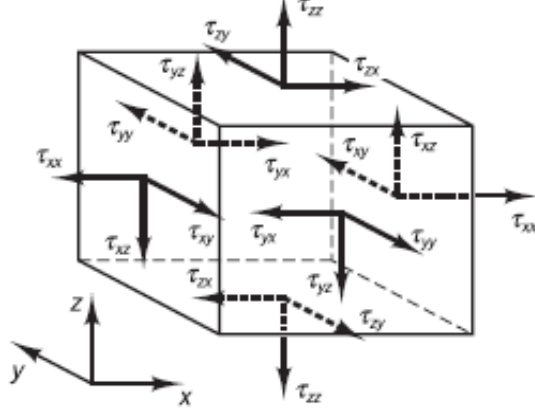


Figure 2.3: Stress components on three faces of fluid element.

$\tau$  shown in Figure 2.3, (Source: Versteeg & Malalasekera, 2007). To indicate the direction of the viscous stresses, the usual suffix notation  $\tau_{ij}$  is applied for the stress component acts in the  $j$  direction on a surface normal to the  $i$ -direction.

The  $x$ -,  $y$ -,  $z$ - components of the momentum equation are respectively established by setting the rate of change of momentum of the fluid particle equal to the total force in the  $x$ -,  $y$ -,  $z$ - direction on the element due to surface stresses (pressure and viscous stress) plus the rate of increase of  $x$ -,  $y$ -,  $z$ - momentum due to body forces are given as (Versteeg & Malalasekera, 2007)

$$\rho \frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial(\tau_{yx})}{\partial y} + \frac{\partial(\tau_{zx})}{\partial z} + S_x \quad (2.4.4)$$

$$\rho \frac{Dv}{Dt} = \frac{\partial(\tau_{xy})}{\partial x} + \frac{\partial(-p + \tau_{yy})}{\partial y} + \frac{\partial(\tau_{zy})}{\partial z} + S_y \quad (2.4.5)$$

$$\rho \frac{Dw}{Dt} = \frac{\partial(\tau_{xz})}{\partial x} + \frac{\partial(\tau_{yz})}{\partial y} + \frac{\partial(-p + \tau_{zz})}{\partial z} + S_z \quad (2.4.6)$$

In a Newtonian fluid the viscous stresses are proportional to the rates of deformation. Thus, the two constants of proportionality which relate stresses to linear deformations, the dynamic viscosity ( $\mu$ ), and the second viscosity ( $\varrho$ ) which relate stresses to the volumetric deformation are involved in three-dimensional form of Newton's law of

viscosity for compressible flows. Moreover, the nine viscous stress components, of which six are independent, are found as (Moukalled et al., 2016).

$$\begin{aligned}\tau_{xx} &= 2\mu \frac{\partial u}{\partial x} + \varrho \text{div} \mathbf{u}, & \tau_{yy} &= 2\mu \frac{\partial v}{\partial y} + \varrho \text{div} \mathbf{u} \\ \tau_{zz} &= 2\mu \frac{\partial w}{\partial z} + \varrho \text{div} \mathbf{u}, & \tau_{xy} = \tau_{yx} &= \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \tau_{xz} = \tau_{zx} &= \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), & \tau_{yz} = \tau_{zy} &= \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)\end{aligned}\quad (2.4.7)$$

The second viscosity  $\varrho$  is not much recognized as its effect is small in practice. For gases a sound practical approximation can be found by taking the value  $\varrho = \frac{-2}{3}\mu$  (Schlichting, 1979). The mass conservation equation is  $\text{div} \mathbf{u} = 0$  for incompressible fluid, and the viscous stresses are just twice the dynamic viscosity times the local rate of linear deformation. Substituting the above shear stresses (2.4.7) into (2.4.4)-(2.4.6) gives the well-known *Navier-Stokes equations*, named after the two nineteenth-centuries scientists who derived them independently as (Moukalled et al., 2016; Versteeg & Malalasekera, 2007).

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ 2\mu \frac{\partial u}{\partial x} + \varrho \text{div} \mathbf{u} \right] + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] + S_x \quad (2.4.8)$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[ 2\mu \frac{\partial v}{\partial y} + \varrho \text{div} \mathbf{u} \right] + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] + S_y \quad (2.4.9)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[ 2\mu \frac{\partial w}{\partial z} + \varrho \text{div} \mathbf{u} \right] + S_z \quad (2.4.10)$$

### 2.4.3 Internal Energy Equation

The effect of surface forces on the motion of a homogeneous fluid may be determined by considering the internal energy balance for the fluid of volume  $V$  contained with

in a specified material surface. Also, work is done on this mass of the fluid of both volume and surface forces and it may be gaining heat transfer across the boundary (Cengel & Ghajar, 2015). Thus, the energy equation is derived from the *first law of thermodynamics*, which states that the rate of increase of energy of a fluid particle is equal to the sum of the rate of work done on the fluid particle and the rate of heat addition to the fluid particle.

The rate of change of energy of a fluid particle per unit volume is obtained by

$$\rho \frac{DE}{Dt} \quad (2.4.11)$$

The rate of work done on the fluid particle in the element by a surface force is given as the product of the force and velocity component in the direction of the force. Hence, total rate of work done on the fluid particle by surface stresses is given as (Moukalled et al., 2016):

$$\begin{aligned} -div(p\mathbf{u}) + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} \\ \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \end{aligned} \quad (2.4.12)$$

On the other hand, for the heat flux vector  $\mathbf{q} = \langle q_x, q_y, q_z \rangle$ , the overall rate of heat added to the fluid particle per unit volume due to heat flow across element boundaries is related to the local temperature gradient with Fourier's law of heat conduction as (Versteeg & Malalasekera, 2007; Cengel & Ghajar, 2015)

$$-div\mathbf{q} = div(kgradT) \quad (2.4.13)$$

where  $k$  is constant of proportionality which is known as thermal conductivity of the fluid.

The specific energy  $E$  of a fluid element is often defined as the sum of internal (thermal) energy  $i$ , kinetic energy  $\frac{1}{2}(u^2 + v^2 + w^2)$ , and gravitational potential energy (Edward et al., 2005). This study considered the gravitational force as a body force,

which does a work on the fluid element as it moves through the gravitational field. Thus, the energy equation is (Moukalled et al., 2016; Versteeg & Malalasekera, 2007).

$$\begin{aligned} \rho \frac{DE}{Dt} = & -div(p\mathbf{u}) + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} \\ & + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + div(kgradT) + S_E \end{aligned} \quad (2.4.14)$$

where  $E = i + \frac{1}{2}(u^2 + v^2 + w^2)$ , and  $S_E$  is the rate of increase of energy due to sources.

Extracting the changes of the kinetic energy from the above equation of specific energy  $E$ , the equation of internal energy  $i$  for Newtonian model with viscous stresses are obtained after some rearrangement as (Versteeg & Malalasekera, 2007):

$$\frac{D(\rho i)}{Dt} = -pdiv(\mathbf{u}) + div(kgradT) + \Phi_v + S_i \quad (2.4.15)$$

where  $S_i$  is the rate of increase of internal energy due to sources, and for the special case of an incompressible fluid we have  $i = c_p T$ , where  $c_p$  is the specific heat capacity and  $div\mathbf{u} = 0$ . Moreover,  $\Phi_v$  is the dissipation function, which describes all the effects of viscous stresses in the internal energy, after considerable algebra, given as:

$$\begin{aligned} \Phi_v = & 2\mu \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] + \\ & \mu \left[ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 \right] + \varrho(div\mathbf{u})^2 \end{aligned} \quad (2.4.16)$$

Note that  $\Phi_v$  represents a source of internal energy due to deformation work on the fluid particle which causes the motion and changed into internal energy or heat. So, its effect is considered in most of the problems we have addressed in this study.

#### 2.4.4 Initial and Boundary Conditions

The equations of continuity, momentum, energy, and concentration, composed with the suitable constitutive associations afford a comprehensive mathematical description of a Newtonian fluid flow (Cengel & Ghajar, 2015). To solve this complex set

of governing equations, we have to identify a proper set of boundary and initial conditions for the flow problem being examined. The main set of unknowns for which a result is required generally consists of the three components of a velocity, density, pressure, temperature, and concentration of the fluid flow. However, in the study, since incompressible nanofluid is considered, density to be fixed. A comprehensive discussion of the needed boundary conditions is contingent on the exact nature of the problem, the approximations used, and the set of equations to be solved (Edward et al., 2005). Although most of the boundary conditions related with the fluid velocity field are usually the no-slip, no-penetration conditions, for the study as well we have considered slip velocity and permeability conditions. Moreover, for temperature and concentration distribution also we have employed jump and convective boundary conditions. In an unsteady flow problem, the initial conditions precede the form of the stipulation of the spatial dispersion of the unknowns such as velocities, temperature, and concentration of the nanofluids flow at an initial instant of time. The choice of suitable boundary and initial conditions will be revealed for each problem in chapters 4-9, where the governing equations of the flow problems were studied with numerical solutions.

## 2.5 Nanofluids

Fluids of different types are usually used as heat transporters in heat transfer applications. Such applications where heat transfer fluids have an important role are heat exchanging systems in power stations, cooling and heating systems in buildings, vehicles air conditioning system in transportation, and cooling systems of most of the processing plants (Ali et al., 2018). In the aforementioned applications, the heat transfer fluids thermal conductivity has a strong influence on the efficiency of the heat transfer process and with it the overall efficiency of the system. For this reason, researchers have continuously worked on developing advanced heat transfer fluids that

have significantly higher thermal conductivities than common used fluids (Choi & Eastman, 1995). In 1995, Choi was the first researcher who developed a newly innovative class of heat transfer fluids and gave a name for such type of fluids the term "nanofluids" (Ali et al., 2018; Choi & Eastman, 1995). He prepared this fluids on suspending nanoscale particles of metallic origin with an average particle size of less than 100 nm into conventional heat transfer fluids. Choi noticed that the addition of 1vol% of nanoparticles to the usual fluids increases the thermal conductivity of the fluid up to approximately twice.

Commonly used nanoparticles for nanofluids formulation are aluminium (Al), copper (Cu), silver (Ag), iron (Fe), titanium (Ti), silicon (Si), zinc (Zn), magnesium (Mg), carbon nanotubes (CNTs), graphene, graphene oxide, and diamond. Water, ethylene glycol (EG), EG-H<sub>2</sub>O and oils mixtures are frequently used base fluids for nanofluid formulation (Ali et al., 2018). Nanofluid can be formed from nanoparticles of single element such as Cu, Fe, and Ag, single element oxide such as CuO, Cu<sub>2</sub>O, Al<sub>2</sub>O<sub>3</sub>, and TiO<sub>2</sub>, alloys like Cu-Zn, Fe-Ni, and Ag-Cu, multi element oxides like CuZnFe<sub>4</sub>O<sub>4</sub>, NiFe<sub>2</sub>O<sub>4</sub>, and ZnFe<sub>2</sub>O<sub>4</sub>, metal carbides for example, SiC, B<sub>4</sub>C, and ZrC, metal nitrides for example, SiN, TiN, and AlN, and carbon materials such as graphite, carbon nanotubes, and diamond suspended in water, ethanol, EG, oil, and refrigerants (Gupta et al., 2018)

In nanofluids flow, nanoparticles do not follow the fluid streamlines passively. In fact, there are some reasons that induce a slip velocity between the nanoparticles and the base fluid. Movements of nanoparticles have significant impact on rheological and thermophysical properties of the nanofluids (Azizian et al., 2013). Therefore, investigating the nanoparticles motion is critical for evaluating the performance of nanoparticles inclusion to the base fluid as a heat transfer medium. Since the nanoparticles are very small (100 nm), Buongiorno (2006) attempted to explain the increase in the thermal conductivity of such fluids and developed a model that took into account

the particle Brownian motion and thermophoresis. He concluded that brownian and thermophoretic diffusivity are the main slip mechanisms in nanofluids.

### 2.5.1 Effective Thermophysical Properties of Nanofluids

The main challenges of nanofluids are their stability and operational performance. Nanofluids stability is significantly important in order to maintain their thermophysical properties after fabrication for a long period of time. Due to the interaction between the particles themselves and between the particles and the surrounding liquid, nanofluids have poor stability (Dey et al., 2017). Nanofluids are formed with entirely different thermophysical properties such as density, specific heat capacity, thermal conductivity, convective heat transfer, thermal diffusivity, and viscosity (Ali et al., 2018). Because of this, they are considered superior to their base fluid. To differentiate the thermophysical properties of the base fluid and the fabricated nanofluid, the term "effective" is normally used to describe the thermophysical properties of nanofluids.

The *effective density* of a nanofluid can be theoretically calculated using its nanoparticle density ( $\rho_s$ ) and base fluid density ( $\rho_{bf}$ ) since it is expected to be a mixed property of both, nanoparticles and base fluid. Employing mixing theory, nanofluid effective density ( $\rho_{nf}$ ), is defined as (Ali et al., 2018).

$$\rho_{nf} = \phi_s \rho_s + (1 - \phi_s) \rho_{bf} \quad (2.5.1)$$

where  $\phi_s = \frac{V_{np}}{V_{np} + V_{bf}}$  is particle volumetric fraction for  $V_{np}$  and  $V_{bf}$  are the nanoparticles volume and base fluid volume, respectively.

The *effective specific heat* of a nanofluid  $(c_p)_{nf}$  is a significant property that influences the heat transfer rate of a nanofluid. It is the extent of heat required to rise the temperature of one gram of nanofluid by one degree Celsius. To compute the

effective specific heat of nanofluids, two main effective specific heat models were recommended. From, these we have employed the first model which was recommended by Pak and Cho (1998), based on the volume concentration of nanoparticles and the liquid-particle mixture formula given as (Ali et al., 2018).

$$(c_p)_{nf} = \phi_s(c_p)_s + (1 - \phi_s)(c_p)_{bf} \quad (2.5.2)$$

where  $(c_p)_s$  and  $(c_p)_{bf}$  are the specific heat capacity of the nanoparticles and the basefluid, respectively.

*Nanofluid effective viscosity* is defined as the ratio of the shear stress to the shear rate, and it is a measure of the tendency of the suspension to resist the flow. The advantage related with nanofluids heat enhancement is responded by the upswing in effective viscosity instigated from the added nanoparticles in the base fluid. This growth in viscosity leads to higher pressure losses and hence lifts the pumping power demands. The key parameters that affect the effective viscosity are the particles type, particle shape, base fluid viscosity, nanoparticles concentration, temperature, pressure, shear rate, and pH value. The effective viscosity ( $\mu_{nf}$ ) used for this study was first driven from the base fluid viscosity ( $\mu_{bf}$ ) by Einstein (1906), to calculate the effective viscosity of suspensions of spherical solids, and is given as

$$\mu_{nf} = \mu_f(1 + 2.5\phi_s) \quad or \quad \mu_{nf} = \frac{\mu_{bf}}{(1 - \phi_s)^{2.5}} \quad (2.5.3)$$

Until now, this model is the only existing common correlation that can calculate the effective viscosity of nanofluids of low concentration.

*Effective Thermal Conductivity:* One of the basic ideas of nanofluids is the enrichment of the thermal conductivity related to common fluids, which improves the fluid convective heat transfer. Various theoretical and experimental works have been carried out to study the variations in nanofluids thermal conductivity. Using the thermal conductivities of both nanoparticles ( $k_s$ ) and base fluid ( $k_{bf}$ ), Maxwell model

was developed in 1881 for the first time to predict the effective thermal conductivity ( $k_{nf}$ ) of solid-liquid dispersion. The model considers the two phases (solid and liquid) of the nanofluid and adequately predicts the effective thermal conductivity of nanofluids when the added particles are of spherical shape, low vol percent, and the suspension is at ambient conditions (Ali et al., 2018). Modified Maxwell model that determines the effective thermal conductivity and employed for some problems of this study is given as

$$\frac{k_{nf}}{k_{bf}} = \frac{k_s + (n-1)k_{bf} - (n-1)\phi_s(k_{bf} - k_s)}{k_s + (n-1)k_{bf} + \phi_s(k_{bf} - k_s)} \quad (2.5.4)$$

where  $n$  is the shape factor, and  $n = \frac{3}{\varphi}$  and  $\varphi = 0.5$  for cylindrical particles or  $\varphi = 1$  for spherical particles.

Also, the effective thermal conductivity of nanofluid  $k_{nf}$ , the Xue's model which is effective for spherical and elliptical shape is utilized as follows (Alsagri et al., 2018).

$$\frac{k_{nf}}{k_f} = \frac{1 - \phi_s + 2\phi_s \left( \frac{k_s}{k_s - k_{bf}} \right) \ln \left( \frac{k_s + k_{bf}}{2k_{bf}} \right)}{1 - \phi_s + 2\phi_s \left( \frac{k_{bf}}{k_s - k_{bf}} \right) \ln \left( \frac{k_s + k_{bf}}{2k_{bf}} \right)} \quad (2.5.5)$$

Despite the fact that, also there are some models that can predict the effective thermal conductivity of nanofluids flow problems, only the above two models are employed for this study.

## 2.5.2 Hybrid Nanofluids

Nowadays, much effort has been given to develop new experimental work on obtaining well-defined and highly stable nanoparticles which enhance the thermal conductivity of base fluid. The new classes of fluids prepared with hybrid (two or more) nanoparticles are known as hybrid nanofluids. Several studies show that hybrid nanoparticles have the ability to improve the heat transfer rates more effectively with low cost as compared to single nanoparticles. Contingent on the composite materials matrix, the

hybrid nanoparticles can be divided into three categories, metal matrix, ceramic matrix and polymer matrix nanocomposites. Some examples of metal matrix nanocomposites are  $\text{Al}_2\text{O}_3/\text{Cu}$ ,  $\text{Al}_2\text{O}_3/\text{Ni}$ ,  $\text{MgO}/\text{Fe}$ ,  $\text{Al}/\text{CNT}$ ,  $\text{Mg}/\text{CNT}$ ,  $\text{Al}_2\text{O}_3/\text{Fe-Cr}$ , and  $\text{ND}/\text{Ni}$ , while  $\text{Al}_2\text{O}_3/\text{SiO}_2$ ,  $\text{Al}_2\text{O}_3/\text{Ni}$ ,  $\text{CNT}/\text{Fe}_3\text{O}_4$ ,  $\text{Al}_2\text{O}_3/\text{SiC}$ , and  $\text{Al}_2\text{O}_3/\text{CNT}$  are among some examples of ceramic matrix nanocomposites. Besides, examples of polymer matrix nanocomposites are polymer/layered double hydroxides, polymer/CNT, thermoplastic/thermoset polymer/layered silicates, and polyester/ $\text{TiO}_2$  (Sundar et al., 2014).

Some studies show that hybrid nanofluids have significantly improved thermal and mechanical properties than regular fluids or single nanoparticles-based nanofluids (Sundar et al., 2014). These hybrid nanofluids have significant applications in various engineering and biomedical fields such as, manufacturing, transportation, nuclear safety, military, in modern electronic devices and supercomputers, in pharmaceutical and drug delivery (Manjunatha et al., 2019). Due to the mentioned benefits, many scholars have interested in hybrid nanofluids investigation. Using the polymer method, Shi et al. (2010) synthesized  $\text{CNT}/\text{Fe}_3\text{O}_4$  hybrid nanoparticles. Sundar et al. (2017) experimentally synthesized  $\text{MWCNT}/\text{Fe}_3\text{O}_4$  water based hybrid nanofluids, and he found hybrid nanofluids create higher thermal conductivity and heat transfer enhancement than single nanoparticles-based nanofluids. Similarly, he observed the viscosity of  $\text{MWCNT}-\text{Fe}_3\text{O}_4$  water based hybrid nanofluids significantly improved as compared to base fluid. An experimental study on the thermal properties of  $\text{MWCNT}-\text{SiO}_2/\text{EG}$  was conducted by Esfe et al. (2018) and they reported that hybrid nanofluid are more efficient and economical compared with nanofluid. Also, Esfe et al. (2018) analyzed the effects of temperature and volume fraction on thermal conductivity of  $\text{SWCNT}-\text{Al}_2\text{O}_3/\text{EG}$  hybrid nanofluid. Their result indicated that the highest enhancement of thermal conductivity is 41.2 % which was realized at the volume fraction of 2.5% and temperature of  $50^\circ\text{C}$ .

### 2.5.3 Carbon Nanotubes

Nowadays, heat transfer of carbon nanofluids have received considerable attention as a result of their extensive applications in the fields of nanotechnology and medicine. Nanoparticles of carbon nanotubes (CNTs) were first discovered by Lijima (1991). They have a remarkable conductivity which supports them to form a network of conductive tubes. Because of their configuration, some CNTs are good conductor of heat and electricity whereas others act as semiconductors. CNTs have 15 times thermal conductivity and 1000 times capability of copper, and 200 times power and 5 times resistance of steel (Khalid et al, 2018). Carbon nanotubes have significance applications in nanotechnology, hardware, optics, energy storage, biomedical, ceramic, thermal defense and in various fields of material sciences and engineering (Hayat et al, 2017).

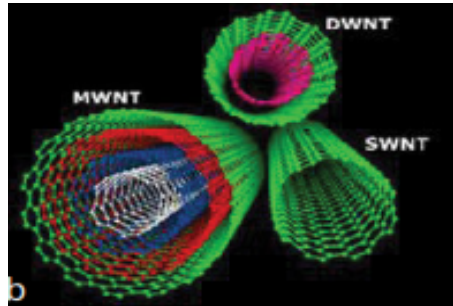


Figure 2.4: Carbon nanotubes structures.

Carbon nanotubes used in nanofluids are generally divided into single wall carbon nanotubes (SWCNTs) and multi wall carbon nanotubes (MWCNTs) contingent on their number of concentric layers of rolled grapheme sheets. SWCNTs contain catalyst for their synthesis and they are prepared from covering layer of grapheme into a unified cylinder whereas MWCNTs can be formed without catalyst and comprise of numerous rolled layers of graphite with complex structure(Khan et al., 2018), as shown in Figure

2.4, (Source: Dawar et al., 2018). MWCNTs are fabricated of coaxial cylinders using the systematic layout among layers, nearby central hollow. The number of layers defines the internal and external diameter of MWCNTs. Internal diameter ranges from 0.4 nm and grows up to a few nanometers, while the external diameter varies from 2 nm to 30 nm. MWCNTs frequently have locked by half-fullerene molecules which have a dome shaped structure in both tips. Axial length of MWCNTs varies between 1 mm and a few centimeters. SWCNTs diameters, likewise, vary from 0.4 to 3 nm, and their length is in the micrometer range (Samadishadlou et al., 2018).

Because of their unique morphology, electronic structural, mechanical properties and innovative physicochemical features, CNTs are the most resourceful material of this century. Moreover, the presence of carbon chains in CNTs does not convey any hazard to the atmosphere (Alsagri et al. 2019). The viscosity of MWCNT-water nanofluid was investigated using simulating equilibrium dynamical model by Jabbari et al. (2019). They reported a growth in viscosity with increases in nanoparticle volume fraction at low temperature. Also, Sreedevi et al. (2019) investigated the MHD effects on fluid transport properties of chemically reacted water-based SWCNT/MWCNT nanofluid over a vertical cone embedded in a porous medium. Mahanthesh et al. (2017) investigated the Marangoni transport of dissipating SWCNT and MWCNT with water nanofluids under the influence of magnetic force and radiation. They reported that the thermal distribution of SWCNT nanoliquid is better than MWCNT nanoliquid. Due to modern applications of Carbon nanotubes in various technologies and biomedical field, nanofluids of CNTs are examined in chapters 6-8 of this paper.

## 2.6 Non-Newtonian Fluids

The analysis of non-Newtonian fluids have attained a significant achievement as a result of its wide applications in many branches of material processing engineering.

Several non-Newtonian models show the nonlinear relationship between stress and rate of strain. The well-known Navier-Stokes equations of flow may not completely describe the noticeable features of non-Newtonian fluids.

The non-Newtonian flows through a porous medium have gained much attention in nowadays research because of its significance in many engineering applications, such as composite materials manufacturing processes, filtration processes, insulation systems, and enhanced oil recovery (Pramanik, 2013).

Casson fluid is among non-Newtonian fluids model because of its unique rheological features in relation to the shear stress-strain liaison. It begins to flow when yield stress is less than the shear stress, and it behaves like an elastic solid when shear stress is less than the yield stress. The nonlinear Casson's constitutive equation which designates the properties of several polymers over a broad range of shear rates was for the first time derived by Casson (1959). Human blood, honey, soup, orange guice and tomato sauce are some examples of fluids that show Casson fluid features. The study of Casson fluid has received much attention by many researchers because of its application in food processing, metallurgy, bio-engineering operations, drilling operations, manufacturing of pharmaceutical products and blood due to its contents such as plasma, fibrinogen and protein (Makanda et al., 2015). Due to its various application, Casson fluid flow model was considered in this study.

## **2.7 Some Properties Affecting Fluid Flow**

### **2.7.1 Magnetohydrodynamics (MHD)**

The basic interaction of the magnetic fields and the flow of electrically conducting fluids such as liquid metals, strong electrolytes and hot ionized gases (plasmas) are core for the emerging of the field magnetohydrodynamic (MHD). MHD is the study of fluid flow in electrically conducting fluids with magnetic properties that affect

fluid flow. When a magnetic field is incident in an electrically conducting fluid, a current is induced. This effect polarizes the fluid and as a result the magnetic field is changed (Makanda et al., 2015). The study of MHD flow has wide applications in technological processes, for example in MHD power generator designs, design of cooling systems, plasma studies, petroleum industries, design for cooling of nuclear reactors, construction of heat exchangers, blood flow measurements, electric motors, flow meters and pumps etc. (Rehman et al. 2017).

Consequently, different studies on MHD boundary layer flow of nanofluid and heat transfer are include the work of Alam et al. (2017) who analyzed unsteady MHD flow and heat transfer of a nanofluid over a moving wedge, and Nageeb et al. (2017) investigated the influences of thermal radiation on nanofluid flow over a stretching sheet in the presence of a magnetic field. Moreover, MHD boundary layer flow of nanofluid over a continuously moving stretching surface was examined by Rasheed et al. (2017). Also, Nagendramma and Leelaratnam (2017) analyzed MHD boundary layer slip flow of a Casson nanofluid through a porous medium over a stretching cylinder. Their result shows that the velocity profile is an increasing function, whereas, temperature and concentration profiles are decreasing functions of the magnetic parameter. Hence, the effects of MHD on boundary layer flow of nanofluid is analyzed in this research.

### **2.7.2 Porous Media**

A porous medium is a region which contains pores that offers resistance to fluid flow. The volume of the fluid to flow across a porous medium is determined by the porosity or permeability of the medium (Makanda et al., 2015). The basic law governing the flow of fluids through porous media is Darcy's Law. The law states that the flow rate is a function of fluid pressure, flow area and elevation. Darcy's law applies only under certain conditions that is, Darcy's law does not hold while inertial forces are dominant.

Under these conditions the fluid flow was described as non-Darcy (Makanda et al., 2015).

Due to its vast applications, many studies have been performed on fluid flow in porous media. These include the works of Pal and Mandal (2015) who investigated flow of nanofluids in porous medium, and Arthur et al. (2015) studied MHD convective boundary layer flow towards a vertical surface in a porous medium. Furthermore, Sheri et al. (2016) studied the transient MHD free convection flow past a vertical plate in a porous medium. Raju et al. (2016) numerically investigated unsteady MHD free convection flow past a vertically inclined porous plate. Unsteady mixed convection flow with magnetic field and radiation over vertical plate in porous medium was analyzed by Rajput et al. (2017). Also, Alama et al. (2016) analyzed unsteady MHD boundary layer flow of nanofluid and heat transfer past a moving wedge in a porous medium. They concluded that presence of porous parameter is helpful to generate heat in the system. Thus, the boundary layer flow of nanofluids embedded in a porous medium is the subject of this study.

### **2.7.3 Viscous Dissipation**

Viscous dissipation is the heat energy that is produced as a result of friction between fluid layers. It changes the temperature distribution by playing a role like an energy source. It leads to affected heat transfer rate and hence sometimes it necessarily to be considered in fluid flow with heat transfer problems (Pal & Mandal, 2015). The effect of viscous dissipation is mostly considered in non-Newtonian fluids such as in power-law fluids (Makanda et al., 2015). In modeling fluid flow, the dimensionless parameters that describe viscous dissipation is the Eckert number ( $Ec$ ). Viscous dissipation plays an important role in natural convection such as polymer processing, geological processes, and in strong gravitational field processes. It plays a practical role in oil products transportation through ducts (Arthur & Seini, 2014b). Many

researchers have analyzed the influences of viscous dissipation on hydrodynamics, these include the works of Wahiduzzaman et al. (2015), and Sheri et al. (2016) to mention few.

There are some studies that analysis the effects of viscous dissipation on boundary layer flows of nanofluids in porous and nonporous media. Pal and Mandal (2015) studied mixed convection of nanofluids flow with heat generation and viscous dissipation. Heat and mass transfer of MHD flow of nanofluids in the presence of viscous dissipation effects was numerically analyzed by Haile and Shankar (2015). They noticed that viscous dissipation parameter enhance the temperature profile, yet it reduce concentration profile. Majety et al. (2016) studied the effect of viscous dissipation on MHD boundary layer flow past a wedge through porous medium. They concluded that viscous dissipation produces heat due to drag between the fluid particles, which cause an increase in fluid temperature, and the heat transfer rate at the boundary surface decreases with increasing in Eckert number. Thus, the effects of viscous dissipation was also considered in this study.

#### **2.7.4 Cattaneo-Christov Heat Flux Model**

The dynamics of heat transfer phenomenon is a subject of special attention due to its abundant applications in engineering and industrial processes including energy production, nuclear reactor cooling, space cooling, biomedical applications such as heat conduction in tissues, magnetic drug targeting, and many others (Abbasi et al., 2016; Shehzad et al., 2016). Fourier was the first who developed the most successful classical heat flux model in continuum mechanics in 1822 (Akbar et al., 2017). One of the main shortcomings of Fourier's law is that it leads to a parabolic energy equation for the temperature field. This means that any initial disturbance is felt instantly throughout the entire medium. That is, the whole system (medium) is instantly influenced by the initial disturbance. This behavior is said to contradict the principle

of causality. In order to overcome this unrealistic feature which is known as the "paradox of heat conduction", various modifications of the Fourier's law have been proposed over the years. Of these the successful and best known is the Maxwell-Cattaneo model which was named after Cattaneo (Christov, 2009).

Cattaneo (1948) modified classical Fourier's law of heat conduction by adding the thermal relaxation time while studying the flow of Maxwell fluid.

$$(1 + \tau_0 \partial_t) \mathbf{q} = -k \nabla T \quad (2.7.1)$$

where  $\mathbf{q}$  is the thermal flux vector,  $T(x, y, z, t)$  is temperature, and  $k > 0$  stands for thermal conductivity of the medium. The minus sign indicates that heat transfer is in the direction of decreasing temperature,  $\partial_t$  is partial time derivative, and the thermal relaxation characteristics time  $\tau_0 > 0$  represents the time lag required to establish steady heat conduction in a volume element once a temperature gradient has been imposed across it.

Christov (2009) further modified the Maxwell-Cattaneo model by replacing the partial time derivative with the Oldroyd's upper-convected derivative in order to preserve the material-invariant formulation.

$$\tau_0 \left[ \frac{\partial \mathbf{q}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{u} + (\nabla \cdot \mathbf{u}) \mathbf{q} \right] + \mathbf{q} = -k \nabla T \quad (2.7.2)$$

consequently, this model is known as the *Cattaneo-Christov heat flux model*.

A comparative study for Cattaneo-Christov heat flux model in boundary layer flow generated by a linear stretching sheet with a consideration of the two classes of viscoelastic fluids namely, second grade fluid and viscoelastic fluid was presented by Hayat et al. (2016). Furthermore, Prakash (2017) discussed the effect of Cattaneo-Christov heat flux model on nonlinear radiative MHD flow of Casson fluid over a semi-infinite stretching surface in the presence of nonlinear thermal radiation.

### 2.7.5 Slip Boundary Conditions

The Naviere-Stokes theory of fluid transport is based on the no-slip boundary condition. However, there are conditions in which no-slip condition is not suitable. Especially, no-slip boundary condition is inappropriate for most non-Newtonian fluids science some polymer melts normally show tiny wall slip and generally it is governed by a non-linear and a monotonic relation between the adhesive friction and the slip velocity (Halim et al., 2016). Fluids offering boundary slip have important applications in various technological and biomedical fields such as purifying of internal cavities and artificial heart valve. Due to this, the slip boundary condition has been considered up to now by a number of researchers. Among these, (Nagendramma, 2017; Tlili, 2018) studied the effects of slip boundary condition on the flow and heat transfer of nanofluid over some geometries. Moreover, the flow of fluid with second order slip boundary condition has been not got much attention so far by the researchers. But, second order slip flow takes place in several engineering applications. Thus, in this study, both first and second order velocity slip boundary conditions were investigated.

# Chapter 3

## Methodology of the Study

### 3.1 Approaches Used in the Study

To conduct this study, the research title was selected after intensive review of related literatures. The relevant information was gathered from different books, journals, articles and web sites. The collected information, concepts, facts and methods used by other authors were understood and examined in detail. Subsequently, the usefulness of concept of the research title area was organized and explored. As a result, the problem of both two- and three-dimensional steady/unsteady laminar boundary layer flow of nanofluids/hybrid nanofluids past some geometrical shapes (wedge, cylinder, disc, and cone) embedded in a porous media with the effects of some parameters was considered for this study.

To highlights the problems under study, the researcher has considered the conservation equation for a general scalar variable  $\Phi$  which can be expressed as ( see Versteeg & Malalasekera, 2007)

$$\frac{\partial(\rho\Phi)}{\partial t} + \nabla \cdot (\rho \mathbf{u} \Phi) = \nabla \cdot (\Gamma \nabla \Phi) + S_{\Phi} \quad (3.1.1)$$

where the velocity vector given by  $\mathbf{u} = \langle u, v, w \rangle$ , and  $\rho$  is density of the fluid. Equation (3.1.1) is the so-called *transport equation for property*  $\Phi$ . It clearly highlights the various transport processes: the transit term and the convective term on the left

hand side, and the diffusive term (  $\Gamma$  = diffusion coefficient) and the source term respectively on the right hand side.

Setting  $\Phi$  equal to 1,  $\mathbf{u}$ ,  $T$ , and  $C$  in equation (3.1.1), and selecting appropriate values for diffusion coefficient  $\Gamma$  and source terms, the special forms of PDEs respectively for mass, momentum, energy, and concentration has been obtained. For incompressible Newtonian nanofluids flow with constant nanofluid properties, these equations are:

$$\nabla \cdot \mathbf{u} = 0 \quad (3.1.2)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho_{nf}} \nabla p + \nu_{nf} \nabla^2 \mathbf{u} + \frac{1}{\rho_{nf}} F_b \quad (3.1.3)$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \alpha_{nf} \nabla^2 T + \frac{1}{(\rho c_p)_{nf}} S_e \quad (3.1.4)$$

$$\frac{\partial C}{\partial t} + (\mathbf{u} \cdot \nabla) C = D_B \nabla^2 C + S_m \quad (3.1.5)$$

where  $\rho_{nf}$ ,  $\nu_{nf}$ ,  $\alpha_{nf}$ ,  $(c_p)_{nf}$  and  $D_B$  are respectively, effective density, kinematic viscosity, thermal diffusivity, specific heat capacity and brownian diffusivity of the nanofluid.  $p$  is pressure term,  $F_b$  is body force,  $T$  is temperature of the nanofluid,  $C$  is concentration of the nanofluid,  $S_e$  is additional terms of heat transfer, and  $S_m$  is additional terms of mass transfer.

Using the above equations (3.1.2)-(3.1.5) with boundary layer flow behaviors of the nanofluids and boundary conditions of the problems; the governing equation (non-linear partial differential equations) was obtained for each problem. Then employing suitable similarity transformations, the governing partial differential equations were transformed into systems of nonlinear higher order ordinary differential equations. Then, the transmuted equations were shown to be controlled by a number of thermo-physical parameters. However, for the features of nanofluids flow and heat transfer, unique rheological model was formulated for Casson non-Newtonian fluid according

to its nature. Moreover, Cattaneo-Christov heat flux model was used in some fluids flow with heat transfer problems.

For this study the numerical technics was employed, and the results of the study were analyzed using graphs and tables. The transformed non-linear terms of ordinary differential equations were linearized, and the Chebychev pseudo-spectral collocation method was applied to it. Consequently, the appropriate MATLAB codes were developed, then the problems were solved numerically using *iterative spectral collocation methods*. The accuracy of the methods were checked against the previously published results. Flow velocity, temperature and concentration profiles with their boundary layer thickness were graphically analyzed via different thermophysical parameters. Moreover, the important physical quantities of engineering aspects in the study such as the skin friction coefficient  $C_f$ , local Nusselt number  $Nu$ , and local Sherwood number  $Sh$  were discussed via different thermophysical parameters. Finally, based on obtained results, conclusions and recommendations were drown.

## 3.2 Spectral Collocation Method

Most of the methods of solving the boundary layer problems are the numerical approach based on the shooting algorithm with Runge-Kutta scheme, finite difference method, spectral homotopy analysis method, finite element Galerkin method, and Newton-Raphson based methods (Motsa et al., 2014a). In recent times, spectral based numerical techniques such as the *Spectral quasilinearization method*, *Spectral local linearization method*, and *Spectral relaxation method* were developed (Motsa et al., 2014; Motsa, 2013a).

The spectral methods can be described as an approximation to the solution of a differential equation using a finite series of smooth basis functions. Spectral methods approximate functions using truncated series of orthogonal functions like the

Chebyshev and Legendre polynomials (Gheorghiu, 2007). Derived from the method of weighted residuals, spectral methods enjoy a so called exponential rate of convergence compared to the polynomial convergence rate of most other numerical methods. The spectral methods transform a function from a physical space to the spectral space, and apply the concept of global representations in obtaining high order approximations (Gottlieb, 2001).

There are three types of spectral methods. These are the collocation spectral method, the Tau spectral method and the Galerkin spectral method. The Tau and Galerkin spectral methods are weighted residual methods while the collocation spectral method is not (Gottlieb, 2001). As reported by Babolian et al. (2007), the Tau and Galerkin methods are applied on expansion coefficients while the collocation methods are applied on physical space values of unknown functions. The conditioning of differentiation matrices in spectral collocation method is explained in (Trefethen, 2000). In spectral collocation method, errors occur in the process of calculating the differentiation matrices for Chebyshev nodes (Baltensperger et al., 1999). Applications of spectral collocation method also includes periodic geometries (Trefethen, 2000). According to Boyd (2001), spectral collocation method is also applicable to flows with shock waves and fronts. Further, spectral collocation method has been applied successfully in multivariable cell population models, in weather forecasting and in many other applications (Mantzaris et al., 2001; Juang et al., 1994).

There are several features that set spectral collocation method apart from other numerical solutions to differential equations. The first is the computational domain. Finite element method (FEM), for example, the overall physical domain is broken up into a number of sub domains and a local basis function is chosen to be non-zero over a couple of sub-intervals within that domain. Conversely, spectral collocation

method chooses a basis function that is global to the entire computational domain and is non-zero except at isolated points. Due to this reason, spectral collocation methods is often referred to as a global numerical method (Thibeault, 2009). The second distinguishing feature of spectral collocation method is the choice of basis functions. Spectral collocation methods selects basis functions that are high degree polynomials like Chebyshev and Legendre polynomials or trigonometric polynomials that are infinitely differentiable. The basis functions for FEM are generally low-order polynomials and local in nature. Because of this, FEM are better suited for complex geometries but suffer from lower accuracy as compared to spectral methods (Thibeault, 2009).

The third feature of spectral collocation method is easy to implement, its accuracy and efficiency, and adaptable to various problems, including variable coefficient and nonlinear differential equations (Motsa et al., 2014a). The widespread use of spectral collocation method has been motivated by its accuracy and efficiency in solving incompressible Navier-Stokes equations (Gottlieb & Hesthaven, 2001). Zhou (2014) also added that spectral collocation method provides more accurate approximations with a relatively small number of unknowns, and so play increasingly important roles in optimizing engineering design and other scientific computations. Motsa et al. (2004a) added that, the interest in using Chebyshev spectral collocation method in solving nonlinear PDEs stems from the fact that these method requires less grid points to achieve accurate results.

### 3.2.1 Collocation Points

The pseudo-spectral collocation technique as described in (Trefethen, 2000) is based on Chebyshev polynomial of first kind  $T_n(x)$  is defined as

$$T_n(x) = \cos(nt) \quad \text{for } t = \arccos(x) \quad n = 0, 1, 2, \dots \quad (3.2.1)$$

on the domain  $-1 \leq x \leq 1$ .

The collocation (grid) points are based on *Chebyshev-Gauss-Lobatto points* that are defined as:

$$x_j = \cos(j\pi/N) \quad -1 \leq x \leq 1; \quad j = 0, 1, 2, \dots, N. \quad (3.2.2)$$

Geometrically, it can be visualize that these collocation points as the projections on  $[-1,1]$  to the x-axis of equally spaced points on the upper half of the unit circle as sketched in Figure 3.1, (Source: Trefethen, 2000). Note that they are numbered from right to left.

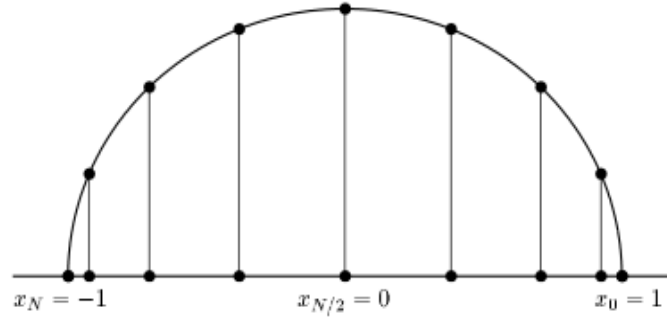


Figure 3.1: Chebyshev collocation points.

### 3.2.2 Differentiation Matrix

The unique aspect of Chebyshev polynomials is that the basis functions and their derivatives can be constructed using trigonometric functions. Also note that, differentiation as defined here, can be achieved in either transform space or in physical space. For this research, to apply the spectral collocation method successfully, all differentiation is done in transform space. The basic idea of the spectral collocation method is to approximate unknown functions (the functions at  $r + 1$ ) in system of

equation in the full domain using an interpolating higher order Lagrange polynomial at the given collocation points as stated by Trefethen (2000). These functions are approximated using the *Lagrange interpolating polynomials* as shown below.

$$f(x) \approx \sum_{k=0}^{N_x} f_k l_k(x_j) \quad (3.2.3)$$

where  $l_k$  represents the cardinal functions and are polynomials of degree  $N$  that satisfy equation

$$l_k(x_j) = \begin{cases} 1 & , \quad k = j \\ 0 & , \quad k \neq j \end{cases} \quad (3.2.4)$$

these are defined by

$$l_k(x_j) = \prod_{j=0; k \neq j}^N \frac{x - x_k}{x_j - x_k} \quad (3.2.5)$$

Interpolating functions are differentiated to approximate derivatives of the function  $f(x)$  as shown below.

$$f^p(x) = \sum_{k=0}^{N_x} f_k l_k^p(x); \quad p = 1, 2, \dots, N. \quad (3.2.6)$$

where  $f^p(x)$  is the  $p^{th}$  derivative of  $f(x)$ .

Functions and their derivatives are evaluated by collocating at selected grid points as follows:

$$f^p(x_j) = \sum_{k=0}^N D_{jk}^p f_k = DF \quad p = 1, 2, 3, \dots \quad (3.2.7)$$

where  $D$  is the differentiation matrix of size  $(N + 1) \times (N + 1)$ , and

$F = [f(x_0), f(x_1), \dots, f(x_N)]^T$  is vector of size  $(N + 1) \times 1$ . The entries of matrix  $D$  are computed as described below by Chebyshev differentiation matrix.

For each  $N \geq 1$ , let the row and columns of the Chebyshev differentiation matrix  $D$  be indexed from 0 to  $N$ . As described by Trefethen (2000), the entries of these matrix are given as follows:

$$(D_N)_{00} = \frac{2N^2 + 1}{6} = -(D_N)_{NN}$$

$$(D_N)_{jj} = \frac{-x_j}{2(1-x_j^2)}; \quad j = 1, 2, \dots, N-1. \quad (3.2.8)$$

$$(D_N)_{jk} = \frac{c_k (-1)^{k+j}}{c_j x_k - x_j}; \quad k \neq j \quad j, k = 1, 2, \dots, N-1.$$

where

$$c_k = \begin{cases} 2 & , \quad k = 0 \quad \text{or} \quad N, \\ 1 & , \quad \text{otherwise} \end{cases}$$

This is expressed best visually and  $D_N$  is shown below in Figure 3.2, (Source: Trefethen, 2000).

|                                     |                                |                                        |
|-------------------------------------|--------------------------------|----------------------------------------|
| $\frac{2N^2+1}{6}$                  | $2 \frac{(-1)^j}{1-x_j}$       | $\frac{1}{2}(-1)^N$                    |
| $\frac{-1}{2} \frac{(-1)^k}{1-x_k}$ | $\frac{(-1)^{j+k}}{x_k - x_j}$ | $\frac{1}{2} \frac{(-1)^{k+N}}{1+x_k}$ |
| $\frac{-1}{2}(-1)^N$                | $-2 \frac{(-1)^{j+N}}{1+x_j}$  | $-\frac{2N^2+1}{6}$                    |

Figure 3.2: Chebyshev differentiation matrix D.

For the above differentiation matrix, the matlab code is developed and it is remind whenever the problems under the study are solved.

### 3.2.3 Spectral Quasilinearization Method (SQLM)

The quasilinearization method is a generalization of the Newton-Rhapson method that was originally used by Bellman and Kalaba (1965) for solving nonlinear boundary value problems. Motsa (2013a) further extended the application of the quasilinearization method by applying spectral collocation method to solve the linearized equation. Reduction of order of equations is not necessary when the spectral quasilinearization method (SQLM) is applied. Rather, nonlinear terms are identified in the system of

equations and linearized using the Taylor series expansion for multiple variables to obtain a linear coupled system of equations as given by Otegbeye (2014). Chebyshev pseudospectral collocation method is applied to solve the linear systems. The systems are solved starting with an initial approximation (or guess that satisfies all boundary conditions) which is used to generate subsequent approximations iteratively until convergence is obtained at some point. The concept of the SQLM is similar to that of the successive linearisation method which is another Newton-Rhapson based method Otegbeye (2014). The limitation of the SQLM is that the coupled high-order system of differential equations may often lead to very large systems of algebraic equations that may require significant computing resources. In addition, the actual process of developing the solution algorithm is time-consuming in comparison to SRM (Motsa et al. 2004a).

To develop the SQLM scheme, as described in Motsa (2013a), it is considered that a system of  $n$  nonlinear ordinary differential equations in  $n$  unknowns functions  $f_i(\xi)$  where  $i = 1, 2, \dots, n$  and  $\xi \in (a, b)$  is the dependent variable. This system can be given as a sum of its linear  $L$  and nonlinear components  $N$  as

$$L[f_1(\xi), f_2(\xi), \dots, f_n(\xi)] + N[f_1(\xi), f_2(\xi), \dots, f_n(\xi)] = 0; \quad i = 1, 2, \dots, n. \quad (3.2.9)$$

subject to separated boundary conditions

$$\begin{aligned} A_i[f_1(a), f_2(a), \dots, f_n(a)] &= K_{a,i} \\ B_i[f_1(b), f_2(b), \dots, f_n(b)] &= K_{b,i} \end{aligned} \quad (3.2.10)$$

where  $K_{a,i}$  and  $K_{b,i}$  are constants, and  $A_i$  and  $B_i$  are linear operators for  $i = 1, 2, \dots, n$ . The vector function  $F_i$  is defined as the vector of the derivatives of the variable  $f_i$  with respect to the dependent variable  $\xi$ , which is given as

$$F_i = [f_i^{(0)}, f_i^{(1)}, \dots, f_i^{(m_i)}] \quad (3.2.11)$$

where  $f_i^{(0)} = f_i$ ,  $f_i^{(p)}$  is the  $p$ th derivative of  $f_i$  with respect to  $\xi$  and  $m_i (i = 1, 2, 3, \dots, n)$  is the highest derivative order of the variable  $f_i$  occurring in the system of equations. Moreover,  $L_i$  and  $N_i$  are respectively defined as linear and nonlinear operator that operates on  $F_i$  for  $i = 1, 2, 3, \dots, n$ . Now, with these definitions, (3.2.9) – (3.2.10) can be written as

$$L_i[F_1, F_2, \dots, F_n] + N_i[F_1, F_2, \dots, F_n] = \sum_{j=1}^n \sum_{p=0}^{m_j-1} \alpha_{i,j}^{[p]} f_j^{(p)} + N_i[F_1, F_2, \dots, F_n] = 0 \quad (3.2.12)$$

where  $f_j^{(p)}$  is derivative of  $f_i (j = 1, 2, 3, \dots, n)$  that exists in the  $i^{th}$  equation for  $i = 1, 2, \dots, n$ , and  $\alpha_{i,j}^{[p]}$  are the constant coefficients of  $f_j^{(p)}$ .

Remarking that, in the governing system (3.2.9), the derivatives in the boundary conditions can at most be one less than derivative of  $f_i$  for each variable  $f_i$ . Then, it is defined that the vector  $\tilde{F}_i$  to be the vector of the derivatives of the variable  $f_i$  with respect to the dependent variable  $\xi$  from 0 up to  $m_i - 1$ , that is,

$$\tilde{F}_i = [f_i^{(0)}, f_i^{(1)}, \dots, f_i^{(m_i-1)}] \quad (3.2.13)$$

The boundary conditions (3.2.10) can be written as

$$\begin{aligned} A_\nu[\tilde{F}_1(a), \tilde{F}_2(a), \dots, \tilde{F}_n(a)] &= \sum_{j=1}^n \sum_{p=0}^{m_j-1} \alpha_{\nu,j}^{[p]} f_j^{(p)}(a) = K_{a,\nu}, \quad \nu = 1, 2, \dots, n_a. \\ B_\sigma[\tilde{F}_1(b), \tilde{F}_2(b), \dots, \tilde{F}_n(b)] &= \sum_{j=1}^n \sum_{p=0}^{m_j-1} \beta_{\sigma,j}^{[p]} f_j^{(p)}(b) = K_{b,\sigma}, \quad \sigma = 1, 2, \dots, n_b. \end{aligned} \quad (3.2.14)$$

where  $n_a$  and  $n_b$  are the entire number of given boundary conditions  $\xi = a$  and  $\xi = b$ , and  $\alpha_{\nu,j}^{[p]}$  and  $\beta_{\sigma,j}^{[p]}$  are the constant coefficients of  $f_j^{(p)}$ , respectively. Noting that the sum  $n_a + n_b$  is equal to the sum of the highest orders of the derivatives corresponding

to the dependent variables  $f_i$ , that is written as

$$n_a + n_b = \sum_{i=1}^n m_i \quad (3.2.15)$$

Consider that the solution  $f_i(\xi)$  of (3.2.12) at the  $(r+1)^{th}$  iteration is  $f_{i,r+1}$ . When the result at the preceding iteration  $f_{i,r}$  is adequately approach to  $f_{i,r+1}$ , then the nonlinear part  $N_i$  of (3.2.12) can be linearized employing the one-term Taylor series for multiple variables. Thus, (3.2.12) can be written as

$$\begin{aligned} & L_i[F_{1,r+1}, F_{2,r+1}, \dots, F_{n,r+1}] + N_i[F_{1,r}, F_{2,r}, \dots, F_{n,r}] \\ & + \sum_{j=1}^n \sum_{p=0}^{m_j} \left( F_{r+1}^{(p)} - F_r^{(p)} \right) \frac{\partial N_i}{\partial f_j^{(p)}} [F_{1,r}, F_{2,r}, \dots, F_{n,r}] = 0 \end{aligned} \quad (3.2.16)$$

subject to boundary condition

$$\begin{aligned} & \sum_{j=1}^n \sum_{p=0}^{m_j-1} \alpha_{\nu,j}^{[p]} f_{j,r+1}^{(p)}(a) = 0, \quad \nu = 1, 2, \dots, n_a. \\ & \sum_{j=1}^n \sum_{p=0}^{m_j-1} \beta_{\sigma,j}^{[p]} f_{j,r+1}^{(p)}(b) = 0, \quad \sigma = 1, 2, \dots, n_b. \end{aligned} \quad (3.2.17)$$

Equation (3.2.16) can be written as

$$\begin{aligned} & L_i[F_{1,r+1}, F_{2,r+1}, \dots, F_{n,r+1}] + \sum_{j=1}^n \sum_{p=0}^{m_j} F_{r+1}^{(p)} \frac{\partial N_i}{\partial f_j^{(p)}} [F_{1,r}, F_{2,r}, \dots, F_{n,r}] \\ & = \sum_{j=1}^n \sum_{p=0}^{m_j} F_r^{(p)} \frac{\partial N_i}{\partial f_j^{(p)}} [F_{1,r}, F_{2,r}, \dots, F_{n,r}] - N_i[F_{1,r}, F_{2,r}, \dots, F_{n,r}] \end{aligned} \quad (3.2.18)$$

The initial approximation, needed to begin the iteration scheme (3.2.18) is selected to be a function that fulfils the boundary conditions (3.2.10). Using the Chebyshev spectral collocation on the recursive iteration scheme (3.2.18) give

$$\sum_{j=1}^n [\Delta_{i,j} + \Pi i, j] F_{j,r+1} = \Psi_{i,r} \quad i, j = 1, 2, 3, \dots, n. \quad (3.2.19)$$

where  $F_{j,r+1} = [f_{i,r+1}(\tau_0), f_{i,r+1}(\tau_1), \dots, f_{i,r+1}(\tau_N)]^t$ ,  $\Delta_{i,j}$ ,  $\Pi_{i,j}$  and  $\Psi_{i,r}$  are respectively given by

$$\begin{aligned}\Delta_{i,j} &= \sum_{p=1}^{m_j} \alpha_{i,j}^p \mathbf{D}^p, \quad \Pi_{i,j} = \sum_{p=1}^{m_j} \left[ \frac{\partial N_i}{\partial f_j^{(p)}} \right]_d \mathbf{D}^p, \quad i, j = 1, 2, 3, \dots, n. \\ \Psi_{i,r} &= \sum_{j=1}^n \sum_{p=0}^{m_j} f_r^{(p)} \frac{\partial N_i}{\partial f_j^{(p)}} [F_{1,r}, F_{2,r}, \dots, F_{n,r}] - N_i[F_{1,r}, F_{2,r}, \dots, F_{n,r}]\end{aligned}\tag{3.2.20}$$

where  $[\dots]_d$  means denote diagonal matrix.

Setting,  $\Lambda = \Delta + \Pi$ , equation (3.2.19) can be written in matrix form as:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \dots \Lambda_{1n} \\ \Lambda_{21} & \Lambda_{22} & \dots \Lambda_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \Lambda_{n1} & \Lambda_{n2} & \Lambda_{nn} \end{bmatrix} \begin{bmatrix} F_{1,r+1} \\ F_{2,r+1} \\ \cdot \\ \cdot \\ \cdot \\ F_{n,r+1} \end{bmatrix} = \begin{bmatrix} \Psi_{1,r} \\ \Psi_{2,r} \\ \cdot \\ \cdot \\ \cdot \\ \Psi_{n,r} \end{bmatrix}\tag{3.2.21}$$

where  $F_{i,r+1}$  and  $\Psi_{1,r}$  are vectors of size  $(N+1) \times 1$  and  $\Lambda_{i,j}$  are  $(N+1) \times (N+1)$  matrices. Beginning from  $F_{i,0}$ , the recursive sequence (3.2.21) is solved iteratively for  $r = 0, 1, 2, 3, \dots$ . Accordingly, the size of the coefficient matrix in (3.2.21) is  $m(N+1) \times m(N+1)$  and the column vector on the right-hand side has the dimension  $m(N+1) \times 1$  as described in Motsa et al. (2013a).

### 3.2.4 Spectral Relaxation Method (SRM)

Spectral relaxation method (SRM) is a newly developed numerical scheme used to solve nonlinear systems of boundary value problems. It is based on simple decoupling and rearrangement of the governing nonlinear equations in a Gauss-Seidel manner (Motsa et al., 2014a). The resulting sequence of equations is integrated using the

Chebyshev spectral collocation method. The actual process of developing the solution algorithm for SRM is time-saving in comparison to SQLM. This is because with SQLM, the process begins with the quasi-linearization step whereas with SRM the iteration scheme is obtained directly by requiring some terms to be evaluated at the current iteration and others at the previous iteration. Consequently, the SRM is easy to implement and computationally efficient. As shown by Motsa et al. (2014a) the accuracy and efficiency of the SRM suggests that the method can be used as a more practical tool for solving unsteady boundary layer flows and complex nonlinear PDEs in some fluid mechanics applications.

To develop the SRM scheme, as described in Motsa et al. (2014), a system of  $n$  nonlinear ordinary differential equations in  $n$  unknown functions  $f_i(\xi)$ ,  $i = 1, 2, \dots, n$  is considered, where  $\xi \in [a, b]$  is the dependent variable. It is defined that a vector  $F_i$  to be a vector of derivatives of the variable  $f_i$  with respect to  $\xi$ .

$$F_i(\xi) = \left[ f_i^{(0)}, f_i^{(1)}, f_i^{(2)}, \dots, f_i^{(m)} \right] \quad (3.2.22)$$

where  $f_i^{(0)} = f_i$ ,  $f_i^{(p)}$  is the  $p$ th derivative of  $f_i$  with respect to  $\xi$  and  $m_i$  ( $i = 1, 2, 3, \dots, n$ ) is the highest derivative order of the variable  $f_i$  which is in the system of equations. The system can be written in terms of  $F_i$  as the sum of linear ( $L_i$ ) and nonlinear components ( $N_i$ ) as

$$L_i[F_1, F_2, \dots, F_n] + N_i[F_1, F_2, \dots, F_n] = G_i(\xi); \quad i = 1, 2, \dots, n. \quad (3.2.23)$$

$G_i$  is a known function of  $\xi$ .

Equation (3.2.23) is solved subject to separated boundary conditions

$$A_\nu[F_1(a), F_2(a), \dots, F_n(a)] = K_{a,\nu} \quad (3.2.24)$$

$$B_\sigma[F_1(b), F_2(b), \dots, F_n(b)] = K_{b,\sigma} \quad (3.2.25)$$

which can be written as

$$\sum_{j=1}^n \sum_{p=0}^{m_j-1} \alpha_{\nu,j}^{[p]} f_j^{(p)}(a) = K_{a,\nu}; \quad \nu = 1, 2, \dots, n_a. \quad (3.2.26)$$

$$\sum_{j=1}^n \sum_{p=0}^{m_j-1} \beta_{\sigma,j}^{[p]} f_j^{(p)}(b) = K_{b,\sigma}; \quad \sigma = 1, 2, \dots, n_b. \quad (3.2.27)$$

where  $\alpha_{\nu,j}^{[p]}$  and  $\beta_{\sigma,j}^{[p]}$  are the constant coefficients of  $f_j^{(p)}$  in the boundary conditions,  $n_a$  and  $n_b$  are the total number of prescribed boundary conditions at  $\xi = a$  and  $\xi = b$  respectively. Starting from the initial approximation  $F_{1,0}, F_{2,0}, \dots, F_{n,0}$  the iterative method is obtained as

$$L_i[F_{1,r+1}, F_{2,r+1}, \dots, F_{n,r+1}] = G_i(\xi) + N_i[F_{1,r}, F_{2,r}, \dots, F_{n,r}]; \quad i = 1, 2, \dots, n. \quad (3.2.28)$$

Where  $F_{1,r+1}$  and  $F_{1,r}$  are the approximation of  $F_i$  at the current and the previous iterations respectively. An appropriate initial approximation  $F_{i,0}$  which satisfy boundary conditions can be started from a system of  $n$  linear decoupled equations can be solved iteratively for  $r = 1, 2, \dots$ . The iterations are repeated until convergence is reached.

The decoupling error can be used to assess the desired convergence. The decoupling error  $E_r$  at the  $(r + 1)^{th}$  iteration is defined by

$$E_r = \text{Max} (||f_{1,r+1} - f_{1,r}||_{\infty}, ||f_{2,r+1} - f_{2,r}||_{\infty}, \dots, ||f_{m,r+1} - f_{m,r}||_{\infty}) \quad (3.2.29)$$

In this method, the Gauss-Seidel relaxation method which is normally used for solving large systems of algebraic equations is incorporated. To implement the spectral collocation method, the differentiation matrix is defined as:

$$\frac{df_i(\xi)_l}{d\xi} = \sum_{k=0}^N \mathbf{D}_{l,k} f_i(\tau_k) = \mathbf{D} F_i; \quad l = 0, \dots, N. \quad (3.2.30)$$

and higher order derivatives are obtained in powers of  $\mathbf{D}$  given by

$$f_j^{(p)} = \mathbf{D}^p F_j \quad (3.2.31)$$

Where  $N + 1$  is the number collocation points,  $\mathbf{D} = 2D/(b - a)$  and

$F = [f(\tau_0), f(\tau_1), \dots, f(\tau_N)]^t$  is the vector function of the collocation points.

Then, the Chebyshev pseudo spectral method is applied to the iteration scheme shown in Eqs. (3.2.28), and the equations gives

$$\sum_{j=1}^n \sum_{p=0}^{m_j} \gamma_{i,j}^{[p]} f_j^{(p)} + N_i[F_1, F_2, \dots, F_n] = G_i \quad (3.2.32)$$

where  $\gamma_{i,j}^{[p]}$  are constant coefficients of  $f_j^{(p)}$ , the derivative of  $f_j$  ( $j = 1, 2, \dots, n$ ) in the  $i^{th}$  equation for  $i = 1, 2, \dots, n$ . The iteration scheme used in (3.2.28) can be expressed as

$$\sum_{j=1}^i \sum_{p=0}^{m_j} \gamma_{i,j}^{[p]} f_{j,r+1}^{(p)} = G_i - \sum_{j=1+i}^n \sum_{p=0}^{m_j} \gamma_{i,j}^{[p]} f_{j,r+1}^{(p)} - N_i[F_{1,r+1}, \dots, F_{i-1,r+1}, F_{i,r}, \dots, F_{n,r}] \quad (3.2.33)$$

for  $i = 1, 2, \dots, n$

Using equ. (3.2.30) in (3.2.33) and the boundary conditions, the spectral Gauss-Seidel relaxation method iteration scheme is given by

$$\sum_{j=1}^i \sum_{p=0}^{m_j} \gamma_{i,j}^{[p]} \mathbf{D}^p F_{j,r+1} = G_i - \sum_{j=1+i}^n \sum_{p=0}^{m_j} \gamma_{i,j}^{[p]} \mathbf{D}^p F_{j,r} - N_i[F_{1,r+1}, \dots, F_{i-1,r+1}, F_{i,r}, \dots, F_{n,r}] \quad (3.2.34)$$

subject to

$$\sum_{j=1}^n \sum_{p=0}^{m_j-1} \alpha_{\nu,j}^{[p]} \sum_{k=1}^N \mathbf{D}_{N,k}^p f_{j,r+1}(\tau_k) = \mathbf{K}_{a,\nu}; \quad \nu = 1, 2, \dots, n_a. \quad (3.2.35)$$

$$\sum_{j=1}^n \sum_{p=0}^{m_j-1} \beta_{\sigma,j}^{[p]} \sum_{k=1}^N \mathbf{D}_{N,k}^p f_{j,r+1}(\tau_k) = \mathbf{K}_{b,\sigma}; \quad \sigma = 1, 2, \dots, n_b. \quad (3.2.36)$$

The substitution of previously known functions decouples the system of equations and an efficient iteration scheme is created giving accurate results.

## Chapter 4

# MHD Boundary Layer Flow of Nanofluid Past a Wedge in a Porous Media with Viscous Effects

### 4.1 Introduction

Fluid flows with convective heat and mass transfer over a wedge shaped bodies is ensured in many thermal engineering applications like crude oil extraction, geothermal systems, thermal insulation, heat exchangers and the storage of nuclear waste, etc. (Nagendramma et al., 2015). A model of steady laminar fluid flow over a wedge was developed for the first time by Falkner and Skan (1931) to illustrate the application of Prandtl's boundary layer theory. Late, Hartree (1937) investigated the same problem with similarity transformation and gave numerical results for wall shear stress for different values of the wedge angle. Eckert (1942) also solved Falkner-Skan flow along an isothermal wedge and presented the first wall heat transfer values. Afterward, the variety of applications and understanding of the physical features of laminar boundary layer flow past a wedge have motivated many researchers ( Sattar, 2011 ; Kandasamy et al., 2013 ; Turkyilmazoglu, 2015) to mention few.

Magnetohydrodynamic (MHD) is the study of fluid flow in electrically conducting fluids with magnetic properties that affect fluid flow characteristics. When a magnetic

field is incident in an electrically conducting fluid, a current is induced. This effect polarizes the fluid and as a result the magnetic field is changed (Makanda et al., 2015). Due to extensive practical applications of MHD in technological processes such as plasma studies, petroleum industries, MHD power generator designs, design for cooling of nuclear reactors, construction of heat exchangers and on the performance of many other systems, there are many studies that considered MHD fluid flow past a wedge. These include the work of Abbasbandy et al. (2014) who examined the effects of MHD in the Falken-Skan flow of Maxwell fluid, and El-Dabe et al. 2015 considered MHD boundary layer flow of non-Newtonian Casson fluid on a moving wedge with heat and mass transfer. Khan et al. (2014) also analyzed MHD laminar boundary layer flow past a wedge with the influence of thermal radiation, heat generation and chemical reaction.

In the past few years, MHD boundary layer flow with heat and mass transfer of nanofluids has become a major topic of modern-day interest. Nanofluids have a significant role in enhancing the heat transfer properties of fluids. The most important properties of nanofluids are enhanced effective fluid thermal conductivity and heat transfer coefficient. Some recent studies on MHD boundary layer flow of nanofluid include the work of the Srinivasacharya et al. (2015) who analyzed the steady laminar MHD flow in a nanofluid over a wedge in the presence of a variable magnetic field. Rasheed et al. 2017) also investigated MHD boundary layer flow of nanofluid over a continuously moving stretching surface. The effects of thermal radiation on mixed convection flow of nanofluid over a stretching sheet in the presence of a magnetic field was studied by Nageeb et al. (2017).

Viscous dissipation effect changes the temperature distribution by playing a role like an energy source, which leads to affected heat transfer rate and hence needs to be considered in heat transfer problems. The analysis of MHD boundary layer flows in porous media with and without the effect of viscous dissipation has been a subject

of several recent papers. Arthur et al. (2014) analyzed hydro magnetic stagnation point flow over a porous stretching surface in the presence of radiation and viscous dissipation. Ramesh et al. (2017) studied the MHD boundary layer flow past a constant wedge within porous media. Heat and mass transfer of MHD flow of nanofluids in the presence of viscous dissipation effects was numerically analyzed by Haile and Shankar (2015). Majety et al. (2016) studied the effect of viscous dissipation on MHD boundary layer flow past a wedge through porous medium. They concluded that viscous dissipation produces heat due to drag between the fluid particles, which cause an increase in fluid temperature.

Most of the standard methods of solving the boundary layer problems are the numerical approach based on the shooting algorithm with the Runge-Kutta scheme, finite difference method, spectral homotopy analysis method, and Newton-Raphson based methods such as the quasilinearization method and the successive linearization method. Recently, spectral based numerical techniques such as the Spectral Quasilinearization Method and Spectral Relaxation Method have been developed (Motsa, 2013; Motsa et al., 2014). As indicated by Motsa et al. (2013b) Chebyshev spectral collocation methods are easy to implement, adaptable to various problems and provide more accurate approximations with a relatively small number of unknowns. Gottlieb and Hesthaven (2001) also added that the wide use of spectral methods has motivated the researcher by their accuracy and efficiency in solving incompressible Navier-Stokes equations. Furthermore, Motsa et al. (2014a) stated that the interest in using Chebyshev spectral collocation methods in solving nonlinear PDEs stems from the fact that these methods require less grid points to achieve accurate results and efficient compared to traditional methods like finite difference and finite element methods.

The present study deals with MHD boundary layer flow past a wedge with heat and mass transfer of Nanofluid embedded in a porous media with viscous dissipation,

brownian motion and thermophoresis effects. Using appropriate similarity transformation, the governing nonlinear partial differential equations are reduced to nonlinear higher order ordinary differential equations. Then, these equations are numerically solved by SQLM. The effects of different parameters on velocity, temperature and concentration fields are investigated and analyzed with the help of their graphical representations.

## 4.2 Mathematical Formulation

In this study it was considered steady two-dimensional, laminar boundary layer flow past a wedge with heat transfer of incompressible electrically conducting nanofluid embedded in a porous media with viscous dissipation effects. The coordinate system is chosen with  $x$  coordinate pointing parallel to the plate in the direction of the flow and  $y$  coordinate pointing towards the free stream, as shown in the Figure 4.1. The wall of the wedge is maintained with uniform and constant temperature  $T_w$  and nanoparticle concentration  $C_w$ .  $T_w$  and  $C_w$  are respectively, greater than the ambient temperature  $T_\infty$  and ambient nanoparticle concentration  $C_\infty$ . In addition, effects of the buoyancy force is not considered since it is assumed that the fluid have constant thermophysical properties with constant and uniform  $T_w$  and  $C_w$  near the wall of the wedge. The subscripts  $w$  and  $\infty$  denotes conditions at wall and ambient respectively.

It was also considered that a constant magnetic field  $B_0$  is applied in the positive  $y$ -direction, normal to the walls of the wedge. The induced magnetic field caused by the motion of electrically conducting fluid is neglected, as it is very small compared to magnetic field. The Navier-Stokes equation of motion including electromagnetic body force or Lorentz force within conductive media is given by

$$F = J_c \times B \quad (4.2.1)$$

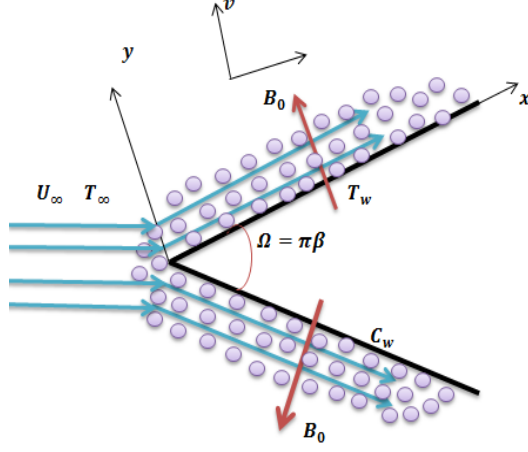


Figure 4.1: Physical model of flow configuration.

where  $J_c$  is the conduction current defined as

$$J_c = \sigma(E_c + \mathbf{u} \times B) \quad (4.2.2)$$

where  $B$  is the magnetic field,  $E_c$  is the electric field,  $\mathbf{u}$  is the velocity vector of the fluid and  $\sigma$  is the electrical conductivity of the fluid (Nicholas, 2010).

Since the induced magnetic field caused by the motion of electrically conducting fluid  $E_c$  is negligible as compared to the magnetic field  $B$ , then

$$J_c = \sigma(\mathbf{u} \times B) = \sigma[(u, v, 0) \times 0, B_y, 0] = \sigma u B_y \hat{k} \quad (4.2.3)$$

Thus, the Lorentz force is given as

$$F = J_c \times B = (0, 0, \sigma u B_y) \times (0, B_y, 0) = -\sigma B_y^2 u \hat{i} = -\sigma B_0^2 u \hat{i} \quad (4.2.4)$$

With the above assumptions and the boundary layer approximations, the continuity, momentum, energy and nanoparticle concentration equations governing steady

two dimensional MHD boundary layer flow past a wedge embedded in a porous media with viscous dissipation effects and constant fluid properties are given as (Srinivasacharya et al., 2015; Haile & Shankar, 2015; Alam et al., 2017).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.2.5)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho_{bf}} \frac{\partial p}{\partial x} + \nu_{bf} \frac{\partial^2 u}{\partial y^2} - \left( \frac{\sigma B_0^2}{\rho_{bf}} + \frac{\nu_{bf}}{K} \right) u \quad (4.2.6)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{bf} \frac{\partial^2 T}{\partial y^2} + \frac{\nu_f}{c_p} \left( \frac{\partial u}{\partial y} \right)^2 + \varsigma \left\{ D_B \left( \frac{\partial T}{\partial y} \frac{\partial C}{\partial y} \right) + \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \right\} \quad (4.2.7)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2} \quad (4.2.8)$$

The appropriate boundary conditions are given as:

$$\begin{aligned} \text{at } y = 0 : \quad & u = 0; \quad v = 0; \quad T = T_w; \quad C = C_w \\ \text{as } y \rightarrow \infty : \quad & u = U_\infty x^m; \quad T \rightarrow T_\infty; \quad C \rightarrow C_\infty \\ \text{at } x = 0 : \quad & u = U_\infty; \quad T = T_\infty \end{aligned} \quad (4.2.9)$$

where  $u$  and  $v$  are respectively, the  $x$  and  $y$  velocity components.  $\rho_{bf}, \nu_{bf}, \alpha_{bf}$  and  $c_p$ , are respectively, density, kinematic viscosity, thermal diffusivity and specific heat capacity of the base fluid.  $K$  is the permeability of porous medium,  $D_B$  is the Brownian diffusion coefficient,  $D_T$  is the thermophoresis diffusion coefficient,  $\varsigma$  is the ratio of the effective heat capacity of the nanoparticle material and the heat capacity of base fluid.

The y-momentum equation implies that the pressure  $p$  in the boundary layer must be equal to that of the free stream for any given  $x$  coordinate. Since the velocity profile is uniform in the free stream, there is no vorticity involved, therefore a simple Bernoulli's equation can be applied in this high Reynolds number (Falkner & Skan, 1931). It is assumed that  $U(x) = U_\infty x^m$  is the fluid velocity at the wedge outside

the boundary layer, where  $U_\infty$  is the free stream velocity. For a uniform stream, the pressure term in the momentum equation (4.2.6) becomes (Falkner & Skan, 1931; Nageeb et al., 2017).

$$-\frac{1}{\rho_{bf}} \frac{\partial p}{\partial x} = U \frac{dU}{dx} + \left( \frac{\sigma B_0^2}{\rho_{bf}} + \frac{\nu_f}{K} \right) U \quad (4.2.10)$$

Substituting (4.2.10) in (4.2.6), the momentum equation written as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu_f \frac{\partial^2 u}{\partial y^2} + \left( \frac{\sigma B_0^2}{\rho_{bf}} + \frac{\nu_f}{K} \right) (U - u) \quad (4.2.11)$$

Here,  $x$  is measured from the tip of the wedge,  $m$  is the Falkner-Skan power-law parameter,  $\beta = \frac{2m}{1+m}$  is the Hartree pressure gradient parameter corresponding to  $\beta = \frac{\Omega}{\pi}$  for the total angle  $\Omega$  of the wedge (see Figure 4.1). Physically,  $m < 0$  corresponding to an adverse pressure gradient (often resulting in boundary layer separation) while  $m > 0$  represents favorable pressure gradient. In the Blasius solution  $m = 0$  corresponding to an angle of attack of zero radians, where  $m = 1$  corresponding to stagnation point flow (Falkner & Skan, 1931).

In order to transform the governing equations (4.2.5)-(4.2.9) into a set of ordinary differential equations, we introduce the stream function  $\psi(x, y)$  such that

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \quad (4.2.12)$$

and, we use the following transformation variables (Ullah et al., 2016; Alam et al., 2017)

$$\begin{aligned} \xi &= y \sqrt{\frac{1+m}{2} \frac{U_\infty}{\nu_{bf}} x^{\frac{m-1}{2}}}; & \psi(x, \xi) &= \sqrt{\frac{2}{1+m} \nu_{bf} U_\infty} x^{\frac{m+1}{2}} f(\xi) \\ f'(\xi) &= \frac{u}{U}; & \theta(\xi) &= \frac{T - T_\infty}{T_w - T_\infty}; & \phi(\xi) &= \frac{C - C_\infty}{C_w - C_\infty} \end{aligned} \quad (4.2.13)$$

where  $\xi$  is a dimensionless similarity variable,  $f(\xi)$  non-dimensional stream function,  $f'(\xi)$  is non-dimensional velocity,  $\theta(\xi)$  is non-dimensional temperature,  $\phi(\xi)$  is non-dimensional nanoparticle concentration.

We find that continuity equation (4.2.5) is identically satisfied, and upon substituting similarity variables into equations (4.2.6)-(4.2.9), we obtain the following system of ordinary differential equations:

$$f''' + ff'' + \beta[1 - f'^2] + \frac{1}{1+m}(M + \kappa)[1 - f'] = 0 \quad (4.2.14)$$

$$\theta'' + Pr [f\theta' + Ec f'^2 + Nb\theta'\phi' + Nt\theta'^2] = 0 \quad (4.2.15)$$

$$\phi'' + LePr (f\phi') + \frac{Nt}{Nb}\theta'' = 0 \quad (4.2.16)$$

The transformed boundary conditions are:

$$\begin{aligned} \text{at } \xi = 0; \quad f = 0; \quad f' = 0; \quad \theta = 1; \quad \phi = 1 \\ \text{as } \xi \rightarrow \infty; \quad f' \rightarrow 1; \quad \theta \rightarrow 0; \quad \phi \rightarrow 0 \end{aligned} \quad (4.2.17)$$

with

$$\begin{aligned} M = \frac{2\sigma B_0^2 x^{1-m}}{\rho_{bf} U_\infty}; \quad \kappa = \frac{2\nu_{bf} x^{1-m}}{K U_\infty}; \quad Ec = \frac{U^2}{c_p(T_w - T_\infty)}; \quad Pr = \frac{\nu_{bf}}{\alpha_{bf}} \\ Le = \frac{\alpha_{bf}}{D_B}; \quad Nb = \frac{\varsigma D_B(C_w - C_\infty)}{\nu_{bf}}; \quad Nt = \frac{\varsigma D_T(T_w - T_\infty)}{\nu_{bf} T_\infty}; \quad Re_x = \frac{Ux}{\nu_{bf}} \end{aligned}$$

where  $M$  is magnetic parameter,  $Re_x$  is local Reynolds number,  $Pr$  is Prandtl number,  $Ec$  is Eckert number,  $\kappa$  is the permeability parameter,  $Le$  is Lewis number,  $Nb$  is the Brownian motion parameter,  $Nt$  is the thermophoretic parameter, and prime (') denotes derivative with respect to  $\xi$ .

The physical quantities of engineering interest in the present study are the skin friction coefficient  $C_f$ , local Nusselt number  $Nu_x$ , and local Sherwood number  $Sh_x$  respectively, defined as

$$C_f = \frac{2\tau_w}{\rho U^2(x)}; \quad Nu_x = \frac{xq_w}{k(T_w - T_\infty)}; \quad Sh_x = \frac{xM_w}{D_B(C_w - C_\infty)}; \quad (4.2.18)$$

where  $\tau_w$ ,  $q_w$  and  $M_w$  are the surface shear stress, the surface heat flux, and surface mass flux, and respectively, given as

$$\tau_w = \mu_f \left( \frac{\partial u}{\partial y} \right)_{y=0}; \quad q_w = -k_f \left( \frac{\partial T}{\partial y} \right)_{y=0}; \quad M_w = -D_B \left( \frac{\partial C}{\partial y} \right)_{y=0} \quad (4.2.19)$$

The non-dimensional skin friction coefficient, Nusselt number, and local Sherwood number are respectively, given as

$$C_f \sqrt{Re_x} = 2\sqrt{\frac{m+1}{2}} f''(0); \frac{Nu_x}{\sqrt{Re_x}} = -\sqrt{\frac{m+1}{2}} \theta'(0); \frac{Sh_x}{\sqrt{Re_x}} = -\sqrt{\frac{m+1}{2}} \phi'(0) \quad (4.2.20)$$

### 4.3 Numerical Method

The system of non-linear ODE (4.2.14)-(4.2.16) subject to the boundary conditions (4.2.17) have been solved numerically using spectral quilinearization method (SQLM). The main idea behind this approach is identifying univariate and multivariate nonlinear terms of function and its derivative in each of the equations, linearizing the terms and applying Chebychev pseudo-spectral collocation method (see Motsa, 2013).

Applying spectral quilinearization method, equations (4.2.14)-(4.2.16) gives the following iterative sequence of linear differential equations:

$$f_{r+1}''' + a_{1,r} f_{r+1}'' - a_{2,r} f_{r+1}' + a_{3,r} f_{r+1} = a_{4,r} \quad (4.3.1)$$

$$\theta_{r+1}'' + b_{1,r} \theta_{r+1}' + b_{2,r} f_{r+1}'' + b_{3,r} f_{r+1}' + b_{4,r} \phi_{r+1}' = b_{5,r} \quad (4.3.2)$$

$$\phi_{r+1}'' + c_{1,r} \phi_{r+1}' + c_{2,r} f_{r+1}'' + c_{3,r} f_{r+1}' + c_{4,r} \theta_{r+1}' = c_{5,r} \quad (4.3.3)$$

where the terms containing  $r+1$  subscripts denote current approximations and terms containing  $r$  subscripts denote previous approximations. The corresponding boundary conditions are:

$$\begin{aligned} f_{r+1}(0) = 0; \quad f_{r+1}'(0) = 0, \quad f_{r+1}'(\infty) \rightarrow 1 \\ \theta_{r+1}(0) = \phi_{r+1}(0) = 1, \quad \theta_{r+1}(\infty) = \phi_{r+1}(\infty) \rightarrow 0 \end{aligned} \quad (4.3.4)$$

where

$$a_{1,r} = f_r; \quad a_{2,r} = 2\beta f_r' + \frac{1}{1+m}(M + \kappa); \quad a_{4,r} = f_r f_r'' - \beta(1 + f_r'^2) - \frac{1}{1+m}(M + \kappa)$$

$$\begin{aligned}
a_{3,r} &= f_r''; \quad b_{1,r} = Pr(f_r + Nb\phi_r' + 2Nt\theta_r'); \quad b_{2,r} = 2PrEc f_r''; \quad b_{3,r} = Pr\theta_r'; \quad b_{4,r} = PrNb\theta_r' \\
b_{5,r} &= PrEc f_r''^2 + PrNb\theta_r'\phi_r' + PrNt\theta_r'^2 + Prf_r\theta_r'; \quad c_{1,r} = Pr(Le f_r - Nt\theta_r') \\
c_{3,r} &= Pr(Le\phi_r' - (Nt/Nb)\theta_r'); \quad c_{4,r} = -Pr(Nt/Nb)(f_r + Nb\phi_r' + 2Nt\theta_r') \\
c_{2,r} &= -2PrEc(Nt/Nb)f_r''; \quad c_{5,r} = Pr\{Le f_r\phi_r' - (Nt/Nb)[Ec f_r''^2 + Nb\theta_r'\phi_r' + Nt\theta_r'^2 + f_r\theta_r']\}
\end{aligned}$$

The physical domain on which the governing system of equations (4.2.14)-(4.2.16) defined in  $[0, \infty)$  is truncated to a domain  $[0, L]$ . Then, the interval  $[0, L]$  is transformed to the interval  $[-1, 1]$  using the linear transformation  $\xi = \frac{1}{2}L(\chi + 1)$ , where  $L$  is large but finite number chosen to represent the behavior of the flow properties of the boundary condition value at infinity. The Gauss-Lobatto points are selected to define the nodes in  $[-1, 1]$  as follows

$$\chi_i = \cos(\pi i/N), \quad i = 0, 1, 2, \dots, N; \quad -1 \leq \chi \leq 1, \quad (4.3.5)$$

where  $N$  is the number of collocation points used.

The Chebyshev differentiation matrix used to approximate the derivatives of the unknown variables is given as

$$\frac{df}{d\xi} = \sum_{p=0}^N \mathbf{D}_{ik} f(\chi_i) = \mathbf{D}F, \quad i = 1, 2, 3, \dots, N \quad (4.3.6)$$

where  $\mathbf{D} = \frac{2D}{L}$ , and the higher order derivatives are found as powers of  $\mathbf{D}$  as

$$F^n(\xi) = \mathbf{D}^n F, \quad (4.3.7)$$

where  $n$  is the order of derivative and  $\mathbf{D}$  is matrix of size  $(N + 1) \times (N + 1)$ .

Applying the spectral method to the system of equations (4.3.1)-(4.3.3), and it can be solved as a coupled matrix:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{bmatrix} F_{r+1} \\ \Theta_{r+1} \\ \Phi_{r+1} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \quad (4.3.8)$$

with transformed boundary condition

$$F_{r+1}(\chi_N) = 0, \quad F_{r+1}(\chi_{N-1}) = 0, \quad F_{r+1}(\chi_0) = 1$$

$$\Theta_{r+1}(\chi_N) = 1, \quad \Theta_{r+1}(\chi_0) = 0; \quad \Phi_{r+1}(\chi_N) = 1, \quad \Phi_{r+1}(\chi_0) = 0 \quad (4.3.9)$$

$$\text{where } B_1 = a_{4,r}; \quad B_2 = b_{5,r}; \quad B_3 = c_{5,r}$$

$$\Lambda_{11} = \mathbf{D}^3 + \text{diag}(a_{1,r})\mathbf{D}^2 - \text{diag}(a_{2,r})\mathbf{D} + \text{diag}(a_{3,r}); \quad \Lambda_{12} = \mathbf{O}; \quad \Lambda_{13} = \mathbf{O}$$

$$\Lambda_{21} = \text{diag}(b_{2,r})\mathbf{D}^2 + \text{diag}(b_{3,r}); \quad \Lambda_{22} = \mathbf{D}^2 + \text{diag}(b_{1,r})\mathbf{D}; \quad \Lambda_{23} = \text{diag}(b_{4,r})\mathbf{D}$$

$$\Lambda_{31} = \text{diag}(c_{2,r})\mathbf{D}^2 + \text{diag}(c_{3,r}); \quad \Lambda_{32} = \text{diag}(c_{4,r})\mathbf{D}; \quad \Lambda_{33} = \mathbf{D}^2 + \text{diag}(c_{1,r})\mathbf{D}$$

$F_{r+1} = [f_{r+1,0}, f_{r+1,1}, \dots, f_{r+1,N}]^t$ ,  $\Theta_{r+1} = [\theta_{r+1,0}, \theta_{r+1,1}, \dots, \theta_{r+1,N}]^t$ ,  
 $\Phi_{r+1} = [\phi_{r+1,0}, \phi_{r+1,1}, \dots, \phi_{r+1,N}]^t$  are vectors of sizes  $(N_x + 1) \times 1$ ,  $\text{diag}(\dots)$  represents a diagonal matrix of vectors, and  $\mathbf{O}$  is a zero vectors of size  $(N_x + 1) \times (N_x + 1)$ .

Choice of the initial guesses from known physical considerations of the flow properties is necessary for obtaining numerical solutions of SQLM. For most boundary layer flow problems, it is well known that the flow velocity tends to the mainstream flow exponentially. Thus, the appropriate initial guesses which satisfy the governing boundary conditions of the present problems (4.2.14)-(4.2.17) are

$$f_0(\xi) = \xi - 1 + e^{-\xi}, \quad f'_0(\xi) = 1 - e^{-\xi} \quad \theta_0(\xi) = \phi_0(\xi) = e^{-\xi} \quad (4.3.10)$$

The boundary conditions are imposed on the individual matrices as follows:

$$M_{11} = \begin{bmatrix} D_{00} & \dots & D_{0N} \\ \hline & \Lambda_{11} & \\ \hline D_{N0} & \dots & D_{NN} \\ \hline 0 & \dots & 1 \end{bmatrix}; M_{12} = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \hline & \Lambda_{12} & \\ \hline 0 & \dots & 0 & 0 \\ \hline 0 & \dots & 0 & 0 \end{bmatrix}; M_{13} = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \hline & \Lambda_{13} & \\ \hline 0 & \dots & 0 & 0 \\ \hline 0 & \dots & 0 & 0 \end{bmatrix} \quad (4.3.11)$$

$$M_{21} = \left[ \begin{array}{cccc} 1 & \dots & 0 & 0 \\ & \Lambda_{21} & & \\ \hline 0 & \dots & 0 & 1 \end{array} \right]; M_{22} = \left[ \begin{array}{cccc} 1 & \dots & 0 & 0 \\ & \Lambda_{22} & & \\ \hline 0 & \dots & 0 & 1 \end{array} \right]; M_{23} = \left[ \begin{array}{cccc} 1 & \dots & 0 & 0 \\ & \Lambda_{23} & & \\ \hline 0 & \dots & 0 & 1 \end{array} \right] \quad (4.3.12)$$

$$M_{31} = \left[ \begin{array}{cccc} 1 & \dots & 0 & 0 \\ & \Lambda_{31} & & \\ \hline 0 & \dots & 0 & 0 \end{array} \right]; M_{32} = \left[ \begin{array}{cccc} 1 & \dots & 0 & 1 \\ & \Lambda_{32} & & \\ \hline 0 & \dots & 0 & 1 \end{array} \right]; M_{33} = \left[ \begin{array}{cccc} 1 & \dots & 0 & 0 \\ & \Lambda_{33} & & \\ \hline 0 & \dots & 0 & 1 \end{array} \right] \quad (4.3.13)$$

$$R_1 = \left[ \begin{array}{c} \frac{1}{B_1} \\ \\ \hline 0 \\ 0 \end{array} \right]; \quad R_2 = \left[ \begin{array}{c} 0 \\ \\ \hline B_2 \\ 1 \end{array} \right]; \quad R_3 = \left[ \begin{array}{c} 0 \\ \\ \hline B_3 \\ 1 \end{array} \right]; \quad (4.3.14)$$

$$F_{r+1} = \left[ \begin{array}{c} \frac{f_{r+1}, 0}{\cdot} \\ \cdot \\ \cdot \\ \cdot \\ \hline \frac{f_{r+1}, N-1}{f_{r+1}, N} \end{array} \right]; \quad \Theta_{r+1} = \left[ \begin{array}{c} \frac{\theta_{r+1}, 0}{\cdot} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hline \theta_{r+1}, N \end{array} \right]; \quad \Phi_{r+1} = \left[ \begin{array}{c} \frac{\phi_{r+1}, 0}{\cdot} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hline \phi_{r+1}, N \end{array} \right] \quad (4.3.15)$$

## 4.4 Results and Discussion

Numerical solutions were obtained using SQLM for the velocity, temperature and concentration profiles across the boundary layer for different values of the embeded parameters. The number of collocation points in the space  $\xi$  variable used to generate the results is  $N_x = 40$  in all cases. To ensure the numerical accuracy of the numerical

Table 4.1: Comparison of the SQLM results of  $-f''(0)$  and  $-\theta'(0)$  for some  $m$ .

| $m$    | $-f''(0)$ |         |          |        | $-\theta'(0)$ |          |
|--------|-----------|---------|----------|--------|---------------|----------|
|        | Present   | Ashwini | Watanaba | Ullah  | Present       | Watanaba |
| 0.0000 | 0.46960   | 0.4696  | 0.46960  | 0.4696 | 0.42016       | 0.42015  |
| 0.0141 | 0.50461   | 0.5046  | 0.50461  | 0.5046 | 0.42578       | 0.42578  |
| 0.0435 | 0.56898   | 0.5690  | 0.56898  | 0.5690 | 0.43548       | 0.43548  |
| 0.0909 | 0.65498   | 0.6550  | 0.65498  | 0.6550 | 0.44730       | 0.44730  |
| 0.1429 | 0.73200   | 0.7320  | 0.73200  | 0.7320 | 0.45694       | 0.45693  |
| 0.2000 | 0.80213   | 0.8021  | 0.80213  | 0.8021 | 0.46503       | 0.46503  |
| 0.3333 | 0.92765   | 0.9277  | 0.92765  | 0.9277 | 0.47814       | 0.47814  |
| 1.0000 | 1.23258   | 1.2326  | -        | 1.2326 | -             | -        |

method used, the skin friction coefficient  $-f''(0)$  and local Nusselt number  $-\theta'(0)$  were calculated for different values of Falkner-Skan power-law parameter  $m$ . From Table 4.1, it is observed that the data produced by the SQLM code and those reported by Watanaba (1990), Ashwini and Eswara (2012), and Ullah et al. (2016) are excellent agreement. Thus, we are very much confident that the present results are accurate.

Figure 4.2 shows the variation of velocity profiles for different values of pressure gradient parameter  $\beta$ . It clearly demonstrates that the velocity profile increases with an increase in pressure gradient parameter. This is because of the increment of wedge angle, the fluid moves much slower and decreases velocity boundary layer thickness. Figure 4.3 shows the effect of permeability parameter  $\kappa$  on the velocity profile. It is observed that increase in  $\kappa$  leads to increase the velocity of the nanofluid on the porous surface and decrease its boundary layer thickness. It is also noticed that both pressure gradient parameter  $\beta$  and permeability parameter  $\kappa$  have no significant effect on both nanofluid temperature and concentration.

The effects of magnetic field parameter  $M$  on the nanofluid flow is shown in Figures 4.4-4.6. The magnetic field effect can be viewed as the combination of the Lorentz force  $\frac{\sigma B_0^2}{\rho_{bf}}u$  and the imposed Hartree pressure  $U\frac{\partial U}{\partial x} + \frac{\sigma B_0^2}{\rho_{bf}}U$ . Figure 4.4 reveals that the

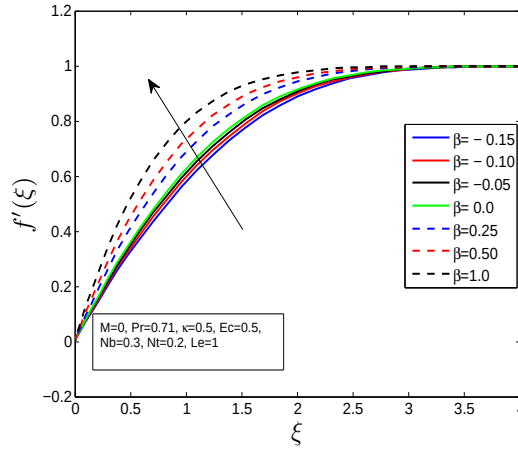


Figure 4.2: Velocity profiles for various values of  $\beta$ .

growth in magnetic field strength leads to increase the fluid velocity profile. However, the temperature and concentration profiles reduce with the increase in magnetic field parameter, as shown in Figure 4.5 and Figure 4.6 respectively. The reason behind this observation is that the pressure gradient due to wedge angle dominates the Lorentz force; and this effect plays the role of assisting force to facilitate the motion of the fluid and the removal of heat in the boundary layer region. Figure 4.7 also describes the concentration profile for different values of Lewis number  $Le$ . It is clearly observed that the concentration profile and its boundary layer thickness reduce considerably as the Lewis number increases. The effect of the Prandtl number on the temperature and concentration is shown in Figures 4.8 and 4.9 respectively. It is depicted that the temperature and concentration profiles and their boundary layer thickness reduce significantly as the Prandtl number increase. Because increasing the Prandtl number tends to reduce the thermal diffusivity of the fluid and causes weak penetration of heat inside the fluid. However, in the region near to the boundary surface, the heat transfer rate increases with an increase in  $Pr$ . This is due to the fact that the temperature gradient at the surface increase.

The effect of viscous dissipation parameter  $Ec$  on the temperature and concentration profiles is presented in Figure 4.10 and 4.11 respectively. The Eckert number

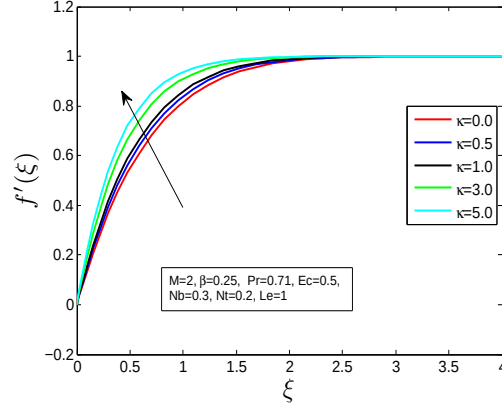


Figure 4.3: Velocity profiles for various values of  $\kappa$ .

express the conversion of kinetic energy into internal energy by work done against the viscous fluid stress. It is observed that the temperature increases significantly from the surface and attains a peak value around  $\xi = 0.5$ , and then decreases in the rest of the region as given in figure 4.10. This implies that the thermal boundary layer becomes thicker with large Eckert number. The concentration profile gradually reduces near the surface up to  $\xi = 1$  and then it increases with an increase in the viscous dissipation parameter  $Ec$  as highlighted in Figure 4.11.

The influences of the Brownian motion parameter  $Nb$  on the nanofluid temperature and concentration profiles is presented in Figures 4.12 and 4.13 respectively. Figure 4.12 reveals that the temperature profile increases with an increase in  $Nb$ , particularly in the region close to the surface. The physics behind this phenomenon is that the increased  $Nb$  increase the thickness of thermal boundary layer, which finally enhances the temperature. Figure 4.13 remarks that increasing the  $Nb$  tends to decrease the concentration profile near the surface. Figures 4.14 and 4.15 respectively, reveal the usual temperature and concentration profiles for various value of

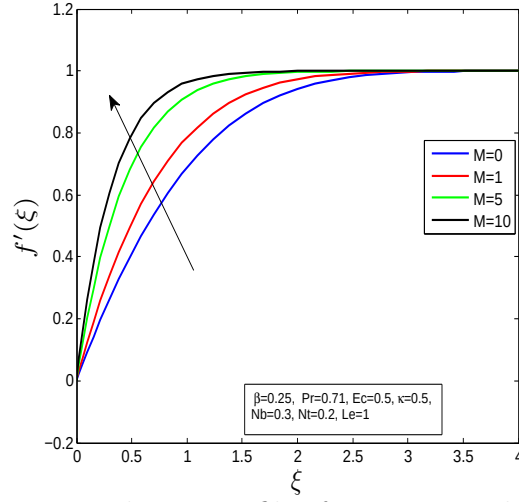


Figure 4.4: Velocity profiles for various values of  $M$ .

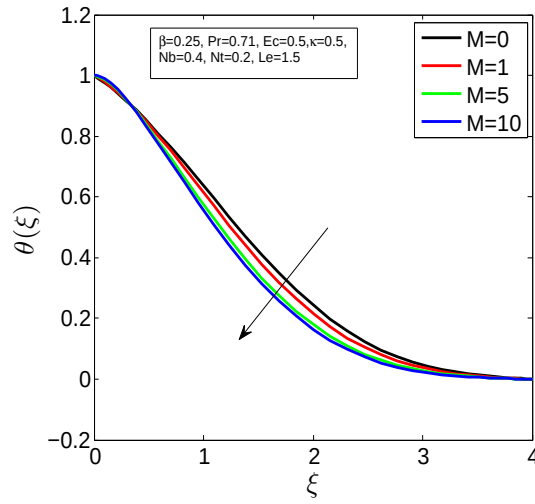


Figure 4.5: Temperature profiles for various values of  $M$ .

thermophoresis parameter  $Nt$ . The thermophoresis force generated by the temperature gradient produce a fast flow and more fluid is heated away from the surface. Consequently, the higher the value of  $Nt$  increase temperature profile and its boundary layer thickness, as given in Figure 4.14. Figure 4.15 reveals that concentration profile decline near the boundary surface until  $\xi = 0.5$  and afterward it increases with an increase in thermophoresis parameter  $Nt$ .

Table 4.2 illustrates the influence of various governing parameters on the skin friction coefficient, local Nusselt number, and local Sherwood number. The skin friction

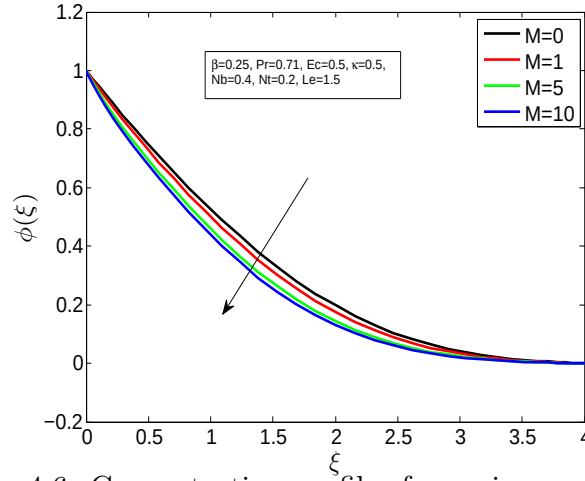


Figure 4.6: Concentration profiles for various values of  $M$ .

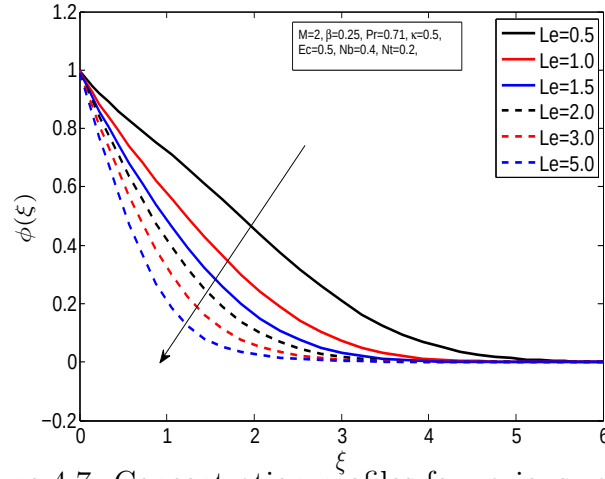


Figure 4.7: Concentration profiles for various values of  $Le$ .

coefficient enhances with increase in pressure gradient, permeability and magnetic parameter. The local Nusselt number is a decreasing function and a local Sherwood number is an increasing function of the pressure gradient parameter, magnetic parameter, Permeability parameter, Prandtl number, Lewis number, thermophoresis parameter and Eckert number.

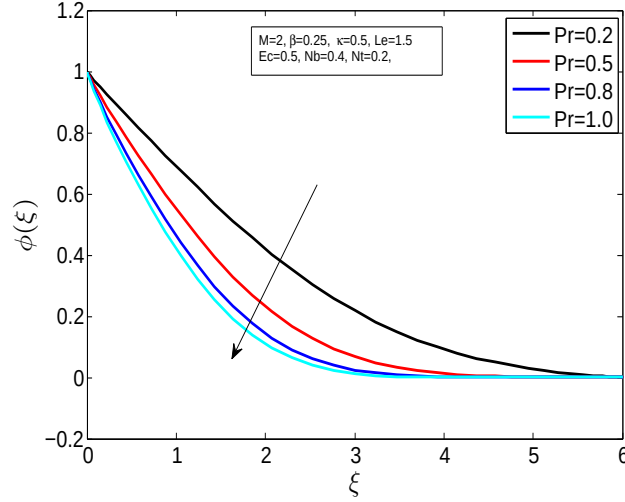


Figure 4.8: Temperature profiles for various values of  $Pr$ .

## 4.5 Conclusion

The problem of steady MHD boundary layer wedge flow with heat and mass transfer of nanofluid past a porous media with viscous dissipation, brownian motion and thermophoresis effects was studied. Using suitable similarity transformations, the governing equations were transformed into a system of non-linear ordinary differential equations and solved numerically employing SQLM. From the above discussions the following conclusions are given:

- The thickness of velocity boundary layer reduces with an increase in pressure gradient, permeability and magnetic parameters.
- Thermal boundary layer thicker with an increase in Eckert number, Brownian motion and thermophoresis parameters.
- Greater the value of Prandtl number, Lewis number, Brownian motion and magnetic parameter reduce the nanofluid concentration profile.
- The skin-friction at the surface enhances with an increase in pressure gradient,

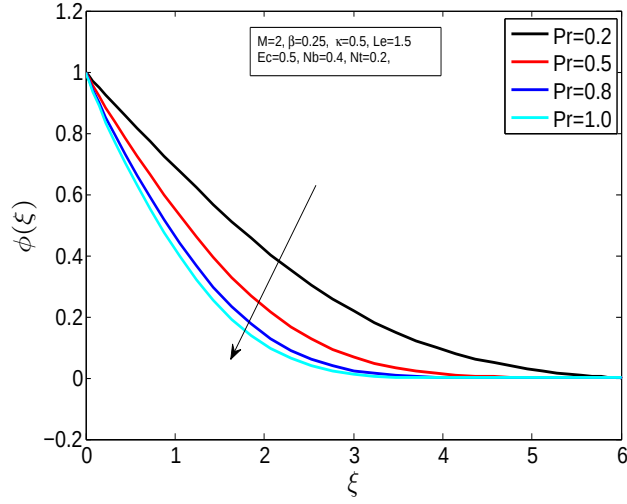


Figure 4.9: Concentration profiles for various values of  $Pr$ .

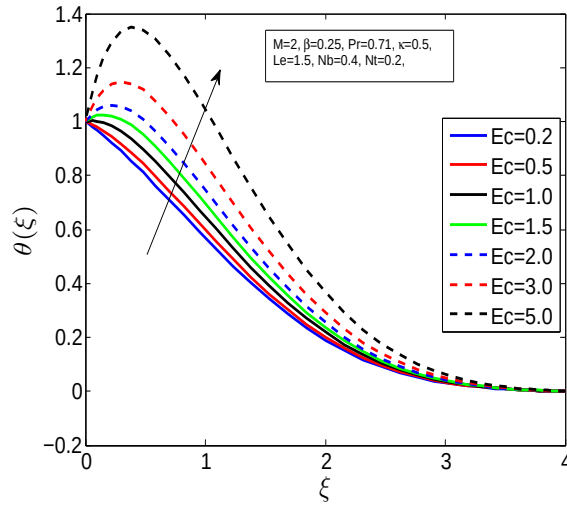


Figure 4.10: Temperature profiles for various values of  $Ec$ .

permeability and magnetic parameters.

- The local Nusselt number is a decreasing function whereas a local Sherwood number is an increasing function of the pressure gradient parameter, magnetic parameter, Permeability parameter, thermophoresis parameter and Eckert number.

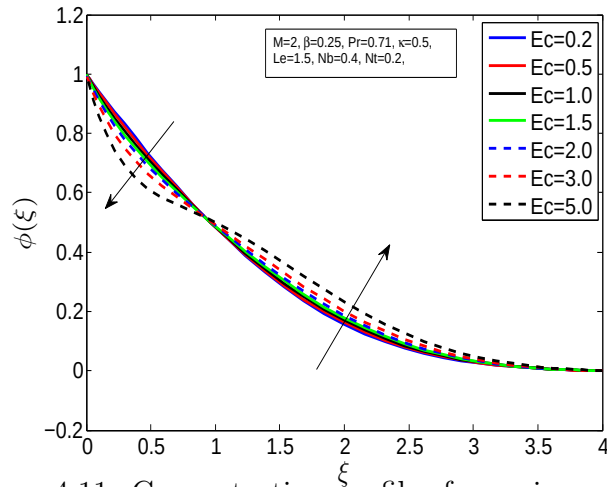


Figure 4.11: Concentration profiles for various values of  $Ec$ .

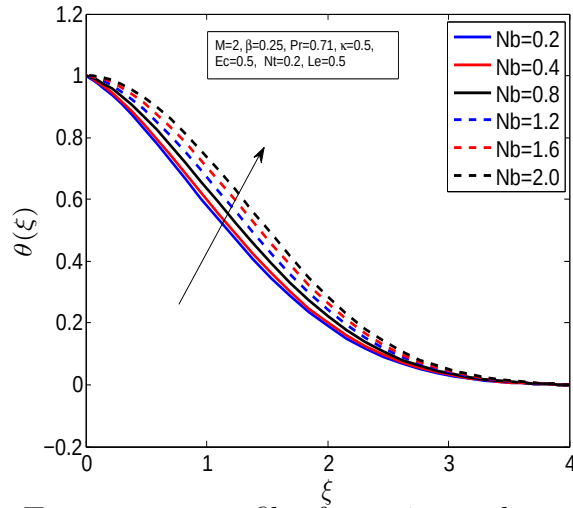


Figure 4.12: Temperature profiles for various values of  $Nb$ .

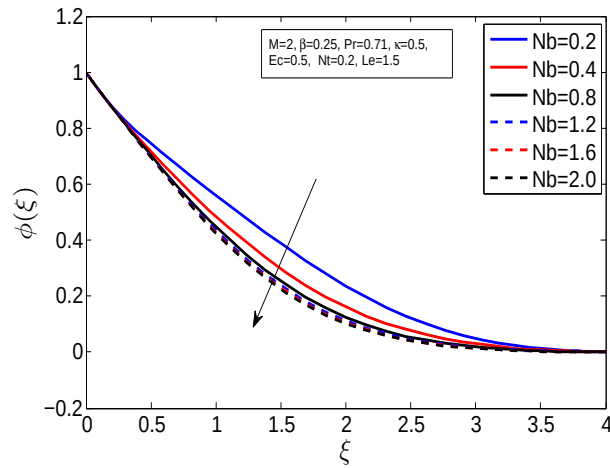


Figure 4.13: Concentration profiles for various values of  $Nb$ .

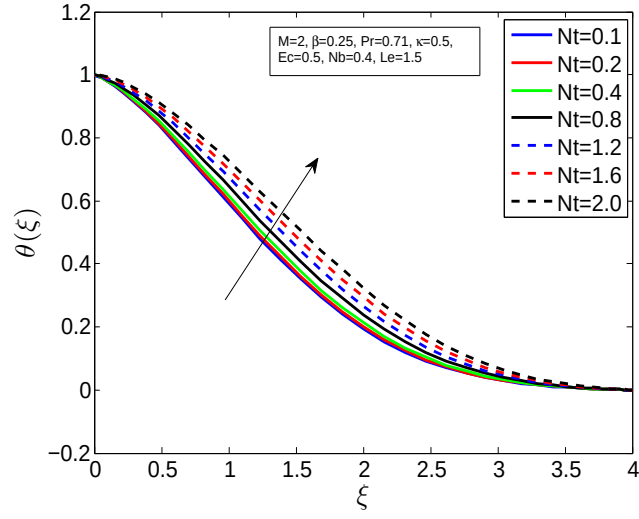


Figure 4.14: Temperature profiles for various values of  $Nt$ .

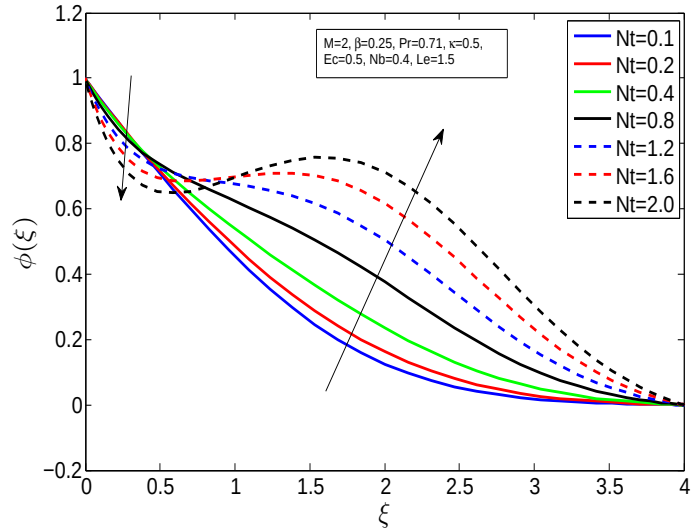


Figure 4.15: Concentration profiles for various values of  $Nt$ .

Table 4.2: Computations of  $-f''(0)$ ,  $-\theta'(0)$  and  $-\phi'(0)$  for various parameters

| $\beta$ | $M$ | $\kappa$ | $Pr$ | $Ec$ | $Nb$ | $Nt$ | $Le$ | $-f''(0)$ | $-\theta'(0)$ | $-\phi'(0)$ |
|---------|-----|----------|------|------|------|------|------|-----------|---------------|-------------|
| 0.25    | 2.0 | 0.5      | 0.5  | 0.5  | 0.4  | 0.2  | 1.5  | 1.642832  | 0.204098      | 0.564850    |
| 0.5     |     |          |      |      |      |      |      | 1.649256  | 0.202453      | 0.567923    |
| 1.0     |     |          |      |      |      |      |      | 1.662020  | 0.199180      | 0.573925    |
| 0.25    | 1.0 |          |      |      |      |      |      | 1.352138  | 0.226508      | 0.522468    |
|         | 3.0 |          |      |      |      |      |      | 1.889814  | 0.182951      | 0.600156    |
|         | 5.0 |          |      |      |      |      |      | 2.306224  | 0.144147      | 0.658486    |
|         | 2.0 | 0.2      |      |      |      |      |      | 1.561229  | 0.210692      | 0.553049    |
|         |     | 0.5      |      |      |      |      |      | 1.642832  | 0.204098      | 0.564850    |
|         |     | 0.8      |      |      |      |      |      | 1.720614  | 0.197619      | 0.576034    |
|         |     | 0.5      | 0.1  |      |      |      |      | 1.642832  | 0.246705      | 0.317615    |
|         |     |          | 0.5  |      |      |      |      | 1.642832  | 0.204098      | 0.564850    |
|         |     |          | 1.0  |      |      |      |      | —         | 0.092267      | 0.815738    |
|         |     |          | 0.5  | 0.1  |      |      |      | —         | 0.352789      | 0.497440    |
|         |     |          |      | 0.5  |      |      |      | —         | 0.204098      | 0.564850    |
|         |     |          |      | 1.0  |      |      |      | —         | 0.017948      | 0.649245    |
|         |     |          |      | 0.5  | 0.2  |      |      | —         | 0.225556      | 0.580703    |
|         |     |          |      |      | 0.4  |      |      | —         | 0.204098      | 0.564850    |
|         |     |          |      |      | 0.8  |      |      | —         | 0.1640511     | 0.556243    |
|         |     |          |      |      | 0.4  | 0.1  |      | —         | 0.211295      | 0.546857    |
|         |     |          |      |      |      | 0.2  |      | —         | 0.204098      | 0.564850    |
|         |     |          |      |      |      | 0.4  |      | —         | 0.190125      | 0.608716    |
|         |     |          |      |      |      | 0.2  | 1.0  | —         | 0.208463      | 0.479245    |
|         |     |          |      |      |      |      | 1.5  | —         | 0.204098      | 0.564850    |
|         |     |          |      |      |      |      | 2.0  | —         | 0.201059      | 0.635752    |

## Chapter 5

# MHD Slip Flow of a Casson Nanofluid over a Stretching Cylinder Embedded in a Porous Medium with Cattaneo-Christov Heat Flux

### 5.1 Introduction

The analysis of non-Newtonian fluids have achieved a remarkable success as a result of its extensive applications in many branches of material process engineering. By their nature, various non-Newtonian fluids do not express the linear relationship concerning stress and rate of strain. Hence, the well-known Navier-Stokes equations of flow may not fully describe the prominent features of non-Newtonian fluids. Some non-Newtonian fluids like human blood, honey, soup, orange juice and tomato sauce start to move when yield stress is less than the shear stress. However, they behave like an elastic solid when yield stress is more than the shear stress. Because of its distinctive features, these fluids are categorized as nonlinear Casson's fluid model. The nonlinear Casson's constitutive equation which designates the properties of several polymers over a broad range of shear rates was first derived by Casson (1959). The

study of Casson fluid in several flow fields has got attention by numerous researchers because of its application in food processing, metallurgy, bio-engineering operations, drilling operations, manufacturing of pharmaceutical products and biological fluids (Pramanik, 2013). For example, Pramanik (2013), and Makanda et al. (2015) analyzed the flow of Casson fluid and heat transfer over a porous medium with incidence of thermal radiation.

Similarly, in the past two decades, nanotechnology is dignified as one of the most important means of innovation for industrial uprising. Different studies related to the rheology of nanofluids revealed that nanofluid has both Newtonian and non-Newtonian behavior. Nanofluids consist of nanoscale particles in usual liquid-carrier fluids and enhances the thermal performance and conductivity of base fluids. Due to its unique chemical and physical features, nanofluids are used in various thermal flow systems like in microchips, in microelectronics, food processing, transportation, biomedicine, space technology, and solid state lightening (Ibrahim & Hindebu, 2019). Furthermore, the study of nanofluids flow over cylindrical geometry has received considerable attention by different researchers. Some of these include Noghrehabadi et al.(2014), and Rehman et al. (2017) who worked on the flow of nanofluids over a stretching cylinder. Moreover, Malik et al. (2014) examined boundary layer flow of Casson nanofluid due to exponentially stretching cylinder, whereas MHD slips flow of a Casson nanofluid through a porous medium over a stretching cylinder was addressed by Nagendramma and Leelaratnam (2017). They reported that nanoparticles improve the heat transfer rate of the base fluid and nanofluids magnetic susceptibility. Also, they showed that inclusion of magnetic field in the Casson nanofluid flow declines the flow velocity while it enhances temperature and concentration of the fluid.

The dynamics of heat transfer phenomenon is a focus of modern-day due to its abundant applications in residential, industrial production, engineering and life sciences. Some of these applications include energy production, nuclear reactor cooling,

space cooling, biomedical applications such as heat conduction in tissues and magnetic drug targeting (Khan et al., 2015; Shehzad et al., 2016). Thus, to overwhelm its side effects and to use efficiently, scholars have clarified the phenomenon of heat transfer and suggested different heat flux models. Initially, Fourier developed the most popular classical heat flux model in continuum mechanics (Akber et al., 2017). One of the major limitations of this model is that it leads to a parabolic energy equation for the temperature field which results the whole system is instantly affected by initial disturbance. This behavior is identified as the paradox of heat conduction and it contradicts the law of causality. To avoid this unrealistic feature, first Cattaneo (1948) improved Fourier's law by adding thermal relaxation time that tolerates the heat flux. Then, Christov (2009) more modified Cattaneo heat flux model from Maxwell-Cattaneo's model. Besides, the uniqueness of Cattaneo Christov model was tested by Tibllo and Zampoli (2001). Currently, much attention has been shown in the study of Cattaneo-Christov heat flux mode. Some of these inquiries include (Makinde et al., 2017; Kundu et al., 2018; Gangadhar et al., 2018). From the studies, they reported that greater value of thermal relaxation parameter leads to reduce the temperature field.

The Naviere-Stokes model of fluid transport is based on the no-slip boundary condition. Nevertheless, there are conditions in which no-slip situation is not suitable. Particularly, no-slip boundary condition is not suitable for most non-Newtonian fluids, as some polymer melts normally express tiny wall slip and usually it is governed by anon-linear and a monotonic relation between the adhesive friction and the slip velocity (Halim et al., 2016). Fluids posing boundary slip have significant applications in different technological and biomedical fields such as purifying of internal cavities and artificial heart valve, inexpensive lubricating, optical coatings, refrigeration equipment, and in many industrial processes (Malik et al., 2017). Due to this, the slip boundary condition has been considered up to now by different researchers.

For example, Nagendramma and Leelaratnam (2017), and Tlili (2018) considered the effects of slip boundary condition on the flow and heat transfer of nanofluid over various geometries.

From the literature cited above, it appears that no analysis has been yet published concerning MHD boundary layer slip flow of a Casson nanofluid over stretching cylinder embedded in a porous medium with Cattaneo-Christov heat flux model and variable thermal conductivity under the considered boundary conditions. The developed model has various applications in the cosmetics industries, and biomedical fields such as purifying of internal cavities and drug delivery. The work considered in this paper was based on the work put forward by Nagendramma and Leelaratnam (2017) in the study of momentum, heat and mass transfer in Casson nanofluid. We extend the model to consider the Cattaneo-Christov heat and mass flux models, thermal and solutal slip effects. The model equations of the flow were solved numerically by using Spectral Relaxation Method (SRM). SRM converges fast and provides more accurate approximations with less grid points (Motsa, 2013). Effects of embedded parameters such as Casson fluid parameter, normally applied magnetic field parameter, permeability parameter, Brownian motion and thermophoresis parameters, thermal and solutal relaxation parameters, thermal conductivity parameter, and wall slip factors on velocity, temperature, concentration, as well as skin friction coefficient, local heat and mass transfer rates were presented in graphical and tabular forms. Furthermore, the accuracy of the present result was confirmed against the previously published results.

## 5.2 Mathematical Formulation

We considered two dimensional, MHD and axisymmetric laminar boundary layer steady state flow of incompressible Casson nanofluid over a horizontally stretching cylinder of radius  $a$ . Since Casson nanofluid is moving along the x-axis direction as

the cylinder is stretching, the x-axis is along the axis of cylinder and the radial coordinate  $r$  is taken perpendicular to the axis of cylinder. It was assumed that the stretching cylinder has the linear axial velocity  $u_w(x) = u_0(x/L)$ , wall temperature  $T_w(x) = T_\infty + T_0(x/L)$  and wall concentration  $C_w(x) = C_\infty + C_0(x/L)$  where  $u_0$  is the reference velocity,  $L$  is the characteristic length,  $T_0$  and  $C_0$  are, respectively constant reference temperature and concentration, whereas  $T_\infty$  and  $C_\infty$  are the ambient temperature and concentration, respectively where  $T_\infty < T_w(x)$ , and  $C_\infty < C_w(x)$ . Furthermore, velocity, thermal and concentration slip conditions, and transpiration effects are considered at the surface of the cylinder. It was also assumed that uniform strength of magnetic field  $B_0$  is perpendicularly applied in the radial direction. To reduce the complexity of mathematical computation, the induced magnetic field caused by the motion of electrically conducting nanofluid was neglected since it is very small compared to applied magnetic field  $B_0$  (see Figure 5.1).

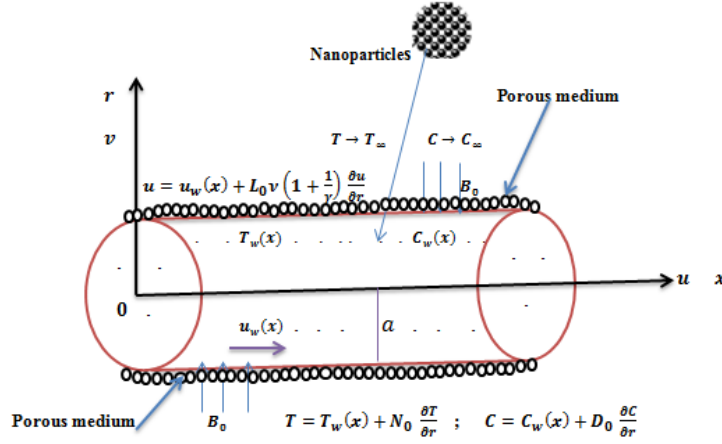


Figure 5.1: Physical model and coordinate system

The rheological equation of state for an isotropic and incompressible flow of a Casson fluid can be expressed as follows (Nagendramma & Leelaratnam 2017).

$$\tau_{ij} = \begin{cases} 2(\mu_\gamma + \frac{P_y}{\sqrt{2\pi}})e_{ij} & \text{if } \pi > \pi_c; \\ 2(\mu_\gamma + \frac{P_y}{\sqrt{2\pi_c}})e_{ij} & \text{if } \pi < \pi_c \end{cases} \quad (5.2.1)$$

where  $e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$  is the rate of strain tensor,  $\mu_\gamma$  is the Casson fluid viscosity coefficient,  $\pi = e_{ij}e_{ij}$  is the product of the components of the deformation rate tensor with itself,  $\pi_c$  is the critical value of this product following the non-Newtonian model,  $P_y$  is the yield stress of the fluid and  $u_i$  is the velocity component.

The Cattaneo-Christov diffusion model that characterize the thermal and concentration diffusion with relaxation of heat flux  $\mathbf{q}$  and mass flux  $J$  respectively, can be written as (Khan et al. 2015; Ibrahim & Hindebu, 2019).

$$\lambda_e \left[ \frac{\partial \mathbf{q}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{u} + (\nabla \cdot \mathbf{u}) \mathbf{q} \right] + \mathbf{q} = -k \nabla T \quad (5.2.2)$$

$$\lambda_c \left[ \frac{\partial J}{\partial t} + \mathbf{u} \cdot \nabla J - J \cdot \nabla \mathbf{u} + (\nabla \cdot \mathbf{u}) J \right] + J = -D_B \nabla C \quad (5.2.3)$$

where  $\lambda_e$  and  $\lambda_c$  denote the relaxation time of heat flux and mass flux respectively,  $k$  is the thermal conductivity,  $D_B$  is the Brownian diffusion coefficient, and  $\mathbf{u}$  is the velocity vector.

For steady incompressible flow, equations (5.2.2) and (5.2.3)) are simplified to

$$\lambda_e [\nabla \cdot ((\nabla \cdot \mathbf{q}) \mathbf{u})] + \nabla \cdot \mathbf{q} = -k \nabla^2 T \quad (5.2.4)$$

$$\lambda_c [\nabla \cdot ((\nabla \cdot J) \mathbf{u})] + \nabla \cdot J = -D_B \nabla^2 C \quad (5.2.5)$$

The above equations (5.2.4) and (5.2.4) respectively, give Fourier's law of heat conduction and Fick's law of concentration diffusion when  $\lambda_e = \lambda_c = 0$ .

Employing equations (5.2.4) and (5.2.5) with the above assumptions, the governing equations of the flow problem for mass, momentum, thermal energy, and concentration are given as (Nagendramma & Leelaratnam, 2017; Ibrahim & Hindebu, 2019).

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0 \quad (5.2.6)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \left(1 + \frac{1}{\gamma}\right) \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r}\right) - \left(\frac{\sigma B_0^2}{\rho} + \frac{\nu}{K}\right) u \quad (5.2.7)$$

$$\begin{aligned} \lambda_e \left[ u^2 \frac{\partial^2 T}{\partial x^2} + v^2 \frac{\partial^2 T}{\partial r^2} + 2uv \frac{\partial^2 T}{\partial x \partial r} + u \frac{\partial u}{\partial x} \frac{\partial T}{\partial x} + u \frac{\partial v}{\partial x} \frac{\partial T}{\partial r} + v \frac{\partial u}{\partial r} \frac{\partial T}{\partial x} + v \frac{\partial v}{\partial r} \frac{\partial T}{\partial r} \right] \\ + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \alpha \frac{\partial T}{\partial r} \right) + \frac{\nu}{(c_p)} \left(1 + \frac{1}{\gamma}\right) \left( \frac{\partial u}{\partial r} \right)^2 \quad (5.2.8) \\ + \varsigma \left\{ D_B \left( \frac{\partial T}{\partial r} \frac{\partial C}{\partial r} \right) + \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial r} \right)^2 \right\} \\ \lambda_c \left[ u^2 \frac{\partial^2 C}{\partial x^2} + v^2 \frac{\partial^2 C}{\partial r^2} + 2uv \frac{\partial^2 C}{\partial x \partial r} + u \frac{\partial u}{\partial x} \frac{\partial C}{\partial x} + u \frac{\partial v}{\partial x} \frac{\partial C}{\partial r} + v \frac{\partial u}{\partial r} \frac{\partial C}{\partial x} + v \frac{\partial v}{\partial r} \frac{\partial C}{\partial r} \right] \\ u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial r} = \frac{D_B}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C}{\partial r} \right) + \frac{D_T}{r T_\infty} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \quad (5.2.9) \end{aligned}$$

The appropriate boundary conditions are written as:

$$\begin{aligned} \text{at } r = a : \quad u = u_w(x) + L_0 \nu \left(1 + \frac{1}{\gamma}\right) \frac{\partial u}{\partial r}; \quad v = v_0; \\ T = T_w(x) + N_0 \frac{\partial T}{\partial r}; \quad C = C_w(x) + D_0 \frac{\partial C}{\partial r} \quad \forall x \quad (5.2.10) \\ \text{as } r \rightarrow \infty : \quad u = 0; \quad T \rightarrow T_\infty; \quad C \rightarrow C_\infty \end{aligned}$$

where  $u$  and  $v$  are velocity of the  $x$  and  $r$  directions respectively.  $\rho$ ,  $\nu$ ,  $c_p$  and  $\sigma$  are respectively density, kinematic viscosity, specific heat capacity and electrical conductivity of the Casson nanofluid.  $\gamma = \mu_\gamma \sqrt{2\pi_c/P_y}$  is the non-Newtonian Casson parameter,  $K$  is the permeability of porous medium,  $\alpha = \alpha_\infty (1 + \epsilon[(T - T_\infty)/\Delta T])$  is the variable thermal diffusivity of the fluid,  $\alpha_\infty$  is the thermal diffusivity of the ambient fluid,  $\epsilon$  is variable thermal conductivity parameter,  $D_B$  is the Brownian diffusion coefficient,  $D_T$  is the thermophoresis diffusion coefficient,  $\varsigma = (\rho c_p)_s/(\rho c_p)_{bf}$  is the ratio of the effective heat capacity of the nanoparticle and the base fluid,  $v_0$  is the suction/blowing velocity, and  $L_0$ ,  $N_0$  and  $D_0$  are the velocity, thermal, and solutal slips coefficients respectively.

To transform highly non-linear and parabolic PDEs into a system of ODEs, we introduce the stream function  $\psi(x, r)$  such that

$$u = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad v = -\frac{1}{r} \frac{\partial \psi}{\partial x} \quad (5.2.11)$$

and dimensionless similarity variables based on (Ibrahim & Hindebu, 2019).

$$\xi = \frac{r^2 - a^2}{2a} \sqrt{\frac{u_0}{\nu L}}; \psi = \sqrt{u_w \nu x a} f(\xi); \theta(\xi) = \frac{T - T_\infty}{T_w - T_\infty}; \phi(\xi) = \frac{C - C_\infty}{C_w - C_\infty} \quad (5.2.12)$$

We find that continuity equation (5.2.6) is satisfied, and equations (5.2.7) - (5.2.10) are transformed into a dimensionless system of non-linear ODEs:

$$(1 + \frac{1}{\gamma}) [(1 + 2\omega\xi)f''' + 2\omega f''] + f f'' - f'^2 - (Ha^2 + \kappa) f' = 0 \quad (5.2.13)$$

$$\begin{aligned} \frac{1}{Pr} (1 + 2\omega\xi)(1 + \epsilon\theta)\theta'' + \frac{2\omega}{Pr} (1 + \epsilon\theta)\theta' + \frac{\epsilon}{Pr} (1 + 2\omega\xi)\theta'^2 - \alpha_t (f^2\theta'' + f f'\theta') \\ + f\theta' + Ec(1 + \frac{1}{\gamma})(1 + 2\omega\xi)f''^2 + (1 + 2\omega\xi)[Nb\theta'\phi' + Nt\theta'^2] = 0 \end{aligned} \quad (5.2.14)$$

$$(1 + 2\omega\xi)\phi'' + 2\omega\phi' + Le f\phi' - Le\alpha_c (f^2\phi'' + f f'\phi') + \frac{Nt}{Nb} [(1 + 2\omega\xi)\theta'' + 2\omega\theta'] = 0 \quad (5.2.15)$$

subject to the transformed boundary conditions:

$$\begin{aligned} \text{at } \xi = 0; \quad f = s, \quad f'(0) = 1 + (1 + \frac{1}{\gamma})\delta_v f''(0); \\ \theta(0) = 1 + \delta_t \theta'(0); \quad \phi(0) = 1 + \delta_c \phi'(0) \\ \text{as } \xi \rightarrow \infty; \quad f' \rightarrow 0; \quad \theta \rightarrow 0; \quad \phi \rightarrow 0 \end{aligned} \quad (5.2.16)$$

where  $\omega = \sqrt{\frac{\nu L}{a^2 u_0}}$  is a curvature parameter while  $1 + 2\omega\xi = \frac{r^2}{a^2}$ ,  $Ha = \sqrt{\frac{\sigma L B_0^2}{\rho u_0}}$  is Hartman number (magnetic parameter),  $\kappa = \frac{\nu L}{K u_0}$  is permeability parameter,  $Pr = \frac{\nu}{\alpha_\infty}$  is a Prandtl number,  $\alpha_t = \lambda_e \frac{u_0}{L}$  is the thermal relaxation parameter,  $Ec = \frac{u_w^2}{(c_p)(T_w - T_\infty)}$  is the Eckert number,  $Nb = \frac{\tau D_B (C_w - C_\infty)}{\nu}$  is the Brownian motion parameter,  $Nt = \frac{\tau D_T (T_w - T_\infty)}{\nu T_\infty}$  is the thermophoresis parameter,  $Le = \frac{\alpha}{D_B}$  is a Lewis number,

$\alpha_c = \lambda_c \frac{u_0}{L}$  is the solutal relaxation parameter,  $s = -v_0 \sqrt{\frac{L}{\nu u_0}}$  is the suction (for  $s > 0$ ) or blowing (for  $s < 0$ ) parameter,  $\delta_v = L_0 \sqrt{\frac{\nu u_0}{L}}$  is the velocity slip parameter,  $\delta_t = N_0 \sqrt{\frac{u_0}{\nu L}}$  and  $\delta_c = D_0 \sqrt{\frac{u_0}{\nu L}}$  are respectively the thermal slip factor and solutal slip factor.

The most important dimensionless physical quantities for the practical problem in engineering the skin friction coefficient  $C_f$ , Nusselt number  $Nu$ , and Sherwood number  $Sh$  at the surface of the cylinder are respectively defined as:

$$C_f = \frac{2\tau_w}{\rho u_w^2}, \quad Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \quad Sh_x = \frac{xJ_w}{D_B(C_w - C_\infty)} \quad (5.2.17)$$

where for Casson fluid the surface shear stress  $\tau_w = -\mu \left(1 + \frac{1}{\gamma}\right) \left(\frac{\partial u}{\partial r}\right)_{r=a}$ , the surface heat flux  $q_w = -k \left(\frac{\partial T}{\partial r}\right)_{r=a}$ , and the surface mass flux  $J_w = -D_B \left(\frac{\partial C}{\partial r}\right)_{r=a}$ . Thus, the dimensionless skin friction coefficient, local Nusselt number, and local Sherwood Number are respectively, given by:

$$\frac{1}{2} \sqrt{Re_x} C_f = - \left(1 + \frac{1}{\gamma}\right) f''(0), \quad \frac{Nu_x}{\sqrt{Re_x}} = -\theta'(0), \quad \frac{Sh_x}{\sqrt{Re_x}} = -\phi'(0) \quad (5.2.18)$$

where  $Re_x$  is local Reynolds number defined by  $Re_x = \frac{u_w x}{\nu}$

### 5.3 Method of Solutions

The systems of coupled ODEs defined in (5.2.13) - (5.2.15) are highly nonlinear and it is difficult to find their analytical solutions. Thus, they have been solved numerically employing a powerful numerical technique, called Spectral Relaxation Method (SRM). The SRM algorithm first decouples the system of equations. Then, an iteration scheme is established from the decoupled equations by assessing linear terms at the  $r + 1$  current iteration level. To employ this method, first the order of the equation

(5.2.13) is reduced by setting  $g = f'$ , then using the concept of the Gauss-Seidel relaxation method, equations (5.2.13) - (5.2.16) can be given in compact form as:

$$f'_{r+1} = g_r \quad (5.3.1)$$

$$a_{1,r}g''_{r+1} + a_{2,r}g'_{r+1} - a_{3,r}g_{r+1} = a_{4,r} \quad (5.3.2)$$

$$b_{1,r}\theta''_{r+1} + b_{2,r}\theta'_{r+1} = b_{3,r} \quad (5.3.3)$$

$$c_{1,r}\phi''_{r+1} + c_{2,r}\phi'_{r+1} = c_{3,r} \quad (5.3.4)$$

subject to corresponding boundary condition:

$$\begin{aligned} f_{r+1}(0) &= s, \quad g_{r+1}(0) = 1 + \left(1 + \frac{1}{\gamma}\right)\delta_v g'_{r+1}(0) \\ \theta_{r+1}(0) &= 1 + \delta_t \theta'_{r+1}(0), \quad \phi_{r+1}(0) = 1 + \delta_c \phi'_{r+1}(0) \\ f'_{r+1}(\infty) &= 0, \quad \theta_{r+1}(\infty) = 0, \quad \phi_{r+1}(\infty) = 0 \end{aligned} \quad (5.3.5)$$

where

$$\begin{aligned} a_{1,r} &= (1 + \frac{1}{\gamma})(1 + 2\omega\xi), \quad a_{2,r} = 2\omega(1 + \frac{1}{\gamma}) + f_{r+1}, \quad a_{3,r} = Ha^2 + \kappa, \quad a_{4,r} = g_r^2 \\ b_{1,r} &= (1 + 2\omega\xi)(1 + \epsilon\theta_{r+1}) - Pr\alpha_t f_{r+1}^2 \\ b_{2,r} &= 2\omega(1 + \epsilon\theta_{r+1}) + Pr[f_{r+1} - \alpha_t f_{r+1}g_{r+1} + (1 + 2\omega\xi)Nb\phi'_r] \\ b_{3,r} &= -[PrEc(1 + \frac{1}{\gamma})(1 + 2\omega\xi)g_{r+1}^2 + (\epsilon(1 + 2\omega\xi) + PrNt(1 + 2\omega\xi))\theta_r'^2] \\ c_{1,r} &= (1 + 2\omega\xi) - Le\alpha_c f_{r+1}^2, \quad c_{2,r} = 2\omega + Le(f_{r+1} - \alpha_c f_{r+1}g_{r+1}) \\ c_{3,r} &= -\frac{Nt}{Nb}[(1 + 2\omega\xi)\theta_{r+1}'' + 2\omega\theta_{r+1}'] \end{aligned}$$

where the terms  $r+1$  and  $r$  are at the current and previous iteration levels respectively.

Equations (5.3.1)-(5.3.4) are solved using Chebyshev pseudospectral method. The unknown functions  $f$ ,  $g$ ,  $\theta$  and  $\phi$  are defined by Chebyshev interpolating polynomials with Gauss-Lobatto points as follows (Trefethen, 2000).

$$\chi_j = \cos(\pi j/N), \quad j = 0, 1, 2, \dots, N; \quad -1 \leq \chi \leq 1 \quad (5.3.6)$$

where  $N$  is the number of collocation points used. For large scale parameter, the semi infinite interval  $[0, \infty)$  is changed to  $[0, L]$ . The physical domain  $[0, L]$  is converted into interval  $[-1, 1]$  by employing linear transformation

$$\xi = \frac{1}{2}(\chi + 1), \quad -1 \leq \chi \leq 1 \quad (5.3.7)$$

The system of equations are discretized using spectral collocation method. The differentiation matrix  $\mathbf{D} = \frac{2D}{L}$  is used to approximate derivatives of unknown functions where  $D$  is  $(N + 1) \times (N + 1)$  Chebyshev differentiation matrix. Finally, the system of equations (5.3.1)-(5.3.4) can be solved by means of matrices phase by phase as follows.

$$A_1 F_{r+1} = B_1; \quad f_{r+1}(\chi_N) = s \quad (5.3.8)$$

$$A_2 G_{r+1} = B_2; \quad g_{r+1}(\chi_N) - (1 + \frac{1}{\gamma})\delta_v g'_{r+1}(\chi_N) = 1, \quad g_{r+1}(\chi_0) = 0 \quad (5.3.9)$$

$$A_3 \Theta_{r+1} = B_3; \quad \theta_{r+1}(\chi_N) - \delta_t \theta'_{r+1}(\chi_N) = 1, \quad \theta_{r+1}(\chi_0) = 0 \quad (5.3.10)$$

$$A_4 \Phi_{r+1} = B_4; \quad \phi_{r+1}(\chi_N) - \delta_c \phi'_{r+1}(\chi_N) = 1, \quad \phi_{r+1}(\chi_0) = 0 \quad (5.3.11)$$

where

$$A_1 = \mathbf{D}, \quad B_1 = g_r,$$

$$A_2 = a_{1,r} \mathbf{D}^2 + \text{diag}(a_{2,r}) \mathbf{D} - a_{3,r} \mathbf{I}, \quad B_2 = a_{4,r},$$

$$A_3 = b_{1,r} \mathbf{D}^2 + \text{diag}(b_{2,r}) \mathbf{D}, \quad B_3 = b_{3,r},$$

$$A_4 = c_{1,r} \mathbf{D}^2 + \text{diag}(c_{2,r}) \mathbf{D}, \quad B_4 = c_{3,r}$$

And

$$F_{r+1} = [f_{r+1}(\chi_0), f_{r+1}(\chi_1), \dots, f_{r+1}(\chi_N)]^t,$$

$$G_{r+1} = [g_{r+1}(\chi_0), g_{r+1}(\chi_1), \dots, g_{r+1}(\chi_N)]^t,$$

$$\Theta_{r+1} = [\theta_{r+1}(\chi_0), \theta_{r+1}(\chi_1), \dots, \theta_{r+1}(\chi_N)]^t \text{ and}$$

$$\Phi_{r+1} = [\phi_{r+1}(\chi_0), \phi_{r+1}(\chi_1), \dots, \phi_{r+1}(\chi_N)]^t$$

are vectors of sizes  $(N + 1) \times 1$ , and  $\mathbf{I}$  is the identity matrix of size  $(N + 1) \times (N + 1)$ .

When SRM is used the choice of an initial guess is very essential. The suitable initial guesses which satisfy the governing boundary conditions are:

$$f_0(\xi) = s + \frac{1 - e^{-\xi}}{1 + (1 + \frac{1}{\gamma})\delta_v}, \quad \theta_0(\xi) = \frac{e^{-\xi}}{1 + \delta_t}, \quad \phi_0(\xi) = \frac{e^{-\xi}}{1 + \delta_c} \quad (5.3.12)$$

The boundary conditions are imposed on the individual matrices as follows:

$$M_1 = \left[ \begin{array}{c} A_1 \\ \hline 0 \quad - \quad - \quad - \quad 1 \end{array} \right]; \quad F_{r+1} = \left[ \begin{array}{c} f_{r+1,0} \\ \cdot \\ \cdot \\ \cdot \\ \hline f_{r+1,N} \end{array} \right]; \quad R_1 = \left[ \begin{array}{c} B_1 \\ \hline s \end{array} \right] \quad (5.3.13)$$

$$M_2 = \left[ \begin{array}{c} 1 \quad - \quad - \quad - \quad 0 \\ \hline A_2 \\ \hline 0 \quad - \quad - \quad - \quad 1 \end{array} \right]; \quad G_{r+1} = \left[ \begin{array}{c} g_{r+1,0} \\ \cdot \\ \cdot \\ \cdot \\ \hline g_{r+1,N} \end{array} \right]; \quad R_2 = \left[ \begin{array}{c} 0 \\ \hline B_2 \\ \hline 1 + (1 + \frac{1}{\gamma})\delta_v g'_{r+1}(\chi_N) \end{array} \right] \quad (5.3.14)$$

$$M_3 = \left[ \begin{array}{c} 1 \quad - \quad - \quad - \quad 0 \\ \hline A_3 \\ \hline 0 \quad - \quad - \quad - \quad 1 \end{array} \right]; \quad \Theta_{r+1} = \left[ \begin{array}{c} \theta_{r+1,0} \\ \cdot \\ \cdot \\ \cdot \\ \hline \theta_{r+1,N} \end{array} \right]; \quad R_3 = \left[ \begin{array}{c} 0 \\ \hline B_3 \\ \hline 1 + \delta_t \theta'_{r+1}(\chi_N) \end{array} \right] \quad (5.3.15)$$

$$M_4 = \left[ \begin{array}{c} 1 \quad - \quad - \quad - \quad 0 \\ \hline A_4 \\ \hline 0 \quad - \quad - \quad - \quad 1 \end{array} \right]; \quad \Phi_{r+1} = \left[ \begin{array}{c} \phi_{r+1,0} \\ \cdot \\ \cdot \\ \cdot \\ \hline \phi_{r+1,N} \end{array} \right]; \quad R_4 = \left[ \begin{array}{c} 0 \\ \hline B_4 \\ \hline 1 + \delta_c \phi'_{r+1}(\chi_N) \end{array} \right] \quad (5.3.16)$$

## 5.4 Results and Discussion

In present study, the non-Newtonian Casson nanofluid flow, heat and mass transfer over stretching cylinder in non-Darcy porous medium in the presence of Cattaneo-Christov heat flux model was investigated under the influence of normally applied magnetic field. The effects of variable thermal conductivity, viscous dissipation and wall slip factors are also considered. Since the model equations of the flow have non-linearity of higher order, thus numerical technique spectral relaxation method (SRM) was employed with number of collocation points  $N = 200$  and the scaled parameter  $L = 10$  to acquire more accurate results. Values of the parameters are fixed throughout the computations as  $\gamma = 2, \omega = 0.5, \delta_v = 0.3, Ha = 1, \kappa = 1, Pr = 5, Le = 3, s = 0.1, \delta_t = \delta_c = 0.1, Ec = 0.5, Nt = 0.1, Nb = 0.5, \alpha_t = \alpha_c = 0.1, \epsilon = 0.2$ , unless it is specified. All these parameters are chosen arbitrarily being careful to stay within the acceptable range of the Casson nanofluid.

Table 5.1: SRM Solutions Convergence

| Order of iteration | $-f''(0)$  | $-\theta'(0)$ | $-\phi'(0)$ |
|--------------------|------------|---------------|-------------|
| 01                 | 0.68965517 | 1.08327217    | 1.95884272  |
| 05                 | 1.94321950 | 0.56190873    | 2.08343131  |
| 08                 | 1.94324749 | 0.56187085    | 2.08343058  |
| 09                 | 1.94324745 | 0.56187075    | 2.08343061  |
| 10                 | 1.94324745 | 0.56187074    | 2.08343061  |
| 11                 | 1.94324745 | 0.56187074    | 2.08343061  |
| 15                 | 1.94324745 | 0.56187074    | 2.08343061  |
| 20                 | 1.94324745 | 0.56187074    | 2.08343061  |

The numerical scheme was performed for various values of flow parameters to obtain the velocity, temperature, concentration profiles as well as the skin friction, local heat transfer and local mass transfer coefficients. Convergence of SRM solutions was achieved at few order of approximations. Presented values in Table 5.1 tested the

10<sup>th</sup> order of approximations for  $-f''(0)$ , and  $-\phi'(0)$ , and 11<sup>th</sup> order of approximations for  $-\theta'(0)$  are enough . The accuracy of solution was confirmed by equating with the previously published results. Currently calculated values of local heat transfer  $Nu_x$  and local mass transfer  $Sh_x$  coefficients for different values of Brownian motiom Nb and thermophoresis Nt parameters were compared with previously published data of the literatures (Khan & Pop, 2010; Ittedi et al., 2017; Nagendramma & Leelaratnam, 2017; Ibrahim & Hindebu, 2019); and the results have a very sound agreement up to desired significant digits, as shown in Table 5.2.

Table 5.2: Comparison of  $Nu_x$  and  $Sh_x$  for  $N_b$  and  $N_t$

| $Nu_x$ |       |        |             |        |         |          |
|--------|-------|--------|-------------|--------|---------|----------|
| $N_t$  | $N_b$ | Khan   | Nagendramma | Ittedi | Ibrahim | Present  |
| 0.1    | 0.1   | 0.9524 | 0.952336    | 0.9523 | 0.9524  | 0.952373 |
|        | 0.2   | 0.5056 | 0.505557    | 0.5055 | 0.5055  | 0.505579 |
|        | 0.3   | 0.2522 | 0.252135    | 0.2521 | 0.2521  | 0.252155 |
|        | 0.4   | 0.1194 | 0.119396    | 0.1194 | 0.1193  | 0.119405 |
|        | 0.5   | 0.0543 | 0.054249    | 0.0542 | 0.0541  | 0.054253 |
| 0.3    | 0.1   | 0.5201 | 0.520092    | 0.5201 | 0.5201  | 0.520018 |
|        | 0.2   | 0.2731 | 0.273101    | -      | -       | 0.273058 |
|        | 0.3   | 0.1355 | 0.135514    | -      | -       | 0.135495 |
|        | 0.4   | 0.0641 | 0.064082    | -      | -       | 0.064078 |
|        | 0.5   | 0.0290 | 0.029137    | -      | -       | 0.029132 |
| $Sh_x$ |       |        |             |        |         |          |
| 0.1    | 0.1   | 2.1294 | -           | 2.1291 | 2.1294  | 2.129389 |
|        | 0.2   | 2.3819 | -           | 2.3817 | 2.3820  | 2.381866 |
|        | 0.3   | 2.4100 | -           | 2.4099 | 2.4101  | 2.410014 |
|        | 0.4   | 2.3997 | -           | 2.3995 | 2.3998  | 2.399647 |
|        | 0.5   | 2.3836 | -           | 2.3834 | 2.3837  | 2.383567 |
| 0.2    | 0.1   | 2.2740 | -           | 2.2736 | 2.2741  | 2.274016 |
| 0.3    |       | 2.5286 | -           | 2.5280 | 2.5288  | 2.528827 |

The behavior of physical parameters on velocity  $f'(\xi)$ , temperature  $\theta(\xi)$ , and concentration  $\phi(\xi)$  profiles are interpreted here. Figures 5.2-5.4 show the change of

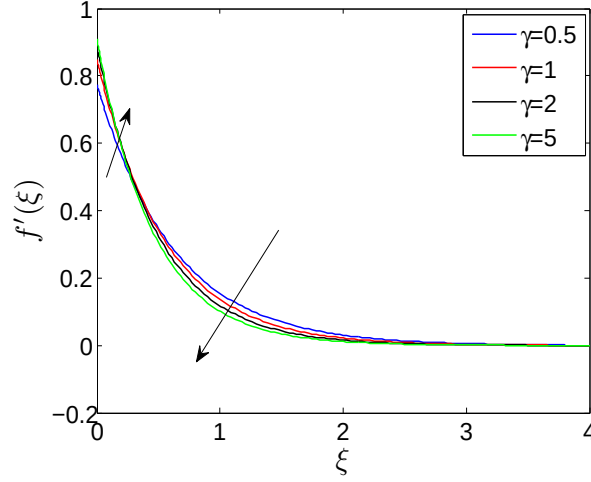


Figure 5.2: Velocity profiles for various values of  $\gamma$ .

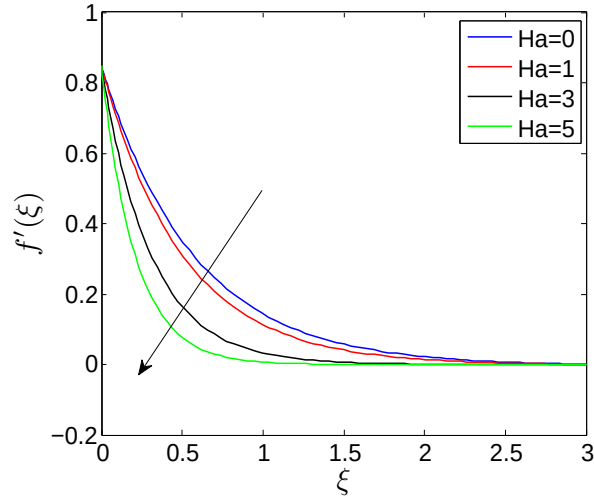


Figure 5.3: Velocity profiles for various values of  $Ha$ .

velocity distribution for increasing values of Casson parameter  $\gamma$ , Hartman number  $Ha$ , and velocity slip parameter  $\delta_v$ . As shown in Figure 5.2, increments in the value of Casson fluid parameter  $\gamma$  tends to enhance the velocity profile near the boundary surface. In fact, increases in Casson fluid parameter tends to reduce in yield stress of fluid, and this helps the Casson fluid to flow easily. However, due to the incidence of magnetic field, the reverse effect is resulted in thinning the momentum boundary layer. Similar result which supports this finding was found by Makanda et al. (2015). Figure 5.3 shows that velocity field and momentum boundary layer thickness are

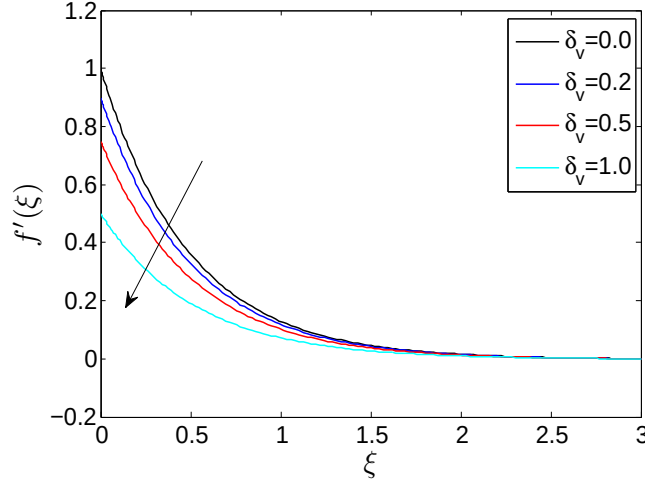


Figure 5.4: Velocity profiles for various values of  $\delta_v$ .

considerably declined with greater the value of Hartman number  $Ha$ . Actually, for greater values of Hartman number, the Lorentz force acts as a retarding force which resist the flow of Casson nanofluid, and this reduces the velocity field of the fluid. Also, as revealed in Figure 5.4, velocity profile diminishes when the value of velocity slip parameter  $\delta_v$  enhances, and the highest effect is noticed near the wall of the cylinder. Indeed, a growth in the values of velocity slip parameter  $\delta_v$  creates the friction force, which slows down the flow of fluid. Similar trend of Casson nanofluid flow was observed as pointed out in Nagendramma and Leelaratnam (2017), and Ibrahim and Hindebu (2019).

The effects of various embedded parameters on temperature profile are presented in Figures 5.5-5.12. Figures 5.5 and 5.6 indicate the influences of Casson fluid parameter  $\gamma$  and magnetic field parameter  $Ha$  on temperature profile, respectively. The temperature profile and thermal boundary layer thickness tend to fall with bigger value of  $\gamma$ . But, it is noted that temperature profile and thermal boundary layer thickness rise as the value of  $Ha$  grows. This is due to the fact that growing in  $Ha$  enhances the Lorentzian force and more heat is produced and boots the temperature

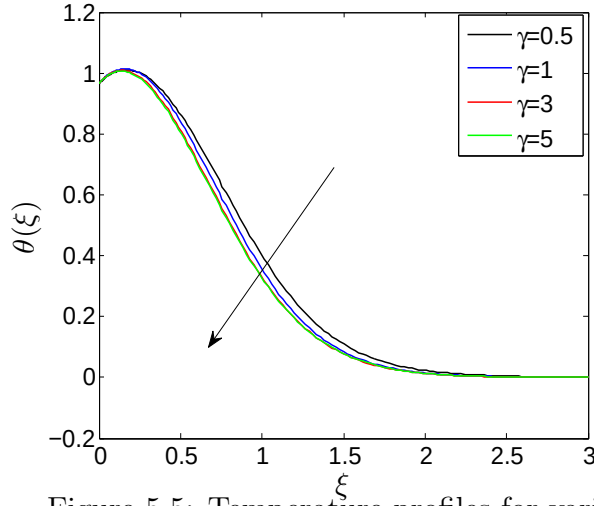


Figure 5.5: Temperature profiles for various values of  $\gamma$ .

field. The effect of Eckert number  $Ec$  and thermal slip factor  $\delta_t$  on temperature profile are presented in Figures 5.7-5.8, respectively. As revealed from the graphs, the temperature field grows and thermal boundary layer thicker with higher the values of  $Ec$ . Because increasing in  $Ec$  it raises the kinetic energy. Consequently, the kinetic energy is converted into the thermal energy, and this results in rises the temperature of the fluid. The temperature profile of the nanofluid reduces with more the value of  $\delta_t$  while the highest effect is noticed at the wall of the cylinder. Moreover, peak temperature is allied with no-slip condition ( $\delta_t = 0$ ).

The influences of Brownian motion  $Nb$  and thermophoresis  $Nt$  on temperature field are presented in Figures 5.9 and 5.10, respectively. Brownian motion is the random motion of dispersed nanoparticles in the base fluid results from the collision of nanoparticles with the fast moving molecules in the base fluid. Physically, larger the  $Nb$  matches to the substantial motion of nanoparticles, and this boosts the nanoparticles kinetic energy that supplement for more heat to produce. Thus, more the value of  $Nb$  tends to enhance the temperature profile and the thickness of thermal boundary layer. Likewise, the temperature profile growths as the values of  $Nt$  improves. Since physically too much thermophoretic effect permeates higher thermal diffusivity of the

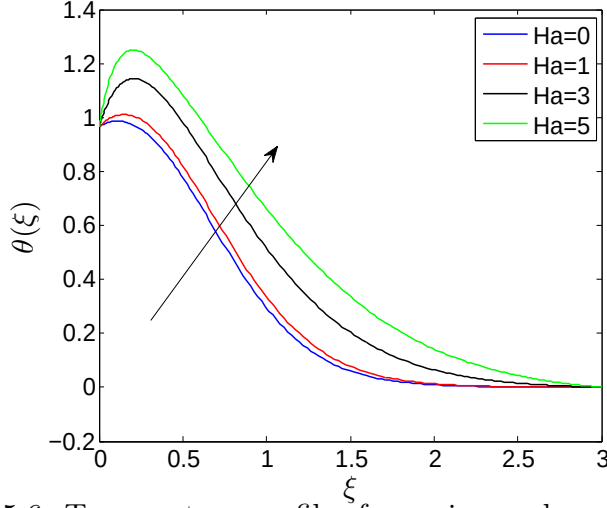


Figure 5.6: Temperature profiles for various values of  $Ha$ .

nanofluid, it increases the thermal boundary layer thickness. For most parameters, similar results were found by Nagendramma and Leelaratnam (2017), and Ibrahim and Hindebu (2019).

Figure 5.11 demonstrates the variation of thermal conductivity parameter  $\epsilon$  on temperature profile. Near the wall of cylinder, the temperature profile reduces for stronger thermal conductivity. But, apart from the wall, reverse behavior is perceived due to the effect of viscous dissipations. The upshot of thermal relaxation parameter  $\alpha_t$  on temperature distribution is demonstrated in Figure 5.12. Except near the wall, temperature profile and thermal boundary layer thickness reduce as  $\alpha_t$  increases. Thus, temperature in Cattaneo-Christov heat flux model is more reduced than in the classical Fourier model ( $\alpha_t = 0$ ).

The variation of Lewis number  $Le$ , solutal relaxation parameter  $\alpha_c$ , and solutal slip factor  $\delta_c$  on concentration distribution are shown in Figures 5.13-5.15. Figure 5.13 displays that concentration profiles reduce as the Lewis number increase. This is due to the fact that Lewis number is the ratio of thermal diffusion to mass diffusion, and greater the  $Le$  adds a less Brownian diffusion coefficient which reduces the thickness of

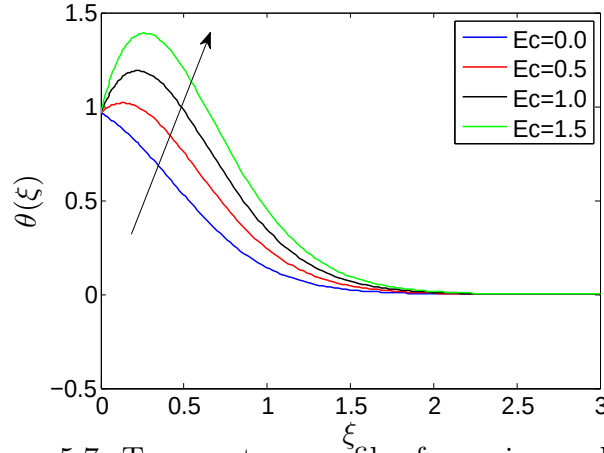


Figure 5.7: Temperature profiles for various values of  $Ec$ .

concentration boundary layer. Figure 5.14 reveals that greater the value of solutal slip factor leads to the decline of concentration profiles with the highest effect recognized at the wall of the cylinder. Moreover, as displayed in Figure 5.15 concentration profiles and its boundary layer thickness drop with greater values of solutal relaxation parameter.

Table 5.3 depicts the variation of skin friction coefficient, local heat and mass transfer rates for various embedded parameters such as Casson parameter  $\gamma$ , permeability parameter  $\kappa$ , magnetic parameter  $Ha$ , curvature parameter  $\omega$ , velocity slip factor  $\delta_v$  and suction parameter  $s$  when other parameters are constant. As it is seen from the table, skin friction coefficient is improved with increasing the values of  $\beta$ ,  $\kappa$ ,  $Ha$ ,  $\omega$  and  $s$ . Actually, the opposition force rises and skin friction force is improved in the boundary layer flow as the values of these parameters growth in the flow region. Similarly, the result reveals that the improvement in the value of  $\beta$ ,  $\kappa$ ,  $Ha$ , and  $\delta_v$  are forced to reduce the local heat and mass transfer rate while the opposite trend is observed for the value of  $\omega$  and  $s$ . Similar result was reported by Nagendramma and Leelaratnam (2017).

Table 5.4 describes the effects of the physical parameters viz. Brownian motion

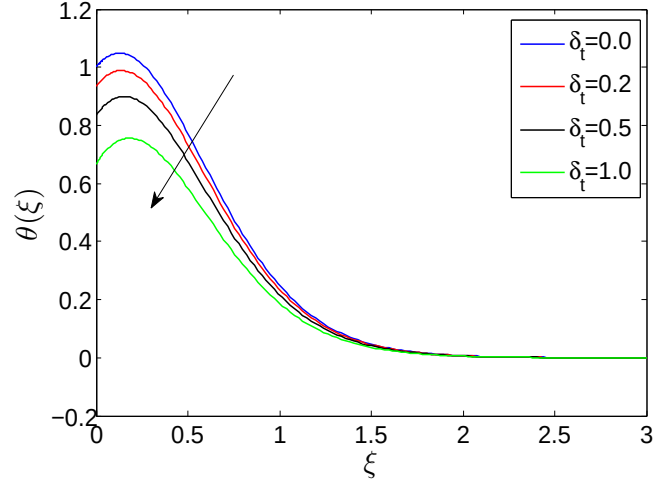


Figure 5.8: Temperature profiles for various values of  $\delta_t$ .

$Nb$ , thermophoresis  $Nt$ , Eckert number  $Ec$ , variable thermal conductivity  $\epsilon$ , thermal relaxation  $\alpha_t$ , concentration relaxation  $\alpha_c$ , thermal slip factor  $\delta_t$ , and solutal slip factor  $\delta_c$  on the local heat transfer and local mass transfer coefficients fixing the others parameters constant. The local Nusselt number is a decreasing function of Brownian motion, thermophoresis, Eckert number, thermal relaxation, concentration relaxation and thermal slip factor while it is an increasing function of variable thermal conductivity and solutal slip factor. Since the Brownian motion influences the nanoparticles to move from high concentration areas to low ones, the impacts of both the Brownian motion and thermophoresis tend to move the nanoparticles away from the stretching cylinder. Thus, the growth of  $Nb$  or  $Nt$  intensifies the diffusion of nanoparticles into the boundary layer, consequently it reduces the local Nusselt number. Also, it is illustrated that the local Sherwood number increases with increasing in Brownian motion parameter, thermophoresis parameter, Eckert number, thermal relaxation parameter and concentration relaxation parameter while the reverse tend is revealed for variable thermal conductivity, thermal slip factor, and solutal slip factor.

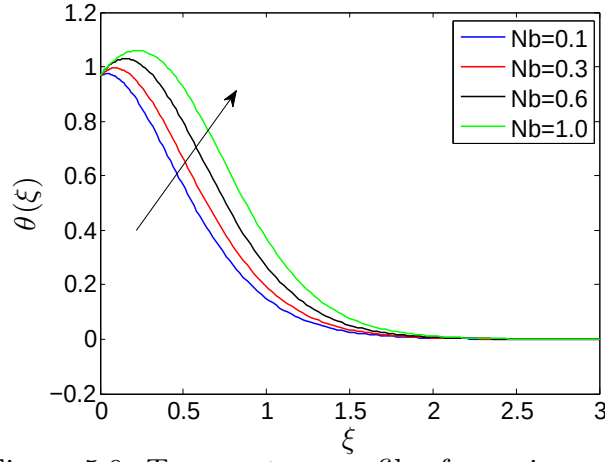


Figure 5.9: Temperature profiles for various values of  $Nb$ .

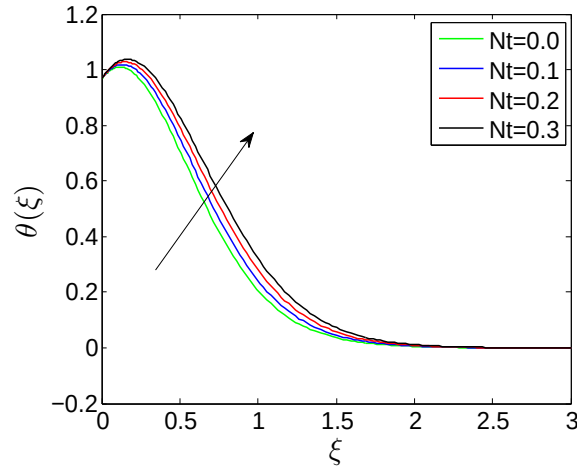


Figure 5.10: Temperature profiles for various values of  $Nt$ .

## 5.5 Conclusion

In this paper, the flow of Casson nanofluid and heat transfer over stretching cylinder embedded in a porous medium with variable thermal conductivity and CattaneoChristov heat flux model is studied. The effects of the embedded physical parameters are taken into account. From obtained results, the following observations are identified.

- Velocity profile declines with more values of magnetic field, velocity slip, and permeability parameters.

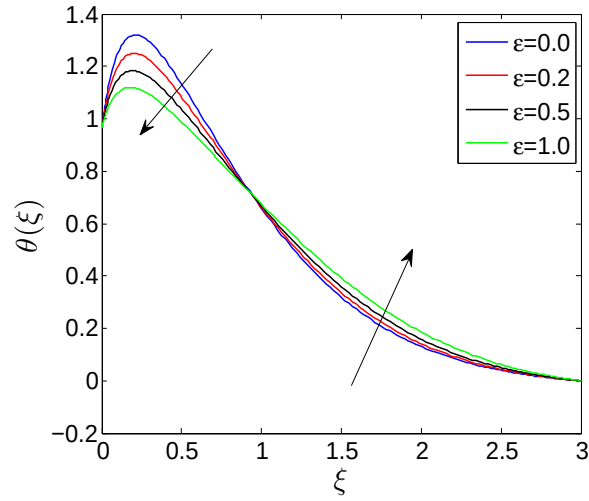


Figure 5.11: Temperature profiles for various values of  $\epsilon$ .

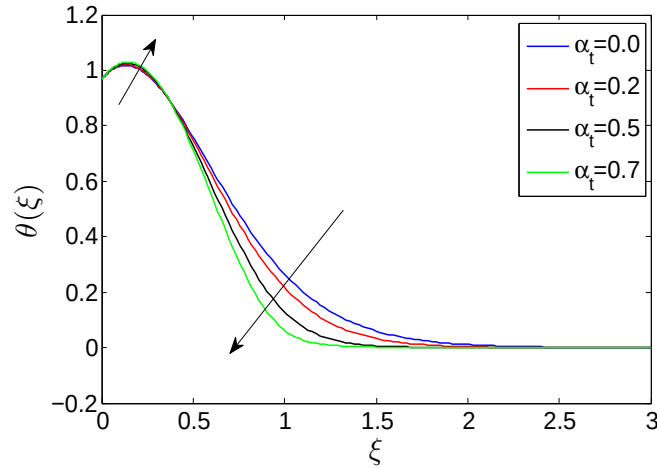


Figure 5.12: Temperature profiles for various values of  $\alpha_t$ .

- Thermal boundary layer tends to thicker with greater the values of Eckert number, magnetic field, permeability, thermal conductivity, Brownian motion, and thermophoresis parameters.
- Magnetic field and permeability parameters have positive effects while Casson fluid parameter, Lewis number, solutal relaxation, and solutal slip parameters have negative effects on concentration profile.
- Skin friction force develops in the boundary layer flow with more values of Casson fluid, magnetic field, and suction parameters.

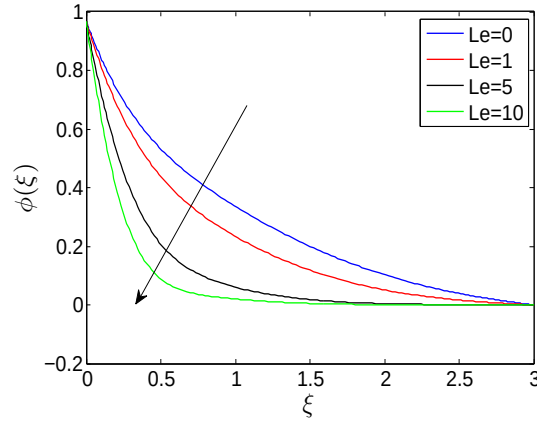


Figure 5.13: Concentration profiles for various values of  $Le$ .

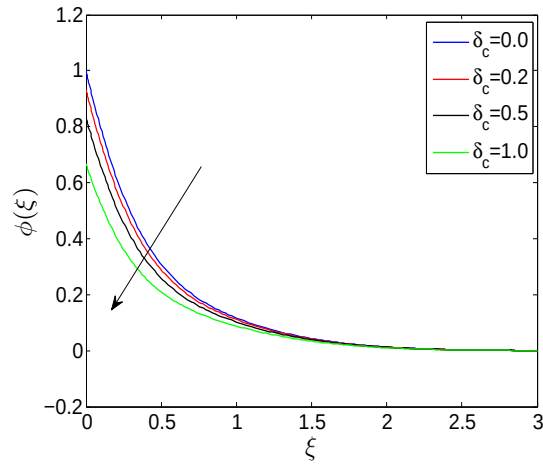


Figure 5.14: Concentration profiles for various values of  $\delta_c$ .

- Surface heat transfer rate weakens as Eckert number, Casson fluid, magnetic field, and thermal relaxation parameters rise.
- Surface mass transfer rate reduces with higher values of Casson fluid, magnetic field, and velocity slip parameters.

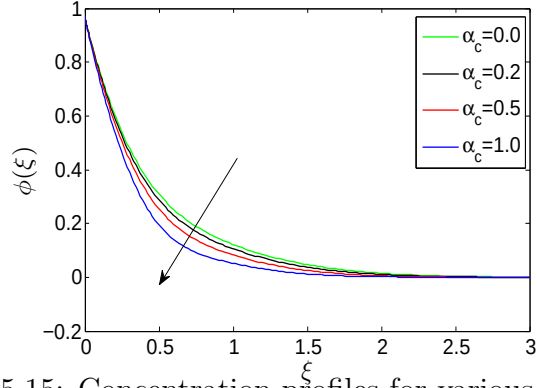


Figure 5.15: Concentration profiles for various values of  $\alpha_c$ .

Table 5.3: Values of  $-f''(0)$ ,  $-\theta'(0)$  and  $-\phi'(0)$  for some parameters

| $\gamma$ | $\kappa$ | $Ha$ | $\omega$ | $\delta$ | $s$  | $-f''(0)$ | $-\theta'(0)$ | $-\phi'(0)$ |
|----------|----------|------|----------|----------|------|-----------|---------------|-------------|
| 1        | 1        | 1    | 0.5      | 0.3      | 0.1  | 1.7354    | 0.5652        | 2.1002      |
| 5        |          |      |          |          |      | 2.1247    | 0.5580        | 2.0665      |
| 10       |          |      |          |          |      | 2.2009    | 0.5563        | 2.0591      |
| 2        | 0        |      |          |          |      | 1.7127    | 0.5697        | 2.1171      |
|          | 1        |      |          |          |      | 1.9432    | 0.5619        | 2.0834      |
|          | 5        |      |          |          |      | 2.6279    | 0.5388        | 1.9968      |
|          | 1        | 1    |          |          |      | 1.9432    | 0.5619        | 2.0834      |
|          |          | 2    |          |          |      | 2.4791    | 0.5438        | 2.0140      |
|          |          | 5    |          |          |      | 4.5896    | 0.4833        | 1.8325      |
|          |          | 1    | 0.1      |          |      | 1.4738    | 0.4637        | 1.3970      |
|          |          |      | 0.5      |          |      | 1.9432    | 0.5619        | 2.0834      |
|          |          |      | 1.0      |          |      | 2.6513    | 0.6994        | 3.0520      |
|          |          |      | 0.5      | 0.0      |      | 2.0448    | 0.5698        | 2.1092      |
|          |          |      |          | 0.3      |      | 1.9432    | 0.5619        | 2.0834      |
|          |          |      |          | 0.5      |      | 1.8760    | 0.5565        | 2.0661      |
|          |          |      |          | 0.3      | -0.1 | 1.8582    | 0.3854        | 1.5947      |
|          |          |      |          |          | 0.2  | 1.9867    | 0.6519        | 2.3558      |
|          |          |      |          |          | 0.5  | 2.1206    | 0.9043        | 3.3550      |

Table 5.4: Values of  $-\theta'(0)$  and  $-\phi'(0)$  for some parameters

| $Nt$ | $Nb$ | $Ec$ | $\epsilon$ | $\alpha_t$ | $\alpha_c$ | $\delta_t$ | $\delta_c$ | $-\theta'(0)$ | $-\phi'(0)$ |
|------|------|------|------------|------------|------------|------------|------------|---------------|-------------|
| 0.0  | 0.5  | 0.0  | 0.2        | 0.1        | 0.1        | 0.1        | 0.1        | 0.6782        | 1.9900      |
| 0.1  |      |      |            |            |            |            |            | 0.5619        | 2.0834      |
| 0.3  |      |      |            |            |            |            |            | 0.3860        | 2.3524      |
| 0.1  | 0.1  |      |            |            |            |            |            | 1.5275        | 1.5991      |
|      | 0.3  |      |            |            |            |            |            | 0.9505        | 2.0331      |
|      | 0.5  |      |            |            |            |            |            | 0.5619        | 2.0834      |
|      | 0.5  | 0.0  |            |            |            |            |            | 0.5619        | 2.0834      |
|      |      | 0.5  |            |            |            |            |            | -1.1449       | 2.3930      |
|      |      | 1.0  |            |            |            |            |            | -2.9092       | 2.7143      |
|      |      | 0.0  | 0.0        |            |            |            |            | 0.5245        | 2.0917      |
|      |      |      | 0.2        |            |            |            |            | 0.5619        | 2.0834      |
|      |      |      | 0.5        |            |            |            |            | 0.5978        | 2.0750      |
|      |      |      | 0.2        | 0.0        |            |            |            | 0.5649        | 2.0818      |
|      |      |      |            | 0.1        |            |            |            | 0.5618        | 2.0834      |
|      |      |      |            | 0.5        |            |            |            | 0.5443        | 2.0915      |
|      |      |      |            | 0.1        | 0.0        |            |            | 0.5738        | 2.0487      |
|      |      |      |            |            | 0.1        |            |            | 0.5619        | 2.0834      |
|      |      |      |            |            | 0.5        |            |            | 0.5151        | 2.2319      |
|      |      |      |            |            | 0.1        | 0.0        |            | 0.5668        | 2.0845      |
|      |      |      |            |            |            | 0.2        |            | 0.5569        | 2.0824      |
|      |      |      |            |            |            | 0.5        |            | 0.5421        | 2.0793      |
|      |      |      |            |            |            | 0.1        | 0.0        | 0.5541        | 2.1048      |
|      |      |      |            |            |            |            | 0.2        | 0.5697        | 2.0690      |
|      |      |      |            |            |            |            | 0.5        | 0.5938        | 1.9976      |

## Chapter 6

# Unsteady Mixed Convection flow of Nanofluid Past a Moving Wedge Embedded in a Porous Media

### 6.1 Introduction

Different types fluids of are frequently used as heat transporters in heat transfer applications. For this reason, researchers have continuously worked to develop innovative heat transfer fluids that have significantly greater thermal conductivity than usual used fluids. In 1995, Choi was the first scholar who developed a newly pioneering type of heat transfer fluids and for such form of fluids he gave the term "nanofluids". He prepared these fluids on sagging nanoscale particles of metallic basis with a normal particle magnitude of less than 100 nm into usual heat transfer fluids. Choi noticed that the addition of 1vol percent of nanoparticles to the usual fluids enhances the thermal conductivity of the fluid nearly twice (Choi & Eastman, 1995).

Nanofluids have been getting high attention in modern years not only due to their higher in thermal conductivity but also potentially worthwhile in many modern-day applications. These including microelectronics, fuel cells, food processing, biomedicine, power generation, ventilation, engine vehicle management, domestic refrigerator, and heat exchanger ( Salahuddin et al., 2017). Nagendramma et al. (2015) described

that the flow of fluid with heat and mass transfer past a wedge body is occurred in several thermal engineering uses such as heat exchangers, geothermal systems, crude oil extraction, thermal insulation, and the storage of nuclear waste. Srinivasacharya et al. (2015) studied MHD flow of a nanofluid past a wedge with a variable magnetic field. Their study revealed that the heat and mass transfer rate are reduced with an increment of pressure gradient parameter. The flow of Cu-Water and Ag-Water nanofluids past a porous wedge with the influences of MHD was considered by Xu and Chen (2017). They showed that dual solutions happen for the negative pressure gradient ( $m < 0$ ). Also, they added that the incidence of flow departure has numerous adverse effects which leads to an enhancement of the drag force on a body absorbed in a flow.

A porous medium is an area which comprises pores that offers opposition to the flow of fluid. The volume of the fluid to flow through a porous medium is resolute by the porosity of the medium. The basic law governing the flow of fluids through porous media is Darcy's Law. The law states that the flow rate of fluid is determined by the fluid pressure, flow area and elevation. Darcy's law does not hold while inertial forces are dominant. Under these conditions the flow of fluid is termed as non-Darcy (Makanda et al., 2015). The flow of fluid in porous media is vital due to its application in material sciences and engineering such as Petroleum industries, seepage of water in riverines, filtration and purification of chemical processes and many others. The analysis of boundary layer flow past a wedge in porous media has been a subject of several recent papers. For example, MHD boundary layer flow past a fixed wedge within porous media was considered by Ramesh et al. (2017). Alama et al. (2017) analyzed unsteady MHD boundary layer flow of nanofluid and heat transfer past a moving wedge in a porous medium. They concluded that presence of porous parameter is helpful to generate heat in the system.

Moreover, convection heat transfer flow, the effect of thermal radiation, viscous

dissipation, and chemical reaction cannot be ignored as they have a direct impact on heat transfer. Mixed convection flow is vital when the buoyancy force is created due to the temperature variation between the solid surface and free stream grows, and which in turn considerably affect the flow and the thermal fields (Rosali et al., 2016 ). Mixed convection flow past a wedge shaped bodies plays an important role in various engineering processes. This flow arises in different transport processes such as heat exchangers, solar collectors, nuclear reactors, ground water pollution, and electronic equipment. In line with this, Pal and Mandal (2015) also conducted a mixed convection flow of nanofluids with the effect of heat generation and viscous dissipation.

Viscous dissipation is the heat energy that is created as a result of friction between fluid layers. It alters the temperature flow distribution by acting like an energy source which affects the amount of heat transfer. Srinivasasai et al., (2015) analyzed the effects of viscous dissipation on MHD flow of nanofluid past a fixed wedge embedded in porous media. They noticed that increases in Eckert number leads to enhance the temperature profile while it leads to reduce concentration profile near the boundary surfaces.

It is remarkable that radiation heat transfer can occur between two bodies separated by a medium colder than both bodies. For example, solar radiation reaches the surface of the earth after passing through cold air layers at high altitudes (Yunus & Ghajar, 2015). Considering the effects of thermal radiation in fluid flow gives rise to a theoretically challenging problem. When radiation emission is incident on fluid flow, it affects certain fluid properties (Chamkha et al., 2013). At high temperatures, radiation significantly affects temperature distribution and heat transfer. The effect of thermal radiation on mixed convection flow of nanofluid flow over a stretching sheet was examined by Nageeb et al. (2017). They revealed that heat transfer rate increases with more the values of thermal radiation parameter while for mass transfer

rate the opposite trend was observed.

Mass convection is the way of mass transfer between a surface and a moving fluid that encompasses both mass diffusion and bulk fluid motion (Yunus & Ghajar, 2015). Chemical reactions are happened in numerous mass transfer problems, and it results in the generation of a species. The flow of fluid is also affected by chemical reactions that take place in it. Makanda et al. (2015) stated that while chemical reactions take place either energy is used or dissolute causing in processes called endothermic or exothermic reactions that affect the properties of fluid. The effect of viscous dissipation, thermal radiation, and chemical reaction on MHD flow of nanofluids past a wedge through porous medium was analyzed by Majety and Gangadhar (2016). Their result revealed that greater the viscous dissipation parameter cause an increase in fluid temperature. They added that mass transfer rate at the boundary surface increases with increases in Lewis number and chemical reaction parameter while it reduces with increases in thermal radiation parameter.

The study of unsteady mixed convection heat and mass transfer flow of nanofluid past a moving wedge embedded in a porous medium has been not given much consideration so far. The aim of the present study is to analyze the effects of nanoparticle volume fraction, and some physical parameters on mixed convection heat and mass transfer flow of nanofluid past a moving wedge embedded in a porous medium with convective boundary condition. The model equations are numerically solved by SQLM which was proposed by Motsa (2013). Spectral method has the characteristics of fast convergence, adaptable to various problems, and more accurate with a relatively small number of unknowns as reported by Motsa et al. (2013). The effects of different parameters on velocity, temperature, and concentration fields as well as the skin friction coefficient, local Nusselt number, and local Sherwood number were analyzed with the help of table and graphical representations.

## 6.2 Mathematical Formulation

We considered unsteady two-dimensional laminar mixed convection heat and mass transfer boundary layer flow of nanofluid past a moving wedge embedded in a porous medium. It was assumed that the coordinate system is chosen with  $x$  corresponding to the plane in the course of the flow and  $y$  indicating towards the free stream as shown in Figure 6.1. It was also presumed that wedge is moving with velocity  $u_w(x, t) = \frac{U_0}{1-\zeta t}x^m$ , and the velocity of the potential flow is  $U(x, t) = \frac{U_\infty}{1-\zeta t}x^m$ . The wall of the wedge is kept with variable temperature  $T_w(x, t) = T_\infty + T_0(\frac{x}{1-\zeta t})^{2m}$ , and variable nanoparticle concentration  $C_w(x, t) = C_\infty + C_0(\frac{x}{1-\zeta t})^{2m}$ , where  $T_0$ ,  $C_0$ ,  $T_\infty$ , and  $C_\infty$  are reference temperature, reference concentration, ambient temperature, and ambient nanoparticle concentration respectively.

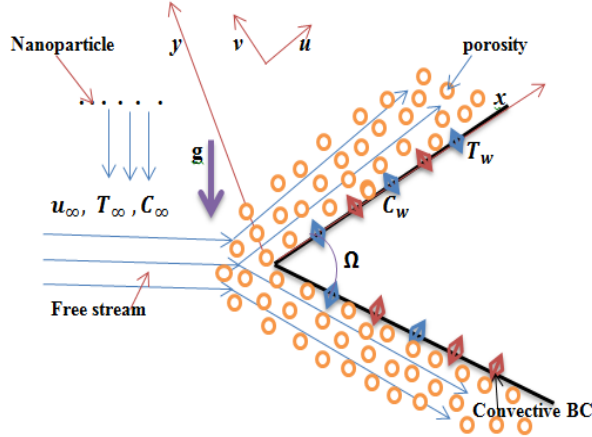


Figure 6.1: Physical model and coordinate system

Furthermore, the buoyancy force caused by temperature and concentration variation inside moving fluid are considered and taken in momentum equation. The effects of thermal radiation, viscous dissipation, chemical reaction, and convective boundary condition are also encompassed in present study. Here,  $U_0 > 0$ ,  $U_\infty \geq 0$ , and  $\zeta \geq 0$  are constants,  $t$  is time and  $\zeta t < 1$ ,  $x$  is measured from the tip of the wedge,  $m$

is the Falkner-Skan power-law parameter,  $\beta = \frac{2m}{1+m}$  is the Hartree pressure gradient parameter corresponding to  $\beta = \frac{\Omega}{\pi}$  for the total angle  $\Omega$  of the wedge.

Using the above assumptions with the boundary layer approximations, the continuity, momentum, energy, and concentration equations governing unsteady two dimensional mixed convection heat and mass transfer boundary layer flow past a moving wedge embedded in a porous medium with thermal radiation, viscous dissipation, chemical reaction effects, and constant fluid properties are given as (Ullah et al., 2016; Nageeb et al., 2017)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6.2.1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho_{nf}} \frac{\partial p}{\partial x} + \nu_{nf} \frac{\partial^2 u}{\partial y^2} - \frac{\nu_{nf}}{K} u + g [\beta_t (T - T_\infty) + \beta_c (C - C_\infty)] \sin \frac{\Omega}{2} \quad (6.2.2)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{nf} \frac{\partial^2 T}{\partial y^2} + \frac{\nu_{nf}}{(c_p)_{nf}} \left( \frac{\partial u}{\partial y} \right)^2 - \frac{1}{(\rho c_p)_{nf}} \frac{\partial q_r}{\partial y} + \frac{\rho_{bf} D_m K_T}{C_s (\rho c_p)_{nf}} \frac{\partial^2 C}{\partial y^2} \quad (6.2.3)$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_m K_T}{T_m} \frac{\partial^2 T}{\partial y^2} - k_r (C - C_\infty) \quad (6.2.4)$$

The suitable boundary conditions are given as:

$$\begin{aligned} t < 0 : \quad u &= 0; \quad v = 0; \quad T = T_\infty; \quad C = C_\infty \quad \forall x, y \\ t \geq 0 : \quad u &= u_w(x, t) = \frac{U_0}{1 - \zeta t} x^m; \quad v = 0; \quad k_{bf} \frac{\partial T}{\partial y} = -h_f (T_w - T) \\ &D_m \frac{\partial C}{\partial y} = -h_s (C_w - C) \quad \text{at } y = 0 \\ u &\rightarrow U(x, t) = \frac{U_\infty}{1 - \zeta t} x^m; \quad T \rightarrow T_\infty; \quad C \rightarrow C_\infty \quad \text{as } y \rightarrow \infty \end{aligned} \quad (6.2.5)$$

where  $u$  and  $v$  denote the  $x$  and  $y$  velocity components respectively.  $\rho_{bf}$ ,  $\rho_{nf}$ ,  $\nu_{nf}$ ,  $\alpha_{nf}$ , and  $(c_p)_{nf}$  are density of base fluid, effective density, kinematic viscosity, thermal diffusivity, and specific heat capacity of the nanofluid respectively. Likewise,  $g$ ,  $\beta_t$ ,  $\beta_c$ ,  $K$ ,  $q_r$ ,  $D_m$ ,  $K_T$ ,  $C_s$ ,  $T_m$ ,  $D_B$  and  $k_r$  are respectively, acceleration due to gravity,

volumetric coefficient of thermal expansion, volumetric coefficient of concentration expansion, permeability of porous medium, radiation heat flux, mass diffusivity of the concentration, thermal diffusion ratio, concentration susceptibility, mean fluid temperature, Brownian diffusion coefficient, and rate of chemical reaction. Moreover,  $h_f(x, t) = h_0 x^{\frac{m-1}{2}} (1 - \zeta t)^{-1/2}$  and  $h_s(x, t) = h_1 x^{\frac{m-1}{2}} (1 - \zeta t)^{-1/2}$  are convective heat and mass transfer coefficients respectively where  $h_0$  and  $h_1$  being constants. The expression  $U(x, t)$ ,  $u_w(x, t)$ ,  $T_w(w, t)$ ,  $C_w(x, t)$ ,  $h_f(x, t)$ , and  $h_s(x, t)$  are usable for  $t > \zeta^{-1}$ .

Since  $U(x)$  is the velocity of the fluid outside the boundary layer, the velocity profile is uniform in the free stream. For a uniform stream, the pressure term in the momentum equation (6.2.2) becomes

$$-\frac{1}{\rho_{nf}} \frac{\partial p}{\partial x} = \frac{dU}{dt} + U \left( \frac{dU}{dx} + \frac{\nu_{nf}}{K} \right) \quad (6.2.6)$$

The radiative heat flux in the y-direction is very high as compared to the x-direction. Employing the Rosseland approximation, the radiative heat flux  $q_r$  is given as (Ibrahim & Shanker, 2014; Nageeb et al., 2017).

$$q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y} \quad (6.2.7)$$

where  $\sigma^*$  is the Stefan-Boltzmann constant and  $k^*$  is the mean absorption coefficient. It is assumed that the temperature variation within the flow is very small and the term  $T^4$  can be expressed as a linear function of temperature. Expanding  $T^4$  in a Taylor series about a free stream temperature  $T_\infty$ , and neglecting higher-order terms beyond the first degree in  $(T - T_\infty)$ , we obtain

$$T^4 \cong 4T_\infty^3 T - 3T_\infty^4 \quad (6.2.8)$$

Hence, replacing equation (6.2.8) into (6.2.7), we obtain

$$q_r = -\frac{16T_\infty^3 \sigma^*}{3k^*} \frac{\partial T}{\partial y} \quad (6.2.9)$$

Substituting equation (6.2.6) into (6.2.2) and equation (6.2.9) into (6.2.3), the momentum and energy equations are written respectively, as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{dU}{dt} + U \frac{dU}{dx} + \nu_{nf} \frac{\partial^2 u}{\partial y^2} + \frac{\nu_{nf}}{K} (U - u) + g [\beta_t (T - T_\infty) + \beta_c (C - C_\infty)] \sin \frac{\Omega}{2} \quad (6.2.10)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{nf} \frac{\partial^2 T}{\partial y^2} + \frac{\nu_{nf}}{(c_p)_{nf}} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{1}{(\rho c_p)_{nf}} \frac{16\sigma^* T_\infty^3}{3k^*} \frac{\partial^2 T}{\partial y^2} + \frac{\rho_{bf} D_m K_T}{C_s (\rho c_p)_{nf}} \frac{\partial^2 C}{\partial y^2} \quad (6.2.11)$$

Nanofluids are formed with entirely different thermophysical properties like density, specific heat capacity, thermal conductivity, convective heat transfer, thermal diffusivity, and viscosity. These thermophysical properties of nanofluids are considered as superior to their basefluid and the term "effective" is used to describe them (Ali et al., 2018).

The nanofluid effective density  $\rho_{nf}$ , effective specific heat capacity  $(c_p)_{nf}$ , effective dynamic viscosity  $\mu_{nf}$ , effective kinematic viscosity  $\nu_{nf}$ , and effective thermal diffusivity  $\alpha_{nf}$  are given as (Yirga & Shankar, 2016; Ali et al., 2018).

$$\begin{aligned} \rho_{nf} &= \phi_s \rho_s + (1 - \phi_s) \rho_{bf}; & (\rho c_p)_{nf} &= \phi_s (\rho c_p)_s + (1 - \phi_s) (\rho c_p)_{bf} \\ \mu_{nf} &= \frac{\mu_f}{(1 - \phi_s)^{2.5}}; & \nu_{nf} &= \frac{\mu_{nf}}{\rho_{nf}} = \frac{\nu_{bf}}{\phi_1}; & \alpha_{nf} &= \frac{k_{nf}}{(\rho c_p)_{nf}} = \frac{k_{nf}}{\phi_2 (\rho c_p)_{bf}} \end{aligned} \quad (6.2.12)$$

where  $\phi_s$ ,  $\rho_s$ ,  $\rho_{bf}$ ,  $(c_p)_s$ ,  $(c_p)_{bf}$ ,  $\mu_{bf}$ , and  $\nu_{bf}$  are respectively, the nanoparticle volume fraction, density of nanoparticle, density of base fluid, specific heat capacity of nanoparticle, specific heat capacity of base fluid, dynamic viscosity, and kinematic viscosity of the base fluid, where the nanoparticle volume fraction  $\phi_1$  and  $\phi_2$  are defined as

$$\phi_1 = (1 - \phi_s)^{2.5} \left[ 1 - \phi_s + \phi_s \frac{\rho_s}{\rho_{bf}} \right]; \quad \phi_2 = 1 - \phi_s + \phi_s \frac{(\rho c_p)_s}{(\rho c_p)_{bf}} \quad (6.2.13)$$

The effective thermal conductivity  $k_{nf}$  is defined using the thermal conductivities of both nanoparticles  $k_s$  and basefluid  $k_f$  as (Yirga & Shankar, 2016; Ali et al., 2018).

$$k_{nf} = k_{bf} \frac{(k_s + 2k_{bf}) - 2\phi_s(k_{bf} - k_s)}{(k_s + 2k_{bf}) + \phi_s(k_{bf} - k_s)} \quad (6.2.14)$$

Table 6.1: Thermo-physical properties of base fluid and nanoparticles

| Physical properties         | $H_2O$ | $Ag$   | $Cu$   | $Al_2O_3$ |
|-----------------------------|--------|--------|--------|-----------|
| $c_p(J/kgK)$                | 4179   | 235    | 385    | 765       |
| $\rho(kg/m^3)$              | 997.1  | 10500  | 8933   | 3970      |
| $k(W/mK)$                   | 0.613  | 429    | 401    | 40        |
| $\alpha \times 10^7(m^2/s)$ | 1.47   | 1738.6 | 1163.1 |           |
| $\beta \times 10^5(K^{-1})$ | 21     | 1.89   | 1.67   | 0.85      |

The above model in equation (6.2.14) considers the two phases (solid and liquid) of the nanofluid and adequately foresees the effective thermal conductivity of nanofluids when the added particles are of spherical shape, low volume percent, and the suspension is at ambient conditions (Ali et al., 2018).

Also, it is assumed that the base fluid and the nanoparticles are in thermal equilibrium and no slip take place between them. The thermophysical properties of the nanofluid for  $Cu$ ,  $Ag$ , and  $Al_2O_3$  nanoparticles and  $H_2O$  as base fluid are given in Table 6.1 (Yirga & Shankar, 2016; Nageeb et al., 2017).

To transform the governing equations (6.2.1) - (6.2.5) into a dimensionless forms of a system of coupled ordinary differential equations, we introduce the stream function  $\psi$  which is defined by the relations

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \quad (6.2.15)$$

and the subsequent similarity transformation (Ullah et al., 2016).

$$\begin{aligned} \xi = y \sqrt{\frac{1+m}{2} \frac{U_\infty}{\nu_{bf}(1-\zeta t)}} x^{\frac{m-1}{2}}; \quad \psi(\xi) = \sqrt{\frac{2}{1+m} \frac{\nu_{bf} U_\infty}{1-\zeta t}} x^{\frac{m+1}{2}} f(\xi) \\ \theta(\xi) = \frac{T - T_\infty}{T_w - T_\infty}; \quad \phi(\xi) = \frac{C - C_\infty}{C_w - C_\infty} \end{aligned} \quad (6.2.16)$$

where  $\xi$  is a dimensionless similarity variable,  $f(\xi)$ ,  $f'(\xi)$ ,  $\theta(\xi)$ , and  $\phi(\xi)$  are correspondingly non-dimensional stream function, velocity, temperature, and nanoparticle concentration.

We find that continuity equation (6.2.1) is identically fulfilled and upon substituting similarity variables into equations (6.2.2) - (6.2.5), we obtain the following system of coupled nonlinear ordinary differential equations:

$$f''' + \phi_1 \left[ f f'' + \beta (1 - (f')^2) + \frac{A}{m+1} (2 - \xi f'' - 2f') \right] + \frac{2}{1+m} \phi_1 (Gr_t \theta + Gr_c \phi) \sin \frac{\Omega}{2} + \frac{1}{m+1} \kappa (1 - f') = 0 \quad (6.2.17)$$

$$\left( \frac{1}{Pr} \frac{k_{nf}}{k_{bf}} + Nr \right) \theta'' + \phi_2 \left[ f \theta' - 2\beta f' \theta - A \left( \frac{\xi}{1+m} \theta' + 2\beta \theta \right) \right] + D_f \phi'' + \frac{Ec}{(1 - \phi_s)^{2.5}} (f'')^2 = 0 \quad (6.2.18)$$

$$\frac{1}{Sc} \phi'' + f \phi' - 2\beta f' \phi - A \left( \frac{\xi}{1+m} \phi' + 2\beta \phi \right) + Sr \theta'' - \frac{2}{1+m} \Upsilon \phi = 0 \quad (6.2.19)$$

The transformed boundary conditions are given as:

$$f(0) = 0; \quad f'(0) = \lambda; \quad \theta'(0) = -Bi_1(1 - \theta(0)); \quad \phi'(0) = -Bi_2(1 - \phi(0))$$

$$f'(\infty) \rightarrow 1; \quad \theta(\infty) \rightarrow 0; \quad \phi(\infty) \rightarrow 0 \quad (6.2.20)$$

where

$$\beta = \frac{2m}{1+m}; \quad A = \frac{\zeta}{U_\infty} x^{1-m}; \quad \kappa = \frac{2\nu_{nf}}{KU} x; \quad Gr_t = \frac{g\beta_t(T_w - T_\infty)}{U^2} x$$

$$Gr_c = \frac{g\beta_c(C_w - C_\infty)}{U^2} x; \quad Pr = \frac{\nu_{bf}}{\alpha_{bf}}; \quad Sc = \frac{\nu_{bf}}{D_B}; \quad Ec = \frac{U^2}{(c_p)_{bf}(T_w - T_\infty)}$$

$$Nr = \frac{16\sigma^* T_\infty^3}{3k^* \nu_{bf} (\rho c_p)_{nf}}; \quad Sr = \frac{D_m K_T}{\nu_{bf} T_m} \frac{T_w - T_\infty}{C_w - C_\infty}; \quad D_f = \frac{\rho_{bf} D_m K_T}{\nu_{bf} C_s (\rho c_p)_{nf}} \frac{C_w - C_\infty}{T_w - T_\infty}$$

$$\Upsilon = \frac{k_r}{U} x^{-1}; \quad \lambda = \frac{U_0}{U_\infty}; \quad Bi_1 = \frac{h_0}{k_{bf}} \left( \frac{2\nu_{bf}}{(1+m)U_\infty} \right)^{1/2}; \quad Bi_2 = \frac{h_1}{D_B} \left( \frac{2\nu_{bf}}{(1+m)U_\infty} \right)^{1/2}$$

Here,  $\phi_s$ ,  $m$ ,  $\beta$ ,  $\lambda$ ,  $A$ ,  $\kappa$ ,  $Gr_t$ ,  $Gr_c$ ,  $Pr$ ,  $Ec$ ,  $Sc$ ,  $D_f$ ,  $Nr$ ,  $Sr$ ,  $\Upsilon$ ,  $Bi_1$  and  $Bi_2$  are respectively, the nanoparticles volume fraction, Falkner-Skan power-law parameter, Hartree pressure gradient parameter, moving wedge parameter, unsteady parameter, permeability parameter, local temperature Grashof number, local concentration

Grashof number. Prandtl number, Eckert number, Schmidt number, Dufour number, thermal radiation parameter, Soret number, scaled chemical reaction parameter, Biot numbers and prime (') denotes derivative with respect to  $\xi$ .

The skin friction coefficient  $C_f$ , local Nusselt number  $Nu_x$ , and local Sherwood number  $Sh_x$  are the three important physical quantities and defined as

$$C_f = \frac{2\tau_w}{\rho_{bf}U^2(x,t)}; \quad Nu_x = \frac{xq_w}{k_{bf}(T_w - T_\infty)}; \quad Sh_x = \frac{xM_w}{D_B(C_w - C_\infty)}; \quad (6.2.21)$$

where  $\tau_w$ ,  $q_w$  and  $M_w$  are correspondingly the surface shear stress, the surface heat flux, and surface mass flux which are given as:

$$\tau_w = -\mu_{nf} \left( \frac{\partial u}{\partial y} \right)_{y=0}; \quad q_w = \left[ -k_{nf} \left( \frac{\partial T}{\partial y} \right) + (q_r)_w \right]_{y=0}; \quad M_w = -D_B \left( \frac{\partial C}{\partial y} \right)_{y=0} \quad (6.2.22)$$

Using equations (6.2.16), (6.2.21), and (6.2.22) the non-dimensional skin friction coefficient, local Nusselt number, and local Sherwood number are respectively found by:

$$\begin{aligned} \frac{1}{2}(1 - \phi_s)^{2.5} \sqrt{\frac{2}{1+m}} C_f \sqrt{Re_x} &= -f''(0) \\ \left( \frac{k_{bf}}{k_{nf}} \sqrt{\frac{2}{1+m}} + Nr \right) \frac{Nu_x}{\sqrt{Re_x}} &= -\theta'(0); \quad \sqrt{\frac{2}{1+m}} \frac{Sh_x}{\sqrt{Re_x}} = -\phi'(0) \end{aligned} \quad (6.2.23)$$

where  $Re_x$  is local Reynolds number defined by  $Re_x = \frac{Ux}{\nu_f}$

### 6.3 Numerical Method

In this section, the system of non-linear ODE (6.2.17) - (6.2.19) subject to the boundary conditions (6.2.20) are solved numerically employing SQLM, the following iterative scheme of linear differential equations are found. (Motsa, 2013; Motsa et al.,

2013).

$$f_{r+1}''' + \phi_1 \left( f_r - \frac{A}{1+m} \xi \right) f_{r+1}'' - \left( 2\phi_1(\beta f_r' + \frac{A}{1+m}) + \frac{k}{1+m} \right) f_{r+1}' + \phi_1 f_r'' f_{r+1} + \frac{2}{1+m} \phi_1 (Gr_t \theta_{r+1} + Gr_c \phi_{r+1}) \sin \frac{\Omega}{2} = \phi_1 (f_r f_r'' - \beta(1 + (f_r')^2)) - \frac{1}{1+m} (2A\phi_1 + k) \quad (6.3.1)$$

$$\left( \frac{1}{Pr} \frac{k_{nf}}{k_f} + Nr \right) \theta_{r+1}'' + \phi_2 \left( f_r - \frac{A}{1+m} \xi \right) \theta_{r+1}' - 2\beta\phi_2(f_r' + A)\theta_{r+1} + \phi_2 \theta_r' f_{r+1} - 2\beta\phi_2 \theta_r f_{r+1}' + \frac{2Ec}{(1-\phi_s)^{2.5}} f_r'' f_{r+1}'' + D_f \phi_{r+1}'' = \phi_2 (f_r \theta_r' - 2\beta f_r' \theta_r) + \frac{Ec}{(1-\phi_s)^{2.5}} (f_r'')^2 \quad (6.3.2)$$

$$\frac{1}{Sc} \phi_{r+1}'' + \left( f_r - \frac{A}{1+m} \xi \right) \phi_{r+1}' - 2 \left( \beta(f_r' + A) + \frac{1}{1+m} \Upsilon \right) \phi_{r+1} + \phi_r' f_{r+1} - 2\beta\phi_r f_{r+1}' + Sr \theta_{r+1}'' = f_r \phi_r' - 2\beta f_r' \phi_r \quad (6.3.3)$$

with transformed corresponding boundary conditions

$$f_{r+1}(0) = 0; \quad f_{r+1}'(0) = \lambda; \quad \theta_{r+1}'(0) = -Bi_1(1 - \theta(0)); \quad \phi_{r+1}'(0) = -Bi_2(1 - \phi(0))$$

$$f_{r+1}'(\infty) \rightarrow 1; \quad \theta_{r+1}(\infty) \rightarrow 0; \quad \phi_{r+1}(\infty) \rightarrow 0 \quad (6.3.4)$$

where the terms  $r+1$  and  $r$  are at the current and previous iteration levels respectively.

Equations (6.3.1)-(6.3.3) are solved using the Chebyshev pseudo-spectral method. The unknown functions are given by Chebyshev interpolating polynomials with Gauss-Lobatto points that are defined by (Trefethen, 2000)

$$\chi_j = \cos(\pi j/N), \quad j = 0, 1, 2, \dots, N; \quad -1 \leq \chi \leq 1 \quad (6.3.5)$$

where  $N$  is the number of collocation points used. For the convenience of numerical computations, the semi-infinite physical domain  $[0, \infty]$  is approximated by the truncated domain. Using the linear transformation  $\chi = \frac{2\xi}{L} - 1$ , the interval  $[0, L]$  is transformed into the interval  $[-1, 1]$  where  $L$  is large but finite number chosen

to represent the behavior of the flow properties of the boundary condition value at infinity.

Discretized the system of equations (6.3.1)-(6.3.3) using spectral collocation method. The differentiation matrix  $\mathbf{D} = \frac{2D}{L}$  is used to approximate derivatives of unknown variables, where  $D$  is  $(N + 1) \times (N + 1)$  Chebyshev differentiation matrix ( for more information see Trefethen, 2000). The system of equations (6.3.1)-(6.3.3) are then solved as a coupled matrix:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{bmatrix} F_{r+1} \\ \Theta_{r+1} \\ \Phi_{r+1} \end{bmatrix} = \begin{bmatrix} B_{1,r} \\ B_{2,r} \\ B_{3,r} \end{bmatrix} \quad (6.3.6)$$

with transformed boundary condition

$$\begin{aligned} F_{r+1}(\chi_N) &= 0, \quad F_{r+1}(\chi_{N-1}) = \lambda, \quad F_{r+1}(\chi_0) = 1 \\ \Theta_{r+1}(\chi_N) &= -Bi_1(1 - \theta(0)), \quad \Theta_{r+1}(\chi_0) = 0 \\ \Phi_{r+1}(\chi_N) &= -Bi_2(1 - \phi(0)), \quad \Phi_{r+1}(\chi_0) = 0 \end{aligned} \quad (6.3.7)$$

where

$$\begin{aligned} \Lambda_{11} &= \mathbf{D}^3 + \phi_1 \text{diag}[f_r - \frac{A}{1+m}\xi] \mathbf{D}^2 - 2\beta\phi_1 \text{diag}[f'_r] \mathbf{D} - \frac{1}{1+m}(2\phi_1 A + k) \mathbf{D} + \phi_1 \text{diag}[f''_r] \\ \Lambda_{12} &= \frac{2}{1+m} \phi_1 Gr_t \sin \frac{\Omega}{2} \mathbf{I}; \quad \Lambda_{13} = \frac{2}{1+m} \phi_1 Gr_c \sin \frac{\Omega}{2} \mathbf{I} \\ \Lambda_{21} &= \frac{2Ec}{(1-\phi_s)^{2.5}} \text{diag}[f''_r] \mathbf{D}^2 - 2\beta\phi_2 \text{diag}[\theta_r] \mathbf{D} + \phi_2 \text{diag}[\theta'_r] \\ \Lambda_{22} &= \left( \frac{1}{Pr} \frac{k_{nf}}{k_f} + Nr \right) \mathbf{D}^2 + \phi_2 \text{diag}[f_r - \frac{A}{1+m}\xi] \mathbf{D} - 2\beta\phi_2 \text{diag}[f'_r + A]; \quad \Lambda_{23} = D_f \mathbf{D}^2 \\ \Lambda_{31} &= -2\beta \text{diag}[\phi_r] \mathbf{D} + \text{diag}[\phi'_r]; \quad \Lambda_{32} = S_r \mathbf{D}^2 \\ \Lambda_{33} &= \frac{1}{Sc} \mathbf{D}^2 + \text{diag}[f_r - \frac{A}{1+m}\xi] \mathbf{D} - 2\beta \text{diag}[f'_r + A] - \frac{2}{1+m} \Upsilon \mathbf{I} \\ B_1 &= \phi_1 [f''_r f_r - \beta(1 + (f'_r)^2)] - \frac{1}{1+m} (2A\phi_1 + k) \\ B_2 &= \phi_2 (f_r \theta'_r - 2\beta f'_r \theta_r) + \frac{Ec}{(1-\phi_s)^{2.5}} (f''_r)^2; \quad B_3 = fr\phi'_r - 2\beta f'_r \phi_r \end{aligned}$$

$F_{r+1} = [f_{r+1,0}, f_{r+1,1}, \dots, f_{r+1,N}]^t$ ,  $\Theta_{r+1} = [\theta_{r+1,0}, \theta_{r+1,1}, \dots, \theta_{r+1,N}]^t$ ,  $\Phi_{r+1} = [\phi_{r+1,0}, \phi_{r+1,1}, \dots, \phi_{r+1,N}]^t$  are vectors of sizes  $(N+1) \times 1$ ,  $diag(\dots)$  represents a diagonal matrix of vectors, and  $\mathbf{I}$  is an identity matrix of size  $(N+1) \times (N+1)$ .

When SQLM is used the choice of an initial guess is very essential. The success of the scheme depends prominently on how good this guess is to give the most accurate solution. The suitable initial guesses that satisfy the governing boundary conditions of the boundary layer problems (6.2.17)-(6.2.20) are:

$$f_0(\xi) = \xi - 1 + e^{-(1-\lambda)\xi}; \quad \theta_0(\xi) = \frac{Bi_1}{1 + Bi_1} e^{-\xi}; \quad \phi_0(\xi) = \frac{Bi_2}{1 + Bi_2} e^{-\xi} \quad (6.3.8)$$

## 6.4 Results and Discussion

The results of this research are found solving the governing equations numerically employing SQLM. The results of the study are created using the number of collocation points  $N = 50$  in the space  $\xi$  and the scaled parameter  $L = 6$  in all cases. For values of different governing physical parameters the solutions are found for the velocity, temperature, and concentration profiles. The results are also obtained for important physical quantities like the skin friction coefficient, local Nusselt number, and local Sherwood number for various values of governing physical parameters. To obtain convergent and stable solutions, the values of the parameters are fixed throughout the computations as  $\phi_s = 0.1, \beta = 1, A = 0.8, \kappa = 0.5, Pr = 6.2, Nr = 0.5, Sc = 1, Sr = 0.3, Df = 0.2, Grt = 0.01, Grc = 0.01, m = \beta/(2 - \beta), Ec = 0.0, Yc = 0.3, \lambda = 0.0, Bi_1 = 0.5, Bi_2 = 0.5$ , unless it is specified. The local Nusselt number  $-\theta'(0)$  is equated with previously available results of other scholars for normal fluid ( $\phi_s = 0$ ) to confirm the numerical precision of the numerical technique used.

Table 6.2 displays the local Nusselt number for some values of Prandtl number  $Pr$  which is compared with the results of Yih (1999), Chamkha and Mujtaba (2003),

Ullah et al. (2016), and Nageeb et al. (2017) when the values of the parameters are constant. Clearly, a very good agreement is realized which in turn confirm the accuracy of the present numerical method used.

Figures 6.2-6.3 show the effect of nanoparticle volume fraction  $\phi_s$  on the velocity and concentration profiles for *Ag* water based nanofluids for both steady and unsteady flow. Increasing the nanoparticle volume fraction leads to increase the fluid velocity profiles while it decreases the concentration profiles for both steady and unsteady flow. Greater concentration distributions is observed in the steady flow than unsteady flow whereas the opposite trend is reported for the velocity distribution. Further, it is noticed that greater the value of nanoparticle volume fraction leads to decline the velocity and concentration boundary layer thickness. Also, Nageeb et al. (2017) observed similar forms of nanofluid flow.

Figures 6.4-6.6 respectively display the influence of the unsteadiness parameter on velocity, temperature, and concentration profiles for both *Ag* and *Al<sub>2</sub>O<sub>3</sub>* water based nanofluids. It is demonstrated that the velocity profiles are increasing function while the temperature and concentration profiles are decreasing function of the unsteadiness parameter which leads to reduce the thickness of the momentum, thermal, and concentration boundary layers. Also, it is observed that there is lower temperature and concentration distributions, and higher velocity distribution in *Ag* water

Table 6.2: Comparison of the SQLM results of  $-\theta'(0)$  when  $\phi_s = 0$  for some  $Pr$ .

| $Pr$   | $-\theta'(0)$ |          |        |          |          |
|--------|---------------|----------|--------|----------|----------|
|        | Chamkha       | Yih      | Ullah  | Nageeb   | Present  |
| 1      | 0.33217       | 0.332057 | 0.3320 | 0.332057 | 0.332057 |
| 10     | 0.72831       | 0.728141 | 0.7281 | 0.728141 | 0.728140 |
| 100    | 1.57218       | 1.571831 | 1.5718 | 1.571658 | 1.571754 |
| 1000   | 3.38809       | 3.387083 | 3.3881 | 3.396962 | 3.392166 |
| 10,000 | 7.30080       | 7.297402 | 7.3102 | 7.351156 | 7.298394 |

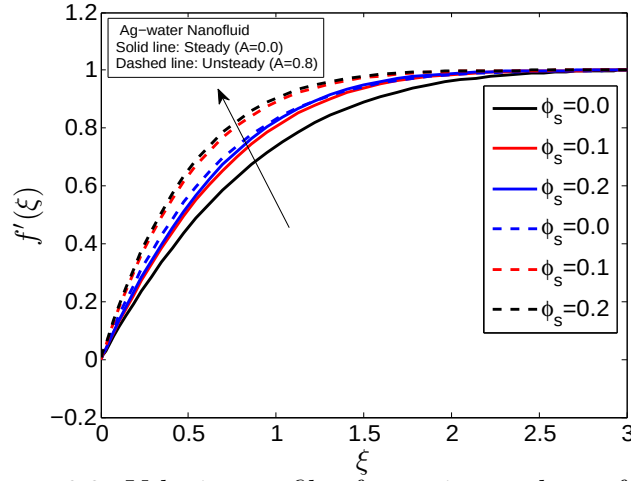


Figure 6.2: Velocity profiles for various values of  $\phi_s$ .

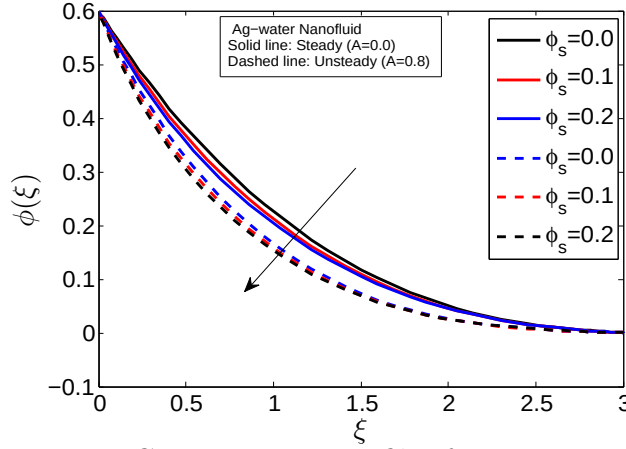


Figure 6.3: Concentration profiles for various values of  $\phi_s$

based nanofluid than in  $Al_2O_3$  water based nanofluid for more value of unsteadiness parameter.

Figures 6.7-6.8 show the effect of thermal radiation parameter  $Nr$  and Eckert number  $Ec$  respectively on the temperature profiles. Figure 6.7 indicates that as  $Nr$  increases near the boundary surface the temperature profiles increase correspondingly up to a certain range, and after that it shows the opposite trend. This result is qualitatively analogous with those results obtained by Nageeb et al. (2017). Figure 6.8 shows that the temperature profiles increase near the boundary surface and attain a peak value for large  $Ec$ , then reduces in the rest of the region. This suggests that

the thermal boundary layer thicker with great value of  $Ec$ . A comparable result was found in the literature of Ullah et al. (2015), Yirga and Shankar (2015).

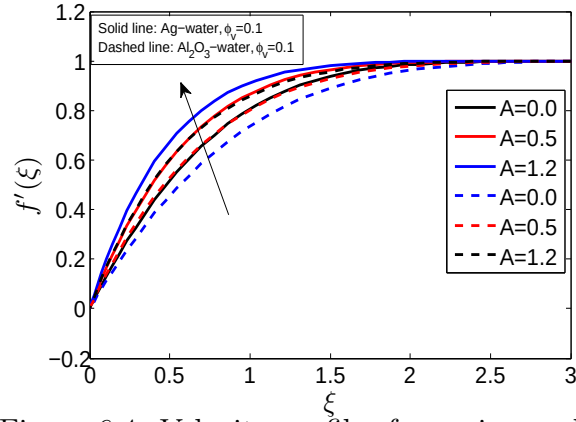


Figure 6.4: Velocity profiles for various values of  $A$ .

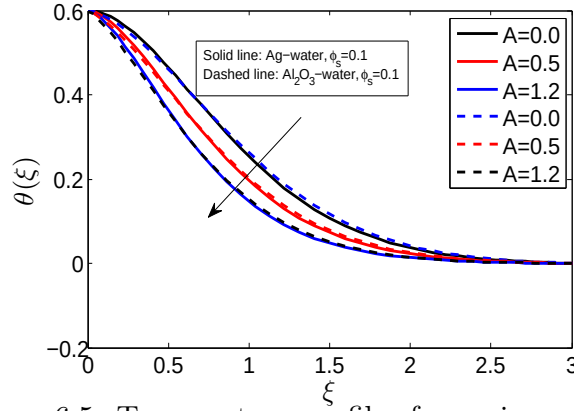


Figure 6.5: Temperature profiles for various values of  $A$

The effect of Dufour number  $Df$  on the temperature profiles is shown in Figure 6.9. Greater the value of  $Df$  leads to an increment of the temperature profiles. Dufour effect is the heat transfer brought by volume fraction gradients and substantial because of the density variance in the flow system. Thus, the figure displays the effect of the concentration gradient on the thermal energy flux in the flow region. Since temperature profile increases as Dufour number increases, it suggests that the effect of the alignment gradient on temperature is also increased which in turn leads to a

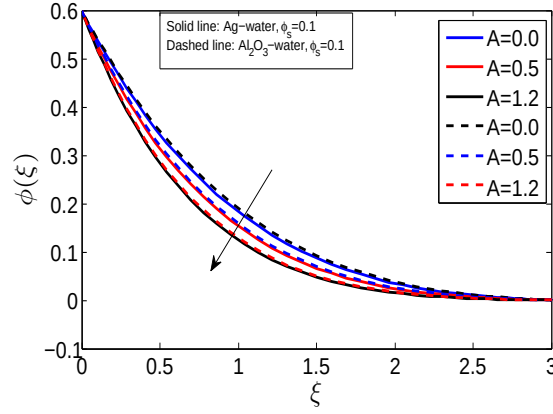


Figure 6.6: Concentration profiles for various values of  $A$

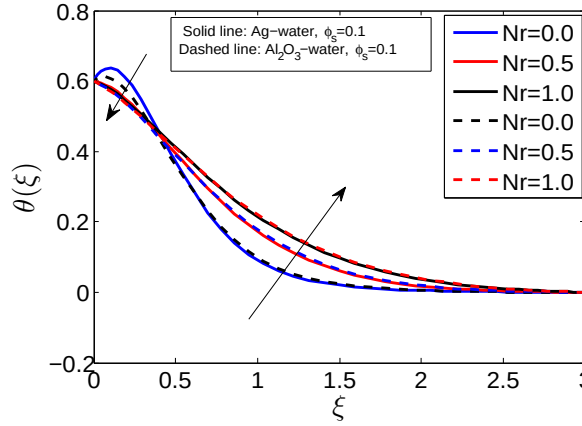


Figure 6.7: Temperature profiles for various values of  $Nr$

heating of the boundary layer region. Figure 6.10 shows the concentration profiles reduce with an increment in chemical reaction parameter. Similar trends are observed for both Dufour number and chemical reaction parameters as pointed out in Nageeb et al. (2017).

Figures 6.11-6.12 reveal the concentration profiles for various value of Schmidt number  $Sc$  and Soret number  $Sr$  respectively. Figure 6.11 demonstrates that the concentration profiles and its boundary layer thickness are a decreasing function of the Schmidt number. Figure 6.12 also shows that increasing Soret number increase the concentration profiles and its boundary layer thickness except near the boundary

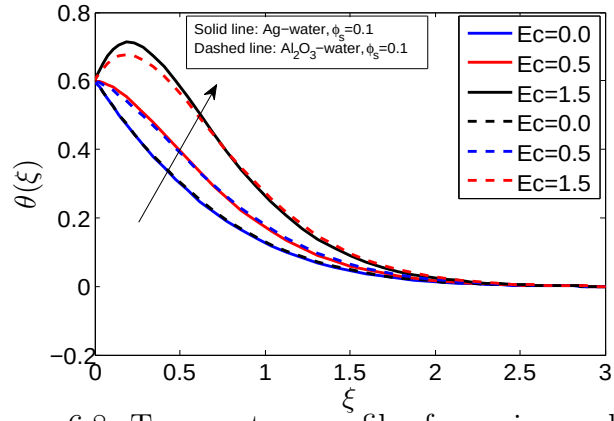


Figure 6.8: Temperature profiles for various values of  $Ec$

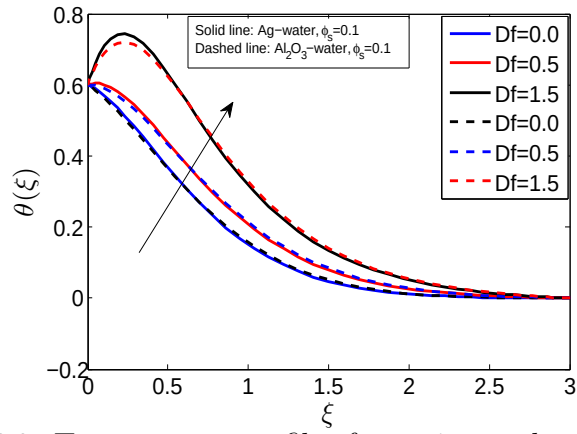


Figure 6.9: Temperature profiles for various values of  $Df$

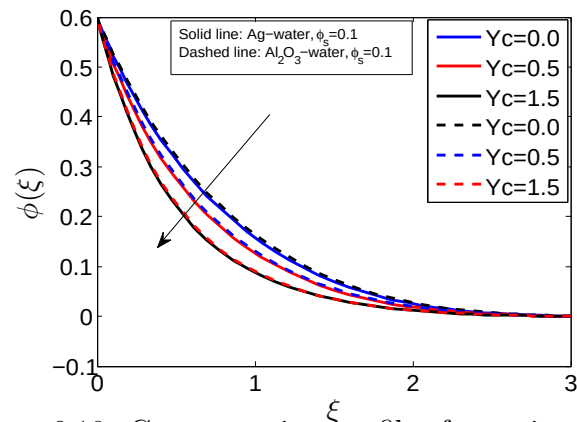


Figure 6.10: Concentration  $\xi$  profiles for various values of  $Yc$

surface. This enrichment of concentration profiles is due to the mass flux generated by the temperature gradient.

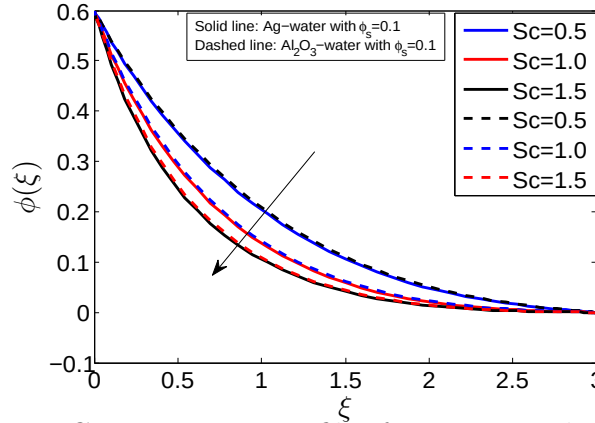


Figure 6.11: Concentration profiles for various values of  $Sc$

Figures 6.13-6.14 show the effect of the moving wedge parameter  $\lambda$  on velocity and temperature profiles. Figure 6.13 displays that the velocity profiles are increasing function of the moving wedge parameter  $\lambda$ . This implies that when wedge moves (for non-zero values of  $\lambda$ ) the fluid velocity at the surface of the wedge is no longer equal to the initial fluid velocity. Increasing  $\lambda$  enhances the velocity profiles while it decreases the velocity boundary layer thickness. Figure 6.14 also reveals that as  $\lambda$  increases the temperature profiles and its boundary layer thicknesses decrease.

Figures 6.15-6.16 display the effect of Biot numbers  $Bi_1$  and  $Bi_2$  on temperature and nanoparticle concentration fields correspondingly. Figure 6.15 shows that the increment of  $Bi_1$  leads to increase the sheet surface and the nanofluid temperature. The intensity of convective heating on the sheet surface increases as the value of  $Bi_1$  increases. This leads to the growth of convective heat transfer from the hot fluid on the lower surface of the sheet to the nanofluid on the upper surface which intern enhance the thickness of thermal boundary layers. Figure 6.16 displays that the sheet surface and nanofluid concentration increase with an increases in  $Bi_2$ , and this leads to enhance the concentration boundary layers thickness.

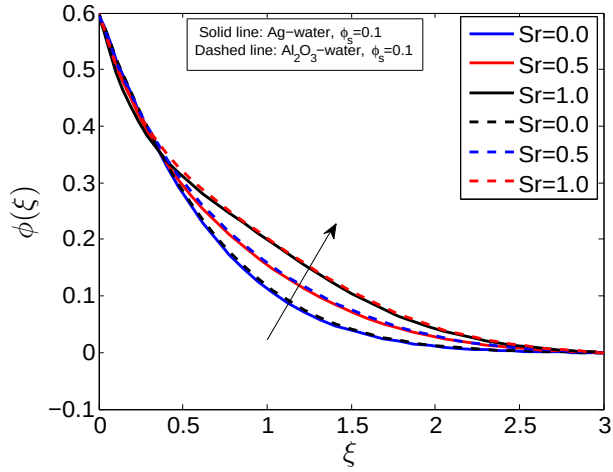


Figure 6.12: Concentration profiles for various values of  $Sr$

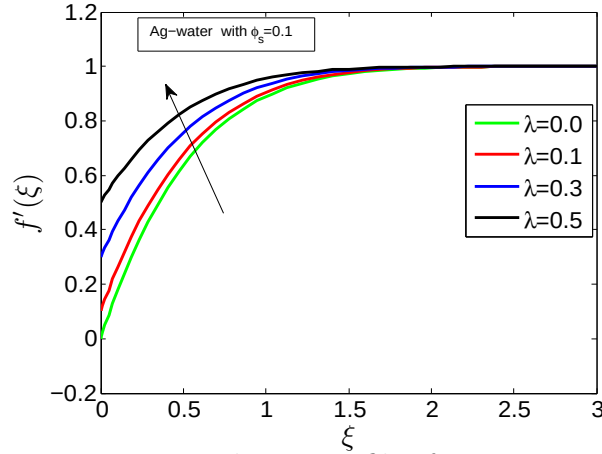


Figure 6.13: Velocity profiles for various values of  $\lambda$

Table 6.3 depicts the skin friction coefficient, local Nusselt number, and local Sherwood number for  $Ag$  and  $Al_2O_3$  water based nanofluids for various values of unsteadiness parameter  $A$  and nanoparticle volume fraction  $\phi_s$  when the values of the rest parameter remain unchanged. Here, it is shown that for  $Ag$  and  $Al_2O_3$  water based nanofluids the skin friction coefficient, local Nusselt number, and local Sherwood number are increased with increasing value of  $A$ . For  $Ag$  water based nanofluids, as  $\phi_s$  increases, the skin friction coefficient and local Sherwood number increase while the local Nusselt number reduces. Again, for  $Al_2O_3$  water based nanofluids, the skin

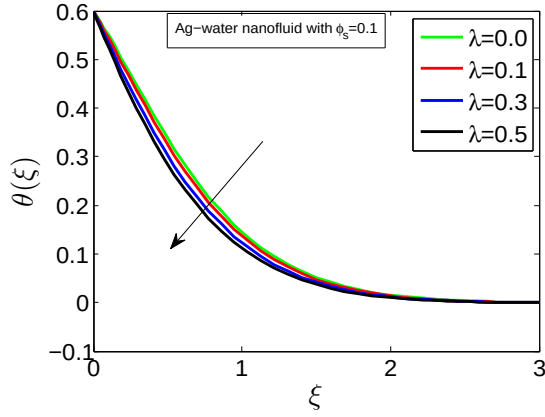


Figure 6.14: Temperature profiles for various values of  $\lambda$

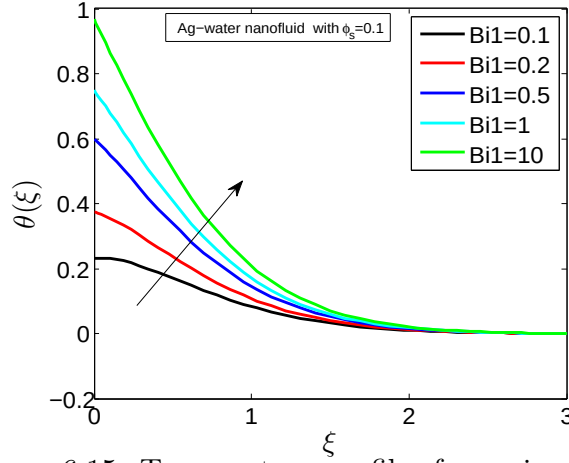


Figure 6.15: Temperature profiles for various values of  $Bi_1$

friction coefficient, local Nusselt number, and local Sherwood number decrease with increasing value of  $\phi_s$ . Analogous results were found in the literature of Yirga and Shankar (2015).

The effects of moving wedge parameter  $\lambda$ , Biot numbers  $Bi_1$  and  $Bi_2$  on the skin friction coefficient, local Nusselt number, and local Sherwood number are disclosed in Table 6.4 for  $Ag$  and  $Al_2O_3$  water based nanofluids. The result reveals that as the value of  $\lambda$  increases the skin friction coefficient is reduced while the local Nusselt number and local Sherwood number are enhanced. It is also presented that greater the values of Biot number  $Bi_1$  and Biot number  $Bi_2$  leads to increase the local Nusselt number and local Sherwood number respectively.

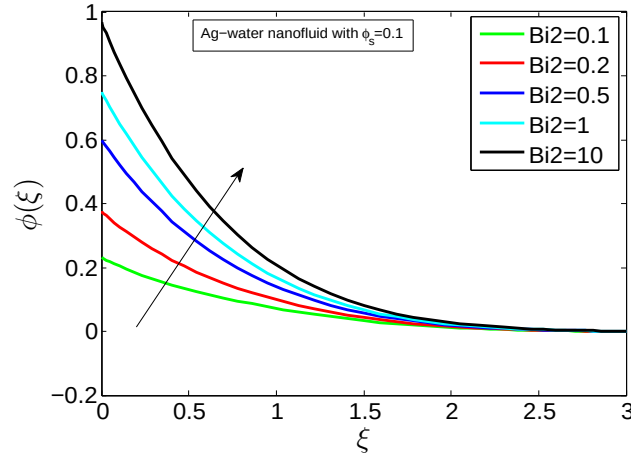


Figure 6.16: Concentration profiles for various values of  $Bi_2$

Table 6.3: Values of  $-f''(0)$ ,  $-\theta'(0)$  and  $-\phi'(0)$  for  $A$  and  $\phi_s$

| $A$ | $\phi_s$ | $-f''(0)$ |           | $-\theta'(0)$ |           | $-\phi'(0)$ |           |
|-----|----------|-----------|-----------|---------------|-----------|-------------|-----------|
|     |          | $Ag$      | $Al_2O_3$ | $Ag$          | $Al_2O_3$ | $Ag$        | $Al_2O_3$ |
| 0.0 | 0.05     | 0.5352    | 0.4721    | 0.6370        | 0.6148    | 0.3660      | 0.3528    |
|     | 0.1      | 0.5753    | 0.4690    | 0.6039        | 0.5716    | 0.3737      | 0.3521    |
|     | 0.2      | 0.6057    | 0.4489    | 0.5199        | 0.4843    | 0.3791      | 0.3475    |
| 0.2 | 0.05     | 0.8832    | 0.7791    | 0.7158        | 0.6935    | 0.4000      | 0.3874    |
|     | 0.1      | 0.9494    | 0.7740    | 0.6748        | 0.6430    | 0.4073      | 0.3868    |
|     | 0.2      | 0.9995    | 0.7408    | 0.5752        | 0.5419    | 0.4124      | 0.3823    |
| 0.8 | 0.05     | 1.5215    | 1.3422    | 0.8151        | 0.7930    | 0.4415      | 0.4299    |
|     | 0.1      | 1.6355    | 1.3334    | 0.7636        | 0.7330    | 0.4480      | 0.4293    |
|     | 0.2      | 1.7219    | 1.2762    | 0.6436        | 0.6140    | 0.4526      | 0.4252    |

## 6.5 Conclusion

Unsteady mixed convection boundary layer flow of nanofluid past a moving wedge with heat and mass transfer embedded in porous media was studied. The flow of  $Ag$  and  $Al_2O_3$  water based nanofluids are considered and assumed that the added nanoparticle are of spherical shape and the nanoparticle volume fraction can be actively controlled at the boundary surface. The effects of nanoparticle volume fraction, unsteadiness parameter, pressure gradient parameter, permeability parameter, thermal radiation parameter, viscous dissipation parameter, chemical reaction parameter,

Table 6.4: Values of  $-f''(0)$ ,  $-\theta'(0)$  and  $-\phi'(0)$  for  $\lambda, Bi_1, Bi_2$

|           |        |        | $-f''(0)$ |           | $-\theta'(0)$ |           | $-\phi'(0)$ |           |
|-----------|--------|--------|-----------|-----------|---------------|-----------|-------------|-----------|
| $\lambda$ | $Bi_1$ | $Bi_2$ | $Ag$      | $Al_2O_3$ | $Ag$          | $Al_2O_3$ | $Ag$        | $Al_2O_3$ |
| 0.0       | 0.5    | 0.5    | 0.5802    | 0.4897    | 0.4845        | 0.4639    | 0.3064      | 0.2940    |
| 0.2       |        |        | 0.5446    | 0.4517    | 0.6051        | 0.5928    | 0.3499      | 0.3391    |
| 0.5       |        |        | 0.4031    | 0.3311    | 0.75377       | 0.7532    | 0.4046      | 0.3978    |
| 0.0       | 0.1    |        | 0.5802    | 0.4897    | 0.1863        | 0.1784    | 0.3064      | 0.2940    |
|           | 1.0    |        | 0.5802    | 0.4897    | 0.6056        | 0.5799    | 0.3064      | 0.2940    |
|           | 10     |        | 0.5802    | 0.4897    | 0.7815        | 0.7482    | 0.3064      | 0.2940    |
|           | 0.5    | 0.1    | 0.5802    | 0.4897    | 0.4845        | 0.4639    | 0.1179      | 0.1131    |
|           |        | 1.0    | 0.5802    | 0.4897    | 0.4845        | 0.4639    | 0.3830      | 0.3675    |
|           |        | 10     | 0.5802    | 0.4897    | 0.4845        | 0.4639    | 0.4942      | 0.4742    |

Soret number, Dufour numbers, moving wedge parameter, and Biot numbers were also analyzed. The governing equations were numerically solved using Spectral Quasilinearization Methods. From the above discussions the following conclusions are drawn:

- The velocity profiles increase with more values of  $\phi_s, A, \beta, \kappa$ , and  $\lambda$ .
- The temperature profiles and its boundary layer thickness increase with increasing the value of  $\phi_s, Nr, Ec, Df, Bi_1$ , and  $Bi_2$ .
- Enhancing the value of  $\phi_s, A, \beta, Sc, Yc, Bi_1$ , and  $Bi_2$  lead to reduce the concentration profiles and its boundary layer thickness.
- The skin friction coefficient increases with an increment in  $A, \beta, \kappa, Nr, Ec, Df, Sr$ , and  $\lambda$ .
- The local Nusselt number is reduced with increasing the value of  $\kappa, Ec, \Upsilon, Df, Sr, \lambda$ , and  $Bi_1$ .
- The local Sherwood number increases when the values of  $A, \beta, \kappa, Ec, \Upsilon, Df, Sr, \lambda$ , and  $Bi_2$  increase.

## Chapter 7

# MHD Slip Flow of CNTs-Ethylene glycol Nanofluid due to a Stretchable Rotating Disk with Cattaneo-Christov Heat Flux Model

### 7.1 Introduction

Nanofluids have gained remarkable attention of researchers due to their inspiring heat transfer in various industrial and engineering applications. Common working fluids such as water, engine oils and ethylene glycol have restricted thermal performances which limit their usage in modern-day cooling applications. Nanofluids consist of nanoscale particles such as copper, alumina, carbides, nitrides, metal oxides, graphite, and carbon nanotubes which enhance the thermal conductivity of base fluids (Ahmad et al., 2017; Mahanthesh et al., 2019). These nanofluids have widespread applications in modern systems of heating and cooling, solar cells, generation of new fuels, hybrid-powered engines, cancer therapy, drug delivery and medicine (Aziz et al., 2018). Due to these various applications of nanofluids, many studies associated with the flow of nanofluids have been conducted. For instance, Prasannakumara

et al. (2017) examined the boundary layer flow and heat transfer of fluid particles suspension with nanoparticles over a nonlinear stretching sheet. Khan et al.(2017) analyzed three-dimensional steady MHD flow of Powell-Eyring nanofluid with convective and nanoparticles mass flux conditions. Also, Khan et al. (2017) investigated unsteady three-dimensional Sisko magneto-nanofluid flow with heat absorption and temperature dependent thermal conductivity. Recently, three-dimensional bioconvection nanofluid flow from a bi-axial stretching sheet was presented by Amirsom et al.(2019).

In the last two decades, heat transfer of carbon nanofluids have received considerable attention as a result of their extensive applications in the fields of nanotechnology and medicine. Nanoparticles of carbon nanotubes (CNTs) were first discovered by Lijima (1991). Carbon nanotubes are allotropes of carbon with a tube shaped nanostructure such as frames of carbon atoms with diameter ranges from 1 to 100 nm. They have a remarkable conductivity which supports them to form a network of conductive tubes. Because of their configuration, some CNTs are good conductor of heat and electricity whereas others act as semiconductors. CNTs have 15 times thermal conductivity and 1000 times capability of copper, and 200 times power and 5 times resistance of steel (Khalid et al., 2018). Carbon nanotubes have significance applications in nanotechnology, hardware, optics, energy storage, biomedical, ceramic, thermal defense and in various fields of material sciences and engineering (Hayat et al., 2017). Carbon nanotubes used in nanofluids are generally divided into single wall carbon nanotubes (SWCNTs) and multi wall carbon nanotubes (MWCNTs) contingent on their number of concentric layers of rolled grapheme sheets. SWCNTs contain catalyst for their synthesis and they are prepared from covering layer of grapheme into a unified cylinder whereas MWCNTs can be formed without catalyst and comprise of numerous rolled layers of graphite with complex structure (Khan et al., 2018).

Because of their unique morphology, electronic structural, mechanical properties

and innovative physicochemical features, CNTs are the most resourceful material of this century. Moreover, the presence of carbon chains in CNTs does not convey any hazard to the atmosphere (Alsagri et al., 2019). Having all these implication of CNTs in mind, Kumaresan et al. (2012) experimentally studied the heat transfer characteristics of nanofluids containing CNTs, and they disclosed nanofluids at very low nanoparticle volume fraction enhance high heat transfer rate. Heat transfer performance of CNTs nanofluids flow through a horizontal tube was examined by Ding et al. (2006). They obtained essential progress of the convective heat transfer which mainly depends on Reynolds number and solid volume fraction of CNTs. The synthetic engine oil and ethylene glycol thermal conductivity improvement in the presence of MWCNTs was designated by Liu et al. (2005). They revealed that CNTs-ethylene glycol nanofluid have better thermal conductivity as compare to ethylene glycol base fluid. Also, Mahanthesh et al. (2017) investigated the Marangoni transport of dissipating SWCNT and MWCNT with water nanofluids under the influence of magnetic force and radiation. They reported that the thermal distribution of SWCNT nanoliquid is better than MWCNT nanoliquid.

Rotating disk induced flow of fluids have significance applications in numerous engineering and industrial sectors such as rotating machinery, electric power generating system, computer storage devices, air cleaning machines, crystal growth processes, gas turbine rotors, food processing, medical equipment and others. Due to these, immense papers have been published regarding the flow of fluid due to rotating disk. For example, Hayat et al. (2017) and Mustafa (2017) analyzed MHD flow of nanofluid by a rotating disk with partial slip effects. Nanofluid flow near a stretchable rotating disk with axial magnetic field and convective conditions was analyzed by Mushtaq and Mustafa (2017). Also, Imtiaz et al. (2016) studied the radiative flow of CNTs between stretchable rotating disks with convective conditions. Recently, the impacts of exponential space dependent heat source on MHD slip flow of SWCNT and MWCNT

nanoliquids past a stretchable rotating disk was investigated by Mahanthesh et al. (2019). Their result established that the thermal field for SWCNT-nanoliquid is higher than MWCNT-nanoliquid.

Motivated by the above cited literatures, the purpose of this study is to examine the flow of CNTs with ethylene glycol nanofluids due to stretchable rotating disk embedded in a porous medium with Cattaneo-Christov heat flux model. Governing equations of the flow are highly nonlinear coupled differential equations and solved numerically by employing spectral SQLM. Spectral collocation method converges fast and provides more accurate approximations with less grid points. To the best of the researcher's knowledge, no analysis has been published so far in this direction. Particularly, examining the effect of Cattaneo Christov heat flux model on SWCNTs and MWCNTs with ethylene glycol ( $C_2H_6O_2$ ) nanofluids flow and heat transfer using SQLM would be the contribution of this paper to the existing body of knowledge. Moreover, the effects of embedded parameters such as nanoparticle volume fraction, axially applied magnetic field, stretching factor, velocity and thermal slip factors on velocity and temperature fields as well as skin friction coefficient and local heat transfer rate are analyzed, and the results are presented and discussed using graphs and tables.

## 7.2 Mathematical Description of Problem

We considered steady and incompressible flow of nanofluid due to stretchable rotating disk embedded in a porous medium. We assumed a non-rotating cylindrical coordinate frame  $(r, \theta, z)$ , and the velocity components  $(u, v, w)$  represent in the directions of increasing  $(r, \theta, z)$ . The disk is rotated along  $z$ -axis with angular velocity  $\Omega$ , and also, it is stretched in the radial direction with stretching rate  $s$ . The nanofluid flow is exposed to magnetic field of uniform strength  $B_0$  acts parallel to  $z$ -direction (see Figure 7.1). Induced magnetic field caused by the motion of electrically conducting

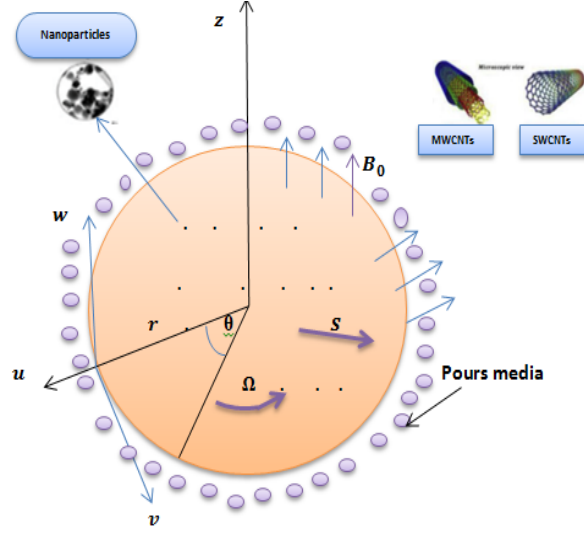


Figure 7.1: Physical model of the flow problem.

nanofluid was not considered because it is very small compared to  $B_0$ . The surface temperature of the disk and the ambient temperature are respectively denoted by  $T_w$  and  $T_\infty$ . Moreover, effects of velocity and thermal slip boundary conditions were considered, and Cattaneo-Christov heat flux model was used to analyze the heat transfer.

Assuming  $\frac{\partial p}{\partial r} = \frac{\partial p}{\partial z} = 0$ , the governing equations for mass, momentum and thermal energy transfer of Casson nanofluid flow past a stretchable rotating disk with axially applied magnetic field and slip boundary conditions are given as (Aziz et al., 2018).

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad (7.2.1)$$

$$\rho_{nf} \left( u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} \right) = \mu_{nf} \left( 1 + \frac{1}{\gamma} \right) \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2} \right) - \left( \sigma_{nf} B_0^2 - \frac{\mu_{nf}}{K} \right) u \quad (7.2.2)$$

$$\rho_{nf} \left( u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} \right) = \mu_{nf} \left( 1 + \frac{1}{\gamma} \right) \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2} \right) - \left( \sigma_{nf} B_0^2 - \frac{\mu_{nf}}{K} \right) v \quad (7.2.3)$$

$$(\rho c_p)_{nf} \left( u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right) = -\nabla \cdot \mathbf{q} \quad (7.2.4)$$

with boundary conditions

$$\begin{aligned} u &= rs + L_0 \left( 1 + \frac{1}{\gamma} \right) \frac{\partial u}{\partial z}, \quad v = r\Omega + L_0 \left( 1 + \frac{1}{\gamma} \right) \frac{\partial v}{\partial z} \\ w &= 0; \quad T = T_w + N_0 \frac{\partial T}{\partial z} \quad \text{at } z = 0 \\ u &\rightarrow 0, \quad v \rightarrow 0, \quad T \rightarrow T_\infty \quad \text{as } z \rightarrow \infty \end{aligned} \quad (7.2.5)$$

where  $L_0$  and  $N_0$  are the velocity slip and thermal slip factors respectively.

In this study, even if nanofluids with two-component model were considered, we assumed that there is no agglomeration of CNTs nanoparticles within the nanofluid, and the base fluid and CNTs are assumed to be in thermal equilibrium and no slip occurs between them. Therefore, the heat flux vector  $\mathbf{q}$  satisfy the Cattaneo-Christov diffusion model can be adapted for nanofluid and given as (Akbar et al., 2017)

$$\lambda_t \left[ \frac{\partial \mathbf{q}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{u} + (\nabla \cdot \mathbf{u}) \mathbf{q} \right] + \mathbf{q} = -k \nabla T \quad (7.2.6)$$

where  $\lambda_t$  is the relaxation time of heat flux,  $k$  is the thermal conductivity, and  $\mathbf{u}$  is the velocity vector. The above equation (7.2.6) gives Fourier's law when  $\lambda_t = 0$ .

For steady incompressible flow, equation (7.2.6) is simplified into

$$\lambda_t [\nabla \cdot ((\nabla \cdot \mathbf{q}) \mathbf{u})] + \nabla \cdot \mathbf{q} = -k \nabla^2 T, \quad (7.2.7)$$

Substituting equation (7.2.7) into (7.2.4) and simplifying, we get energy equation

$$\begin{aligned} u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} &= \alpha_{nf} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) \\ -\lambda_t \left[ u^2 \frac{\partial^2 T}{\partial r^2} + w^2 \frac{\partial^2 T}{\partial z^2} + 2uw \frac{\partial^2 T}{\partial r \partial z} + \left( u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) \frac{\partial T}{\partial r} + \left( u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) \frac{\partial T}{\partial z} \right] \end{aligned} \quad (7.2.8)$$

The non-Newtonian Casson parameter is defined as  $\gamma = \mu_\gamma \sqrt{2\pi_c/P_y}$  whereas the rheological equation of state for the Cauchy stress tensor of Casson fluid was given in equation (5.2.1).

The thermo-physical properties of the nanoparticles (SWCNTs and MWCNTs) and base fluid (ethylene glycol) are given in Table 7.1 (Cengel & Ghajar, 2015; Khalid et al., 2018).

Table 7.1: Thermo-physical properties of SWCNTs, MWCNTs and ethylene glycol at temperature  $20^0c$

| Physical properties                        | Ethylene glycol( $C_2H_6O_2$ ) | SWCNT         | MWCNT                |
|--------------------------------------------|--------------------------------|---------------|----------------------|
| Density, $\rho(kg/m^3)$                    | 1109                           | 2600          | 1600                 |
| Specific Heat, $c_p(J/kgK)$                | 2382                           | 425           | 796                  |
| Thermal Conductivity, $k(W/mK)$            | 2.84                           | 6600          | 3000                 |
| Electrical Conductivity, $\sigma(Sm^{-1})$ | 0.01485                        | $10^6 - 10^7$ | $1.9 \times 10^{-4}$ |

Be aware of the base fluid and the nanoparticles are in thermal equilibrium and no slip take place between them. The nanofluid effective density  $\rho_{nf}$ , specific heat capacity  $(\rho c_p)_{nf}$ , dynamic viscosity  $\mu_{nf}$ , and electrical conductivity  $\sigma_{nf}$  are given as (Khalid et al., 2018; Alsagri et al., 2019).

$$\begin{aligned}
\rho_{nf} &= \phi_s \rho_s + (1 - \phi_s) \rho_{bf}, & (\rho c_p)_{nf} &= \phi_s (\rho c_p)_s + (1 - \phi_s) (\rho c_p)_{bf} \\
\mu_{nf} &= \frac{\mu_f}{(1 - \phi_s)^{2.5}} & \frac{\sigma_{nf}}{\sigma_{bf}} &= 1 + \frac{3 \left( \frac{\sigma_s}{\sigma_{bf}} - 1 \right) \phi_s}{\left( \frac{\sigma_s}{\sigma_{bf}} + 2 \right) - \left( \frac{\sigma_s}{\sigma_{bf}} - 1 \right) \phi_s}
\end{aligned} \tag{7.2.9}$$

where  $\phi_s$  is the nanoparticle volume fraction;  $\rho_s$ ,  $(c_p)_s$ , and  $\sigma_s$  are respectively density, specific heat capacity, and electrical conductivity of CNTs whereas  $\rho_{bf}$ ,  $(c_p)_{bf}$ ,  $\mu_{bf}$ , and  $\sigma_{bf}$  are respectively density, specific heat capacity, dynamic viscosity, and electrical conductivity of the base fluid (ethylene glycol). The nanoparticle volume fraction  $\phi_1$ ,

$\phi_2$  and  $\phi_3$  are defined as

$$\begin{aligned}\phi_1 &= (1 - \phi_s)^{2.5} \left( 1 - \phi_s + \phi_s \frac{\rho_s}{\rho_{bf}} \right), & \phi_2 &= 1 - \phi_s + \phi_s \frac{(\rho c p)_s}{(\rho c p)_{bf}} \\ \phi_3 &= \left( 1 + \frac{3 \left( \frac{\sigma_s}{\sigma_{bf}} - 1 \right) \phi_s}{\left( \frac{\sigma_s}{\sigma_{bf}} + 2 \right) - \left( \frac{\sigma_s}{\sigma_{bf}} - 1 \right) \phi_s} \right) \left( \frac{\rho_{bf}}{(1 - \phi_s) \rho_{bf} + \phi_s \rho_s} \right)\end{aligned}\quad (7.2.10)$$

For effective thermal conductivity of CNTs nanofluid  $k_{nf}$ , the Xue's model which is effective for spherical and elliptical shape is utilized as follows (Alsagri et al., 2019).

$$\frac{k_{nf}}{k_{bf}} = \frac{1 - \phi_s + 2\phi_s \left( \frac{k_s}{k_s - k_{bf}} \right) \ln \left( \frac{k_s + k_{bf}}{2k_{bf}} \right)}{1 - \phi_s + 2\phi_s \left( \frac{k_{bf}}{k_s - k_{bf}} \right) \ln \left( \frac{k_s + k_{bf}}{2k_{bf}} \right)} \quad (7.2.11)$$

We introduce the non-dimensional variables based on Akbar et al., (2017), and Aziz et al. (2018).

$$\begin{aligned}u &= r\Omega f'(\xi), & v &= r\Omega h(\xi), & w &= -(2\Omega\nu_{bf})^{1/2} f(\xi), \\ \theta(\xi) &= \frac{T - T_\infty}{T_w - T_\infty}, & \xi &= z \left( \frac{2\Omega}{\nu_{bf}} \right)^{1/2}\end{aligned}\quad (7.2.12)$$

Hence, the continuity equation (7.2.1) is satisfied and the transformed equations for momentums and energy are found as follows.

$$2 \left( 1 + \frac{1}{\gamma} \right) f''' + \phi_1 [2ff'' - f'^2 + h^2 - \phi_3 M f'] + \kappa f' = 0 \quad (7.2.13)$$

$$2 \left( 1 + \frac{1}{\gamma} \right) h'' + \phi_1 [2fh' - 2f'h - \phi_3 M h] + \kappa h = 0 \quad (7.2.14)$$

$$\theta'' + \phi_2 Pr \frac{k_{bf}}{k_{nf}} [f\theta' - \alpha_t (f^2\theta'' + ff'\theta')] = 0 \quad (7.2.15)$$

with transformed boundary conditions:

$$\begin{aligned}f &= 0, & f'(0) &= A_s + \left( 1 + \frac{1}{\gamma} \right) \delta_v f''(0), & h(0) &= 1 + \left( 1 + \frac{1}{\gamma} \right) \delta_v h'(0) \\ \theta(0) &= 1 + \delta_t \theta'(0), & f'(\infty) &= h(\infty) = \theta(\infty) \rightarrow 0.\end{aligned}\quad (7.2.16)$$

where  $M = \frac{\sigma_f B_0^2}{\rho_{bf} \Omega}$  is the magnetic parameter,  $\kappa = \frac{r \nu_{bf}}{K \Omega}$  is permeability parameter,  $Pr = \frac{\nu_{bf}}{\alpha_{bf}}$  is a Prandtl number,  $\alpha_t = 2\lambda_t \Omega$  is the thermal relaxation parameter,  $A_s = \frac{s}{\Omega}$  is the scaled stretching parameter,  $\delta_v = L_0 \sqrt{\frac{2\Omega}{\nu_{bf}}}$  is the velocity slip parameter,  $\delta_t = N_0 \sqrt{\frac{2\Omega}{\nu_{bf}}}$ , is the thermal slip parameter.

The radial and tangential directions shear stress at the surface of the disc for Casson nanofluid are defined as

$$\tau_{zr} = -\mu_{nf} \left(1 + \frac{1}{\gamma}\right) \left(\frac{\partial u}{\partial z}\right)_{z=0}, \quad \tau_{z\theta} = -\mu_{nf} \left(1 + \frac{1}{\gamma}\right) \left(\frac{\partial v}{\partial z}\right)_{z=0} \quad (7.2.17)$$

The total dimensionless form of surface shear stress at the surface of the disc is

$$\tau_w = \sqrt{\tau_{zr}^2 + \tau_{z\theta}^2} = \frac{\mu_{bf}}{(1 - \phi_s)^{2.5}} \left(1 + \frac{1}{\gamma}\right) r \sqrt{\frac{2\Omega^3}{\nu_{bf}}} (f''(0)^2 + h'(0)^2)^{1/2} \quad (7.2.18)$$

The surface heat flux of the disc for Casson nanofluid is given as

$$q_w = -k_{nf} \left(\frac{\partial T}{\partial z}\right)_{z=0} = -k_{nf} \sqrt{\frac{2\Omega}{\nu_f}} (T_w - T_\infty) \theta'(0), \quad (7.2.19)$$

The practical problem in engineering, the skin friction coefficient  $C_f$  and the Nusselt number  $Nu$  (local heat transfer rate) of the disc are defined as

$$C_f = \frac{\tau_w}{\rho_f (r\Omega)^2}, \quad Nu_r = \frac{r q_w}{k_{bf} (T_w - T_\infty)}, \quad (7.2.20)$$

Thus, the dimensionless normalized skin friction coefficient and local heat transfer rate are given by

$$\sqrt{Re_r} C_f = \frac{\sqrt{2}}{(1 - \phi_s)^{2.5}} \left(1 + \frac{1}{\gamma}\right) (f''(0)^2 + h'(0)^2)^{1/2}, \quad (7.2.21)$$

and

$$\frac{Nu_r}{\sqrt{Re_r}} = -\sqrt{2} \left( \frac{1 - \phi_s + 2\phi_s \left(\frac{k_s}{k_s - k_{bf}}\right) \ln \left(\frac{k_s + k_{bf}}{2k_{bf}}\right)}{1 - \phi_s + 2\phi_s \left(\frac{k_{bf}}{k_s - k_{bf}}\right) \ln \left(\frac{k_s + k_{bf}}{2k_{bf}}\right)} \right) \theta'(0), \quad (7.2.22)$$

where  $Re_r$  is the local Reynolds number defined by  $Re_r = \frac{(\Omega r)r}{\nu_{bf}}$ .

### 7.3 Numerical Method of Solutions

The dimensionless system of non-linear ODEs with boundary conditio (7.2.13) - (7.2.16) were numerically solved using spectral quasilinearization method (SQLM). Applying SQLM, the system of nonlinear ordinary differential equations (7.2.13) - (7.2.16) give the following iterative scheme of linear differential equations

$$a_{1,r}f_{r+1}''' + a_{2,r}f_{r+1}'' - a_{3,r}f_{r+1}' + a_{4,r}f_{r+1} + a_{5,r}h_{r+1} = a_{6,r} \quad (7.3.1)$$

$$b_{1,r}h_{r+1}'' + b_{2,r}h_{r+1}' - (b_{3,r} + b_{4,r})h_{r+1} + b_{5,r}f_{r+1}' + b_{6,r}f_{r+1} = b_{7,r} \quad (7.3.2)$$

$$(c_{1,r} - c_{2,r})\theta_{r+1}'' + c_{3,r}\theta_{r+1}' - c_{4,r}f_{r+1}' + c_{5,r}f_{r+1} = c_{6,r} \quad (7.3.3)$$

boundary conditions

$$\begin{aligned} f_{r+1} = 0, \quad f_{r+1}'(0) = A_s + \left(1 + \frac{1}{\gamma}\right) \delta_v f_{r+1}''(0), \quad h_{r+1}(0) = 1 + \left(1 + \frac{1}{\gamma}\right) \delta_v h_{r+1}'(0), \\ \theta_{r+1}(0) = 1 + \delta_t \theta_{r+1}'(0), \quad f_{r+1}'(\infty) = h_{r+1}(\infty) = \theta_{r+1}(\infty) \rightarrow 0. \end{aligned} \quad (7.3.4)$$

where the terms  $r + 1$  and  $r$  are at the current and previous iteration levels respectively, and the coefficients are given as follows

$$\begin{aligned} a_{1,r} &= 2\left(1 + \frac{1}{\gamma}\right), \quad a_{2,r} = 2\phi_1 f_r, \quad a_{3,r} = 2\phi_1 f_r' + \phi_1 \phi_3 M + \kappa, \\ a_{4,r} &= 2\phi_1 f_r'', \quad a_{5,r} = 2\phi_1 h_r, \quad a_{6,r} = 2f_r f_r'' - f_r'^2 + h_r^2 \\ b_{1,r} &= 2\left(1 + \frac{1}{\gamma}\right), \quad b_{2,r} = 2\phi_1 f_r, \quad b_{3,r} = 2\phi_1 f_r', \quad b_{4,r} = \phi_1 \phi_3 M + \kappa \\ a_{5,r} &= -2\phi_1 h_r, \quad a_{6,r} = 2\phi_1 h_r, \quad a_{7,r} = 2\phi_1 (f_r h_r' - f_r' h_r) \\ c_{1,r} &= \frac{k_{nf}}{k_f} \frac{1}{\phi_2 P r}, \quad c_{2,r} = \alpha f_r^2, \quad c_{3,r} = f_r - \alpha f_r f_r', \quad c_{4,r} = \alpha f_r \theta_r' \\ c_{5,r} &= \theta_r' - 2\alpha f_r \theta_r'' - \alpha f_r' \theta_r', \quad c_{6,r} = f_r \theta_r' - 2\alpha (f_r^2 \theta_r'' + f_r f_r' \theta_r') \end{aligned}$$

Applying the spectral collocation method to the system of equations (7.3.1)-(7.3.3), and it can be solved as a coupled matrix:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{bmatrix} F_{r+1} \\ H_{r+1} \\ \Theta_{r+1} \end{bmatrix} = \begin{bmatrix} B_{1,r} \\ B_{2,r} \\ B_{3,r} \end{bmatrix}, \quad (7.3.5)$$

with boundary condition

$$\begin{aligned} F_{r+1}(\chi_N) &= 0, \quad F'_{r+1}(\chi_{N-1}) - \left(1 + \frac{1}{\gamma}\right) \delta_v F''_{r+1}(\chi_{N-1}) = A_s, \\ F'_{r+1}(\chi_0) &= 0; \quad H_{r+1}(\chi_N) - \left(1 + \frac{1}{\gamma}\right) \delta_v H'_{r+1}(\chi_N) = 1, \quad H_{r+1}(\chi_0) = 0, \\ \Theta_{r+1}(\chi_N) - \delta_v \theta'_{r+1}(\chi_N) &= 1, \quad \Theta_{r+1}(\chi_0) = 0. \end{aligned} \quad (7.3.6)$$

where

$$\begin{aligned} \Lambda_{11} &= a_{1,r} \mathbf{D}^3 + [a_{2,r}]_d \mathbf{D}^2 - [a_{3,r}]_d \mathbf{D} + [a_{4,r}]_d, \quad \Lambda_{12} = [a_{5,r}]_d, \quad \Lambda_{13} = \mathbf{O}; \quad B_{1,r} = a_{6,r} \\ \Lambda_{21} &= [b_{5,r}]_d \mathbf{D} + [b_{6,r}]_d, \quad \Lambda_{22} = b_{1,r} \mathbf{D}^2 + [b_{2,r}]_d \mathbf{D} - [b_{3,r}]_d - b_{4,r} \mathbf{I}, \quad \Lambda_{23} = \mathbf{O}; \quad B_{2,r} = b_{7,r} \\ \Lambda_{31} &= [c_{4,r}]_d \mathbf{D} + [c_{5,r}]_d, \quad \Lambda_{32} = \mathbf{O}, \quad \Lambda_{33} = (c_{1,r} \mathbf{I} - [c_{2,r}]_d) \mathbf{D}^2 + [c_{3,r}]_d \mathbf{D}; \quad B_{3,r} = c_{6,r} \\ F_{r+1} &= [f_{r+1,0}, f_{r+1,1}, \dots, f_{r+1,N}]^t, \quad H_{r+1} = [h_{r+1,0}, h_{r+1,1}, \dots, h_{r+1,N}]^t, \\ \Theta_{r+1} &= [\theta_{r+1,0}, \theta_{r+1,1}, \dots, \theta_{r+1,N}]^t \text{ are vectors of size } (N+1) \times 1, \quad [\dots]_d \text{ denotes a diagonal matrix, } \mathbf{I} \text{ and } \mathbf{O} \text{ are respectively, identity and zero matrices of size } (N+1) \times (N+1). \end{aligned}$$

Starting with the following suitable initial approximations, the iteration schemes are used iteratively to find  $f_{r+1}$ ,  $h_{r+1}$  and  $\theta_{r+1}$  when  $r = 0, 1, 2, \dots$

$$f_0(\xi) = \frac{1 - e^{-A_s \xi}}{1 + (1 + \frac{1}{\gamma}) \delta_v}, \quad h_0(\xi) = \frac{e^{-\xi}}{1 + (1 + \frac{1}{\gamma}) \delta_v}, \quad \theta_0(\xi) = \frac{e^{-\xi}}{1 + \delta_t}. \quad (7.3.7)$$

## 7.4 Results and Discussion

Numerical results for CNTs with ethylene glycol nanofluids flow due to a stretchable rotating disk embedded in a porous medium with Cattaneo-Christov heat flux mode are stated here. The spectral quasilinearization method (SQLM) was used

for the numerical computation with the number of collocation points  $N = 40$  in space  $\xi$  and the scaled parameter  $L = 10$ . The physical properties of ethylene glycol ( $C_2H_6O_2$ ), SWCNTs and MWCNTs were employed from Table 6.1. The convergence of the SQLM solution and the stability of the results were tested. Table 7.2 and Table 7.3 revealed that the convergence of solutions for both SWCNTs and MWCNTs with ethylene glycol nanofluids are achieved at 5<sup>th</sup> order of approximations for all radial wall stress  $-f''(0)$ , tangential wall stress  $-h'(0)$ , normalized skin friction coefficient  $\sqrt{f''(0)^2 + h'(0)^2}$ , and local Nusselt number  $-\theta'(0)$ .

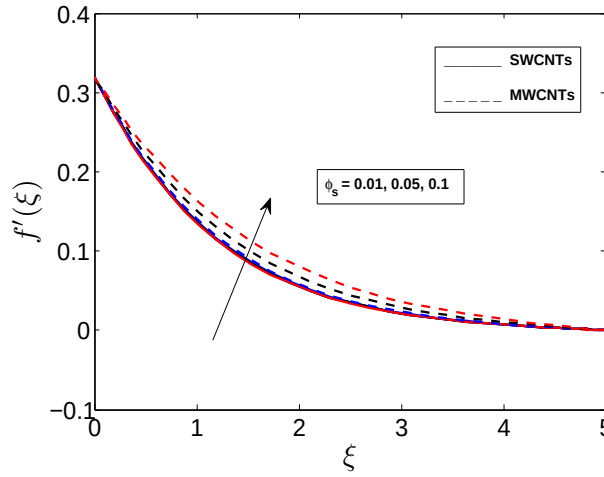
Table 7.2: SQLM Convergence for *SWCNTs* –  $C_2H_6O_2$  nanofluid.

| Order | $-f''(0)$  | $-h'(0)$   | $\sqrt{f''(0)^2 + h'(0)^2}$ | $-\theta'(0)$ |
|-------|------------|------------|-----------------------------|---------------|
| 01    | 0.23689710 | 0.89154598 | 0.922482783                 | 0.67685416    |
| 02    | 0.23359837 | 0.89107446 | 0.92519082                  | 0.67565587    |
| 03    | 0.23359687 | 0.89107467 | 0.92118505                  | 0.67567940    |
| 04    | 0.23359687 | 0.89107467 | 0.92118487                  | 0.67567940    |
| 05    | 0.23359687 | 0.89107467 | 0.92118487                  | 0.67567940    |

Table 7.4 encompasses the computations of normalized skin friction coefficient and local Nusselt number for changing values of important involved parameters such as nanoparticle volume fraction, scaled stretching parameter, Casson fluid parameter, velocity slip parameter, and thermal slip parameter for both SWCNTs and MWCNTs with ethylene glycol nanofluids. Here, it is shown that the normalized skin friction coefficient grows as nanoparticle volume fraction ( $\phi_s$ ) increases from 0.01 to 0.1 for SWCNTs nanofluid. For both SWCNTs and MWCNTs nanofluids normalized skin friction coefficient enhances as scaled stretching parameter and Casson fluid parameter growth. Further, a decreasing trend in normalized skin friction coefficient is perceived for increasing value of velocity slip parameter. Furthermore, from Table 7.4 we perceived that for both SWCNTs and MWCNTs nanofluids the local Nusselt number shows an increasing tendency for higher values of scaled stretching parameter

Table 7.3: SQLM Convergence for  $MWCNTs - C_2H_6O_2$  nanofluid

| Order | $-f''(0)$  | $-h'(0)$   | $\sqrt{f''(0)^2 + h'(0)^2}$ | $-\theta'(0)$ |
|-------|------------|------------|-----------------------------|---------------|
| 01    | 0.22142446 | 0.80709704 | 0.83691960                  | 0.72862155    |
| 02    | 0.21820148 | 0.80799670 | 0.83694119                  | 0.74087932    |
| 03    | 0.21819376 | 0.80799811 | 0.83694054                  | 0.74082968    |
| 04    | 0.21819374 | 0.80799812 | 0.83694054                  | 0.74082974    |
| 05    | 0.21819374 | 0.80799812 | 0.83694054                  | 0.74082974    |


 Figure 7.2: Radial Velocity profiles for various values of  $\phi_s$ .

and Casson fluid parameter. However, it shows a decreasing tendency if the value of Casson fluid parameter is greater than 0.5. It is also noted that the local Nusselt number is reduced for greater values of nanoparticle volume fraction, velocity slip and thermal slip parameters for both SWCNTs and MWCNTs nanofluids. For some parameters comparable result was established in the literatures of Mushtaq and Mustafa, (2017), and Aziz et al. (2018).

The study of important embedded parameters on radial velocity  $f'(\xi)$ , tangential velocity  $h(\xi)$ , and temperature  $\theta(\xi)$  profiles are plotted in Figures 6.2-6.12 for  $\phi_s = 0.05, A_s = 0.5, M = 5, \kappa = 0.5, \delta_v = 0.3, \gamma = 0.5, Pr = 7.3, \delta_t = 0.2$  and  $\alpha_t = 0.3$ . Figures 7.2 and 7.3 represent the effects of nanoparticle volume fraction ( $0 < \phi_s < 0.1$ ) on the radial velocity and temperature profiles respectively for the ethylene

Table 7.4: Values of  $\sqrt{f''(0)^2 + h'(0)^2}$  and  $-\theta'(0)$  for  $SWCNTs - C_2H_6O_2$  and  $MWCNTs - C_2H_6O_2$  nanofluid.

|          |       |            |          |            | $\sqrt{f''(0)^2 + h'(0)^2}$ |              | $-\theta'(0)$ |              |
|----------|-------|------------|----------|------------|-----------------------------|--------------|---------------|--------------|
| $\phi_s$ | $A_s$ | $\delta_t$ | $\gamma$ | $\delta_v$ | <i>SWCNT</i>                | <i>MWCNT</i> | <i>SWCNT</i>  | <i>MWCNT</i> |
| 0.01     | 0.5   | 0.2        | 0.5      | 0.3        | 0.918047                    | 0.900415     | 0.902380      | 0.920289     |
| 0.05     |       |            |          |            | 0.921185                    | 0.836940     | 0.67568       | 0.740830     |
| 0.1      |       |            |          |            | 0.926017                    | 0.761583     | 0.505748      | 0.604996     |
| 0.05     | 0.1   |            |          |            | 0.859218                    | 0.772487     | 0.301925      | 0.341617     |
|          | 0.5   |            |          |            | 0.921185                    | 0.836940     | 0.675679      | 0.740830     |
|          | 0.8   |            |          |            | 0.994724                    | 0.911512     | 0.882447      | 0.958942     |
| 0.05     | 0.5   | 0.2        |          |            | 0.921185                    | 0.836940     | 0.675679      | 0.740830     |
|          |       | 0.5        |          |            | 0.921185                    | 0.836940     | 0.656374      | 0.719663     |
|          |       | 1.0        |          |            | 0.921185                    | 0.836940     | 0.626539      | 0.686951     |
| 0.05     | 0.5   | 0.2        | 0.1      |            | 0.386210                    | 0.350375     | 0.636355      | 0.691414     |
|          |       |            | 0.5      |            | 0.921185                    | 0.836940     | 0.675679      | 0.740830     |
|          |       |            | 5        |            | 1.559282                    | 1.417469     | 0.635683      | 0.707745     |
| 0.05     | 0.5   | 0.2        | 0.5      | 0.1        | 1.017835                    | 0.925215     | 0.824280      | 0.885015     |
|          |       |            |          | 0.3        | 0.921185                    | 0.836940     | 0.675679      | 0.740830     |
|          |       |            |          | 0.5        | 0.855123                    | 0.775726     | 0.584179      | 0.649354     |

glycol ( $C_2H_6O_2$ ) based nanofluid with SWCNTs and MWCNTs. It is shown that for both SWCNTs and MWCNTs, as the value of  $\phi_s$  increases, the radial velocity and temperature profiles of the nanofluid increase. It is also recognized that higher velocity distribution in the MWCNTs nanofluid than SWCNTs nanofluid whereas the the opposite trend is observed for temperature distribution. This is due to the fact that SWCNTs have higher density than MWCNTs.

Figures 7.4-7.6 respectively show the radial velocity and temperature profiles for changed value of scaled stretching parameter  $A_s$ . The parameter  $A_s$  is given as  $A_s = s/\Omega$ , and it measures a radial stretching rate. Figure 7.4 reveals that as parameter  $A_s$  grows the radial velocity momentum boundary layer becomes thicker. Actually, the radial stretching rate raises as  $A_s$  enhances, and it accelerates radially outward flow. Figure 7.5 depicts that, contrary to the radial velocity, the angular velocity and its boundary layer thickness decreases as the value of  $A_s$  grows. From Figure 7.6, it is also

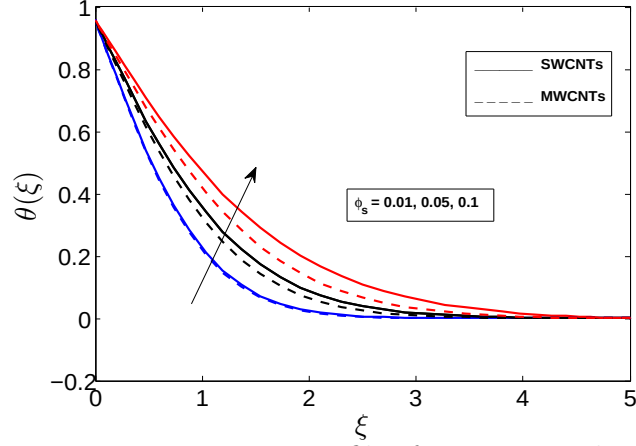


Figure 7.3: Temperature profiles for various values of  $\phi_s$ .

noticed that temperature profile declines and thermal boundary layer become thinner as the value of  $A_s$  enhances. Therefore, in practical applications, radial stretching of the disk is helpful to improve the cooling process of the rotating disk. A related result was established in the literature of Mushtaq and Mustafa (2017).

The effect of magnetic parameter  $M$  on tangential velocity and temperature profiles are demonstrated in Figures 7.7 and 7.8 respectively. For both SWCNTs and MWCNTs nanofluids, tangential velocity declines for increasing value of  $M$ . This is expected, since magnetic field acting in the axial direction provides resistance force which accountable for dropping the radial and tangential fluid velocities. In Figure 7.8 an enhancing tendency of temperature profile is perceived as  $M$  increases. Hence, incidence of magnetic field raises the thermal boundary layer thickness as temperature increases. It is also noticed lower velocity and higher temperature distributions in the SWCNTs nanofluid than MWCNTs nanofluid. This is due to the fact that SWCNTs have higher electrical conductivity than MWCNTs. Analogous result was also found in Mushtaq and Mustafa, (2017), and Aziz et al. (2018).

Figure 7.9 shows the influence of Casson fluid parameter  $\gamma$  on radial velocity profile. Near the boundary surface the velocity profile shows an increasing tendency

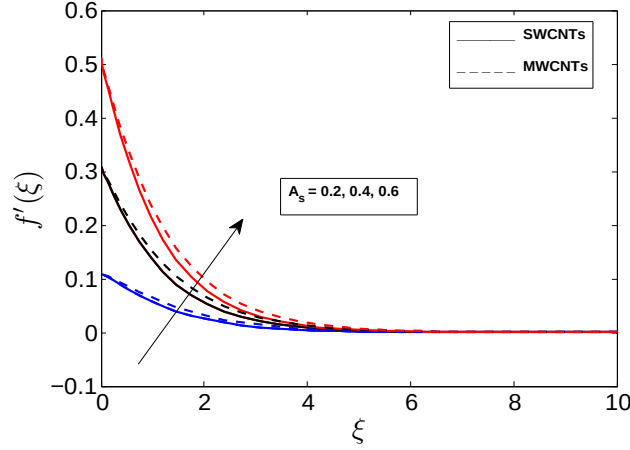


Figure 7.4: Radial Velocity profiles for various values of  $A_s$ .

as  $\gamma$  increases. This is anticipated since increasing the Casson fluid parameter  $\gamma$  tends to reduce in yield stress, and the fluid flows easily. However, as radial velocity faraway from the boundary surface, the reverse effect resulting in the thinning of the momentum boundary layer is happened due to the presence of the magnetic field effect.

Figure 7.10 illustrates the effect of velocity slip parameter  $\delta_v$  on tangential velocity profile for both SWCNTs and MWCNTs nanofluids. The tangential velocity profile shows diminishing trend for increasing value of  $\delta_v$ . In slip condition, since the fluid velocity near the surface of the sheet is no more equal to the velocity of stretchable rotating disk, the momentum boundary layer thickness slows down as  $\delta_v$  growths. Figure 7.11 represents greater the thermal slip parameter  $\delta_t$  leads to the drop of temperature profile for both SWCNTs and MWCNTs nanofluids with the highest effect recognized at the wall of the disk. Thus, an increases in the thermal slip factor  $\delta_t$  reduces the heat transfer from the surface of the disk to the fluid. Figure 7.12 shows the effect of the thermal relaxation parameter  $\alpha_t$  on temperature distribution. It reveals that the temperature profile reduces va increasing value of  $\alpha_t$ . Physically, thermal relaxation time is the time required by the fluid particles to transfer heat

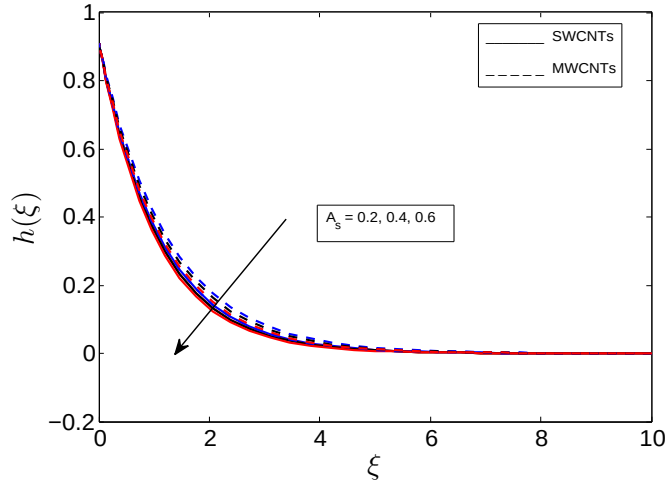


Figure 7.5: Tangential velocity profiles for various values of  $A_s$ .

energy to its adjacent particles. Consequently, as  $\alpha_t$  raises the material particles need extra time to transfer heat to its adjacent particles and this leads to less transfer of heat from the disk to the fluid.

The important physical quantities, the normalized skin friction coefficient  $C_f$  and the local heat transfer rate  $Nu$  are presented in Figures 7.13-7.16 for embedded parameters  $\phi_s = 0.05$ ,  $A = 0.5$ ,  $M = 5$ ,  $\kappa = 0.5$ ,  $\delta_v = 0.3$ ,  $\gamma = 0.5$ ,  $Pr = 7.3$ ,  $\delta_t = 0.2$  and  $\alpha_t = 0.3$ . Figure 7.13 depicts the variation of normalized skin friction coefficient with magnetic field parameter  $M$  and nanoparticle volume fraction  $\phi_s$ . The normalized skin friction coefficient is improved with increasing values of both  $M$  and  $\phi_s$ . From Figure 7.14, it is also observed that the normalized skin friction coefficient is increased with higher values of scaled stretching parameter  $A_s$  and Casson fluid parameter  $\gamma$ . Form both Figures, it is also recognized that there is higher normalized skin friction coefficient in the SWCNTs nanofluid than MWCNTs nanofluid.

The variation of local heat transfer rate with  $M$  and  $\phi_s$  are demonstrated in Figure 7.15 for both SWCNTs and MWCNTs nanofluids. The local heat transfer rate is decreased with greater value of  $M$ . On the other hand, the local heat transfer rate

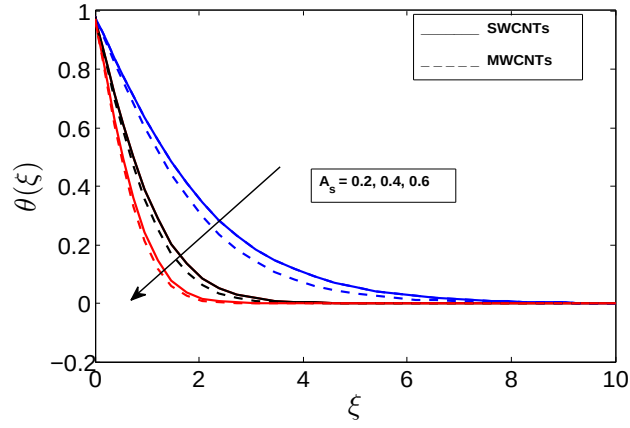


Figure 7.6: Temperature profiles for various values of  $A_s$ .

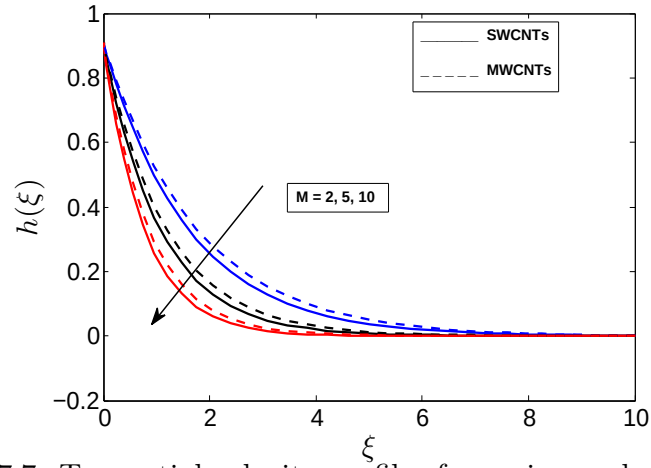


Figure 7.7: Tangential velocity profiles for various values of  $M$ .

improves with rise values of  $M$  and  $\phi_s$ . Further, Figure 7.16 clearly illustrates that the local heat transfer rate is enhanced with greater values of both Prandtl number  $Pr$  and thermal relaxation parameter  $\alpha_t$ . In addition, lower local heat transfer rate in the SWCNTs nanofluid than MWCNTs nanofluid is reported.

## 7.5 Conclusion

In this study, CNTs with ethylene glycol nanofluid flow and heat transfer due to stretchable rotating disk embedded in a porous medium with Cattaneo-Christov heat

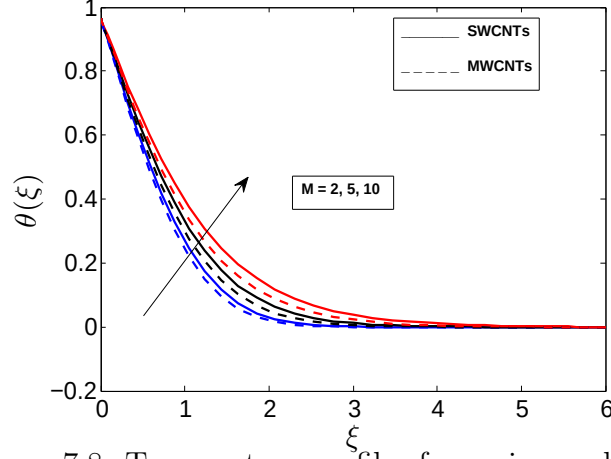


Figure 7.8: Temperature profiles for various values of  $M$ .

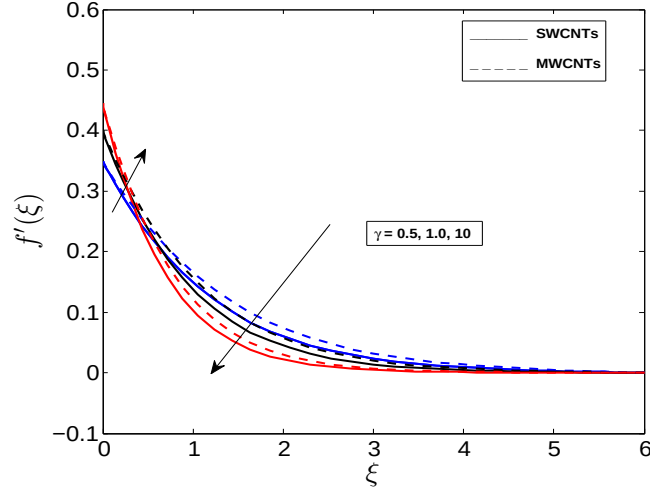


Figure 7.9: Radial velocity profiles for various values of  $\gamma$ .

flux model is examined. Effects of normally applied magnetic field, wall velocity and thermal slip conditions are considered. The governing equations of the flow is solved numerically employing Spectral quasilinearization method (SQRM). From the main outcomes of the present observation the following conclusions are drawn.

- Higher radial and tangential velocities distributions are observed in the MWCNTs nanofluid than SWCNTs nanofluid whereas the trend is opposite for temperature distribution.

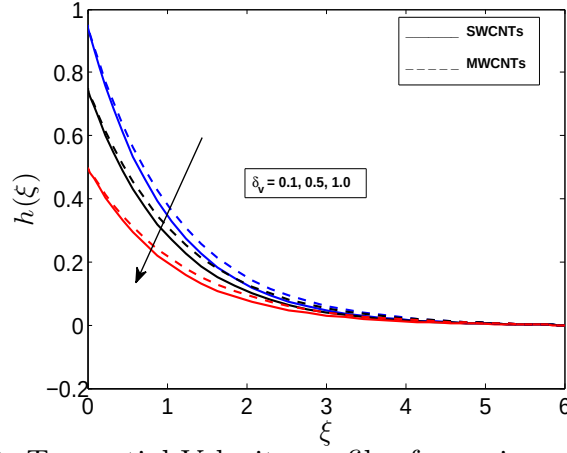


Figure 7.10: Tangential Velocity profiles for various values of  $\delta_v$ .

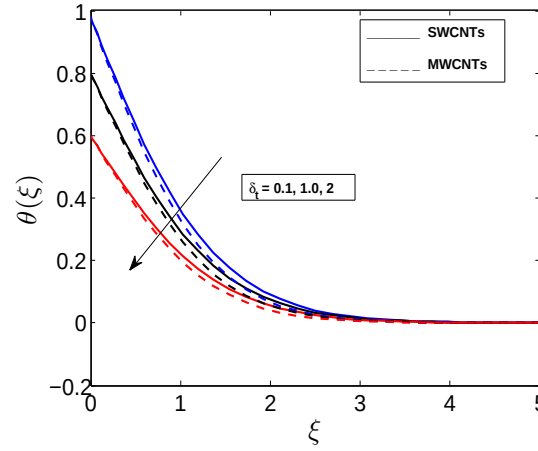


Figure 7.11: Temperature profiles for various values of  $\delta_t$ .

- Normalized skin friction coefficient enhances for both SWCNTs and MWCNTs nanofluids as scaled stretching parameter, magnetic parameter, and Casson fluid parameter growth.
- Local heat transfer rate enhances for both SWCNTs and MWCNTs nanofluids for higher values of scaled stretching parameter, Prandtl number and thermal relaxation parameter.
- The higher incidence of magnetic field raises the temperature field, and it improves the thermal boundary layer thickness.
- The momentum boundary layer thickness thinner as velocity slip growth.

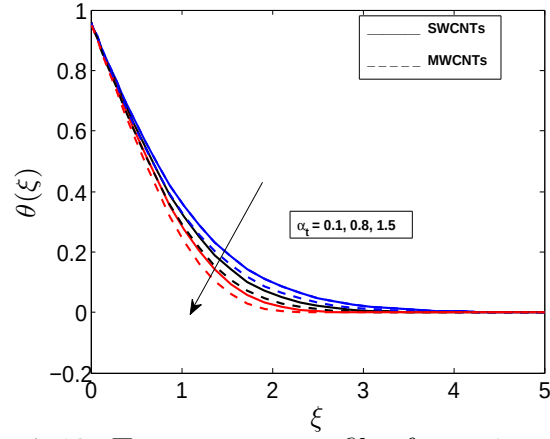


Figure 7.12: Temperature profiles for various values of  $\alpha_t$ .

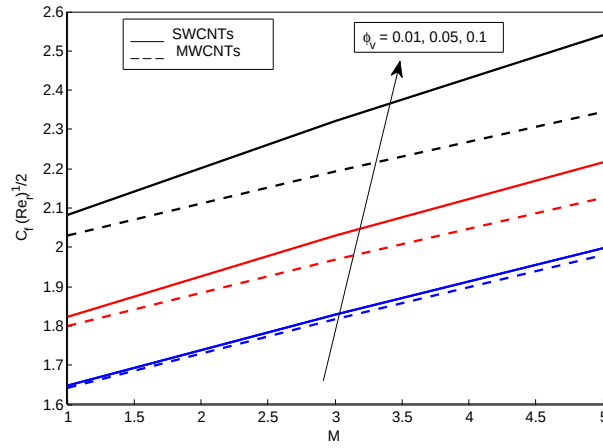


Figure 7.13: Skin friction coefficient for various values of  $M$  and  $\phi_s$ .

- Increases in the thermal slip factor tends to drop the temperature distribution at the wall of the disk, and it reduces the heat transfer from the surface of the disk to the fluid.
- Increasing in scaled stretching parameter thicker the momentum boundary layer while it thinner the thermal boundary layer. Moreover, radial stretching of the disk is helpful to improve the cooling process of the rotating disk in practical applications.

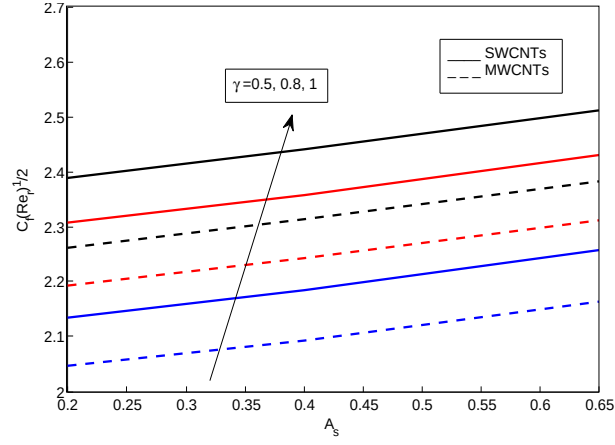


Figure 7.14: Skin friction coefficient for various values of  $A$  and  $\gamma$ .

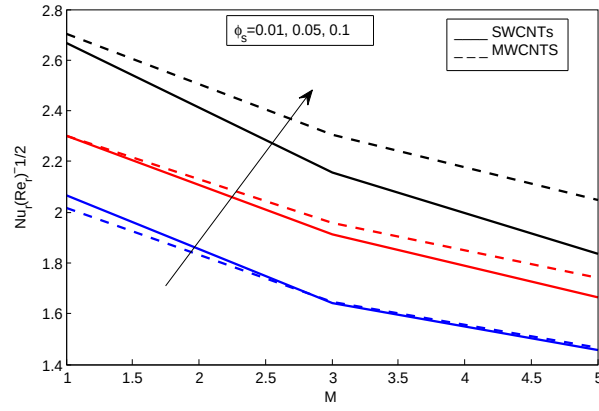


Figure 7.15: Local heat transfer rate for various values of  $M$  and  $\phi_s$ .

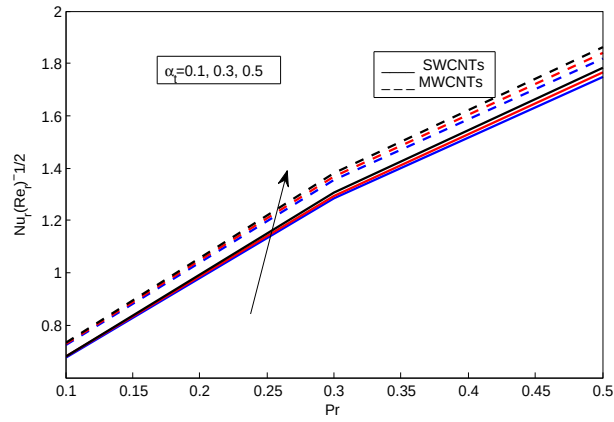


Figure 7.16: Local heat transfer rate for various values of  $Pr$  and  $\alpha_t$ .

## Chapter 8

# Natural Convection Flow of CNTs-Magnetite/Water Hybrid Nanofluids Due to Stretching Surface with Second Order Slip Flow and Variable Viscosity

### 8.1 Introduction

Carbon nanotubes (CNTs) have important potential applications in numerous fields of engineering and biomedicine. They are allotropes of carbon prepared in cylindrical tubes of graphite with nanometer in diameter and a few millimeters in length (Hirlekar et al., 2019). CNTs are generally divided into single wall carbon nanotubes (SWCNTs) and multi wall carbon nanotubes (MWCNTs) depending on their number of concentric layers of rolled grapheme sheets. SWCNTs contain catalyst for their synthesis and encompass a particular graphene sheet encircled around and form a cylinder. MWCNTs are concentrically nested cylinders of grapheme sheets and they can be formed without catalyst. The inner and outer diameter of MWCNTs is determined by their number of layers (Khalid et al., 2018). Carbon nanotubes are

resourceful material of the modern technologies because of their special thermal, electrical, and mechanical characteristics, unique morphology, innovative physicochemical features, and the presence of carbon chains in CNTs does not convey any hazard to the atmosphere (Alsagri et al., 2019).

Modern applications of Carbon nanotubes in the biomedical field for drug delivery, cancer therapy and other application in medicines have attracted different researchers in studying the thermophysical properties and flow of CNTs nanofluids (Nasir et al., 2018). CNTs have encouraging vigorous surface area that permits them to be functionalised and ornamented by numerous kinds of chemical and biological matters. Due to their needle like shape, CNTs have potential to infiltrate more easily into cell walls and move among various bodies' tissues (Cai et al., 2005). Also, they have the potential to carry drugs in the organism and have the ability to target and destroy specific cancer cells without harm the healthy cells (Srinivasan, 2008). Moreover, in the long run, carbon nanotubes with friendly enzymes can be used as enzymatic biosensors that might concurrently sense and measure a multiplicity of biological molecules (Singh et al., 2012).

Carbon nanotubes, however, do not have ample magnetic properties which limit their power for uses in biomedical applications. Furthermore, the week solubility of CNTs in aqueous solutions and challenging in added solvents limits their applications in the biomedical and other technological fields (Yang et al., 2009). Consequently, to overawe the above mentioned limitations of CNTs, researchers have considered different approaches of CNTs nanoparticles preparations. Among these, magnetic delivery of CNTs is shown the best promising approach due to its capability to use a located magnet inside the body or an external magnet in several diagnostic and therapeutic agents at targeted tissues (Samadishadlou et al., 2018). Also, due to the non-hostile behavior of magnetic fields, employing of magnetic CNTs in biomedical applications reduce its side effects to adjacent tissues (Mody et al., 2014). In view of

these, the hybrid nanoparticles with CNTs mostly received attention in the biotechnological and biomedical fields are CNTs with magnetite ( $Fe_3O_4$ ). This is due to the fact that, CNTs- $Fe_3O_4$  nanoparticles advances targeting efficiency and enables magnetically engaged delivery. In addition, the magnetite nanocrystals inside the CNTs advance its drug loading ability, preserve it from agglomeration, and improve its chemical stability (Masotti & Caporali, 2013).

Much effort has been given to develop new experimental work on obtaining well-defined and highly stable hybrid nanoparticles which enhance the thermal conductivity of base fluid (Suresh et al., 2011; Momin, 2013). For example,  $Fe_3O_4$  exhibits more chemical inertness and stability, but it has lower thermal conductivity than CNTs. Hybrid nanofluids are fluids prepared with two or more different nanometer size materials (hybrid nanoparticles). Different studies show that hybrid nanofluids have significantly improved thermal and mechanical properties than regular fluids or single nanoparticles-based nanofluids (Sundar, 2014). These hybrid nanofluids have significant applications in various engineering and biomedical fields such as, manufacturing, transportation, nuclear safety, military, in modern electronic devices and supercomputers, in pharmaceutical and drug delivery (Manjunatha et al., 2019). Due to the mentioned benefits, many scholars have interested in hybrid nanofluids investigation. Using the polymer method, Shi et al. (2010) synthesized CNT/ $Fe_3O_4$  hybrid nanoparticles. Currently, Hayat et al. (2017) numerically analyzed the convective flow of magnetite- $Fe_3O_4$  nanoparticles due to curved stretching sheet. Also, Sundar et al. (2017) experimentally synthesized MWCNT/ $Fe_3O_4$  water based hybrid nanofluids, and he found hybrid nanofluids create higher thermal conductivity and heat transfer enhancement than single nanoparticles-based nanofluids. Similarly, he observed the viscosity of MWCNT- $Fe_3O_4$  water hybrid nanofluids significantly improved as compared to base fluid. Further, he suggested that more analysis and experiments are needed to fully realize means of improving heat transfer of hybrid nanofluids.

Various nanofluid flow studies have considered with constant physical properties of fluid. However, the viscosity of nanofluid may change significantly with temperature and play an indispensable role in nanofluids flow. For example, the heat produced via internal friction rises the temperature which consecutively affects the stickiness of the nanofluid. Thus, to precisely evaluate the flow nature of nanofluid, it is essential to consider this disparity of viscosity with temperature. Currently, Kuttan et al. (2019) is among researchers that take into account viscosity with temperature-dependent and analyzed the flow of fluid over a flat surface. So far, there is scarce of information on CNTs- $Fe_3O_4$  hybrid nanofluids flow considering the effects of temperature-dependent viscosity.

The Naviere-Stokes theory of fluid transport is based on the no-slip boundary condition. But there are situations in which no-slip condition is inappropriate. Typically, no-slip boundary condition is not suitable for most non-Newtonian fluids, science some polymer melts normally show tiny wall slip and generally it is governed by a non-linear and a monotonic relation between the adhesive friction and the slip velocity (Halim et al., 2016). Fluids offering boundary slip condition have important applications in some technological and biomedical fields, for instance, in expensive lubricating, optical coatings, refrigeration equipment, purifying of artificial heart valve, internal cavities, and in others industrial processes. As a result, the effect of slip boundary condition has been considered by some researchers. Among these, Nagendramma et al., (2017), and Tlili et al. (2018) studied the effects of slip boundary condition on the flow and heat transfer of nanofluid over some geometries. The above-mentioned studies considered only the first order slip flow. The flow of fluid with second order slip boundary condition has been not got much attention so far by the researchers. But, the second order slip flow takes place in several engineering applications. Few of the studies considered fluid flow and heat transfer over stretching sheet with a second slip condition (Ibrahim, 2017). Yet, there is limited information about the

effect of second order slip condition on  $CNTs - Fe_3O_4/H_2O$  hybrid nanofluids flow over stretching surface. Thus, one of the intention of this study is to fill the above indicated knowledge gaps.

Motivated by the above cited literature survey, the purpose of this study is to analyze  $CNTs - Fe_3O_4/H_2O$  hybrid nanofluids flow and heat transfer due to stretching surface. The impact of temperature dependent viscosity, free convection and second order slip condition are also considered. The governing equations are solved numerically employing the fast convergent and accurate technique namely, spectral quasilinearization method (SQLM). The effects of governing parameters on hybrid nanofluids flow velocity and temperature distributions are discussed and presented in tables and graphs as well. To the best of our knowledge, yet, no analysis has been published in this direction. The developed model has potential applications in the biomedical fields such as cancer therapy, drug delivery, enzymatic biosensors, and etc. Considering the effects of temperature dependent viscosity and second order slip condition on  $CNTs - Fe_3O_4/H_2O$  hybrid nanofluids flow and heat transfer, and computing it by means of SQLM makes this study novel and different from the former studies.

## 8.2 Mathematical Description of the Problem

We consider unsteady two-dimensional incompressible viscous flow of  $CNTs - Fe_3O_4$  with  $H_2O$  hybrid nanofluids past semi-infinite linearly stretching surface embedded in a porous medium. The flow is situated to  $y > 0$ , where  $x$  and  $y$  are respectively, in the direction of flow and normal to the surface, as shown in Figure 8.1. Also, we considered that the base fluid and the hybrid nanoparticles to be in thermal equilibrium so that no slip occurs between them. It is worth to mention that, to develop the targeted hybrid nanofluids  $(CNTs - Fe_3O_4)/H_2O$ , initially, CNTs is dispersed into base fluid, then  $Fe_3O_4$  is scattered in  $CNTs/H_2O$  nanofluids. In the

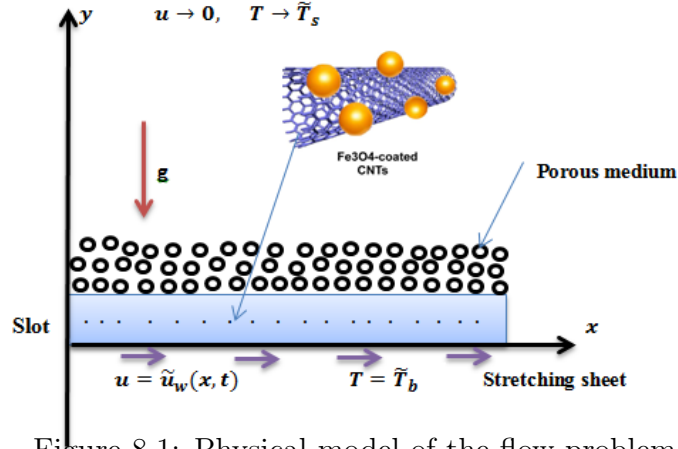


Figure 8.1: Physical model of the flow problem.

beginning, at time  $t = 0$ , both the surface and hybrid nanofluids are at rest with uniform temperature  $T_\infty$ . For time  $t > 0$ , the surface begins linearly stretching with velocity  $u_w = \frac{u_0}{1-\zeta t}x$ , where  $u_0$  is constant ( $u_0 > 0$  for a stretching and  $u_0 < 0$  for a shrinking surface),  $\zeta \geq 0$  is constant,  $t$  is time and  $\zeta t < 1$ . At the same time the surface temperature is upturned to  $T_w$  which is then kept constant.

With foregoing assumptions and Boussinesq approximation, the transport equations of hybrid nanofluid boundary layer flow are establish as follows (Alsagri et al., 2019; Manjunatha et al., 2019).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (8.2.1)$$

$$\rho_{hnf} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu_{hnf} \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \frac{\partial}{\partial y} (\mu_{hnf}) - \frac{\mu_{hnf}}{K} u + g(\rho\beta_t)_{hnf} (T - T_\infty) \quad (8.2.2)$$

$$(\rho c_p)_{hnf} \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k_{hnf} \frac{\partial^2 T}{\partial y^2} + Q_0 (T - T_\infty) \quad (8.2.3)$$

The initial and boundary conditions ewith second order velocity slip and thermal slip

are given as

$$\begin{aligned}
t < 0 : \quad u &= 0, \quad v = 0, \quad T = T_\infty \quad \forall_{x,y} \\
t \geq 0 : \quad u(x, t) &= u_w + u_{slip}, \quad v = 0, \quad T = T_w + N_0 \frac{\partial T}{\partial y} \quad \text{at } y = 0 \quad (8.2.4) \\
t \geq 0 : \quad u(x, t) &\rightarrow 0, \quad v \rightarrow 0, \quad T \rightarrow T_\infty \quad \text{as } y \rightarrow \infty
\end{aligned}$$

where  $u$  and  $v$  are the hybrid nanofluid velocity components in the  $x$  and  $y$  orientations respectively;  $\rho_{hnf}$ ,  $\mu_{hnf}$ ,  $k_{hnf}$ ,  $\sigma_{hnf}$ ,  $(\rho c_p)_{hnf}$  are hybrid nanofluid effective density, dynamic viscosity, thermal conductivity, electrical conductivity, and specific heat capacity respectively;  $g$ ,  $\beta_{hnf}$ ,  $Q_0$ ,  $N_0$  are gravitational acceleration, hybrid nanofluid volume expansion coefficient, heat generation/absorption rate and thermal slip factor respectively.

The temperature-dependent viscosity of the base fluid is given as (Manjunatha et al., 2019).

$$\mu_f = \mu_0 e^{-\alpha_v \theta(\xi)} \quad (8.2.5)$$

where  $\mu_0$  is reference viscosity of base fluid, and  $\alpha_v$  is variable viscosity parameter (note that,  $\alpha_v < 0$  for gases and  $\alpha_v > 0$  for liquids).

The slip velocity  $u_{slip}$  at the surface, where Wu's (2008) slip velocity equation is usable for random Knudsen number,  $Kn$ , and employed by researcher Ibrahim (2017), is specified as

$$u_{slip} = a \frac{\partial u}{\partial y} + b \frac{\partial^2 u}{\partial y^2} \quad (8.2.6)$$

where  $a$  and  $b$  are constants given as

$$a = \frac{2}{3} \left[ \frac{3 - \alpha_m l^2}{\alpha_m} - \frac{3}{2} \frac{1 - l^2}{K_n} \right] \lambda_m, \quad b = -\frac{1}{4} \left[ l^4 + \frac{2}{K_n^2} (1 - l^2) \right] \lambda_m^2 \quad (8.2.7)$$

$K_n$  is Knudsen number,  $l = \min[\frac{1}{K_n}, 1]$ ,  $\lambda_m$  is the molecular mean free path, and  $\alpha_m$  is the momentum accommodation coefficient with  $0 \leq \alpha_m \leq 1$ .

The thermo-physical characteristics (such as, density, specific heat capacity, thermal conductivity, electrical conductivity, and volumetric thermal expansion coefficient) of the nanoparticles for CNTs (SWCNTs and MWCNTs), magnetite ( $Fe_3O_4$ ) and base fluid are given in Table 8.1 (Sundar et al., 2014; Hayat et al., 2017; Alsagri et al., 2019).

Table 8.1: Thermo-physical properties of base fluid, CNTs, and  $Fe_3O_4$

| Physical properties    | $H_2O$ | SWCNT  | MWCNT                | $Fe_3O_4$ |
|------------------------|--------|--------|----------------------|-----------|
| $\rho(kg/m^3)$         | 997.1  | 2600   | 1600                 | 5180      |
| $c_p(J/kgK)$           | 4179   | 425    | 796                  | 670       |
| $k(W/mK)$              | 0.613  | 6600   | 3000                 | 9.7       |
| $\sigma(S/m)$          | 0.05   | $10^6$ | $1.9 \times 10^{-4}$ | 25000     |
| $\beta(10^{-5}K^{-1})$ | 21     | 27     | 44                   | 1.05      |

The nanofluid effective density  $\rho_{nf}$ , specific heat capacity  $(\rho c_p)_{nf}$ , dynamic viscosity  $\mu_{nf}$ , thermal conductivity  $k_{nf}$ , electrical conductivity  $\sigma_{nf}$ , and thermal expansion coefficient  $(\rho\beta_t)_{nf}$  are given in Table 8.2 (Alsagri et al., 2019).

where  $\phi_{s_1}$  is the CNTs nanoparticle volume fraction;  $\rho_{s_1}$ ,  $(\rho c_p)_{s_1}$ ,  $k_{s_1}$ ,  $(\rho\beta_t)_{s_1}$  and  $\sigma_{s_1}$  are respectively density, specific heat capacity, thermal conductivity, volumetric thermal expansion coefficient, and electrical conductivity of CNTs, whereas,  $\rho_{bf}$ ,  $(\rho c_p)_{bf}$ ,  $\mu_{bf}$ ,  $\nu_{bf}$ ,  $(\rho\beta_t)_{bf}$  and  $\sigma_{bf}$  are respectively density, specific heat capacity, dynamic viscosity, kinematic viscosity, volumetric thermal expansion coefficient, and electrical conductivity of the base fluid.

The models of hybrid nanofluid effective density  $\rho_{hnf}$ , specific heat capacity  $(\rho c_p)_{hnf}$ , dynamic viscosity  $\mu_{hnf}$ , thermal conductivity  $k_{hnf}$ , electrical conductivity  $\sigma_{hnf}$ , and thermal expansion coefficient  $(\rho\beta_t)_{hnf}$  are given in Table 8.3 (Alsagri et al., 2019; Manjunatha et al., 2019).

where  $\phi_{s_2}$  is the  $Fe_3O_4$  nanoparticle volume fraction;  $\rho_{s_2}$ ,  $(\rho c_p)_{s_2}$ ,  $k_{s_2}$ ,  $(\rho\beta_t)_{s_2}$  and  $\sigma_{s_2}$ , are respectively, density, specific heat capacity, thermal Conductivity, volumetric

Table 8.2: Thermo-physical Model of Mono Nanofluid

| Thermophysical Properties                  | Nanofluid ( $CNTs - H_2O$ )                                                                                                                                                                                                                                                            |
|--------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Density ( $kg/m^3$ )                       | $\rho_{nf} = \phi_{s1}\rho_{s1} + (1 - \phi_{s1})\rho_{bf}$                                                                                                                                                                                                                            |
| Heat capacity ( $J/kgK$ )                  | $(\rho c_p)_{nf} = \phi_{s1}(\rho c_p)_{s1} + (1 - \phi_{s1})(\rho c_p)_{bf}$                                                                                                                                                                                                          |
| Viscosity ( $Ns/m^2$ )                     | $\mu_{nf} = \frac{\mu_{bf}}{(1 - \phi_{s1})^{2.5}}$                                                                                                                                                                                                                                    |
| Thermal Conductivity ( $J/K$ )             | $\frac{k_{nf}}{k_{bf}} = \frac{1 - \phi_{s1} + 2\phi_{s1} \left( \frac{k_{s1}}{k_{s1} - k_{bf}} \right) \ln \left( \frac{k_{s1} + k_{bf}}{2k_f} \right)}{1 - \phi_{s1} + 2\phi_{s1} \left( \frac{k_{bf}}{k_{s1} - k_{bf}} \right) \ln \left( \frac{k_{s1} + k_{bf}}{2k_{bf}} \right)}$ |
| Electrical Conductivity ( $S/m$ )          | $\sigma_{nf} = \sigma_{bf} \left[ 1 + \frac{3 \left( \frac{\sigma_{s1}}{\sigma_{bf}} - 1 \right) \phi_{s1}}{\left( \frac{\sigma_{s1}}{\sigma_{bf}} + 2 \right) - \left( \frac{\sigma_{s1}}{\sigma_{bf}} - 1 \right) \phi_{s1}} \right]$                                                |
| Thermal Expansion Coefficient ( $K^{-1}$ ) | $(\beta_t)_{nf} = (1 - \phi_{s1})(\rho\beta_t)_{bf} + \phi_{s1}(\rho\beta_t)_{s1}$                                                                                                                                                                                                     |

thermal expansion coefficient, and electrical conductivity of  $F_3O_4$ .

The continuity equation (8.2.1) is identically satisfied by introducing the following non-dimensional variables based on Manjunatha et al. (2019).

$$\xi = \sqrt{\frac{u_0}{\nu_{bf}(1 - \zeta t)}}y; \quad u = \frac{u_0}{1 - \zeta t}x f'(\xi); \quad v = -\sqrt{\frac{u_0 \nu_{bf}}{1 - \zeta t}}f(\xi); \quad \theta(\xi) = \frac{T - T_\infty}{T_w - T_\infty} \quad (8.2.8)$$

where  $f'(\xi)$  is a dimensionless velocity and  $\theta(\xi)$  is the dimensionless temperature. Hence, the transformed equations for momentum and energy are given as follows.

$$f'''(\xi) + \phi_1 \phi_2 \left[ f(\xi) f''(\xi) - A \left( f'(\xi) + \frac{\xi}{2} f''(\xi) \right) - f'^2(\xi) + \phi_3 Gr_x \theta(\xi) \right] - \alpha_v \theta'(\xi) f''(\xi) - \kappa f'(\xi) = 0 \quad (8.2.9)$$

$$\frac{k_{hnf}}{k_{bf}} \theta''(\xi) + \phi_4 Pr \left[ f(\xi) \theta'(\xi) - A \left( 2\theta(\xi) + \frac{\xi}{2} \theta'(\xi) \right) \right] + Pr H \theta(\xi) = 0 \quad (8.2.10)$$

The transformed boundary conditions are:

$$f = 0, \quad f'(0) = 1 + \delta_{v1} f''(0) + \delta_{v2} f'''(0); \quad \theta(0) = 1 + \delta_t \theta'(0), \quad \xi \geq 0. \quad (8.2.11)$$

$$f'(\infty) \rightarrow 0; \quad \theta(\infty) \rightarrow 0, \quad \xi \geq 0.$$

Table 8.3: Thermo-physical Model of Hybrid Nanofluid

| Thermophysical Properties       | Hybrid nanofluid $((CNTs - Fe_3O_4)/H_2O)$                                                                                                                                                                                                                                                 |
|---------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Density $(kg/m^3)$              | $\rho_{hnf} = (1 - \phi_{s2})[(1 - \phi_{s1})\rho_{bf} + \phi_{s1}\rho_{s1}] + \phi_{s2}\rho_{s2}$                                                                                                                                                                                         |
| Heat capacity $(J/kgK)$         | $(\rho c_p)_{hnf} = (1 - \phi_{s2})[(1 - \phi_{s1})(\rho c_p)_{bf} + \phi_{s1}(\rho c_p)_{s1}]$<br>$+ \phi_{s2}(\rho c_p)_{s2}$                                                                                                                                                            |
| Viscosity $(Ns/m^2)$            | $\mu_{hnf} = \frac{\mu_{bf}}{(1 - \phi_{s1})^{2.5}(1 - \phi_{s2})^{2.5}}$                                                                                                                                                                                                                  |
| Thermal Conductivity $(J/K)$    | $\frac{k_{hnf}}{k_{nf}} = \frac{1 - \phi_{s2} + 2\phi_{s2} \left( \frac{k_{s2}}{k_{s2} - k_{nf}} \right) \ln \left( \frac{k_{s2} + k_{nf}}{2k_{nf}} \right)}{1 - \phi_{s2} + 2\phi_{s2} \left( \frac{k_{nf}}{k_{s2} - k_{nf}} \right) \ln \left( \frac{k_{s2} + k_{nf}}{2k_{nf}} \right)}$ |
| Electrical Conductivity $(S/m)$ | $\sigma_{hnf} = \sigma_{nf} \left[ 1 + \frac{3 \left( \frac{\sigma_{s2}}{\sigma_{nf}} - 1 \right) \phi_{s2}}{\left( \frac{\sigma_{s2}}{\sigma_{nf}} + 2 \right) - \left( \frac{\sigma_{s2}}{\sigma_{nf}} - 1 \right) \phi_{s2}} \right]$                                                   |
| Thermal Expa. Coeff. $(K^{-1})$ | $(\beta_t)_{hnf} = (1 - \phi_{s2})[(1 - \phi_{s1})(\beta_t)_{bf} + \phi_{s1}(\beta_t)_{s1}] + \phi_{s2}(\beta_t)_{s2}$                                                                                                                                                                     |

where

$$\phi_1 = (1 - \phi_{s2})\{1 - \phi_{s1} + \phi_{s1} \left( \frac{\rho_{s1}}{\rho_{bf}} \right)\} + \phi_{s2} \left( \frac{\rho_{s2}}{\rho_{bf}} \right)$$

$$\phi_2 = (1 - \phi_{s1})^{2.5}(1 - \phi_{s2})^{2.5}$$

$$\phi_3 = (1 - \phi_{s2})\{1 - \phi_{s1} + \phi_{s1} \left( \frac{(\beta_t)_{s1}}{(\beta_t)_{bf}} \right)\} + \phi_{s2} \left( \frac{(\beta_t)_{s2}}{(\beta_t)_{bf}} \right)$$

$$\phi_4 = (1 - \phi_{FO})\{1 - \phi_{CNTs} + \phi_{s1} \left( \frac{(\rho c_p)_{s1}}{(\rho c_p)_{bf}} \right)\} + \phi_{s2} \left( \frac{(\rho c_p)_{s2}}{(\rho c_p)_{bf}} \right)$$

$$A = \frac{\zeta}{u_0} \text{ is unsteady parameter, } \kappa = \frac{\nu_f(1-\zeta t)}{K u_0} \text{ is permeability parameter, } Pr = \frac{\nu_f(\rho c_p)_f}{k_{bf}}$$

$$\text{is a Prandtl number, } H = \frac{(1-\zeta t)Q_0}{u_0(\rho c_p)_{bf}} \text{ is heat generation/absorption rate parameter,}$$

$$Gr_x = \frac{g\beta_t(T_w - T_\infty)}{x u_0^2} \text{ is Grashof number, } \delta_{v1} = a \sqrt{\frac{u_0}{\nu_{bf}}} \text{ is the first order velocity slip}$$

$$\text{parameter, } \delta_{v2} = b \frac{u_0}{\nu_{bf}} \text{ is the second order velocity slip parameter, } \delta_t = N_0 \sqrt{\frac{u_0}{\nu_{bf}}} \text{ is the}$$

thermal slip parameter.

The shearing stress at the wall for hybrid nanofluid is defined as

$$\tau_w = \mu_{hnf} \left( \frac{\partial u}{\partial y} \right)_{y=0} = \frac{u_0 \mu_f}{(1 - \phi_{s1})^{2.5}(1 - \phi_{s2})^{2.5}} \sqrt{\frac{u_0}{\nu_f}} x f''(0), \quad (8.2.12)$$

The surface heat flux at the wall for hybrid nanofluid is given as

$$q_w = -k_{hnf} \left( \frac{\partial T}{\partial y} \right)_{y=0} = -k_{hnf} \sqrt{\frac{u_0}{\nu_{bf}}} (T_w - T_\infty) \theta'(0) \quad (8.2.13)$$

The surface drag and wall heat transfer rates are characterized by the skin friction coefficient  $C_f$  and the local Nusselt number  $Nu$ , respectively, and they are obtained as

$$C_{fx} = \frac{\tau_w}{\rho_{bf} u_0^2}, \quad Nu_x = \frac{x q_w}{k_{bf}(T_w - T_\infty)} \quad (8.2.14)$$

Thus using equations (8.2.12)-(8.2.14), the dimensionless skin friction coefficient and local Nusselt number are given by

$$(Re_x)^{1/2} C_{fx} = \frac{1}{(1 - \phi_{s1})^{2.5} (1 - \phi_{s2})^{2.5}} f''(0) \quad (8.2.15)$$

and

$$(Re_x)^{-1/2} Nu_x = -\frac{k_{hnf}}{k_{bf}} \theta'(0) \quad (8.2.16)$$

where  $Re_x$  is the local Reynolds number defined by  $Re_x = \frac{x u_w}{\nu_{bf}}$

### 8.3 Method of Solutions

In this section, the nonlinear ordinary differential equations (8.2.9) - (8.2.10) were solved numerically by SQLM. Using this method, the system of nonlinear ordinary differential equations in two unknowns  $f$  and  $\theta$  give the following iterative scheme of linear differential equations.

$$f_{r+1}''' + \left( \phi_1 \phi_2 \left( f_r - \frac{A}{2} \xi \right) - \alpha_v \theta_r' \right) f_{r+1}'' - (2\phi_1 \phi_2 f_r' + A + \kappa) f_{r+1}' + \phi_1 \phi_2 f_r'' f_{r+1} + \phi_1 \phi_2 \phi_3 Gr_x \theta_{r+1} - \alpha_v f_r'' \theta_{r+1}' = \phi_1 \phi_2 (f_r f_r'' - (f_r')^2) - \alpha_v \theta_r' f_r'' \quad (8.3.1)$$

$$\begin{aligned} \frac{k_{hnf}}{k_{bf}} \theta_{r+1}'' + \phi_4 Pr \left( f_r - \frac{A}{2} \xi \right) \theta_{r+1}' + Pr (H - 2A\phi_4) \theta_{r+1} + \phi_4 Pr \theta_r' f_{r+1} \\ = \phi_4 Pr f_r \theta_r' \end{aligned} \quad (8.3.2)$$

with boundary conditions

$$\begin{aligned} f_{r+1} = 0, \quad f_{r+1}'(0) = 1 + \delta_{v1} f_{r+1}''(0) + \delta_{v2} f_{r+1}'''(0) \quad \theta_{r+1}(0) = 1 + \delta_t \theta_{r+1}'(0). \\ f_{r+1}'(\infty) \rightarrow 0; \quad \theta_{r+1}(\infty) \rightarrow 0. \end{aligned} \quad (8.3.3)$$

where the terms  $r$  and  $r+1$  are at the previous and current iteration levels respectively.

Equations (8.3.1)-(8.3.2) establish the iterative scheme for the SQLM. A numerical solution was then found employing the Chebyshev spectral collocation method. Beginning with appropriate initial approximations, the iteration schemes were used iteratively to obtain  $f_{r+1}$  and  $\theta_{r+1}$ . Now, applying the spectral collocation method to the system of equations (8.3.1)-(8.3.2), and it can be solved as a coupled matrix:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} F_{r+1} \\ \Theta_{r+1} \end{bmatrix} = \begin{bmatrix} B_{1,r} \\ B_{2,r} \end{bmatrix} \quad (8.3.4)$$

with transformed boundary condition

$$\begin{aligned} F_{r+1}(\chi_N) = 0, \quad F_{r+1}(\chi_{N-1}) = 1 + \delta_{v1} f''_{r+1}(0) + \delta_{v2} f'''_{r+1}(0), \quad F_{r+1}(\chi_0) = 0 \\ \Theta_{r+1}(\chi_N) = 1 + \delta_t \theta'_{r+1}(0), \quad \Theta_{r+1}(\chi_0) = 0 \end{aligned} \quad (8.3.5)$$

where

$$\Lambda_{11} = \mathbf{D}^3 + [\phi_1 \phi_2 (f_r - \frac{A}{2} \xi) - \alpha_v \theta_r]_d \mathbf{D}^2 - [2\phi_1 \phi_2 f'_r + A + \kappa]_d \mathbf{D} + \phi_1 \phi_2 [f''_r]_d$$

$$\Lambda_{12} = \phi_1 \phi_2 \phi_3 Gr_x \mathbf{I} - \alpha_v [f''_r]_d \mathbf{D}$$

$$\Lambda_{21} = \phi_4 Pr [\theta'_r]_d$$

$$\Lambda_{22} = \frac{k_{hnf}}{k_f} \mathbf{D}^2 + \phi_4 Pr [f_r - \frac{A}{2} \xi]_d \mathbf{D} + Pr (H - 2A) \mathbf{I}$$

$$B_{1,r} = \phi_1 \phi_2 [f_r f''_r - (f'_r)^2] - \alpha_v \theta'_r f''_r$$

$$B_{2,r} = \phi_4 Pr f_r \theta'_r$$

$$F_{r+1} = [f_{r+1,0}, f_{r+1,1}, \dots, f_{r+1,N}]^t$$

$$\Theta_{r+1} = [\theta_{r+1,0}, \theta_{r+1,1}, \dots, \theta_{r+1,N}]^t,$$

are vectors of sizes  $(N+1) \times 1$ ,  $\mathbf{I}$  and  $[...]_d$  respectively, represent an identity and a diagonal matrices of size  $(N+1) \times (N+1)$ .

While SQLM is employed, the choice of an initial guesses is crucial. Thus, the proper initial guesses that fulfill the governing equations of the considered flow problem are

$$f_0(\xi) = \frac{1 - e^{-\xi}}{1 + \delta_{v1} - \delta_{v2}}, \quad \theta_0(\xi) = \frac{e^{-\xi}}{1 + \delta_t} \quad (8.3.6)$$

Table 8.4: SQLM Convergence for  $f''(0)$  and  $-\theta'(0)$  of  $CNTs - Fe_3O_4$  with  $H_2O$  hybrid nanofluid

| Order | $f''(0)$   |            | $-\theta'(0)$ |            |
|-------|------------|------------|---------------|------------|
|       | SWCNTs     | MWCNTs     | SWCNTs        | MWCNTs     |
| 01    | 0.81599349 | 1.94970930 | 2.25089425    | 2.33611827 |
| 02    | 1.16333433 | 2.64977392 | 2.32088370    | 2.46277112 |
| 03    | 1.14627566 | 2.55479408 | 2.31803715    | 2.44889062 |
| 04    | 1.14596453 | 2.54710347 | 2.31797633    | 2.44759291 |
| 05    | 1.14596385 | 2.54700561 | 2.31797633    | 2.44757787 |
| 06    | 1.14596386 | 2.54700539 | 2.31797633    | 2.44757785 |
| 07    | 1.14596386 | 2.54700539 | 2.31797633    | 2.44757785 |

## 8.4 Results and Discussion

Natural convection flow of  $CNTs - Fe_3O_4$  with  $H_2O$  hybrid nanofluids due to stretching surface in non-Darcy porous medium in the presence of second order slip and variable viscosity was considered. For this study, it was assumed that 1 percent (0.01 volume) of CNTs, 4 percent (0.04 volume) of  $Fe_3O_4$ , and 95 percent (0.95 volume) of  $H_2O$  in use for the preparation of  $(CNTs - Fe_3O_4) - H_2O$  hybrid nanofluid. The physical properties of  $Fe_3O_4$ , SWCNTs, MWCNTs and base fluid (water) were taken from Table 8.1. The SQLM was employed for the numerical computation to obtain more accurate results. The convergence and stability of the solution for both SWCNTs and MWCNTs hybrid nanofluids were checked in Table 8.4, and it revealed that the 7<sup>th</sup> order of approximation is enough for both skin friction coefficient  $f''(0)$  and local Nusselt number  $-\theta'(0)$ . To confirm the results, in Table 8.5 and Table 8.6 we gave comparisons of  $f''(0)$  and  $-\theta'(0)$  results when  $\phi_{s1} = \phi_{s2} = A = H = k = \alpha_v = \delta_{v2} = Gr_x = 0$  with previously published data of the literatures (Sahoo, 2010; Noghrehabadi et al., 2012; Oyelakin et al., 2017), and (Wang, 1989; Khan & Pop, 2010; Manjunatha et al., 2019) for comparable parameters, and results have a very sound agreement.

Here, in Table 8.7 we presented how the variations in fluids thermal conductivity with nanoparticles volume fraction at room temperature. We perceived that the thermal conductivity of both  $SWCNT - H_2O$  and  $MWCNT - H_2O$  nanofluids are

Table 8.5: Comparison of the present result of  $f''(0)$  with  $\delta_{v_1}$ .

| $\delta_{v_1}$ | Sahoo    | Noghrehabadi et al. | Oyelakin et al. | Present Result |
|----------------|----------|---------------------|-----------------|----------------|
| 0.0            | 1.001154 | 1.000000            | 1.000000        | 1.000000       |
| 0.1            | 0.871447 | 0.872082            | 0.872083        | 0.872083       |
| 0.2            | 0.774933 | 0.776377            | 0.776377        | 0.776377       |
| 0.3            | 0.699738 | 0.701548            | 0.701548        | 0.701548       |
| 0.5            | 0.589195 | 0.591196            | 0.591195        | 0.591196       |
| 1.0            | 0.428450 | 0.430160            | 0.430160        | 0.430160       |
| 2.0            | 0.282893 | 0.283980            | 0.283979        | 0.283981       |
| 5.0            | 0.144430 | 0.144841            | 0.144714        | 0.144842       |
| 10.0           | 0.081091 | 0.081243            | 0.080932        | 0.081245       |
| 20.0           | 0.043748 | 0.043790            | 0.043569        | 0.043792       |

Table 8.6: Comparison of the present result of  $-\theta'(0)$  with  $Pr$ .

| $Pr$  | Wang   | Khan & Pop | Manjunatha et al. | Present Result |
|-------|--------|------------|-------------------|----------------|
| 0.70  | 0.4539 | 0.4539     | -                 | 0.453932       |
| 2.00  | 0.9114 | 0.9113     | 0.91135           | 0.91136        |
| 7.00  | 1.8954 | 1.8954     | 1.89540           | 1.89540        |
| 20.00 | 3.3539 | 3.3539     | 3.35390           | 3.35390        |
| 70.00 | 6.4622 | 6.4621     | -                 | 6.46220        |

enhanced for increasing values of nanoparticle volume fraction, and it is almost doubled when 5 percent of nanoparticles is added to the base fluid. Also, the thermal conductivity of both  $(SWCNT-Fe_3O_4)/H_2O$  and  $(MWCNT-Fe_3O_4)/H_2O$  hybrid nanofluids are more increased for increasing values of hybrid nanoparticle volume fraction, and it is improved nearly by 100 percent when 1 percent of CNTs nanoparticles and 3 percent of  $Fe_3O_4$  nanoparticles are added to the base fluid. These observations are comparable with experimental results of Sundar (2014).

The impacts of important physical parameters on velocity  $f'(\xi)$  and temperature  $\theta(\xi)$  profiles for both  $(SWCNT-Fe_3O_4)/H_2O$  and  $(MWCNT-Fe_3O_4)/H_2O$  hybrid nanofluids are plotted for  $\phi_{s_1} = 0.01$ ,  $\phi_{s_2} = 0.04$ ,  $A = 0.8$ ,  $\delta_{v_1} = 0.02$ ,  $\delta_{v_2} = 0.1$ ,  $Gr_x =$

Table 8.7: Thermal conductivity ( $k_{nf}$ ) of  $CNTs-H_2O$  nanofluid and  $k_{hnf}$  of  $(CNTs-Fe_3O_4)/H_2O$  hybrid nanofluid.

| $k_{nf}(CNTs - H_2O)$ |          |          | $k_{hnf}(\phi_{s_1} = 0.01)$ |          |          |
|-----------------------|----------|----------|------------------------------|----------|----------|
| $\phi_{s_1}$          | $SWCNTs$ | $MWCNTs$ | $\phi_{s_2}$                 | $SWCNTs$ | $MWCNTs$ |
| 0.00                  | 0.613000 | 0.613000 | 0.00                         | 0.719390 | 0.719390 |
| 0.01                  | 0.719390 | 0.709626 | 0.01                         | 1.039870 | 1.040127 |
| 0.02                  | 0.827947 | 0.808217 | 0.02                         | 1.080291 | 1.080812 |
| 0.03                  | 0.938739 | 0.908834 | 0.03                         | 1.121277 | 1.122066 |
| 0.04                  | 1.051836 | 1.011541 | 0.04                         | 1.162837 | 1.163902 |
| 0.05                  | 1.167309 | 1.116402 | 0.05                         | 1.204986 | 1.206332 |

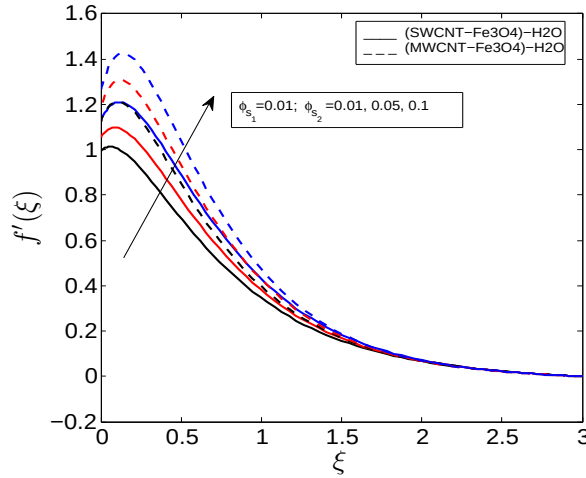


Figure 8.2: Velocity profiles for various values of  $\phi_{s_2}$ .

0.01,  $\kappa = 1$ ,  $Pr = 6.2$ ,  $\delta_t = 0.3$ ,  $\alpha_t = 2$  and  $H = -0.5$ . Figures 8.2 and 8.3 respectively, represented how increasing the nanoparticle volume fraction  $\phi_{s_2}$  parameter impacts the velocity and temperature profiles. It was shown that as the value of  $\phi_{s_2}$  increases, both the velocity and temperature profiles for hybrid nanofluids increase. Due to higher density of SWCNTs than MWCNTs, it was also recognized that higher velocity distribution in the  $(MWCNT - Fe_3O_4)/H_2O$  than  $(SWCNT - Fe_3O_4)/H_2O$  hybrid nanofluids whereas the the opposite trend was perceived for temperature distribution.

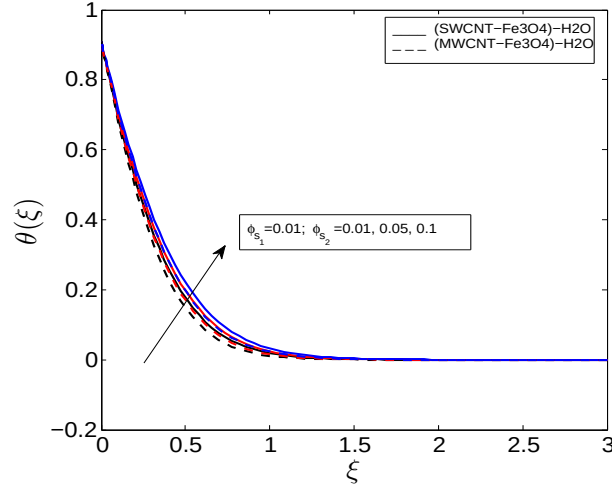


Figure 8.3: Temperature profiles for various values of  $\phi_{s2}$ .

The effects of variable viscosity parameter  $\alpha_v$  on the velocity and temperature profiles of hybrid nanofluids were shown respectively in Figures 8.4 and 8.5. Except near the boundary surface, the velocity profiles show a decreasing tendency as  $\alpha_v$  increases. Also, it indicated that the thickness of the thermal boundary layer is growth for more values of  $\alpha_v$ , and increases the temperature profile. The same pattern of velocity and temperature distribution in hybrid nanofluids flow was observed in Manjunatha et al. (2019).

The impact of Grashof number (natural convection effects) on velocity and temperature profiles were presented in Figures 8.6 and 8.7, respectively. An increment in values of Grashof number tends to enhance the thickness of fluid boundary layer near the boundary surface in which the fluid velocity increases and reaches a maximum, and gradually falls to zero as it far from the surface. Also, it was observed that temperature profile as well as thermal boundary layer thickness gradually reduce as the value of Grashof number raises. Physically, free convection flows are gradually transferred from the stretching surface to the free stream, and growth in Grashof number point out the progress of free convection currents. Similar result in hybrid

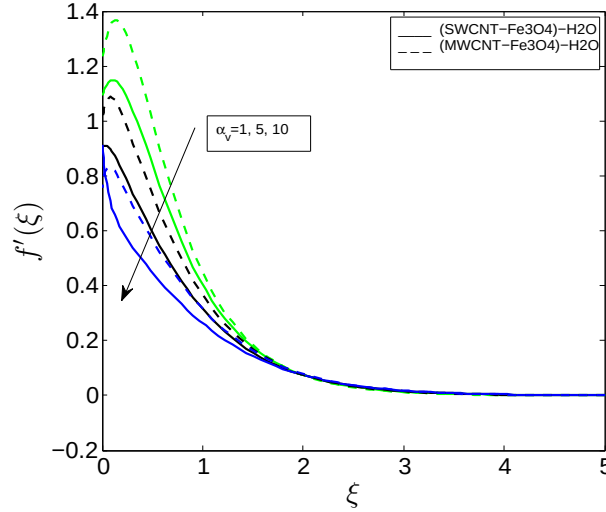


Figure 8.4: Velocity profiles for various values of  $\alpha_v$ .

nanofluids flow was found in Manjunatha et al. (2019).

Figures 8.8 and 8.9 shown the variation of second order velocity slip parameter  $\delta_{v_2}$  on velocity and temperature profiles for both SWCNTs and MWCNTs hybrid nanofluids. Near to the wall of the surface, velocity profile reduces for better value of  $\delta_{v_2}$ , but, at a distance from the wall inverse behavior was observed. Physically, slip parameters  $\delta_{v_2}$  growths the resistance of fluid motion, which decreases the fluid flow field and momentum boundary layer thickness. Furthermore, improved value of  $\delta_{v_2}$  leads to improve temperature profile and thermal boundary layer thickness.

Figures 8.10 and 8.11 illustrate the variation of temperature field with respect to heat absorption parameter  $H$  and Prandtl number  $Pr$ . As shown in Figure 8.10, for both SWCNTs and MWCNTs hybrid nanofluids the temperature profile and thermal boundary layer thickness tends to increases as the value of heat absorption parameter increases. On the other hand, in Figure 8.11, it was illustrated that the temperature profile and thermal boundary layer thickness decrease significantly as the value of Prandtl number increase. Physically, the thermal diffusivity of the fluid tends to decline and effects week diffusion of heat inside the fluid with more Prandtl number.

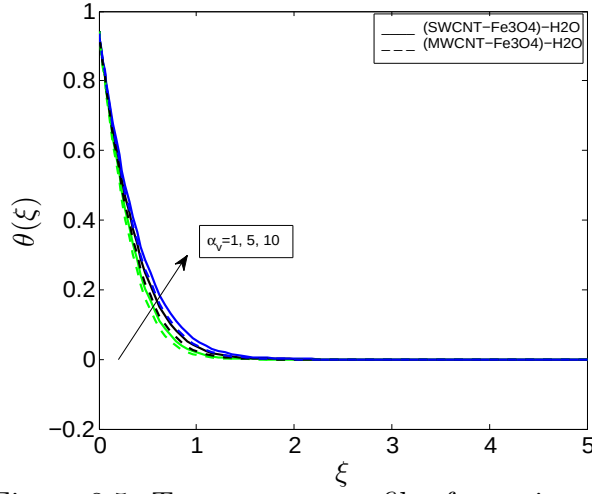


Figure 8.5: Temperature profiles for various values of  $\alpha_v$ .

The physical quantities of interest such as the skin friction coefficient  $C_f$  and the local heat transfer rate  $Nu$  were illustrated in Figures 8.12-8.14 for some parameters. Figure 8.12 shown the skin friction coefficient reduces with higher values of unsteadiness parameter  $A$  while it is improved with nanoparticle volume fraction  $\phi_{s1}$ . Figure 8.13 also presented that the skin friction coefficient is increased with higher values of both thermal dependent viscosity  $\alpha_v$  and second order velocity slip parameter  $\delta_{v2}$ . Form both Figures, also, it was observed that the skin friction coefficient is superior in the  $MWCNTs - Fe3O4/H2O$  hybrid nanofluid than in  $MWCNTs - H2O$  nanofluid. The behavior of local heat transfer rate with nanoparticle volume fraction  $\phi_{s1}$  was reported in Figure 8.14. Clearly, it show that the local heat transfer rate is improved with greater value of  $\phi_{s1}$ . Besides, lower local heat transfer rate in the  $MWCNTs - Fe3O4/H2O$  hybrid nanofluid than in  $MWCNTs - H2O$  nanofluid is reported.

Table 8.7 gives the computations of skin friction coefficient and local Nusselt number for values of important embedded parameters for  $(MWCNTs - Fe_3O_4)/H2O$  hybrid nanofluid with  $\phi_{s1} = 0.01$ . We observed that the skin friction coefficient  $f''(0)$  tends to enhance for superior values of nanoparticle volume fraction ( $\phi_{s2}$ ), Grashof

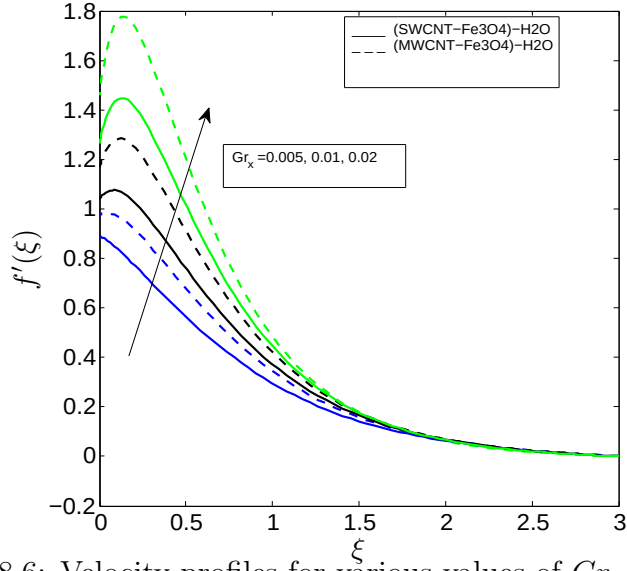


Figure 8.6: Velocity profiles for various values of  $Gr_x$ .

number, second order velocity slip, variable viscosity, and heat absorption parameters. However, the reverse effect occurs for Prandtl number, permeability, first order velocity slip and thermal slip parameters. From Table 8.7, also we found that the local Nusselt number shows an increasing tendency for higher values of Grashof number, second order velocity slip parameter and Prandtl number. Further, it is also noted that the local Nusselt number tends to reduce for greater values of nanoparticle volume fraction, first order velocity slip parameter, variable viscosity parameter, permeability parameter, thermal slip parameter, and heat absorption parameter. Similar result was also found in the literature of Manjunatha et al. (2019).

## 8.5 Conclusion

In this work, we have studied natural convection unsteady flow of CNTs-Magnetite/Water hybrid nanofluids due to stretching surface embedded in a porous medium with the effects of second order slip flow and temperature dependent viscosity. The flow governing problems were solved numerically using SQRM. The main results of present study can be summarized as follows:

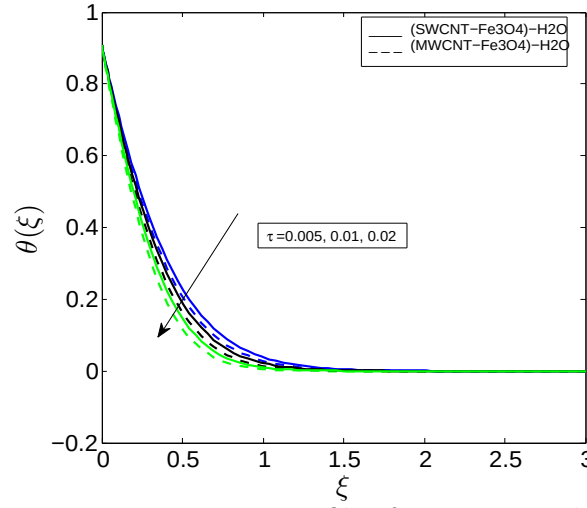


Figure 8.7: Temperature profiles for various values of  $\tau$ .

- The thermal conductivity of  $(CNTs - Fe_3O_4)/H_2O$  hybrid nanofluids is more increasing than  $CNTs - H_2O$  nanofluids for increasing values of hybrid nanoparticle volume fraction, and it is improved nearly by 100 percent when 1 percent of CNTs nanoparticles and 3 percent of  $Fe_3O_4$  nanoparticles are added to the base fluid.
- Both velocity and temperature profiles of  $(CNTs - Fe_3O_4)/H_2O$  hybrid nanofluids flow are enhancing with higher value of nanoparticle volume fraction. Moreover, there is higher velocity distribution in the  $(MWCNT - Fe_3O_4)/H_2O$  than  $(SWCNT - Fe_3O_4)/H_2O$  hybrid nanofluids while the opposite trend is reported for temperature distribution.
- The thickness of thermal boundary layer is growth with more values of temperature dependent viscosity, permeability parameter and second order velocity slip parameter.
- The skin friction coefficient is improving with higher the values of nanoparticle volume fraction, Grashof number, temperature dependent viscosity and second order velocity slip parameter.

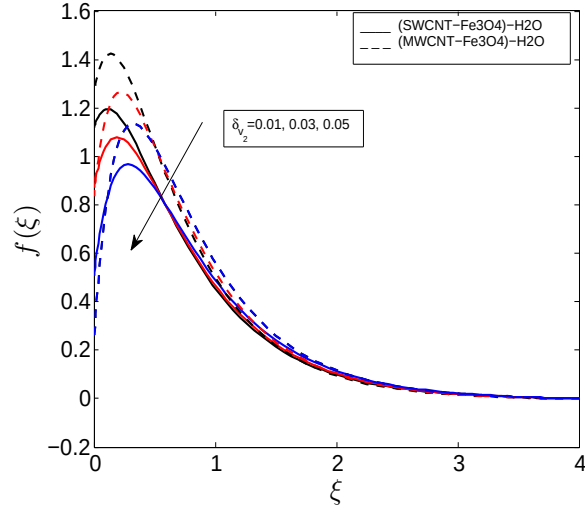


Figure 8.8: Velocity profiles for various values of  $\delta_{v_2}$ .

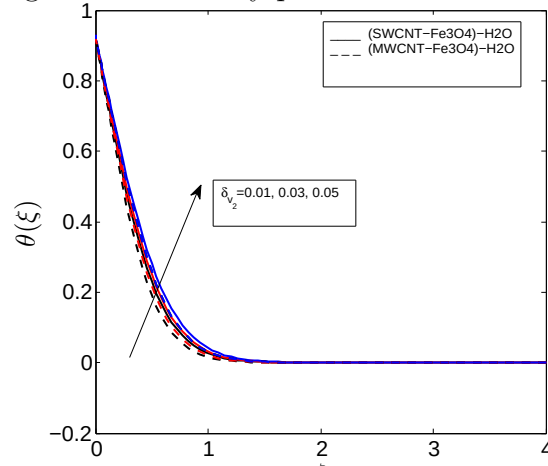


Figure 8.9: Temperature profiles for various values of  $\delta_{v_2}$ .

- Superior the skin friction coefficient and lower the local heat transfer rate are reported in the  $CNTs - Fe_3O_4/H_2O$  hybrid nanofluid than  $CNTs - H_2O$  nanofluid.
- In general, it is observed that hybrid nanofluids are more stable than mono nanofluids, and the obtained results are comparable with earlier experimental results.

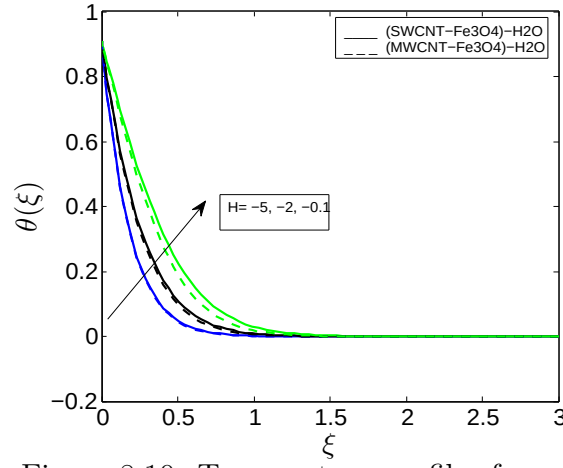


Figure 8.10: Temperature profiles for various values of  $H$ .

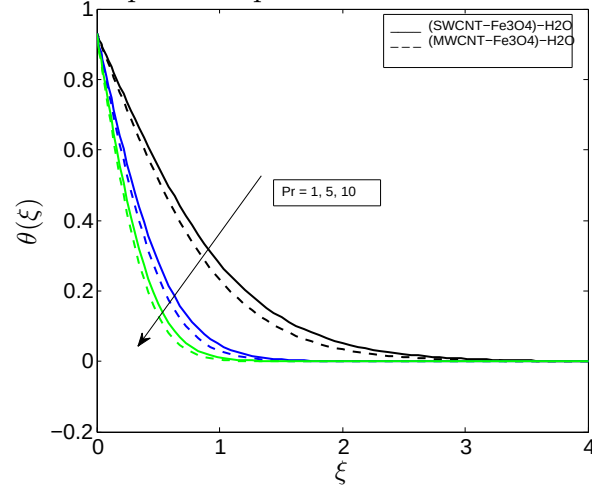


Figure 8.11: Temperature for various values of  $Pr$ .

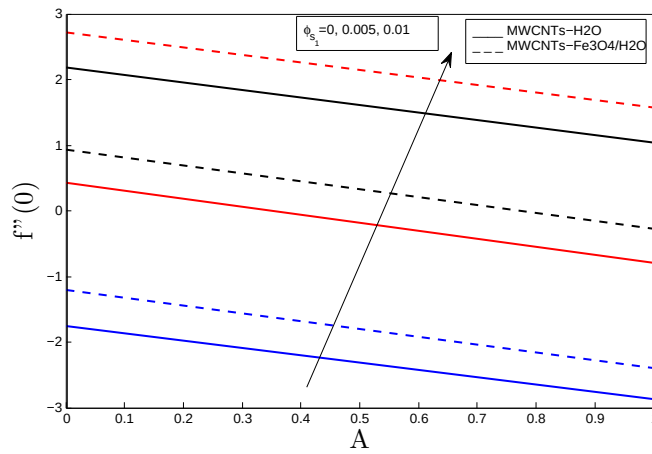


Figure 8.12: Skin friction coefficient  $C_f$  for various values of  $A$  and  $\phi_{s1}$ .

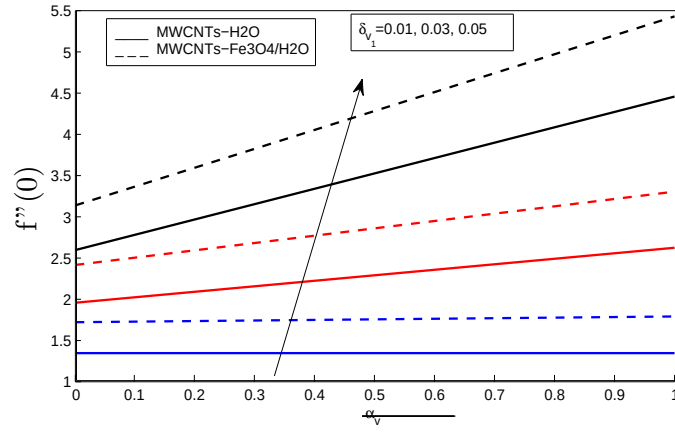


Figure 8.13: Skin friction coefficient  $C_f$  for various values of  $\alpha_v$  and  $\delta_{v_2}$ .

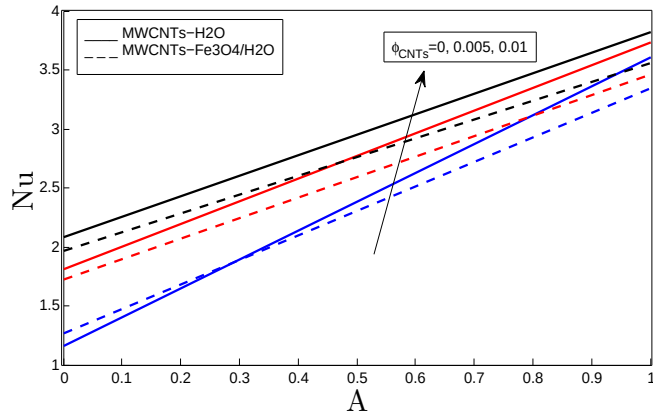


Figure 8.14: Local heat transfer rate for various values of  $A$  and  $\phi_{s_1}$ .

Table 8.8: Values of  $f''(0)$  and  $-\theta'(0)$  of  $(MWCNTs-Fe_3O_4)/H_2O$  hybrid nanofluid with  $\phi_{s_1} = 0.01, A = 0$ .

| $\phi_{s_2}$ | $\alpha_v$ | $\tau$ | $\delta_{v_1}$ | $\delta_{v_1}$ | $\delta_t$ | $\kappa$ | $H$  | $Pr$ | $f''(0)$  | $-\theta'(0)$ |
|--------------|------------|--------|----------------|----------------|------------|----------|------|------|-----------|---------------|
| 0.00         | 2          | 0.01   | 0.2            | 0.01           | 0.3        | 1        | -0.5 | 6.2  | 2.992864  | 2.544203      |
| 0.05         |            |        |                |                |            |          |      |      | 3.869134  | 2.347078      |
| 0.10         |            |        |                |                |            |          |      |      | 4.551694  | 2.181149      |
| 0.04         | 0          |        |                |                |            |          |      |      | 2.666214  | 2.475224      |
|              | 1          |        |                |                |            |          |      |      | 3.118960  | 2.434488      |
|              | 3          |        |                |                |            |          |      |      | 4.602159  | 2.325867      |
|              | 2          | 0.01   |                |                |            |          |      |      | 3.708468  | 2.383574      |
|              |            | 0.02   |                |                |            |          |      |      | 9.102958  | 2.585999      |
|              |            | 0.03   |                |                |            |          |      |      | 14.285720 | 2.715669      |
|              |            | 0.01   | 0.1            |                |            |          |      |      | 3.647700  | 2.386995      |
|              |            |        | 0.3            |                |            |          |      |      | 1.946756  | 2.479578      |
|              |            |        | 0.5            |                |            |          |      |      | 1.318321  | 2.512351      |
|              |            |        | 0.2            | 0.01           |            |          |      |      | 2.547005  | 2.447578      |
|              |            |        |                | 0.02           |            |          |      |      | 3.708468  | 2.383574      |
|              |            |        |                | 0.03           |            |          |      |      | 5.261740  | 2.293187      |
|              |            |        |                | 0.01           | 0.1        |          |      |      | 2.626025  | 2.500755      |
|              |            |        |                |                | 0.3        |          |      |      | 2.547005  | 2.447578      |
|              |            |        |                |                | 0.5        |          |      |      | 2.471471  | 2.396521      |
|              |            |        |                |                | 0.3        | 0.1      |      |      | 3.062998  | 2.513881      |
|              |            |        |                |                |            | 1.0      |      |      | 2.547005  | 2.447578      |
|              |            |        |                |                |            | 5.0      |      |      | 1.172185  | 2.269454      |
|              |            |        |                |                |            | 1.0      | -5.0 |      | 1.433198  | 5.238085      |
|              |            |        |                |                |            |          | -1.0 |      | 2.383106  | 2.868882      |
|              |            |        |                |                |            |          | -0.1 |      | 2.686130  | 2.068741      |
|              |            |        |                |                |            |          | -1.0 | 1    | 3.734532  | 1.497995      |
|              |            |        |                |                |            |          |      | 5    | 2.778049  | 2.230182      |
|              |            |        |                |                |            |          |      | 10   | 2.030652  | 3.010322      |

## Chapter 9

# Mixed Convection Flow of Hybrid Nanofluids due to a Spinning Cone with Variable Viscosity and Thermal Conductivity

### 9.1 Introduction

The wide applications of heat transfer in different sectors of industrial and manufacturing have required the availability of efficient thermal performance techniques. The need for more effective methods of enhancing the thermal efficiency and improving the thermal device performance has attracted the attention of engineers and scientists. In the past decade, different methods of improving thermal performance of working fluids have been adopted. Scattering of nanoparticle metallic structures in the working fluid is considered to be one of the novel and efficient technique. Currently, it is common to use nanofluid in cooling systems, heat exchangers, solar energy and biomedicine (Khanafer & Vafai, 2018). Due to its powerful applications, Reddy and Chamkha (2016) examined the joined effects of MHD and thermal radiation in chemically reacted water-based  $Al_2O_3$  nanofluid over a uniform stretching sheet. The viscosity of *MWCNT*-water nanofluid was investigated using simulating equilibrium

dynamical model by Jabbari et al. (2019). They reported a growth in viscosity with increases in nanoparticle volume fraction at low temperature. Chamkha et al. (2019) investigated  $Fe_3O_4$ -water nanofluid flow over a moveable plane in a parallel stream with high suction. On the other hand, because of their high cost the accessibility of some nanoparticles (like  $Cu$ ,  $Ag$  and  $Au$ ) with better thermal conductivity is limited. Moreover, some unmodified particles might be causes the toxicity risks (Parveen et al., 2016). Owing to these limitations, much effort has been given to develop new experimental work to obtain well-defined and highly stable hybrid nanoparticles which enhance the thermal conductivity of base fluid (Asadi et al., 2018). Accordingly, the new classes of fluids have been prepared with two or more nanoparticles, which are known as hybrid nanofluids. Several studies show that hybrid nanoparticles have the ability to improve the heat transfer rates more efficiently with low cost as compared to single nanoparticles.

Accordingly, an experimental study on the thermal properties of  $MWCNT - SiO_2/EG$  was conducted by Esfe et al. (2018a) and they reported that hybrid nanofluid are more efficient and economical compared with nanofluid. Also, Esfe et al. (2018) analyzed the effects of temperature and volume fraction on thermal conductivity of  $SWCNT - Al_2O_3/EG$  hybrid nanofluid. Their result indicated that the highest enhancement of thermal conductivity was 41.2 percent which was realized at the volume fraction of 2.5 percent and temperature of  $50^{\circ}C$ . Mohebbi et al. (2019) considered the heat transfer efficiency in a ribbed channel with cooling/heating parts in the presence of  $MWCNT - Fe_3O_4$  hybrid nanoparticles. Iqbal et al. (2017) done a comparative investigation on the shape effects of nanoparticles in water-based  $SiO_2$  nanofluid and  $MoS_2 - SiO_2$  hybrid nanofluids. They reported that the blade-shape nanoparticles have high thermal rates compared with other hybrid nanofluids (like cylindrical, platelets, bricks) for hybrid nanofluids.

Thermal oils are usually used as a coolant fluid for energy management and heat

transfer applications in most industries and manufacturing sectors. Moreover, thermal oils are the important fluids in coolant and lubrication applications in the internal combustion engines. The main important functions of these oils would be sealing and cleaning the engines, cooling the moving parts, reducing the friction between moving parts, and inhibiting corrosion (Afrand et al., 2016). In addition, viscosity and thermal conductivity of thermal oils are the two important parameters which play an important role. In view of this, few efforts have been given in studying the rheological behavior of oil-based nanofluids. Eftefaghi et al. (2013) considered the thermal conductivity of the *MWCNT*-engine oil nanofluid in a limited range of solid volume fraction and at the temperature of  $20^{\circ}\text{C}$ . Their results showed that the thermal conductivity of the nanofluid rises as the solid volume fraction increased. Asadi et al. (2018) experimentally investigated the effects of temperature and solid volume fraction on the dynamic viscosity and thermal conductivity of *Mg(OH)/MWCNT*-engine oil hybrid nanolubricant. They indicated that the dynamic viscosity of the nanolubricant rises with a growth in solid concentration whereas it declines with an increase in temperature.

In addition, the thermal conductivity of the nanofluid shows an increasing tendency as the solid concentration and temperature enhanced. The effect of temperature and solid volume fraction on dynamic viscosity of *MWCNT/ZnO*-engine oil nanolubricant was experimentally studied by Asadi and Asadi (2016). They indicated that the nanolubricant exhibits Newtonian behavior over the studied range of temperatures ( $5^{\circ}\text{C} - 50^{\circ}\text{C}$ ) and solid concentrations (0.125 -1 %). The heat transfer efficiency of *Al<sub>2</sub>O<sub>3</sub> - MWCNT*/thermal oil hybrid nanofluid over different temperatures ( $25^{\circ}\text{C} - 50^{\circ}\text{C}$ ) and solid concentrations (0.125-1.5 %) was experimentally studied by Asadi et al. (2018). They revealed that the nanofluid showed Newtonian behavior over the studied range of temperatures and solid concentrations. Also, the dynamic viscosity exhibited increasing trend as the solid concentration improved. The thermal

conductivity showed growing tendency as the temperature and solid concentration increased, and the maximum upgrading is approximately by 45 % at the temperature  $50^{\circ}\text{C}$  and solid concentration 1.5 %.

In recent years, the problem of fluid flow over a cone has great demands in science, engineering and industrial processes such as space technology, astrophysics, energy storage systems, aeronautical science, nuclear power management system, geological science, micro-inverter chips, hospitality and health care systems. Owing to these several applications, Ramzan et al. (2019a) studied magnetized suspended carbon nanotubes based nanofluid flow with bio-convection and entropy generation past a vertical cone. Malik et al. (2016) analyzed the impacts of temperature-dependent viscosity and thermal conductivity on the flow and heat transfer characteristics of a viscous fluid over a rotating vertical cone. Their results indicated that the skin-friction coefficients reduces with more values of variable-viscosity parameter while the heat transfer rate drops for increasing values of Eckert number, variable-viscosity and variable thermal conductivity parameters. Sreedevi et al. (2018) investigated the MHD effects on fluid transport properties of chemically reacted water-based *SWCNT/MWCNT* nanofluid over a vertical cone embedded in a porous medium. Palanisamy and Kumar (2019) experimentally explored the heat transfer and pressure drop of helically coiled tube heat exchanger cone using *MWCNT*/water nanofluids. They reported that for the nanofluids volume concentration of 0.1, 0.3 and 0.5 percents, it was respectively found that the pressure drop of nanofluid is 16, 30 and 42 percents, and local heat transfer rate is 28, 52 and 68 percents higher than water. Also, they confirmed that the prepared *MWCNT*/water nanofluids show good stability and improve the heat transfer with considerable pressure drop. Maraj et al. (2018) performed a comprehensive shape factor analysis of  $\text{MoS}_2\text{-SiO}_2/\text{H}_2\text{O}$  inside an isothermal semi vertical inverted cone with porous boundary. They revealed that skin friction coefficient and Nusselt number increase with an increase in nanoparticles volumetric fractions, and

their maximum value is for hybrid nanofluid having with blade shaped nanoparticles whereas minimum value is for brick shaped nanoparticles. Hanif et al. (2020) examined Entropy analysis for unsteady mixed convection in  $Cu - Fe_3O_4$ /water magneto-hybrid nanofluid flow over an inverted cone surrounded by a porous medium, and they witnessed that heat transfer rates improved with increasing of hybrid nanoparticles volume fraction. Also they reported that heat transfer coefficient reaches higher rates in mixed convectional flow compared with natural convection.

Different studies conformed that the metal oxide nanoparticles like  $SiO_2$ ,  $Al_2O_3$ ,  $TiO_2$ ,  $ZnO$ , and  $Fe_3O_4$  adequately dispersed in working fluids (Afrand, 2017). Furthermore,  $Al_2O_3$  nanoparticles own advanced thermal properties among the stated metal oxide nanoparticles. In their nature CNTs nanoparticles have high thermal conductivity, high aspect ratio, and low specific gravity related to metal oxide nanoparticles (Afrand, 2017). Therefore, using the hybrid nanoparticles of  $CNTs$  and  $Al_2O_3$  might lead to produce hybrid nanofluid with special thermal properties which stay stable for a long time. Moreover, viscous dissipation acts like energy source which influences the distributions of temperature, and affects the heat transfer rates. The significance of viscous dissipation be contingent weather the cone is being heated or cold. Thus, this study is committed to explore the effects of variable viscosity and thermal conductivity together with mixed convection and viscous dissipation effects on the boundary layer flow and heat transfer of  $MWCNTs - Al_2O_3$ /engine oil hybrid nanofluid due to vertically inverted spinning cone embedded in a porous medium. From the earlier studies, so far, no research has been considered the hybrid nanofluid flow and heat transfer of  $MWCNTs - Al_2O_3$ /engine oil due to spinning cone. Owing to the limitations of the former works, this study is devoted to investigate the impacts of variable viscosity, thermal conductivity and nanoparticle volume fraction on velocity field and heat transfer. Besides, influences of different shapes of the nanoparticles on hybrid nanofluids flow is compared. The numerical solution is obtained by using

spectral relaxation method (SRM). The effects of governing flow parameter on hybrid nanofluids velocity field and temperature distributions are provided through graphs and tables.

## 9.2 Mathematical Description of the Problem

We consider unsteady incompressible viscous flow of hybrid nanofluids with variable viscosity  $\mu_{hnf}$  and thermal conductivity  $k_{hnf}$  over a downward pointing rotating cone surrounded by a porous medium in stagnant fluids. It is assumed that the origin of the system is located at the cone vertex, and the fluid flows upward with a uniform velocity  $u_0$  when the cone is rotating with time-dependent angular velocity  $\Omega$ . The velocity components  $u, v$  and  $w$  be in the direction of  $x, y$  and  $z$  respectively, where  $x - axis$  is in tangential direction,  $y - axis$  is along with azimuthal direction, and  $z - axis$  is normal to the cone, as shown in the figure 9.1. Also, the buoyancy forces happen due to temperature and concentration differences, and gravity acts in downward direction along cone's axis. Furthermore, heat transfer is incorporated at the surface of a cone in the presence of thermal radiation and viscous dissipation effects. In the beginning, the temperature is assumed to be constant and equal to ambient ( $T_\infty$ ). With the transit of time, the surface temperature of the cone is raised to  $T_w(x)$  and depends on tangential coordinate where  $T_w > T_\infty$ .

With foregoing assumptions, and employing Boussinesq approximation boundary layer theory with thermal radiation and viscous dissipation effects, the transport equations of hybrid nanofluid flow for momentum, and energy transfer are established as follows (Malik et al., 2016; Maraj et al., 2018; Hanif et al., 2020).

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rw)}{\partial z} = 0 \quad (9.2.1)$$

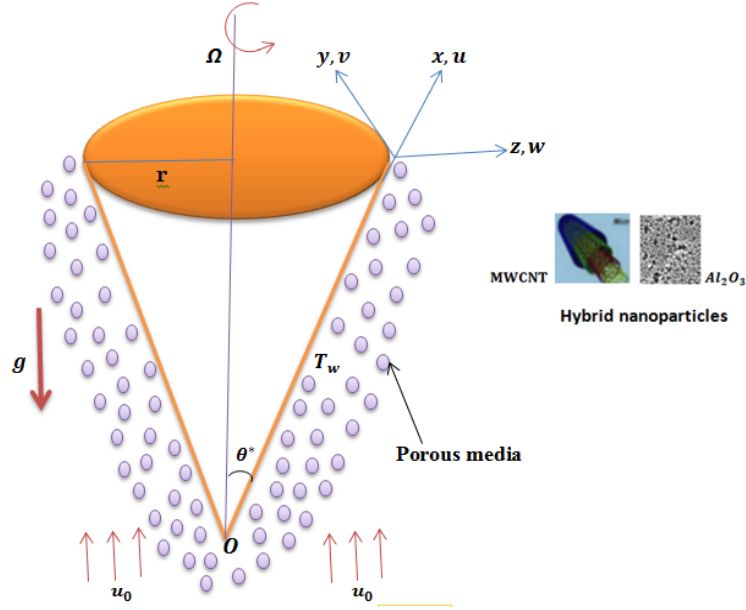


Figure 9.1: Physical model of the problem

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} - \frac{v^2}{x} = \frac{1}{\rho_{hnf}} \frac{\partial}{\partial z} \left( \mu_{hnf} \frac{\partial u}{\partial z} \right) - \frac{\nu_{hnf}}{K} u + g(\beta_t)_{hnf} (T - T_\infty) \cos \theta^* \quad (9.2.2)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + w \frac{\partial v}{\partial z} + \frac{uv}{x} = \frac{1}{\rho_{hnf}} \frac{\partial}{\partial z} \left( \mu_{hnf} \frac{\partial v}{\partial z} \right) - \frac{\nu_{hnf}}{K} v \quad (9.2.3)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial z} = \frac{1}{(\rho c_p)_{hnf}} \frac{\partial}{\partial z} \left( k_{hnf} \frac{\partial T}{\partial z} \right) - \frac{1}{(\rho c_p)_{hnf}} \frac{\partial q_r}{\partial z} + \frac{\mu_{hnf}}{(\rho c_p)_{hnf}} \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right] \quad (9.2.4)$$

where the radius of the cone  $r = x \sin \theta^*$ , the cone vertex angle is  $2\theta^*$ ,  $\rho_{hnf}$ ,  $\mu_{hnf}$ ,  $\nu_{hnf}$ ,  $k_{hnf}$ ,  $(\rho c_p)_{hnf}$ , and  $\beta_{hnf}$  are hybrid nanofluid effective density, dynamic viscosity, kinematic viscosity, thermal conductivity, specific heat capacity, and volumetric thermal expansion coefficient.  $g$ ,  $K$ ,  $q_r$  are acceleration due to gravity, permeability of porous medium, and radiative heat flux respectively.

The initial values and boundary conditions best suite to the physical problem are

(Malik et al., 2016; Hanif et al., 2020).

$$\begin{aligned}
t \leq 0 : \quad & u = 0, \quad v = 0, \quad w = 0, \quad T = T_\infty, \quad \forall x, y, z \\
t > 0 : \quad & u = u_0, \quad v = \frac{1}{1 - \zeta t^*} \Omega x \sin \theta^*, \quad w = 0, \quad T = T_w, \quad \text{at } z = 0. \quad (9.2.5) \\
& u \rightarrow 0, \quad v \rightarrow 0, \quad T \rightarrow T_\infty \quad \text{as } z \rightarrow \infty.
\end{aligned}$$

where  $\zeta$  is unsteadiness parameter,  $t^* = (\Omega \sin \theta^*)t$  denotes dimensionless time.

The temperature-dependent viscosity  $\mu_f$  and thermal conductivity  $k_f$  of the base fluid are given as (Malik et al., 2016).

$$\mu_{bf} = \mu_0(1 - \alpha_v \theta(\xi)), \quad k_{bf} = k_0(1 + \epsilon \theta(\xi)) \quad (9.2.6)$$

where  $\mu_0$ , and  $k_0$  are reference viscosity and thermal conductivity of base fluid respectively,  $\xi$  is dimensionless variable,  $\theta$  is dimensionless temperature,  $\alpha_v = \xi(T_w - T_\infty)$  is variable viscosity parameter (note that,  $\alpha_v < 0$  for gases and  $\alpha_v > 0$  for liquids), and  $\epsilon = -\xi(T_w - T_\infty)$  is variable thermal conductivity parameter.

The radiative heat flux in the  $z$ -direction (normal direction) is very high as compared to that of  $x$ - and  $y$ -directions. In view of Rosseland approximation, the radiative heat flux  $q_r$  is given as (refer chapter 6).

$$q_r = -\frac{16T_\infty^3 \sigma^*}{3k^*} \frac{\partial T}{\partial z} \quad (9.2.7)$$

where  $\sigma^*$  is the Stefan-Boltzmann constant and  $k^*$  is the mean absorption coefficient.

We have considered multi wall carbon nanotubes (*MWCNTs*) and aluminum oxide ( $Al_2O_3$ ) as nanoparticles and Engine oil (EO) as base fluid. Also, the base fluid and the nanoparticles are assumed to be in thermal equilibrium and no slip occurs between them. It worth mentioning that, to produce the targeted hybrid

nanofluid ( $MWCNTs - Al_2O_3/EO$ ), initially, nanoparticles of  $MWCNTs$  scattered with  $Al_2O_3$  and form hybrid nanoparticle, then this hybrid nanoparticle is dispersed into Engine oil. Thus, in this study  $s_1$ ,  $s_2$  and  $bf$  are respectively stand for  $MWCNTs$ ,  $Al_2O_3$  nanoparticles, and base fluid (Engine oil). The thermophysical characteristics such as density, specific heat capacity, thermal conductivity, and volumetric thermal expansion coefficient of nanoparticles and base fluid (Engine oil) at  $25^0c$  are given in Table 9.1 (Asadi et al., 2018).

Table 9.1: Thermophysical properties of Engine oil, MWCNTs, and  $Al_2O_3$

| Physical properties    | Engine oil (EO) | MWCNT | $Al_2O_3$ |
|------------------------|-----------------|-------|-----------|
| $\rho(kg/m^3)$         | 884             | 1600  | 3970      |
| $c_p(J/kgK)$           | 1910            | 796   | 765       |
| $k(W/mK)$              | 0.144           | 3000  | 40        |
| $\beta(10^{-5}K^{-1})$ | 70              | 44    | 0.85      |

The model for nanofluid effective density  $\rho_{nf}$ , specific heat capacity  $(\rho c_p)_{nf}$ , dynamic viscosity  $\mu_{nf}$ , thermal conductivity  $k_{nf}$ , and thermal expansion coefficient  $(\rho\beta_t)_{nf}$  are given in Table 9.2 (Sundar et al., 2017; Maraj et al., 2018).

where  $\phi_{s_1}$  is the MWCNTs nanoparticle volume fraction;  $n_1$ ,  $\rho_{s_1}$ ,  $(\rho c_p)_{s_1}$ ,  $k_{s_1}$ ,  $(\beta_t)_{s_1}$  and  $\sigma_{s_1}$  are respectively shape, density, specific heat capacity, thermal conductivity, dynamic viscosity, and volumetric thermal expansion coefficient of MWCNTs, whereas,  $\rho_{bf}$ ,  $(\rho c_p)_{bf}$ ,  $\mu_{bf}$ , and  $(\beta_t)_{bf}$  are respectively density, specific heat capacity, dynamic viscosity, and volumetric thermal expansion coefficient of the base fluid (Engine oil).

The model for hybrid nanofluid effective density  $\rho_{hnf}$ , specific heat capacity  $(\rho c_p)_{hnf}$ , dynamic viscosity  $\mu_{hnf}$ , thermal conductivity  $k_{hnf}$ , and thermal expansion coefficient  $(\rho\beta_t)_{hnf}$  are given in Table 9.3 (Sundar et al., 2017; Maraj et al., 2018).

where  $\phi_{s_2}$  is the  $Al_2O_3$  nanoparticle volume fraction;  $n_2$ ,  $\rho_{s_2}$ ,  $(\rho c_p)_{s_2}$ ,  $k_{s_2}$ ,  $\sigma_{s_2}$ , and  $(\beta_t)_{s_2}$  are respectively shape, density, specific heat capacity, thermal conductivity,

Table 9.2: Thermophysical Properties Model of Nanofluid

| Thermophysical Properties                  | Nanofluid (MWCNTs-Engine oil)                                                                                                                          |
|--------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------|
| Density ( $kg/m^3$ )                       | $\rho_{nf} = \phi_{s1}\rho_{s1} + (1 - \phi_{s1})\rho_{bf}$                                                                                            |
| Heat capacity ( $J/kgK$ )                  | $(\rho c_p)_{nf} = \phi_{s1}(\rho c_p)_{s1} + (1 - \phi_{s1})(\rho c_p)_{bf}$                                                                          |
| Viscosity ( $Ns/m^2$ )                     | $\mu_{nf} = \frac{\mu_{bf}}{(1 - \phi_{s1})^{2.5}}$                                                                                                    |
| Thermal Conductivity ( $J/K$ )             | $\frac{k_{nf}}{k_{bf}} = \frac{k_{s1} + (n_1 - 1)k_{bf} - (n_1 - 1)\phi_{s1}(k_{bf} - k_{s1})}{k_{s1} + (n_1 - 1)k_{bf} + \phi_{s1}(k_{bf} - k_{s1})}$ |
| Thermal Expansion Coefficient ( $K^{-1}$ ) | $(\beta_t)_{nf} = (1 - \phi_{s1})(\rho\beta_t)_{bf} + \phi_{s1}(\rho\beta_t)_{s1}$                                                                     |

Table 9.3: Thermophysical Properties Model of Hybrid Nanofluid

| Thermophysical Properties         | Hybridnanofluid ( $(MWCNTs - Al_2O_3)/Engine\ oil$ )                                                                                                    |
|-----------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------|
| Density ( $kg/m^3$ )              | $\rho_{hnf} = \phi_{s1}\rho_{s1} + \phi_{s2}\rho_{s2} + (1 - \phi_{s1} - \phi_{s2})\rho_{bf}$                                                           |
| Heat capacity ( $J/kgK$ )         | $(\rho c_p)_{hnf} = \phi_{s1}(\rho c_p)_{s1} + \phi_{s2}(\rho c_p)_{s2} + (1 - \phi_{s1} - \phi_{s2})(\rho c_p)_{bf}$                                   |
| Viscosity ( $Ns/m^2$ )            | $\mu_{hnf} = \frac{\mu_{bf}}{(1 - \phi_{s1} - \phi_{s2})^{2.5}}$                                                                                        |
| Thermal Conductivity ( $J/K$ )    | $\frac{k_{hnf}}{k_{nf}} = \frac{k_{s2} + (n_2 - 1)k_{nf} - (n_2 - 1)\phi_{s2}(k_{nf} - k_{s2})}{k_{s2} + (n_2 - 1)k_{nf} + \phi_{s2}(k_{nf} - k_{s2})}$ |
| Thermal Expa. Coeff. ( $K^{-1}$ ) | $(\beta_t)_{hnf} = (1 - (\phi_{s1} + \phi_{s2}))(\beta_t)_{bf} + \phi_{s1}(\beta_t)_{s1} + \phi_{s2}(\beta_t)_{s2}$                                     |

and volumetric thermal expansion coefficient of  $Al_2O_3$ .

It is suitable to convert the governing partial differential equation into system of non linear ordinary differential equation introducing the following transformations based on (Malik et al., 2016).

$$\begin{aligned}
u &= -\frac{1}{2(1 - \zeta t^*)}\Omega x \sin\theta^* f'(\xi), & v &= \frac{1}{1 - \zeta t^*}\Omega x \sin\theta^* h(\xi) \\
w &= \left(\frac{1}{1 - \zeta t^*}\Omega \nu_f \sin\theta^*\right)^{1/2} f(\xi), & T &= T_\infty + (T_w - T_\infty)\theta(\xi) \\
T_w &= T_\infty + \frac{T_0 - T_\infty}{(1 - \zeta t^*)^2} \frac{x}{L}, & \xi &= \left(\frac{\Omega \sin\theta^*}{\nu_f(1 - \zeta t^*)}\right)^{1/2} z
\end{aligned} \tag{9.2.8}$$

where  $f'(\xi)$  and  $h(\xi)$  are respectively tangential and azimuthal directions dimensionless velocities,  $\theta(\xi)$  is the dimensionless temperature,  $T_0$  is reference temperature, and  $L$  is characteristic length of the cone.

Hence, the continuity equation (9.2.1) is identically satisfied, and the transformed equations for momentum and energy are given as follows.

$$(1 - \alpha_v \theta) f''' - \alpha_v \theta' f'' + \phi_1 \phi_2 \left[ \frac{1}{2} f'^2 - f f'' - 2h^2 \right] - \phi_1 \phi_2 \left[ \zeta(f' + \frac{\xi}{2} f'') + 2\phi_3 \lambda_m \theta \right] - \kappa f' = 0 \quad (9.2.9)$$

$$(1 - \alpha_v \theta) h'' - \alpha_v \theta' h' + \phi_1 \phi_2 \left[ h f' - f h' - \zeta(h + \frac{\xi}{2} h') \right] - \kappa h = 0 \quad (9.2.10)$$

$$\begin{aligned} \frac{1}{Pr} \frac{k_{hnf}}{k_f} (1 + Nr) [(1 + \epsilon \theta) \theta'' + \epsilon \theta'^2] + \phi_4 \left[ \frac{1}{2} f' \theta - f \theta' - \zeta(2\theta + \frac{\xi}{2} \theta') \right] \\ + \frac{1}{\phi_2} (1 - \alpha_v \theta) Ec \left[ \frac{1}{4} f''^2 + h'^2 \right] = 0 \end{aligned} \quad (9.2.11)$$

The non-dimensional form of transformed boundary conditions are:

$$\begin{aligned} f = 0, \quad f'(0) = 0, \quad h(0) = 1, \quad \theta(0) = 1 \\ f'(\infty) \rightarrow 0, \quad h(\infty) \rightarrow 0, \quad \theta(\infty) \rightarrow 0, . \end{aligned} \quad (9.2.12)$$

where the parameters found in the above equations are

$$\begin{aligned} \phi_1 &= (1 - \phi_{s_1} - \phi_{s_2}) + \phi_{s_1} \left( \frac{\rho_{s_1}}{\rho_{bf}} \right) + \phi_{s_2} \left( \frac{\rho_{s_2}}{\rho_{bf}} \right) \\ \phi_2 &= (1 - \phi_{s_1} - \phi_{s_2})^{2.5} \\ \phi_3 &= (1 - \phi_{s_1} - \phi_{s_2}) + \phi_{s_1} \left( \frac{(\beta_t)_{s_1}}{(\beta_t)_{bf}} \right) + \phi_{s_2} \left( \frac{(\beta_t)_{s_2}}{(\beta_t)_{bf}} \right) \\ \phi_4 &= (1 - \phi_{s_1} - \phi_{s_2}) + \phi_{s_1} \left( \frac{(\rho c_p)_{s_1}}{(\rho c_p)_{bf}} \right) + \phi_{s_2} \left( \frac{(\rho c_p)_{s_2}}{(\rho c_p)_{bf}} \right) \\ \kappa &= \frac{\nu_{bf}(1 - \zeta t^*)}{K \Omega x \sin \theta^*} \text{ is permeability parameter, } Ec = \frac{x L (\Omega \sin \theta^*)^2}{(T_0 - T_\infty)(c_p)_{bf}} \text{ is Eckert number, } Nr = \\ &\frac{16 \sigma^* T_\infty^3 L^2}{3 k^* Gr^{(1/2)} \mu_{bf} (\rho c_p)_{bf}} \text{ is thermal radiation parameter, } Pr = \frac{\nu_{bf} (\rho c_p)_{bf}}{k_{bf}} \text{ is a Prandtl num-} \\ &\text{ber, } Gr_L = \frac{g(\beta_t)_{bf} L^3 (T_0 - T_\infty)}{\nu_{bf}^2} \cos \theta^* \text{ is Rayleigh number, } Re_L = \frac{L^2}{\nu_{bf}} \Omega \sin \theta^* \text{ is Reynolds} \\ &\text{number, } \lambda_m = \frac{Gr}{Re_L} \text{ is mixed connection parameter.} \end{aligned}$$

The surface drag and wall heat transfer rates are characterized by the skin friction coefficient  $C_f$  and the local Nusselt number  $Nu$ , respectively. The tangential and azimuthal directions shear stress at the surface of the cone for hybrid nanofluid are defined as (Maraj et al., 2018).

$$\tau_{xz} = \mu_{hnf} \left( \frac{\partial u}{\partial z} \right)_{z=0}, \quad \tau_{yz} = \mu_{hnf} \left( \frac{\partial v}{\partial z} \right)_{z=0} \quad (9.2.13)$$

The physical problem, the tangential and azimuthal directions skin friction coefficients are respectively obtained as (Malik et al., 2016).

$$C_{fx} = \frac{2\tau_{xz}}{\rho_{bf} \left( \frac{x\Omega \sin\theta^*}{1-\zeta t^*} \right)^2}, \quad C_{fy} = \frac{-2\tau_{yz}}{\rho_{bf} \left( \frac{x\Omega \sin\theta^*}{1-\zeta t^*} \right)^2}, \quad (9.2.14)$$

and their dimensionless form are

$$(Re_x)^{1/2} C_{fx} = -\frac{1}{\phi_2} (1 - \alpha_v \theta) f''(0) \quad (9.2.15)$$

and

$$(Re_x)^{1/2} C_{fy} = -\frac{2}{\phi_2} (1 - \alpha_v \theta) h'(0) \quad (9.2.16)$$

The local Nusselt number  $Nu$  (heat transfer rate) for hybrid nanofluid is given as (Maraj et al., 2018).

$$q_w = -\frac{x}{k_{bf}(T_w - T_\infty)} \left[ \left( k_{hnf} \frac{\partial T}{\partial z} + (q_r)_w \right)_{z=0} \right] \quad (9.2.17)$$

and its dimensionless form is

$$(Re_x)^{-1/2} Nu_x = -\left( \frac{k_{hnf}}{k_{bf}} + Nr \right) \theta'(0) \quad (9.2.18)$$

where  $Re_x$  is the local Reynolds number defined by  $Re_x = \frac{\Omega x^2 \sin\theta^*}{\nu_{bf}(1-\zeta t^*)}$

### 9.3 Method of Solutions

The system of non-linear ordinary differential equations with BVP (9.2.9)-(9.2.12) are solved numerically using Spectral Relaxation Method (SRM). The SRM algorithm first decouples the system of equations, and an iteration scheme is established from the decoupled equations by assessing linear terms at the current iteration level  $r + 1$ . To use this method, first, we reduce the order of the equation (9.2.9) by setting  $g = f'$ , and becomes

$$\begin{aligned} (1 - \alpha_v \theta) g'' - \alpha_v \theta' g' + \phi_1 \phi_2 \left[ \frac{1}{2} g^2 - f g' - 2h^2 \right] \\ - \phi_1 \phi_2 \left[ \zeta \left( g + \frac{\xi}{2} g' \right) + 2\phi_3 \lambda_m \theta \right] - \kappa g = 0 \end{aligned} \quad (9.3.1)$$

Employing the concept of the Gauss-Seidel relaxation method, we obtain the following iteration scheme.

$$f'_{r+1} = g_r \quad (9.3.2)$$

$$(1 - \alpha_v \theta_r) g''_{r+1} - \left[ \alpha_v \theta'_r + \phi_1 \phi_2 \left( f_{r+1} + \frac{\xi}{2} \zeta \right) \right] g'_{r+1} - (\phi_1 \phi_2 \zeta + \kappa) g_{r+1} \\ = \phi_1 \phi_2 \left[ \frac{-1}{2} g_r^2 + 2 h_r^2 + 2 \lambda_m \phi_3 \theta_r \right] \quad (9.3.3)$$

$$(1 - \alpha_v \theta_r) h''_{r+1} - \left[ \alpha_v \theta'_r + \phi_1 \phi_2 \left( f_{r+1} + \frac{\xi}{2} \zeta \right) \right] h'_{r+1} + [\phi_1 \phi_2 (g_{r+1} - \zeta) - \kappa] h_{r+1} = 0 \quad (9.3.4)$$

$$\frac{1}{Pr} \frac{k_{hnf}}{k_f} (1 + Nr) (1 + \epsilon \theta_r) \theta''_{r+1} - \phi_4 \left( f_{r+1} + \frac{\xi}{2} \zeta \right) \theta'_{r+1} + \phi_4 \left( \frac{1}{2} g_{r+1} - 2 \zeta \right) \theta_{r+1} \\ = - \frac{1}{Pr} \frac{k_{hnf}}{k_f} (1 + Nr) \epsilon \theta_r'^2 - \frac{Ec}{\phi_2} (1 - \alpha_v \theta_r) \left[ \frac{1}{4} g_{r+1}^2 + h_{r+1}^2 \right] \quad (9.3.5)$$

The non-dimensional form of transformed boundary conditions are:

$$f_{r+1} = 0, \quad g_{r+1}(0) = 0, \quad h_{r+1}(0) = 1, \quad \theta_{r+1}(0) = 1 \\ g_{r+1}(\infty) \rightarrow 0, \quad h_{r+1}(\infty) \rightarrow 0, \quad \theta_{r+1}(\infty) \rightarrow 0, \quad (9.3.6)$$

where the terms  $r$  and  $r+1$  are at the previous and current iteration levels respectively.

A numerical solution is found for (9.3.2)-(9.3.6) using the Chebyshev spectral relaxation method (SRM). The semi-infinite domain is transformed into the interval  $[-1, 1]$  using the linear transformation  $\xi = \frac{1}{2} L_\infty (\chi + 1)$ . Beginning with appropriate initial approximations, the iteration schemes are used iteratively to obtain  $f_{r+1}$ ,  $g_{r+1}$ ,  $h_{r+1}$  and  $\theta_{r+1}$ . Now, the system of equations (9.3.2)-(9.3.5) can be solved as matrix turn by turn as follows:

$$A_1 F_{r+1} = B_{1,r}; \quad f_{r+1}(\chi_N) = 0 \quad (9.3.7)$$

$$A_2 G_{r+1} = B_{2,r}; \quad g_{r+1}(\chi_N) = 0, \quad g_{r+1}(\chi_0) = 0 \quad (9.3.8)$$

$$A_3 H_{r+1} = B_{3,r}; \quad h_{r+1}(\chi_N) = 1, \quad h_{r+1}(\chi_0) = 0 \quad (9.3.9)$$

$$A_4 \Theta_{r+1} = B_{4,r}; \quad \theta_{r+1}(\chi_N) = 1, \quad \theta_{r+1}(\chi_0) = 0 \quad (9.3.10)$$

where

$$A_1 = \mathbf{D}, \quad B_{1,r} = g_r,$$

$$A_2 = (\mathbf{I} - \alpha_v \text{diag}(\theta_r)) \mathbf{D}^2 - \left[ \alpha_v \text{diag}(\theta'_r) + \phi_1 \phi_2 \left( \text{diag}(f_{r+1}) + \frac{\xi(N+1)}{2} \zeta \right) \right] \mathbf{D} \\ - (\phi_1 \phi_2 \zeta + \kappa) \mathbf{I}$$

$$B_{2,r} = \phi_1 \phi_2 \left[ \frac{-1}{2} g_r^2 + 2h_r^2 + 2\lambda_m \phi_3 \theta_j \right]$$

$$A_3 = (\mathbf{I} - \alpha_v \text{diag}(\theta_r)) \mathbf{D}^2 - \left[ \alpha_v \text{diag}(\theta'_r) + \phi_1 \phi_2 \left( \text{diag}(f_{r+1}) + \frac{\xi(N+1)}{2} \zeta \right) \right] \mathbf{D} \\ + \phi_1 \phi_2 (\text{diag}(g_{r+1}) - \zeta \mathbf{I}) - \kappa \mathbf{I}; \quad B_{3,r} = 0,$$

$$A_4 = \frac{1}{Pr} \frac{k_{hnf}}{k_f} (1 + Nr) (\mathbf{I} + \epsilon \text{diag}(\theta_r)) \mathbf{D}^2 - \phi_4 \left( \text{diag}(f_{r+1}) + \frac{\xi(N+1)}{2} \zeta \right) \mathbf{D} \\ + \phi_4 \left( \frac{1}{2} \text{diag}(g_{r+1}) - 2\zeta \mathbf{I} \right)$$

$$B_{4,r} = -\frac{1}{Pr} \frac{k_{hnf}}{k_f} (1 + Nr) \epsilon \theta_r'^2 - \frac{Ec}{\phi_2} (1 - \alpha_v \theta_r) \left[ \frac{1}{4} g_{r+1}'^2 + h_{r+1}'^2 \right]$$

And

$$F_{r+1} = [f_{r+1}(\chi_0), f_{r+1}(\chi_1), \dots, f_{r+1}(\chi_N)]^t,$$

$$G_{r+1} = [g_{r+1}(\chi_0), g_{r+1}(\chi_1), \dots, g_{r+1}(\chi_N)]^t,$$

$$H_{r+1} = [h_{r+1}(\chi_0), h_{r+1}(\chi_1), \dots, h_{r+1}(\chi_N)]^t, \text{ and}$$

$$\Theta_{r+1} = [\theta_{r+1}(\chi_0), \theta_{r+1}(\chi_1), \dots, \theta_{r+1}(\chi_N)]^t$$

are vectors of sizes  $(N+1) \times 1$ , and  $\mathbf{I}$  is the identity matrix of size  $(N+1) \times (N+1)$ .

When SRM is used the choice of an initial guess is very essential. The suitable initial guesses that satisfy the governing boundary conditions of the equations (9.2.9)-(9.2.12) are:

$$f_0(\xi) = 1 - e^{-\xi^2}, \quad h_0(\xi) = e^{-\xi}, \quad \theta_0(\xi) = e^{-\xi} \quad (9.3.11)$$

## 9.4 Results and Discussion

In this section the computational results of the fluid flow velocity and temperature distributions as well as skin friction coefficient and Nusselt number are studied via SRM

Table 9.4: SRM Convergence analysis.

|    | $(Re_x)^{1/2}C_{fx}$ | $\frac{1}{2}(Re_x)^{1/2}C_{fy}$ | $(Re_x)^{-1/2}Nu_x$ |
|----|----------------------|---------------------------------|---------------------|
| 02 | 1.18680784           | 2.62022793                      | 443.29871490        |
| 03 | 1.15420661           | 2.62320299                      | 443.25733564        |
| 05 | 1.15427825           | 2.62320843                      | 443.25737705        |
| 13 | 1.15427826           | 2.62320842                      | 443.25737693        |
| 14 | 1.15427825           | 2.62320843                      | 443.25737703        |
| 15 | 1.15427825           | 2.62320843                      | 443.25737704        |
| 16 | 1.15427825           | 2.62320843                      | 443.25737704        |

for the impacts of more noticeable governing parameters using graphical representations, as shown in Figures 9.2 to 9.18. We considered 0.15 percent (0.0015 volume) of MWCNTs, 0.85 percent (0.0085 volume) of  $Al_2O_3$ , and 99 percent (0.99 volume) of EO for the preparation of  $(MWCNTs - Al_2O_3) - EO$  hybrid nanofluid. The effective thermophysical properties models in use for nanofluid and hybrid nanofluid are respectively employed from Tables 9.2 and 9.3. Similarly, the thermophysical properties of MWCNTs,  $Al_2O_3$ , and base fluid (Engine oil) are obtained from Table 9.1. To attain convergent and stable results the appropriate intervals for the parameters used in this study are  $0.1 \leq \alpha_v \leq 0.9$ ,  $0.1 \leq \lambda_m \leq 0.3$ ,  $1 \leq \kappa \leq 10$ ,  $0 \leq \epsilon \leq 1$ ,  $1 \leq \zeta \leq 10$ ,  $0 \leq Ec \leq 0.01$ ,  $0 \leq Nr \leq 20$ ,  $0 \leq \phi_{s1} \leq 0.002$ , and  $0 \leq \phi_{s2} \leq 0.01$ . Moreover, the convergence analysis is carried out for skin friction coefficients and local Nusselt number, and it depicted that  $16^{th}$  order of approximation is enough for these physical quantities, as shown in Table 9.4. Also, for the validation of the numerical scheme used, the results of skin friction coefficients and local Nusselt number are compared with the previously published results of (Chamkha & Mudhaf, 2005; Saleem & Nadeem, 2015; Malik et al., 2016) for comparable parameters, and it gives a very sound agreement, as depicted in Tables 9.5.

The effects of more significant emerging parameters on both mono and hybrid

Table 9.5: Comparison of skin friction coefficients and local Nusselt number with previous work for  $Pr$  and  $\lambda$ .

| Pr                              | $\lambda_m$ | Saleem | Chamkha | Malik  | Present Result |
|---------------------------------|-------------|--------|---------|--------|----------------|
| $(Re_x)^{1/2}C_{fx}$            |             |        |         |        |                |
| 0.7                             | 0           | 1.0255 | 1.0255  | 1.0253 | 1.0254         |
|                                 | 1           | 2.2010 | 2.2012  | 2.2007 | 2.2008         |
|                                 | 10          | 8.5042 | 8.5041  | 8.5041 | 8.5042         |
| 10                              | 0           | 1.0255 | 1.0256  | 1.0255 | 1.0254         |
|                                 | 1           | 1.5630 | 1.5636  | 1.5626 | 1.5638         |
|                                 | 10          | 5.0820 | 5.0821  | 5.0820 | 5.0819         |
| $\frac{1}{2}(Re_x)^{1/2}C_{fy}$ |             |        |         |        |                |
| 0.7                             | 0           | 0.6154 | 0.6158  | 0.6153 | 0.6158         |
|                                 | 1           | 0.8493 | 0.8496  | 0.8492 | 0.8495         |
|                                 | 10          | 1.3992 | 1.3995  | 1.3990 | 1.3990         |
| 10                              | 0           | 0.6158 | 0.6158  | 0.6157 | 0.6158         |
|                                 | 1           | 0.6835 | 0.6837  | 0.6835 | 0.6838         |
|                                 | 10          | 0.9845 | 0.9840  | 0.9841 | 0.9840         |
| $(Re_x)^{-1/2}Nu_x$             |             |        |         |        |                |
| 0.7                             | 0           | 0.4299 | 0.4299  | 0.4295 | 0.4296         |
|                                 | 1           | 0.6121 | 0.6120  | 0.6121 | 0.6120         |
|                                 | 10          | 1.3992 | 1.0097  | 1.0097 | 1.0098         |
| 10                              | 0           | 1.4111 | 1.4110  | 1.4110 | 1.4110         |
|                                 | 1           | 1.5661 | 1.5662  | 1.5660 | 1.5663         |
|                                 | 10          | 2.3581 | 2.3580  | 2.3581 | 2.3582         |

nanofluids are plotted in Figures 9.2 - 9.11 for the tangential direction velocity  $-f'(\xi)$  and temperature  $\theta(\xi)$  distributions. Unless it is specified, the values for the parameters in use for the entire results to be fixed as  $N = 100, \phi_{s_1} = 0.0015, \phi_{s_2} = 0.0085, \lambda_m = 0.2, \alpha_v = 0.5, Pr = 6450, k = 5, \epsilon = 0.3, \zeta = 2, Ec = 0.01, Nr = 10, n_1 = n_2 = 3$ . It is also found that variation in most of the above embedded parameters have no significant influences on azimuthal direction velocity distribution.

Figure 9.2 depicts the effect of nanoparticle volume fractions  $\phi_{s_1}$  and  $\phi_{s_2}$  on the

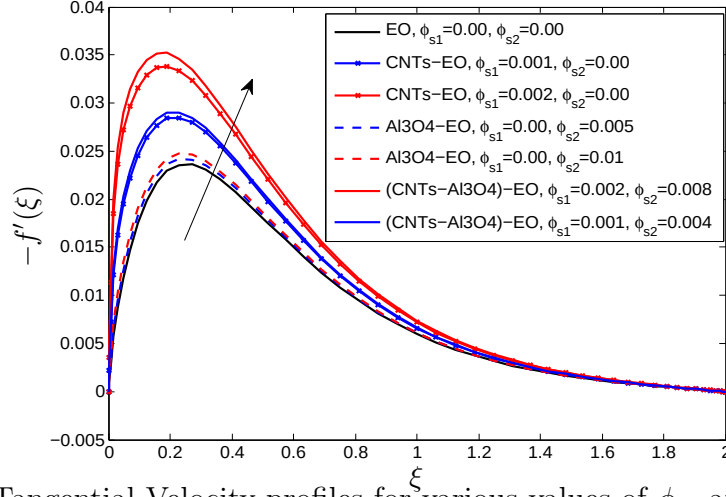


Figure 9.2: Tangential Velocity profiles for various values of  $\phi_{s1}$  and  $\phi_{s2}$ .

tangential velocity of the rotating cone. From the figure, it can be observed that the fluid velocity enhances and it reaches its maximum value, then it turns downward and gradually falls to zero as it far from the surface. Moreover, the velocity profile enhances with an increases in nanoparticle volume fractions. Besides, it can be noted that the fluid flow accelerates more in  $(MWCNTs - Al_2O_3) - EO$  hybrid nanofluid as compare to  $MWCNTs - EO$  and  $Al_2O_3 - EO$  nanofluids. Also, it is revealed that the growth of nanoparticle volume fractions  $\phi_{s1}$  and  $\phi_{s2}$  leads to rise the temperature distribution of hybrid nanofluid, as depicted in Figure 9.3. Figure 9.4 illuminates the influence of the thermal dependent viscosity parameter  $\alpha_v$  on tangential velocity distributions of the fluid flow. A growth in  $\alpha_v$  leads to rise the velocity profile to its highest value, then changes downward and finally it inclines to zero. Also, it is noticed that an improvement in  $\alpha_v$  leads to improve the velocity distribution and this improvement is more in case of hybrid nanofluid than mono nanofluids, as revealed in Figure 9.4. Similar pattern of fluid flow velocity distribution was reported by Malik et al. (2016).

The influence of mixed connection parameter  $\lambda_m$  on velocity profiles is presented

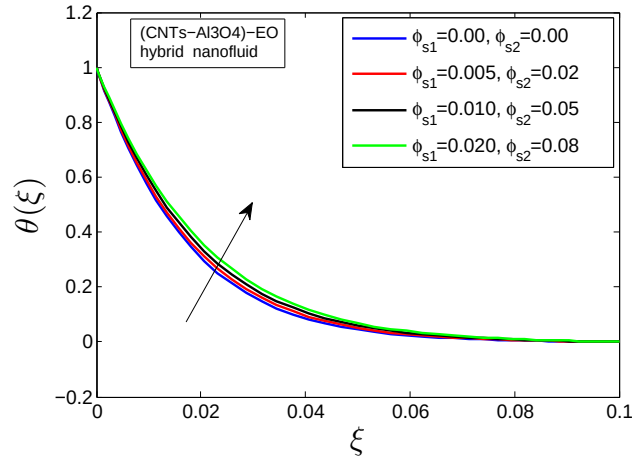


Figure 9.3: Temperature profiles for various values of  $\phi_{s1}$  and  $\phi_{s2}$ .

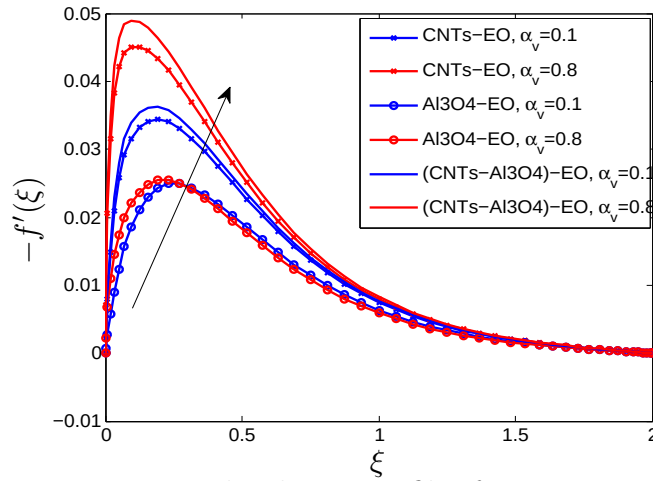


Figure 9.4: Tangential Velocity profiles for various values of  $\alpha_v$ .

in Figure 9.5. An increment in value of  $\lambda_m$  tends to improve the velocity distribution of fluid flow near the boundary surface in which the fluid velocity grows and reaches its peak, and gradually falls to zero. Moreover, the increasing in velocity of hybrid nanofluid is meaningfully more than mono nanofluids. The effect of permeability parameter  $\kappa$  of a porous media on both types of mono and hybrid nanofluids flow is displayed in Figure 9.6. It is shown that the velocity profile declines with an upsurge in  $\kappa$  due to the fact that a porous medium is a region which contains pores that offers resistance to fluid flow. Furthermore, reduction in velocity of hybrid nanofluid

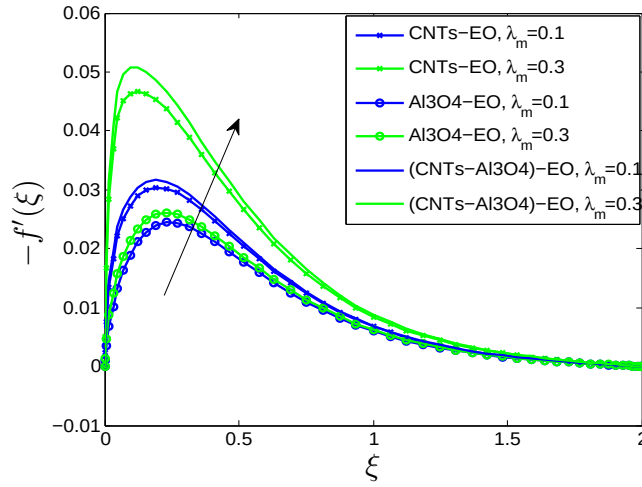


Figure 9.5: Tangential Velocity profiles for various values of  $\lambda_m$ .

is considerably less than mono nanofluids.

The influence of variable thermal conductivity parameter  $\epsilon$  on the velocity and temperature distributions are presented in Figures 9.7 and 9.8 respectively. It is observed that rise in  $\epsilon$  leads to in hastening both velocity and temperature profiles of both nanofluids. Besides, escalation in both velocity and temperature distributions are more in case of hybrid nanofluid than mono nanofluids.

Figures 9.9-9.11 demonstrates the variation of temperature distribution for escalating value of unsteady parameter  $\zeta$ , thermal radiation parameter  $Nr$ , and Eckert number  $Ec$  respectively. As displayed in Figure 9.9, for both mono and hybrid nanofluids the temperature distribution drops with more value of  $\zeta$ . This is due to the fact that during unsteady fluid flow the fluid particles need trembling time to transmit the heat energy to their nearby particles. And, this condition designates the non-conducting manners of the substantial medium which consecutively reduces the temperature distribution in the flow region. From Figure 9.10 it can be shown that for both nanofluids the temperature distributions upswings with an escalation in  $Nr$ . Also, Figure 9.11 confirms that for both mono and hybrid nanofluids temperature distribution rises with an upsurge in Eckert number  $Ec$ . This occurs because

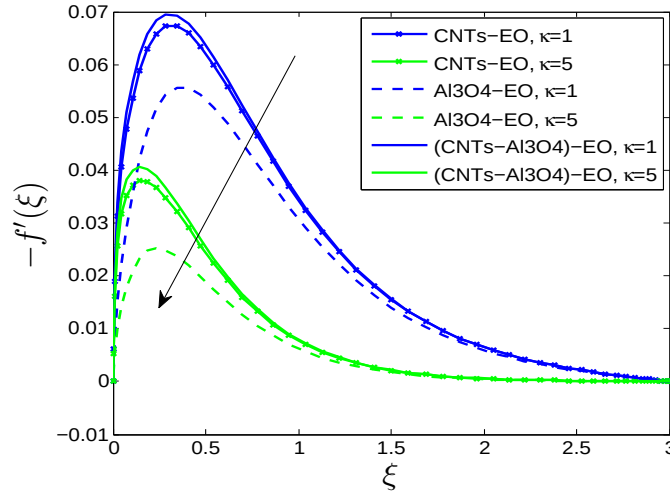


Figure 9.6: Tangential Velocity profiles for various values of  $\kappa$ .

the kinetic energy created due to resistance of viscous forces which described by the viscous dissipative effects in fluid flow leads to rise temperature. Moreover, for both  $Ec$  and  $Nr$  parameters escalation in temperature is more recognized in case of hybrid nanofluid than mono nanofluids, and analogous results are reported by Maraj et al. (2018).

Figures 9.12-9.18 indicate the effects of more visible governing parameters in competition with the unsteady parameter  $\zeta$  on the skin friction coefficients and local Nusselt number. From these figures, it is clear that the tangential skin friction coefficient drops whereas the azimuthal skin friction coefficient and local Nusselt number upswing with unsteady parameter  $\zeta$ . The effect of thermal dependent viscosity parameter  $\alpha_v$  on both tangential and azimuthal skin friction coefficients of the cone are demonstrated in Figures 9.12 and 9.13 respectively. This parameter has a declining effect on both the skin friction coefficients in the flow regime. Moreover, from these figures it can be realized that hybrid nanofluids show significant role in decreasing of the skin friction coefficients than nanofluid.

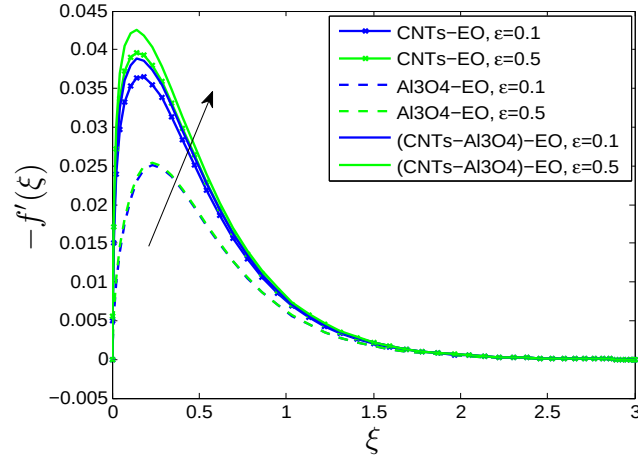


Figure 9.7: Tangential Velocity profiles for various values of  $\epsilon$ .

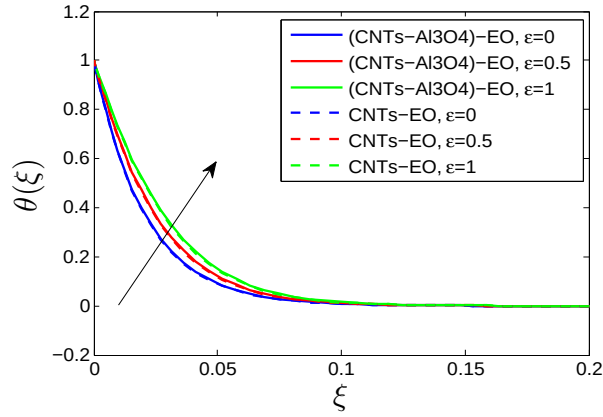


Figure 9.8: Temperature profiles for various values of  $\epsilon$ .

In Figures 9.14-9.15, the influence of the mixed connection parameter  $\lambda_m$  on skin friction coefficients are elaborated. When the parameter  $\lambda_m$  advances, both the tangential and azimuthal skin friction coefficients are growth, and it is evident that the growth of both skin friction coefficients are significantly more for Hybrid nanofluid. The impacts of thermal conductivity parameter  $\epsilon$ , thermal radiation parameter  $Nr$ , and Eckert number  $Ec$  on local Nusselt number are respectively presented in Figures 9.16-9.18. It can be revealed the local Nusselt number is declines with increasing the values of both  $\epsilon$  and  $Ec$  while it is enhances with greater the value of  $Nr$ . Further,

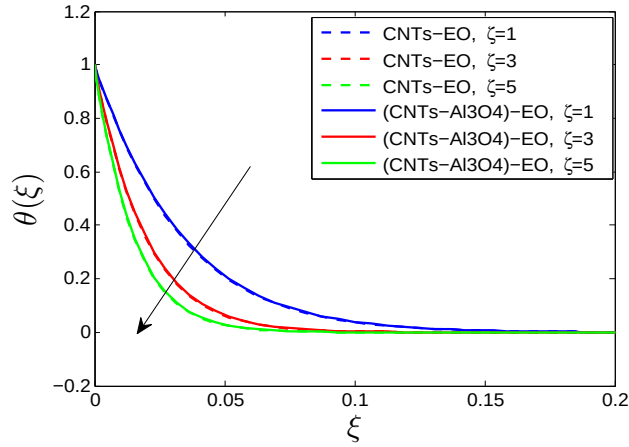


Figure 9.9: Temperature profiles for various values of  $\zeta$ .

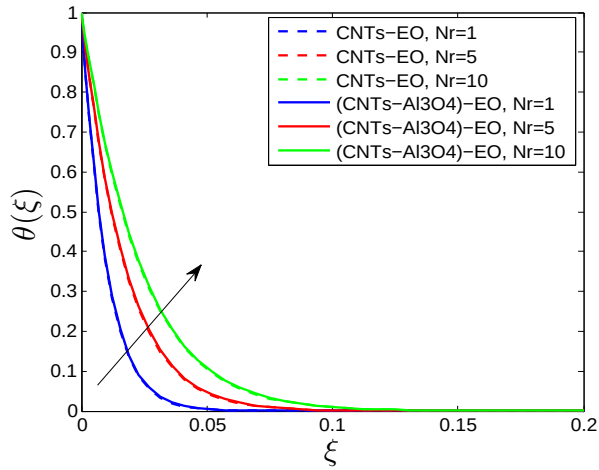


Figure 9.10: Temperature profiles for various values of  $Nr$ .

these responses are significantly high in case of hybrid nanofluid. Also, comparable results were found by Malik et al. (2016) for most of the above governing parameters.

## 9.5 Conclusion

The present work investigated the effects of variable viscosity and thermal conductivity with mixed convection, thermal radiation and viscous dissipation effects on

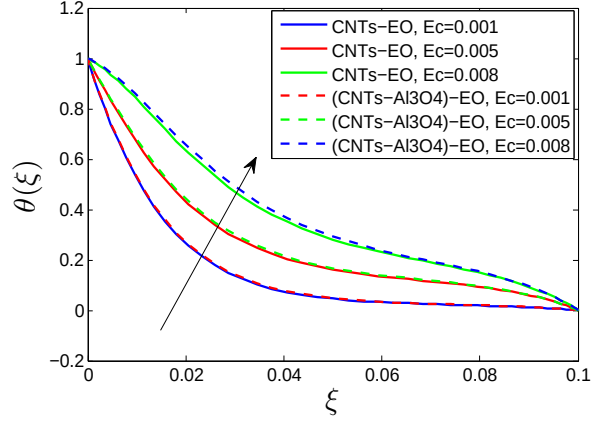


Figure 9.11: Temperature profiles for various values of  $Ec$ .

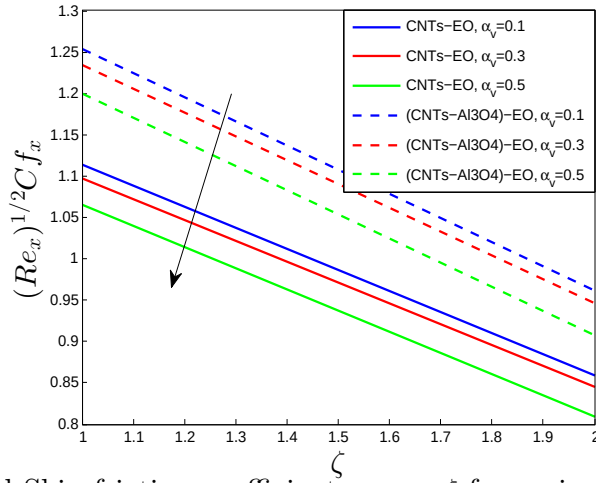


Figure 9.12: Tangential Skin friction coefficient versus  $\zeta$  for various values of  $\alpha_v$ .

*MWCNTs* –  $Al_2O_3$ /engine oil hybrid nanofluid flow due to vertically inverted spinning cone embedded in a porous medium. For the flow problem, the computational solutions are obtained numerically employing the SRM. Thus, from the major results of the study, the following remarks are made:

1. Nanoparticle volume fraction, thermal dependent viscosity, mixed connection and variable thermal conductivity parameters have increasing effect on fluids flow tangential velocity distributions. Moreover, the fluids flow accelerates more in case of hybrid nanofluids than mono nanofluids.

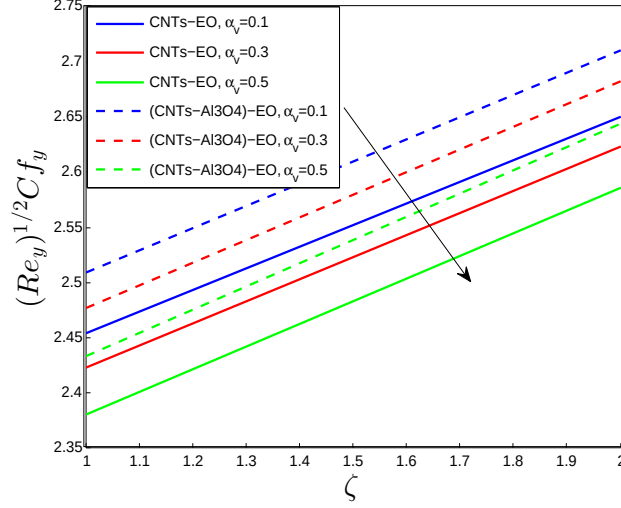


Figure 9.13: Azimuthal Skin friction coefficient versus  $\zeta$  for various values of  $\alpha_v$ .

2. Temperature distribution of the fluids flow improves with greater the values of Eckert number, thermal radiation and variable thermal conductivity parameters. Also, this effect is more recognized in case of hybrid nanofluids than mono nanofluids.
3. With greater unsteady parameter, the tangential skin friction coefficient drops while the azimuthal skin friction coefficient and local Nusselt number enhance.
4. Both the tangential and azimuthal skin friction coefficients growth with more the values of thermal dependent viscosity and mixed connection parameters, and this effect is significantly high for hybrid nanofluids.
5. The local Nusselt number is a decreasing function of both Eckert number and variable thermal conductivity parameter while it is an increasing function of thermal radiation parameter.

In general, from this study, it can be recommended that for practical applications it is better to use hybrid nanofluids of  $(MWCNTs - Al_2O_3)/\text{Engine oil}$  instead of

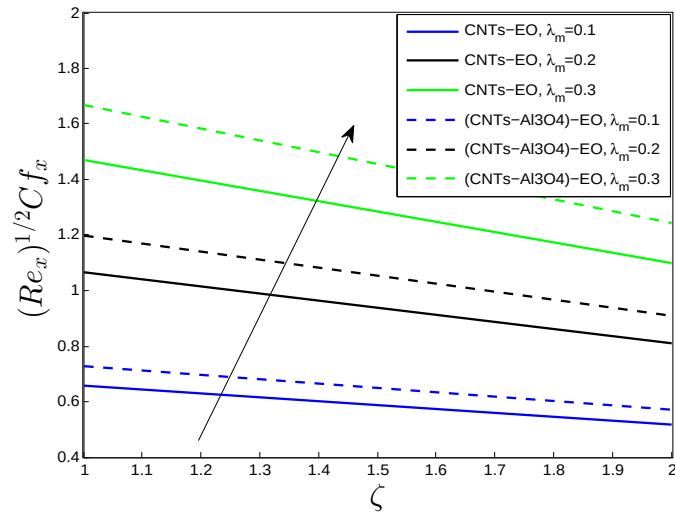


Figure 9.14: Tangential Skin friction coefficient versus  $\zeta$  for various values of  $\lambda_m$ .

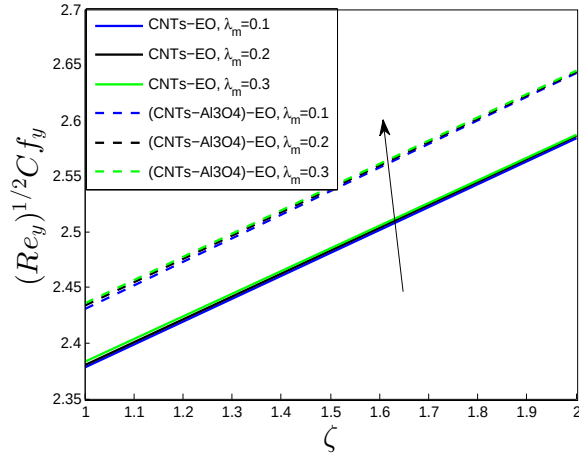


Figure 9.15: Azimuthal Skin friction coefficient versus  $\zeta$  for various values of  $\lambda_m$ .

*MWCNTs*/Engine oil and  $Al_2O_3$ /Engine oil mono nanofluids as the former shows better flow distributions with good stability of thermal properties.

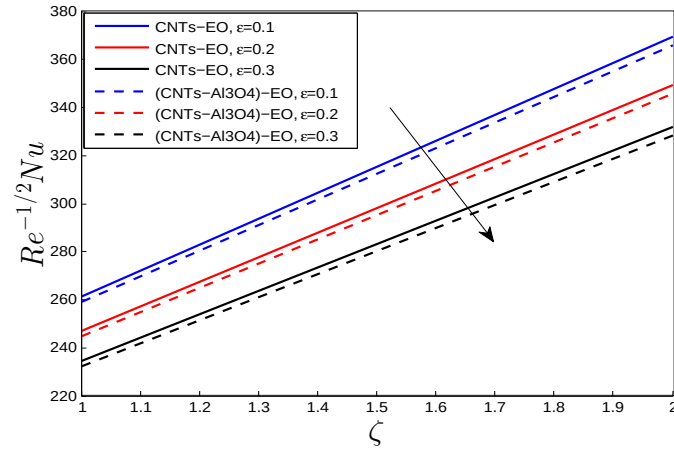


Figure 9.16: Local heat transfer rate versus  $\zeta$  for various values of  $\epsilon$ .

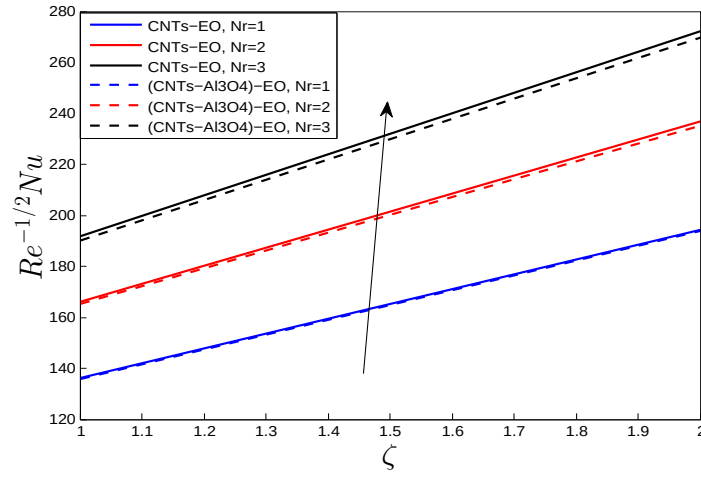


Figure 9.17: Local heat transfer rate versus  $\zeta$  for various values of  $Nr$ .

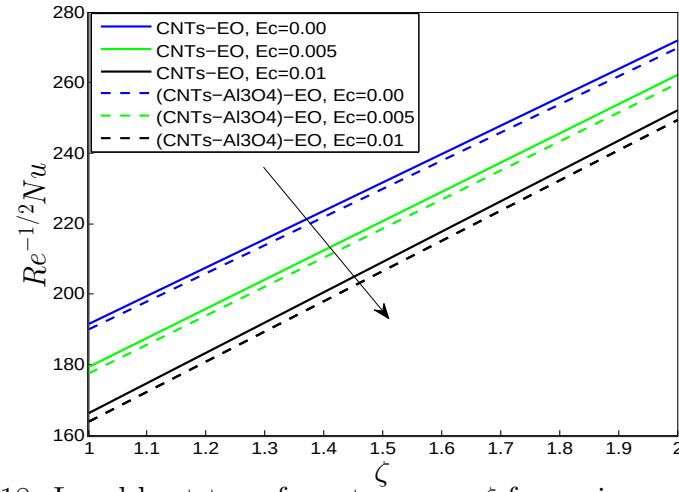


Figure 9.18: Local heat transfer rate versus  $\zeta$  for various values of  $Ec$ .

# Chapter 10

## Conclusion and Recommendation

### 10.1 General Conclusion

In this dissertation, the boundary layer flow of nanofluid past geometrical shape such as wage, cylinder, disc, and cone embedded in porous media have been studied. Some important description of fluid flow such as boundary layer flow, mode of heat and mass transfer, governing equations of fluid flow, basic concepts of nanofluids, and some thermophysical properties affecting fluids flow were highlighted. Also, the basic ideas of spectral collocation method, and the detailed description of the numerical methods used in the study were given. The general observations of the study are summarized below.

From the analysis of the study, it can be concluded that the skin friction coefficient enhances with greater the values of pressure gradient, permeability, magnetic field, unsteadiness, thermal radiation, viscous dissipation, Casson fluid, mixed convection, curvature, temperature dependent viscosity, and second order velocity slip parameters, nanoparticle volume fraction, Soret number, Dufour numbers and Grashof number. The local heat transfer rate is a decreasing function of Eckert number, pressure gradient, magnetic field, permeability, thermophoresis, Casson fluid, unsteadiness,

curvature, viscous dissipation, variable thermal conductivity, thermal relaxation parameters, Soret number, and Dufour numbers. The local mass transfer rate increases with greater the value of unsteadiness, thermal radiation, viscous dissipation parameters, Soret number, and Dufour numbers.

Also, it can be concluded that there are higher velocity fields and lower temperature fields distributions in the MWCNTs nanofluids than SWCNTs nanofluids. The thermal conductivity of hybrid nanofluids is more improved than mono nanofluids for more values of nanoparticle volume fraction. Furthermore, superior the skin friction coefficient and lower the local heat transfer rate are reported in the hybrid nanofluids than mono nanofluids. In general, hybrid nanofluids show better flow distributions with good stability of thermal properties than mono nanofluids, and the obtained results are comparable with earlier experimental results.

As it was observed in the study, the spectral collocation methods are highly accurate, converge faster and computationally efficient when compared to finite differences and others numerical methods. Even though, the dissertation was only limited to fluid flow problems which were solved by spectral collocation methods, these methods can be protracted to other types of differential equations like discrete time equations and difference equations.

In general, from the study it is recommended that for practical applications, it is better to use hybrid nanofluids instead of mono nanofluids as the former exhibits better flow distributions with good stability of thermal properties.

## **10.2 Recommendation for the Future Work**

The dissertation work considered only few hybrid nanofluids flow with heat transfer problems. Yet, there are numerous nanoparticles which are potentially useful in the applications of modern-day nanotechnology and industrial processing. Therefore, Hybrid nanofluid flow with different nanoparticles and base fluids will be further

considered in the future study. Also, due to their progressive flow behavior and their potential uses, turbulent nanofluid flow and multifaced nanofluid flow will be some of the ongoing research area.

# Bibliography

- Abbasbandy, S., Naz, R., Hayat, T. & Alsaedi, A. (2014). Numerical and analytical solutions for Falken-Skan flow of MHD Maxwell fluid. *Applied Mathematics and Computation*, 242, 569-575.
- Abbasi, F.M., Mustefa, M., Shehzad, S.A., Alhuthali, M.S. & Hayat, T. (2016). Analytical study of Cattaneo-Christov heat flux model for a boundary layer flow of Oldroyd-B fluid. *Chin Phys. B.*, 25(1),doi:10.1088/1674-1056/25/1/014701.
- Afrand, M. (2017). Experimental study on thermal conductivity of ethylene glycol containing hybrid nano-additives and development of a new correlation. *Appl. Therm. Eng.*, 110, 1111-1119.
- Afrand, M., Najafabadi, K.N. & Akbari, M. (2016). Effects of temperature and solid volume fraction on viscosity of SiO<sub>2</sub>-MWCNTs/SAE40 hybrid nanofluid as a coolant and lubricant in heat engines. *Appl. Therm. Eng.*, 102, 45-54.
- Ahmad, L., Khan, M. & Khan, W.A. (2017). Numerical investigation of magneto-nanoparticles for unsteady 3D generalized Newtonian liquid flow. *Eur. Phys. J. Plus*, 132, 373.
- Akbar, N.S., Khalique, C.M. & Khan, Z.H. (2017). Cattaneo-Christov heat flux model study for water-based CNT suspended nanofluid past a stretching surface. *INTECH open science open minds, Nanofluid heat and mass trans. in Eng. problems*, <http://dx.doi.org/10.5772/65628>.
- Alam, M.S., Ali, M., Alim, M.A., Munshi, M.J. & Chowdhur, M.Z. (2017). Solution of Falkner- Skan unsteady MHD boundary layer flow and heat transfer past a moving porous wedge in a nanofluid. *Science Direct, Procedia Engineering*, 194, 414-420.
- Ali, N., Teixeira, J.A. & Addali, A. (2018). A review on nanofluids: fabrication, stability, and thermophysical properties. *Journal of Nanomaterials*, <https://>

doi.org/10.1155/2018/6978130.

- Alsagri, A.S., Nasir, S., Gul, T., Nisar, K.S., Islam, S., Shah, Z. & Khan, I. (2019). MHD thin film flow and thermal analysis of blood with CNTs nanofluid. *Coating*, 2019.
- Amirson, N.A., Uddin, M.J., Faisal, Md., Basir, Md., Ismail, A.I.M. & Bg, O.A. (2019). Three-dimensional bioconvection nanofluid flow from a bi-axial stretching sheet with anisotropic slip. *Sains Malaysiana*, 48(5), 1137-1149.
- Arthur, E.M. & Seini, I.Y. (2014). Hydromagnetic stagnation point flow over a porous stretching surface in the presence of radiation and viscous dissipation. *Applied and Computational Mathematics*, 3(5), 191-196.
- Arthur, E.M., Ayando, T. & Seini, Y.I. (2015). MHD convective boundary layer flow towards a vertical surface in a porous medium with radiation, chemical reaction and internal heat generation. *Frontiers in Heat and Mass Transfer (FHMT)*, 6, 21.
- Asadi, A., Asadi, M., Rezaniakolaei, A., Rosendahl, L.A. & Wongwises, S. (2018). An experimental and theoretical investigation on heat transfer capability of Mg (OH) /MWCNT-engine oil hybrid nano-lubricant adopted as a coolant and lubricant fluid. *Applied Thermal Engineering*, 129, 557-586.
- Asadi, A., Asadi, M., Rezaniakolaei, A., Rosendahl, L.A. Afrand, M. & Wongwises, S. (2018). Heat transfer efficiency of Al<sub>2</sub>O<sub>3</sub>-MWCNT/thermal oil hybrid nanofluid as a cooling fluid in thermal and energy management applications: An experimental and theoretical investigation. *International Journal of Heat and Mass Transfer*, 117, 474-486.
- Asadi, M. & Asadi, A. (2016). Dynamic viscosity of MWCNT/ZnO-engine oil hybrid nanofluid: An experimental investigation and new correlation in different temperatures and solid concentrations. *International Communications in Heat and Mass Transfer*, 76, 41-45.
- Asadi, M., Asadi, A. & Aberoumand, S. (2018). An experimental and theoretical investigation on the effects of adding hybrid nanoparticles on heat transfer efficiency and pumping power of an oil-based nanofluid as a coolant fluid. *Int. J. Refrig.*, 89, 83-92.
- Ashwini, G. & Eswara, A.T. (2012). MHD Falkner-Skan boundary layer flow with internal heat generation or absorption. *International Journal of Mathematical and Computational Sciences*, 6(5), 556-559.
- Aziz, A., Alsaedi, A., Muhammad, T. & Hayat, T. (2018). Numerical study for

- heat generation/absorption in flow of nanofluid by a rotating disk. *Results in Physics*, 8, 785-792.
- Azizian, R., Doroodchi, E., McKrell, T., Buongiorno, J., Hu, L.W., & Moghtaderi, B. (2013). Effect of magnetic field on laminar convective heat transfer of magnetite nanofluids. *Int. J. Heat Mass*, 68, 94-109.doi:10.1016/j.ijheatmasstransfer.2013.09.011
- Babolian, E., Bromilow, M., England, R. & Sarav, M. (2007). A modification of pseudospectral method for solving linear ODEs with singularity. *Applied mathematics and computation*, 188(2), 1260-1266.
- Baltensperger, R. & Berrut, J.P. (1999). The errors in calculating the pseudospectral differentiation matrices for chebyshev-gauss-lobatto points. *Computers and Mathematics with Applications*, 37(1), 41-48.
- Bellman, R.E., & Kalaba, R.E. (1965). *Quasilinearization and nonlinear boundary-value problems*. American Elsevier, New York, USA.
- Bergman, T.L., Lavine, A.I., Incropera, F.L., & Dawitt, D.P. (2011). *Fandamental of Heat and Mass Transfer*, 7<sup>th</sup> Edition. R.R. Donnelley, Jefferson, USA. Boyd, J. P. (2001). *Chebyshev and Fourier spectral methods*. Courier Dover Publications.
- Buongiorno, J. (2006). Convective transport in nano fluids. *Journal of Heat Transfer*, 128, 240-250.
- Cai, D., Mataraza, J.M., Qin, Z-H., et al. (2005). Highly efficient molecular delivery into mammalian cells using carbon nanotube spearing. *Nat Meth*, 2, 449.
- Casson, N. (1959). A flow equation for pigment-oil suspensions of the printing ink type. *In. Mill C.C.*, 84-104.
- Cattaneo, C. (1948). *Sulla conuzionedelcalore*. Attisemin. Mat. Fis. Univ. Modena Reggio Emilia, 3, 83-101.
- Cengel, Y.A., & Ghajar, A.J. (2015). *Heat and Mass Transfer: Fundamentals and Applications*, 5<sup>th</sup> Edition. Published by Mcgraw-Hill Education, 2 Penn Plaza, New York.
- Chamkha, A.J. & Al-Mudhaf, A. (2005). Unsteady heat and mass transfer from a rotating vertical cone with a magnetic field and heat generation or absorption effects. *Int. J. Therm. Sci.*,44(3), 267-276.
- Chamkha, A.J., Mujtaba, M., Quadri, A. & Issa, C. (2003). Thermal radiation

- effects on MHD forced convection flow adjacent to a non-isothermal wedge in the presence of a heat source or sink. *Heat Mass Transfer*, 39, 305-312.
- Chamkha. A., Rashad. A., EL-Zahar, E. & EL-Mky, H.A. (2019). Analytical and numerical investigation of Fe<sub>3</sub>O<sub>4</sub>-water nanofluid flow over a moveable plane in a parallel stream with high suction. *Energies*, 12(1), 198.
- Choi, S.U.S., & Eastman, J.A. (1995). *Enhancing Thermal Conductivity of Fluids with Nanoparticles*. Argonne National Lab.
- Christov, C.I. (2009). On frame in different formulation of the Maxwell-Cattaneo model of finite speed heat conduction. *Mechical Research Communication*, 36, 481-486.
- Dawar, A., Shah, Z., Islam, S., Idress, M. & Khan, W. (2018). Magnetohydrodynamic CNTs Casson nanofluid and radiative heat transfer in a rotating channels. *J Phys Res Appl.*, 1, 017-032.
- Dey, D., Kumar, P. & Samantaray, S. (2017). A review of nanofluid preparation, stability, and thermo-physical properties. *Heat Transfer-Asian Research*, 46 (8),1413-1442.
- Ding, Y., Alias, H., Wen, D. & Williams, R.A. (2006). Heat transfer of aqueous suspensions of carbon nanotubes (CNT) nanofluids. *Int. J. Heat Mass Trans*, 49(2), 240-250.
- Eckert, E.R.G. (1942). Die berechnungs/des varmeuberganges in der laminaren grenzschicht um stomter korper. *VDI-Forschungsheft*, 416:1-24.
- Edward, J. Shaughnessy, Jr., Ira, M.K., & James, P.S. (2005). *Introduction to Fluid Mechanics*. New York Oxford, Oxford UNIVERSITY PRESS. Einstein, A., Eine neue Bestimmung der Molekldimensionen. *Annalen der Physik*, 324(2) 289-306.
- El-Dabe, N.T., Ghaly, A.Y., Rizkallah, R.R., Ewis, K.M. & Al-Bareda, A.S. (2015). Numerical solution of MHD boundary layer flow of non-newtonian casson fluid on a moving wedge with heat and mass transfer and induced magnetic field. *Journal of Applied Mathematics and Physics*, 3, 649-663.
- El-Kabeir, S.M.M. & El-Sayed, E.A. (2012). Effects of thermal radiation and viscous dissipation on MHD viscoelastic free convection past a vertical isothermal cone surface with chemical reaction. *International Journal of Energy and Technology*, 4, 1-7.
- Esfe, M.H., Rejvani, M., Karimpour, R. Arani, A.A. (2016). Estimation of thermal conductivity of ethylene glycol-based nanofluid with hybrid suspensio-

- ns of SWCNT-Al<sub>2</sub>O<sub>3</sub> nanoparticles by correlation and ANN methods using experimental data. *J Therm Anal Calorim*, DOI 10.1007/s10973-016-6002-9.
- Esfe, M.H., Esfandeh, S. & Rejvani, M. (2018). Modeling of thermal conductivity of MWCNT-SiO<sub>2</sub> (30: 70 percent)/EG hybrid nanofluid, sensitivity analyzing and cost performance for industrial applications. *J Therm Anal Calorim*, 131(2), 1437-1447.
- Ettefaghi, E., Ahmadi, H., Rashidi, A., Nouralishahi, A. & Mohtasebi, S.S. (2013). Preparation and thermal properties of oil-based nanofluid from multi-walled carbon nanotubes and engine oil as nano-lubricant. *Int. Commun. Heat Mass Transf.*, 46(8), 142-147.
- Falkner, V.M. & Skan, S.W. (1931). Some approximate solutions of the boundary layer equations. *Philos. Mag.*, 12, 865-896.
- Gangadhar, K., Ramana, K.V., Makinde, O.D. & Kuma, B.R. (2018). MHD flow of a Carreau fluid past a stretching cylinder with Cattaneo-Christov heat flux using spectral relaxation method. *Def. Diffu. Forum Subm.*, 387, 91-105.
- Gheorghiu, C. I.(2007). *Spectral methods for differential problems*. Casa Cartiide Stiinta Publishing House, Cluj-Napoca.
- Gottlieb, D. & Hesthaven, J.S. (2001). Spectral methods for hyperbolic problems. *Journal of Computation and Applied Mathematics*, 128, 83-131.
- Gottlieb, S. & Gottlieb, G. (2009). *Spectral methods*. Scholarpedia, 4 (9), 7504, Revision 91796.
- Gupta, N.K., Tiwari, A.K. & Ghosh, S.K. (2018). Heat transfer mechanisms in heat pipes using nanofluids. *A review, Experimental Thermal and Fluid Science*, 90, 84-100.
- Haile, E. & Shankar, B. (2015). Boundary-layer flow of nanofluids over a moving surface in the presence of thermal radiation, viscous dissipation and chemical reaction. *Journal of Applied Mathematics*, 10(2), 952-969.
- Halim, N., Haq, R.U. & Noor, N. (2016). Active and passive controls of nanoparticles in Maxwell stagnation point flow over a slipped stretched surface. *Meccanica*, 1-13.
- Hanif, H., Khan, I. & Shafie, S. (2020). Heat transfer exaggeration and entropy analysis in magneto-hybrid nanofluid flow over a vertical cone: a numerical study. *Journal of Thermal Analysis and Calorimetry*, <https://doi.org/10.1007>

/ s10973-020-09256-z.

- Hartree, D.R. (1937). On an equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer. *Procedia Cambridge Philos. Soc.*, 33, 223-239.
- Hayat, T., Muhammad, T., Shehzad, S.A. & Alsaedi, A. (2017). On magnetohydrodynamic flow of nanofluid due to a rotating disk with slip effect, a numerical study. *Comput Methods Appl Mech Eng.*, 315, 467-477.
- Hayat, T., Anwar, M.S., Farooq, M. & Alsaedi, A. (2014). MHD stagnation point flow of second grade fluid over a stretching cylinder with heat and mass transfer. *International Journal of Nonlinear Science and Numerical Simulation*, 15(6), 365-376.
- Hayat, T., Kiran, A., Imtiaz, M. & Alsaedi, A. (2017). Unsteady flow of carbon nanotubes with chemical reaction and Cattaneo-Christov heat flux model. *Results in Physics*, 7, 823-831.
- Hayat, T., Muhammad, T., Alsaedi, A. & Mustafa, A. (2016). A comparative study for flow of viscoelastic fluids with Cattaneo-Christov heat flux. *PLoS ONE*, 11(5), e0155185.
- Hayat, T., Rashid, M. & Alsaedi, A. (2017). MHD convective flow of magnetite-Fe<sub>3</sub>O<sub>4</sub> nanoparticles by curved stretching sheet. *Results in Physics*, 7, 3107-3115.
- Hirlekar, R., Yamagar, M., Garse, H., Vij, M. & Kadam V. (2009). Carbon nanotubes and its applications: a review. *Asian Journal of Pharmaceutical and Clinical Research*, 2(4), 17-27.
- Ibrahim, W. & Hindebu, B. (2019). MHD boundary layer flow of Eyring-Powell nanofluid past stretching cylinder with Cattaneo-Christov heat flux model. *Nonlinear Engineering*, 8, 303-317.
- Ibrahim, W. & Shanker, B. (2014). Unsteady MHD mixed convective boundary-layer slip flow and heat transfer with thermal radiation and viscous dissipation. *Heat Transfer-Asian Research*, 43 (5).
- Ibrahim, W. (2017). Magnetohydrodynamics (MHD) flow of a tangent hyperbolic fluid with nanoparticles past a stretching sheet with second order slip and convective boundary condition. *Results in Physics*, 2017.
- Ibrahim, W. & Tulu, A. (2019). Magnetohydrodynamic (MHD) boundary layer flow past a wedge with heat transfer and viscous effects of nanofluid embedded in porous media. *Mathematical Problems in Engineering*, <https://doi.org>

/10.1155/2019/4507852.

- Imtiaz, M., Hayat, T., Alsaedi, A. & Ahmad, B. (2016). Convective flow of carbon nanotubes between rotating stretchable disks with thermal radiation effects. *Int. J. Heat Mass Transfer.*, 101, 948-957.
- Iqbal, Z., Maraj, E., Azhar, E. & Mehmood, Z. (2017). A novel development of hybrid (MoS<sub>2</sub>-SiO<sub>2</sub>/H<sub>2</sub>O) nanofluidic curvilinear transport and consequences for effectiveness of shape factors. *J Taiwan Inst Chem Eng.*, 81, 150-158.
- Ittedi, Ramya, D. & Joga, S. (2017). MHD heat transfer of nanofluids over a stretching sheet with slip effects and chemical reaction. *In. J. of Latest Eng. Rese. and Appl.*, 2(8), 10-20.
- Jabbari, F., Rajabpour, A., & Saedodin, S. (2019). Viscosity of carbon nanotube/water nanofluid. *J Therm Anal Calorim*, 135(3), 1787-96.
- Juang, H.M.H. & Kanamitsu, M. (1994). The nmc nested regional spectral model. *Monthly Weather Review*, 122(1), 3-26.
- Kandasamy, R., Muhaimin, I., Khamis, A.B. & Roslan, R.B. (2013). Unsteady Hiemenz flow of Cu-nanofluid over a porous wedge in the presence of thermal stratification due to solar energy radiation. Lie group transformation.-*International Journal of Thermal Science*, 65, 196-205.
- Khalid, A., Khan, I., Khan, A., Shafe, S. & Tlili, I. (2018). Case study of MHD blood flow in a porous medium with CNTs and thermal analysis. *Case Studies in Thermal Engineering*, 12, 374-380.
- Khan, A., Mustefa, M., Hayat, T., & Alsaedi, A. (2015). Numerical study of Cattaneo Christov heat flux model MHD for viscoelastic flow due to an exponentially stretching surface. *PLOS ONE*, 10(9), doi:10.1371/journal.pone.0137363.
- Khan, A., Shah, Z., Islam, S., Dawar, A., Bonyah, E., Ullah, H. & Khan, A. (2018). Darcy-Forchheimer flow of MHD CNTs nanofluid radiative thermal behavior and convective non uniform heat source/sink in the rotating frame with microstructure and inertial characteristics. *AIP Advances*, 8, Article ID 125024.
- Khan, M. Ahmad, L. & Khan, W.A. (2017). Numerically framing the impact of radiation on magneto nanoparticles for 3D Sisko fluid flow. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, 39(11), 4475-4487.
- Khan, M., Irfan, M., Khan, W.A. & Ahmad, L. (2017). Modeling and simulation

- for 3D magneto Eyring-Powell nanomaterial subject to nonlinear thermal radiation and convective heating. *Results in Physics*, 7, 1899-1906.
- Khan, M.S., Karim, I., Islam, M.S. & Wahiduzzaman, M. (0000). MHD boundary layer radiative, heat generating and chemical reacting flow past a wedge moving in a nanofluid. *Nano Convergence*, 1, 20-28.
- Khan, W.A. & Pop, I. (2010). Boundary layer flow of a nanofluid past a stretching sheet. *Internatinal Journal of Heat and Mass Transfer*, 53, 2477-2483.
- Khanafer, K. & Vafai, K. (2018). A review on the applications of nanofluids in solar energy field. *Renew Energy*, 123, 398-406.
- Klaus G., (2017). *Boundary Layer Thoery, 9 th Edition*. Springer VerlagBerlin Heidelberg, Bochum, Nordrhen-Westfalen, Germany.
- Kumaresan, V., Velraj, R. & Das,S.K. (2012). Convective heat transfer characteristics of secondary refrigerant based CNT nanofluids in a tubular heat exchanger. *Int. J. Refrig*, 35(8), 2287-2296.
- Kundu, P.K., Chakraborty, T. & Das, K. (2018). Framing the Cattaneo-Christov heat flux phenomena on CNT- based maxwell nanofluid along stretching sheet with sultiple slips. *Arab Journal of Science and Engineering*, 43, 1177-1188.
- Kuttan, B.A., Manjunatha, S., Jayanthi, S., Gireesha, B.J. & Archana, M. (2019). Effect of variable viscosity on Marangoni convective boundary layer flow of nanofluid in the presence of mixed convection. *Journal of Nanofluids*, 8(4), 845-851.
- Lijima, S. (1991). Helical microtubules of graphitic carbon. *Nature*, 354, 56-58.
- Liu, M.S., Lin, M.C.C., I-Te, H. & Wang, C.C. (2005). Enhancement of thermal conductivity with carbon nanotube for nanofluids. *Int. Commun. Heat Mass Transf*, 32, 1202-1210.
- Mahanthesh, B., Gireesha, B.J., Animasaun, I.L., Muhammad T. & Shashikumar, N.S. (2019). MHD flow of SWCNT and MWCNT nanoliquids past a rotating stretchable disk with thermal and exponential space dependent heat source. *Phys. Scr.*, 94, 11.
- Mahanthesh, B., Gireesha, B.J., Shashikumar, S. & Shehzad, S.A. (2017). Marangoni convective MHD flow of SWCNT and MWCNT nanoliquids due to a disk with solar radiation and irregular heat source. *Physics E*, 94, 25-30.
- Mahanthesh, B., Shehzad, S.A., Gireesha, B.J., Shashikumar, N.S. & Madhu,

- M.(2019). Brinkman-Forchheimer slips flow subject to exponential space and thermal dependent heat source in a microchannel utilizing SWCNT and MWCNT nanoliquids. *Heat Transfer-Asian Research*, 48, 1688-1708.
- Majety, S.S. & Gangadhar, K. (2016). Viscous dissipation effects on radiative MHD boundary layer flow of nano fluid past a wedge through porous medium with chemical reaction. *IOSR Journal of Mathematics*, 12(5), 71-81.
- Makanda, G. (2015). Numerical study of convective fluid flow in porous and non-porous media. (Unpublished PhD Thesis), University of KwaZulu-Natal, Pietermaritzburg.
- Makanda, G., Shaw, S. & Sibanda, P. (2015b). Effects of radiation on MHD free convection of Casson fluid from a horizontal circular cylinder with partial slip in non-Darcy porous medium with viscous dissipation. *Boundary Value Problems*, 9(1), 75.
- Makanda, G., Shaw, S. & Sibanda, P. (2015a) Diffusion of chemically reactive species in Casson fluid over an unsteady stretching surface in porous medium with viscous dissipation. *Mathematical Problems in Engineering*, [https://doi:10.1155/2014/724596](https://doi.org/10.1155/2014/724596)
- Makinde, O.D., Nagendramma, V., Raju, C.S.K. & Leelarathnam, A. (2017). Effects of Cattaneo-Christov heat flux on Casson nanofluid flow past a stretching cylinder. *Def. Diffu. Forum Subm.*, 378, 28-38.
- Malik, M.Y., Jamil, H., Salahuddin, T., Bilal, S., Rehman, K.U. & Z. Mustafa. (2016). Mixed convection dissipative viscous fluid flow over a rotating cone by way of variable viscosity and thermal conductivity. *Results in Physics*, 6, 1126-1135.
- Malik, M.Y., Naseer, M., Nadeem, S. & Rehman, A.A. (2014). The boundary layer flow of Casson nanofluid over a vertical exponentially stretching cylinder. *Applied Nanoscience*, 4(7), 869-873.
- Malik, R., Khan, M., Shafiq, A., Mushtaq, M. & Hussain, M. (2017). An analysis of CattaneoChristov doublediffusion model for Sisko fluid flow with velocity slip. *Result in Physics*, 7, 1232-1237.
- Manjunatha, S., Kuttan, B.A., Jayanthi, S., Chamkha, A. & Giresha B.J. (2019). Heat transfer enhancement in the boundary layer flow of hybrid nanofluids due to variable viscosity and natural convection. *Heliyon*, 5(e01469).
- Mantzaris, N.V., Daoutidis, P., & Srienc, F. (2001). Numerical solution of multi-variable cell population balance models, Spectral methods. *Computers and*

*Chemical Engineering*, 25(11), 1441-1462.

- Maraj, E.N., Iqbal, Z., Azhar, E. & Mehmood, Z. (2018). A comprehensive shape factor analysis using transportation of MoS<sub>2</sub>-SiO<sub>2</sub>/H<sub>2</sub>O inside an isothermal semi vertical inverted cone with porous boundary. *Results in Physics*, 8, 633-641.
- Masotti, A. & Caporali, A. (2013). Preparation of Magnetic Carbon Nanotubes (Mag-CNTs) for Biomedical and Biotechnological Applications. *International Journal of Molecular Sciences*, 14, 24619-24642.
- Mody, V.V., Cox, A., Shah, S., et al. (2014). Magnetic nanoparticle drug delivery systems for targeting tumor. *Appl Nanosci*, 4, 385-392.
- Mohebbi, R., Izadi, M., Delouei, A.A. & Sajjadi, H. (2019). Effect of MWCNT-Fe<sub>3</sub>O<sub>4</sub>/water hybrid nanofluid on the thermal performance of ribbed channel with apart sections of heating and cooling. *J Therm Anal Calorim.*, 135(6):3029-3042.
- Momin, G.G. (2013). Experimental investigation of mixed convection with water Al<sub>2</sub>O<sub>3</sub> and hybrid nanofluid in inclined tube for laminar flow. *IJSTR*, 2, 195-202.
- Motsa, S.S. & Makukula, Z.G. (2013). On spectral relaxation method approach for steady von Karman flow of a Reiner-Rivlin fluid with Joule heating and viscous dissipation. *Cen. Eur. J. Phys.*, 11, 363-374.
- Motsa, S.S. (2013a). A new spectral local linearization method for nonlinear boundary layer flow problems. *Journal of Applied Mathematics*, doi:10. 1155/2013/423628.
- Motsa, S.S. (2013b). A new spectral relaxation method for similarity variable nonlinear boundary layer flow systems. *Chemical Engineering Communications*, 201(2), 241-256.
- Motsa, S.S., Dlamini, G. & Khumalo, M. (2014). Spectral relaxation method and spectral quasilinearization method for solving unsteady boundary layer flow problems. *Advances in Mathematical Physics*, [http://dx.doi.org/ 10. 1155/2014/341964](http://dx.doi.org/10.1155/2014/341964).
- Motsa, S.S., Magagula, V.M. & Sibanda, P. (2014). A bivariate chebyshev spectral collocation quasilinearization method for nonlinear evolution parabolic equations. *Scientific World Journal*. <https://doi:10.1155/2014/581987>.
- Motsa, S.S., Makukula, Z.G. & Shateyi, S. (2013). Spectral local linearisation approach for natural convection boundary layer flow. *Mathematical Problems*

in *Engineering*, <https://doi.org/10.1155/2013/765013>.

- Motsa, S.S., Sibanda, P., Ngnotchouye, J.M. & Marewo, G.T. (2014). A spectral relaxation approach for unsteady boundary-layer flow and heat transfer of a nanofluid over a permeable stretching/shrinking sheet. *Advances in Mathematical Physics*, <https://doi.org/10.1155/2014/564942>.
- Moukalled, F., Mangani, L. & Darwish, M. (2016). *Fluid Mechanics and Its Applications*. Springer International Publishing, Switzerland.
- Mushtaq, A. & Mustafa, M. (2017). Computations for nanofluid flow near a stretchable rotating disk with axial magnetic field and convective conditions. *Results in Physics*, 7, 3137-3144.
- Mustafa, M. (2017). MHD nanofluid flow over a rotating disk with partial slip effects: Buongiorno model. *Int. J. Heat Mass Transfer.*, 108, 1910-1916.
- Mustafa, M., Mushtaq, A., Hayat, T. & Alsaedi, A. (2015) Radiation effects in three dimensional flow over a bi-directional exponentially stretching sheet. *Journal of the Taiwan Institute of Chemical Engineers*, 47, 43-49.
- Nageeb, A., Haroun, H., Mondal, S. & Sibanda, P. (2017). Effects of thermal radiation on mixed convection in a MHD nanofluid flow over a stretching sheet using a spectral relaxation method. *Int. J. of Mathematical, Computational, Physical, Electrical and Computer Engineering*, 11(2).
- Nagendramma, V. & Leelaratnam, A. (2017). MHD boundary layer slips flow of a Casson nanofluid through a porous medium over a stretching cylinder with dissipation and suction/blowing. *Journal of Nanofluids*, 6, 1-10.
- Nagendramma, V., Sreelakshmi, K. & Sarojamma, G. (2015). MHD heat and mass transfer flow over a stretching wedge with convective boundary condition and thermophoresis. *Science Direct, Procedia Engineering*, 127, 963-969.
- Nasir, S., Islam, S., Gul, T., et al. (2018). Three-dimensional rotating flow of MHD single wall carbon nanotubes over a stretching sheet in presence of thermal radiation. *Applied Nanoscience*, 8, 1361-1378.
- Nicholas, A. (2010). *An Introduction to Magnetohydrodynamics*. Battista Stony Brook University, New York.
- Noghrehabadi, A., Pourrajab, R. & Ghalambaz, M. (2012). Effect of partial slip boundary condition on the flow and heat transfer of nanofluids past stretching sheet prescribed constant wall temperature. *Int. J. Ther.Sci.*, 54, 253-261.

- Noghrehabadi, A., Ghalambaz, M., Izadpanahi, E. & Pourrazab, R. (2014). Effect of magnetic field on the boundary layer flow, heat and mass transfer of nanofluids over a stretching cylinder. *Journal of heat and mass transfer research*, 9-16.
- Otegbeye, O. (2014). On decoupled quasi-linearization methods for solving systems of nonlinear boundary value problems. (Unpublished thesis) School of Mathematical Sciences University of KwaZulu Natal, Pietermaritzburg.
- Oyelakin, S., Mondal, S. & Sibanda, P. (2017). Cattaneo-Christov nanofluid flow and heat transfer with variable properties over a vertical cone in a porous medium. *Int. J. Appl. Comput.*, 3(1), S1019-S1034.
- Pak, B.C. & Cho, Y.I. (1998). Hydrodynamic and heat transfer study of dispersed fluids with submicron metallic oxide particles. *Experimental Heat Transfer*, 11(2), 151-170.
- Pal, D., & Mandal, G. (2015.) Mixed convection-radiation on stagnation point flow of nanofluids over a stretching/shrinking sheet in a porous medium with heat generation and viscous dissipation. *Journal of Petroleum Science and Engineering*, 126, 16-25.
- Palanisamy, K. & Kumar, P.C.M. (2019). Experimental investigation on convective heat transfer and pressure drop of cone helically coiled tube heat exchanger using carbon nanotubes/water nanofluids. *Helvion*, 5, e01705.
- Parveen, K., Banse, V. & Ledwani, L. (2016). Green synthesis of nanoparticles: Their advantages and disadvantages. In: AIP Conference Proceedings. *AIP Publishing*, 1724, 020048.
- Prakash, J. (2017). Effect of Cattaneo-Christov heat flux on non-linear radiative MHD flow of Casson fluid induced by a semi-finite stretching surface. *Frontier in heat and mass transfer (FHMT)*, 8(8), doi:10.5098/htm.8.8.
- Pramanik, S. (2013). Casson fluid flow and heat transfer past an exponentially porous stretching surface in the presence of thermal radiation. *Ain Shams Engineering Journal*, 5, 205-212.
- Prasannakumara, B.C., Shashikumar, N.S. & Venkatesh, P. (2017). Boundary layer flow and heat transfer of fluid particle suspension with nanoparticles over a nonlinear stretching sheet embedded in a porous medium. *Nonlinear Engineering*, 6(3), 179-190.
- Rajput, G.R., Patil, V.S. & Jadhav, B.P. (2017). MHD mixed flow of unsteady

- convection with radiation over a vertical porous plate: Lie group symmetry analysis. *Applied and Computational Mechanics*, 11, XXX-YYY. Raju, R.S.,
- Reddy, B.M., Rashidi, M.M. & Gorla, R.S.R. (2016). Application of finite element method to unsteady magnetohydrodynamics free-convection flow past a vertically inclined porous plate including thermal diffusion and diffusion thermo effects. *Journal of Porous Media*, 19(8), 701-722.
- Raju, R.S., Sudhakar, K., & Rangamma, M. (2016). The effects of thermal radiation and heat source on an unsteady MHD free convection flow past an infinite vertical plate with thermal diffusion and diffusion thermo. *Journal of The Institution of Engineers*, 94(2), 175-186.
- Ramesh, B.K., Shreenivas, R.K., Achala, L.N. & Bujurk, N.M. (2017). Similarity solutions of the MHD boundary layer flow past a constant wedge within porous media. *Hindawi, Mathematical Problems in Engineering*, [https://doi:10.1155/2017/1428137](https://doi.org/10.1155/2017/1428137).
- Ramzan, M., Mohammad, M. & Howari, F. (2019). Magnetized suspended carbon nanotubes based nanofluid flow with bio-convection and entropy generation past a vertical cone. *Sci Rep.*, 9(1):1-15.
- Rasheed, H., Rehman, A., Sheikh, N. & Iqbal, S. MHD boundary layer flow of nanofluid over a continuously moving stretching surface. *Applied and Computational Mathematics*, 6(6), 265-270.
- Reddy, P.S. & Chamkha, A.J. (2016). Influence of size, shape, type of nanoparticles, type and temperature of the base fluid on natural convection MHD of nanofluids. *Alex Eng J.*, 55(1), 331-341.
- Reddy, P.S. & Chamkha, A.J. (2016). Soret and Dufour effects on MHD convective flow of Al<sub>2</sub>O<sub>3</sub>-water and TiO<sub>2</sub>-water nanofluids past a stretching sheet in porous media with heat generation/absorption. *Adv Powder Technol*, 27(4), 1207-1218.
- Rehman, K.U., Khan, A.A., Malik, M.Y. & Makinde, O.D. (2017). Thermophysical aspects of stagnation point magnetonanofluid flow yields by an inclined stretching cylindrical surface: a nonNewtonian fluid model. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, 39, 3669-3682.
- Rosali, H., Ishak, A., Nazar, R. & Pop, I. (2016). Mixed convection boundary layer flow past a vertical cone embedded in a porous medium subjected to a convective boundary condition. *Propulsion and Power Research*, 5(2), 118-

- Sahoo Y.Do. (2010). Effects of slip on sheet-driven flow and heat transfer of a third grade fluid past a stretching sheet. *Int. Commun. Heat Mass Trans.*, 37, 1064-1071.
- Salahuddin, T., Hussain, A., Malik, M.Y., Awais, M. & Khan, M. (2017). Carreau nanofluid impinging over a stretching cylinder with generalized slip effects: Using finite difference scheme. *Results in Physics*, 7, 3090-3099.
- Saleem, S. & Nadeem, S. (2015). Theoretical analysis of slip flow on a rotating cone with viscous dissipation effects. *J. Hydrodyn.*, 27(4), 616-23.
- Samadishadlou, M., Farshbaf, M., Annabi, N., Kavetsky, T., Saghfi, S., Akbarzadeh, A. & Mousavi, S.(2018) Magnetic carbon nanotubes: preparation, physical properties, and applications in Biomedicine. *Artificial Cells, Nanomedicine, and Biotechnology*, 46(7), 1314-1330.
- Sattar, M.A. (2011). A local similarity transformation for the unsteady two-dimensional hydrodynamic boundary layer equations of a flow past a wedge. *International Journal of Applied Mathematics and Mechanics*, 7(1), 15-28.
- Schlichting, H. (1979). *Boundary-layer Theory*, 7<sup>th</sup> Edition. McGraw-Hill, New York.
- Shehzad, S.A., Abbasi, F.M., Hayat, T. & Ahmad, B (2016). Cattaneo-Christov heat flux model for third-grade fluid flow towards exponentially stretching sheet. *Appl. Math. Mech. Eng. Ed.*, 37(6), 361768.
- Sheri, S.R., & Raju, R.S. (2016). Transient MHD free convective flow past an infinite vertical plate embedded in a porous medium with viscous dissipation. *Meccanica*, 51(5), 1057-1068.
- Shi, D., Cheng, J.P., Liu, F. & Zhang, X.B. (2010). Controlling the size and size distribution of magnetite nanoparticles on carbon nanotubes. *J Alloy Compd*, 502, 365-370.
- Singh, B.G.P., Baburao, C. & Pispatis, V. (2012). Carbon nanotubes: A novel drug delivery system. *International Journal of Research in Pharmacy and Chemistry*, 2(2), 523-532.
- Sreedevi, P., Reddy, P.S. & Chamkha, A.J. (2018). Magneto-hydrodynamics heat and mass transfer analysis of single and multi-wall carbon nanotubes over vertical cone with convective boundary condition. *Int J Mech Sci.*, 135, 646-655.
- Srinivasacharya, D., Mendu, U. & Venumadhav, K. (2015). MHD boundary layer

- flow of a nanofluid past a wedge. *Science Direct, Procedia Engineering*, 127, 1064-1070.
- Srinivasan, C. (2008). Carbon nanotubes in cancer therapy. *Current Science*, 94, 300.
- Srinivasasai, P., Krishna, K.R., Puppala, N. & Remmy, K. J. (2015). Effects on MHD flow over an exponentially stretching sheet with viscous dissipation and heat generation. *Int. J. of Computer Applications Research*, 5(3), 35-48.
- Sundar, L.S., Sharma, K.V., Singh, M.K. & Sousa A.C.M. (2017). Hybrid nanofluids preparation, thermal properties, heat transfer and friction factor. *Renewable and Sustainable Energy Reviews*, 68, 185-198.
- Sundar, L.S., Singh, M.K. & Sousa A.C.M. (2014). Enhanced heat transfer and friction factor of MWCNT-Fe<sub>3</sub>O<sub>4</sub>/water hybrid nanofluids. *Int Comm Heat Mass Transf*, 52, 73-83.
- Suresh, S., Venkitaraj, K.P., Selvakumar, P. & Chandrasekar, M. (2011). Synthesis of Al<sub>2</sub>O<sub>3</sub>-Cu/water hybrid nanofluids using two step method and its thermophysical properties, Colloid. Surf. A Physicochem. *Eng. Asp*, 388(1e3), 41-48.
- Thibeault, C. M. (2009). Employing the spectral collocation method in the modeling of laminar tube flow dynamics. (Unpublished thesis), Minnesota State University Mankato.
- Tibbilo, V. & Zampoli, A. (2011). Uniqueness result for Cattaneo-Christov heat conduction model applied to incompressible fluids. *Mechical Research Communication*, 38, 77-99.
- Tlili, I., Khan, W.A. & Khan, I. (2018). Multiple slips effects on MHD SA-Al<sub>2</sub>O<sub>3</sub> and SA-Cu non-Newtonian nanofluids flow over a stretching cylinder in porous medium with radiation and chemical reaction. *Results in Physics*, 8, 213-222.
- Trefethen, L.N. (2000). *Spectral methods in MATLAB*. Society for Industrial and Applied Mathematics.
- Turkylmazoglu, M. (2015). Slip flow and heat transfer over a specific wedge, an exactly solvable Falkner-Skan equation. *Journal of Engineering and Mathematics*, 92(1), 73-81.
- Ullah, I., Khan, I. & Shafie, S. (2016). Hydromagnetic Falkner-Skan flow of Casson fluid past a moving wedge with heat transfer. *Alexandria Engineering-*

*Journal*, 55, 2139-2148.

- Versteeg, H. K., & Malalasekera, W. (2007). *An introduction to computational fluid dynamics: The finite volume method 2<sup>nd</sup> Edition*. Pearson Education Limited.
- Wahiduzzaman, M., Khan, Md.S., Biswas, P., Karim, I. & Uddin, M.S. (2015). Viscous dissipation and radiation effects on MHD boundary layer flow of a nanofluid past a rotating stretching sheet. *Applied Mathematics*, 6, 547-567.
- Wang C.Y. (1989). Free convection on a vertical stretching surface. *J. Appl. Math. Mech.*, 69, 418-420.
- Watanabe, T. (1990). Thermal boundary layer over a wedge with uniform suction and injection in forced flow. *Acta Mechanica*, 83, 119-126.
- Wu L. (2008). A slip model for rarefied gas flows at arbitrary Knudsen number. *Applied Physics Letters*, 93(25), 253103.
- Xu, X. & Chen, S. (2017). Dual solutions of a boundary layer problem for MHD nanofluids through a permeable wedge with variable viscosity. *Xu and Chen Boundary Value Problems*, 2017(147), DOI 10.1186/s13661-017-0877-7.
- Yang, D., Yang, F., Hu, J., Long, J., Wang, C., Fu, D. & Ni, Q. (2009). Hydrophilic multi-walled carbon nanotubes decorated with magnetite nanoparticles as lymphatic targeted drug delivery vehicles. *Chem. Commun*, 29, 4447-4449.
- Yih, K.A. (1999). MHD forced convection flow adjacent to a nonisothermal wedge. *Internatinal Communication of Heat and Mass Transfer*, 26(6), 819-827.
- Yirga, Y. & Shankar, Y. B. (2015). MHD flow and heat transfer of nanofluids through a porous media due to a stretching sheet with viscous dissipation and chemical reaction effects. *International Journal for Computational Methods in Engineering Science and Mechanics*, 16(5), 275-284.
- Zhou, J. (2014). The constants in a posteriori error indicator for state-constrained optimal control problems with spectral methods. *Abstracts and Applied Analysis*, <https://doi:10.1155/2014/946026>.

# Appendix

## Publications Arise from the Dissertation

Some Articles from the Dissertation published on international reputable journals.

1. **Ibrahim W, Tulu A.** Magnetohydrodynamic (MHD) boundary layer flow past a wedge with heat transfer and viscous effects of nanofluid embedded in porous media. *Mathematical Problems in Engineering*, Volume 2019, Article ID 4507852, 12 pages, <https://doi.org/10.1155/2019/4507852>.
2. **Tulu A, Ibrahim W.** Numerical analysis of heat and mass transfer flow of nanofluid over a moving wedge using Spectral quasilinearization method. *International Journal of Applied Mathematics and Theoretical Physics*. Vol. 5, No. 4, 2019, pp. 111-117. Doi: 10.11648/j.ijamtp.20190504.13.
3. **Tulu A, Ibrahim W.** MHD slip flow of CNT-Ethylene glycol nanofluid due to a stretchable rotating disk with Cattaneo-Christov heat flux model. *Mathematical Problems in Engineering*, Volume 2020, Article ID 1374658, 13 pages, <https://doi.org/10.1155/2020/1374658>.
4. **Tulu A, Ibrahim W.** Spectral relaxation method analysis of Casson nanofluid flow over stretching cylinder with variable thermal conductivity and Cattaneo-Christov heat flux model. *Heat Transfer*. 2020; 1-23, <https://doi.org/10.1002/htj.21781>.
5. **Tulu A, Ibrahim W.** Mixed convection hybrid nanofluids flow of ( $MWCNTs-Al_2O_3$ )/Engine oil over a spinning cone with variable viscosity and thermal conductivity. *Journal: Heat Transfer*. 2021; <http://dx.doi.org/10.1002/htj.22051>.
6. **Tulu A, Ibrahim W.** Effects of second order slip flow and variable viscosity on natural convection flow of ( $CNTs-Fe_3O_4$ )/Water hybrid nanofluids due to stretching Surface. *Journal: Mathematical Problems in Engineering*, Volume 2021, Article ID 8407194, 18 pages, <https://doi.org/10.1155/2021/8407194>.