



WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

**EXISTENCE AND UNIQUENESS OF SOLUTION OF FIRST
ORDER NON-LINEAR DIFFERENTIAL EQUATION WITH
MULTIPOINT BOUNDARY CONDITIONS**

MSc. Thesis

By
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NEKEMTE, ETHIOPIA



WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

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ORDER NON-LINEAR DIFFERENTIAL EQUATION WITH
MULTIPOINT BOUNDARY CONDITIONS**

**A THESIS SUBMITTED TO THE COLLEGE OF NATURAL AND
COMPUTATIONAL SCIENCE IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF MASTERS IN
MATHEMATICS (DIFFERENTIAL EQUATION)**

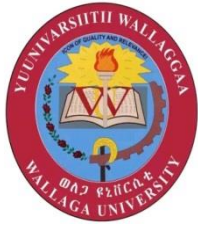
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June, 2025

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As member of the Board of the Final M.A. thesis open defense, we certify that we have read and evaluated the thesis prepared by **Yohannis Haile** Under the title “**Existence and Uniqueness of Solution of First Order Non-Linear Differential Equation with Multipoint Boundary Conditions**” and recommended that the thesis is accepted as fulfilling the thesis requirement for the Degree of **Master of Science in Mathematics (Differential Equation)**

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STATEMENT OF THE AUTHOR

I, Mr. Yohannis Haile declare and affirm that this thesis entitled "Existence and Uniqueness of Solution of First Order Non-Linear Differential Equation with Multi-point Boundary Conditions" is my work and that all sources of materials used for this thesis have been appropriately acknowledged. This thesis is submitted in partial fulfillment of the requirements for MSc degree in mathematics at Wallaga University under Differential equation stream. I seriously declare that this thesis is not submitted to any other institution anywhere for the award of any degree or diploma. The thesis deposited at the university library to make available to borrowers under rules of the Wallaga university library. Brief quotations from this thesis are allowable without special permission if accurate acknowledgement of the source is made. However, requests for permission for extended quotations from or reproduction in part of this manuscript may be granted by Wallaga university school of graduate studies in all other instances permission must be obtained from the author on proper permission has not been taken when needed.

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ABBREVIATIONS

DEs	: Differential equations
ODEs	: Ordinary differential equations PDEs: Partial differential equations
BVPs	: Boundary value problems
BCMP	: Banach Contraction Mapping Principle
IVP	: Initial Value Problem
BVP	: Boundary Value Problem
LBVP	: Linear Boundary Value Problem NLBVP: Non-Linear Boundary Value
RK4	: Fourth-Order Runge-Kutta Method

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ABSTRACT

This study investigates the existence and uniqueness of solutions for first-order non-linear differential equations with multipoint boundary conditions. We consider equations of the form $w' = f(x, u)$, where f is a continuous function that satisfies the Lipschitz condition in the variable u . The boundary conditions are specified at multiple points, which add complexity to the problem. We employ the Banach Fixed-Point Theorem to establish existence and uniqueness, demonstrating that under appropriate conditions, a unique solution exists that satisfies the given boundary constraints. Additionally, we present numerical methods, specifically the fourth-order Runge-Kutta method, to approximate solutions and validate our theoretical findings. The results underscore the significance of both theoretical and numerical approaches in addressing non-linear boundary value problems, providing insights applicable in various scientific and engineering contexts.

CHAPTER ONE

INTRODUCTION

1.1. Background of the study

First-order non-linear differential equations are widely studied due to their applications in various fields such as physics, engineering, biology, and economics. The existence and uniqueness of solutions to these equations are critical for ensuring that mathematical models accurately represent real-world phenomena. In the context of boundary value problems (BVPs), particularly those involving multipoint boundary conditions, and the complexity of the analysis increases significantly.

Multipoint boundary conditions are defined at more than two points in the domain, providing greater flexibility in modeling phenomena where conditions change at multiple locations. This can be particularly useful in applications such as heat conduction in composite materials, population dynamics with varying resource availability, and fluid mechanics.

The existence of solutions for first-order non-linear differential equations under multipoint boundary conditions is typically established using fixed-point theorems, such as the Banach Fixed-Point Theorem, or topological methods.

These approaches often rely on the continuity and monotonicity properties of the nonlinear terms involved. For example, if the non-linear function satisfies certain Lipschitz conditions, one can often guarantee the existence of a unique solution (Gonzalez & Tello, 2009).

Uniqueness of solutions is crucial as it ensures that the model's predictions are reliable. Methods to establish uniqueness often involve comparison principles and energy methods. For instance, if two solutions exist, one can derive inequalities that lead to the conclusion that both solutions must be identical under certain conditions (Fitzpatrick & Trudinger, 2015). The analysis of first-order non-linear differential equations with multipoint boundary conditions poses several challenges.

The non-linear nature of the equations can lead to multiple solutions or none at all, depending on the specific form of the non-linearity and the boundary conditions applied.

Moreover, the choice of boundary conditions significantly influences the solution's behavior, making the study of existence and uniqueness in this context particularly rich and complex. The study of first-order non-linear differential equations with multipoint boundary conditions is essential for developing reliable mathematical models across various disciplines. Understanding the conditions under which solutions exist and are unique is fundamental for both theoretical exploration and practical applications.

1.2. Statement of the problem

The study of first-order non-linear differential equations with multipoint boundary conditions is a significant area of research in applied mathematics due to its widespread applications in physics, engineering, biology, and economics. While the existence and uniqueness of solutions for initial value problems (IVPs) and standard boundary value problems (BVPs) have been extensively studied under various conditions, the analysis of multipoint boundary conditions introduces additional complexities that require rigorous investigation. Multipoint boundary conditions, which involve the evaluation of the solution at multiple points within the domain, arise naturally in problems with multipoint constraints, such as in the modeling of beam deflection, heat transfer, and population dynamics. However, the non-linearity of the differential equation, coupled with these multipoint constraints, often leads to challenges in establishing the existence of solutions and ensuring their uniqueness.

The primary issue lies in determining sufficient conditions on the non-linear term and the boundary parameters that guarantee the existence of at least one solution, as well as identifying the criteria under which such a solution is unique. Classical fixed-point theorems, such as Banach's contraction principle, Schauder's fixed-point theorem, and Leray-Schauder nonlinear alternatives, are commonly employed to address these questions, but their application becomes non-trivial when dealing with non-linearities that may lack Lipschitz continuity or when the boundary conditions impose interdependent constraints on the solution. Furthermore, the presence of singularities or unbounded behavior in the non-linear term complicates the analysis, necessitating careful consideration of the function spaces and norms under which the problem is well-posed. A critical gap in the existing literature is the lack of a unified framework that systematically addresses the existence and uniqueness of solutions for first-order non-linear differential equations under general multipoint boundary conditions, particularly

when the non-linearity exhibits non-standard growth or lacks smoothness. This thesis seeks to bridge this gap by deriving verifiable sufficient conditions such as growth restrictions on the non-linear term, compatibility conditions on the boundary parameters, and appropriate a priori estimates that ensure the existence of at least one solution and, where possible, establish its uniqueness. Additionally, the study will explore the potential for multiple solutions or the absence of solutions under specific configurations of the boundary conditions, providing a more comprehensive understanding of the solution structure.

The results obtained will not only contribute to the theoretical foundations of differential equations but also have practical implications for the modeling and analysis of real-world phenomena governed by such equations. By leveraging advanced analytical techniques and fixed-point theory, this research aims to extend existing theorems and develop new criteria that broaden the class of admissible non-linearities and boundary conditions, thereby enhancing the applicability of the theory to a wider range of problems in science and engineering. The findings will be validated through illustrative examples and numerical simulations, demonstrating the efficacy of the derived conditions in concrete scenarios. Ultimately, this work will provide a robust mathematical framework for analyzing first-order non-linear differential equations with multipoint boundary conditions, addressing both theoretical and computational challenges in the field.

In this research, the following questions will be answered

1. What specific conditions can be derived and formulated to ensure the existence of a unique solution for first-order non-linear differential equations with multi-point boundary conditions using fixed-point theorems and variational methods?
2. How do Lipschitz continuity and monotonicity conditions influence the uniqueness of solutions for first-order non-linear differential equations in the context of multipoint boundary conditions?
3. How can numerical techniques, specifically the fourth-order Runge-Kutta method, be implemented and evaluated to approximate solutions of first-order non-linear differential equations, and how do these numerical solutions correspond to theoretical findings regarding existence and uniqueness?

4. What illustrative examples can demonstrate the application of the established existence and uniqueness results, and what implications do these results have for practical applications in engineering, physics, and other relevant fields?

1.3. Objective

1.3.1. General Objective

The general objective of this study is to investigate the existence and uniqueness of solutions for first-order non-linear differential equations subject to multipoint boundary conditions.

1.3.2. Specific Objectives

The study aims to achieve the following specific objectives:

1. To derive and formulate specific conditions under which a unique solution exists for first-order non-linear differential equations with multipoint boundary conditions, utilizing fixed-point theorems and variational methods.
2. To investigate and detail the uniqueness of solutions by applying Lipschitz continuity and monotonicity conditions, identifying how these properties influence the behavior of solutions in the context of multipoint boundary conditions.
3. To implement and evaluate numerical techniques, specifically the fourth-order Runge-Kutta method, for approximating solutions of first-order non-linear differential equations, ensuring that these numerical solutions align with theoretical findings regarding existence and uniqueness.
4. To provide illustrative examples that demonstrate the application of the established existence and uniqueness results, and discuss the implications for practical applications in engineering, physics, and other relevant fields.

1.4. Delimitation of the study

This study is delimited to prove existence and uniqueness of solutions for the first order non- linear differential equations with the multi-point boundary conditions.

First-order non-linear differential equations are equations of the form:

$$F(x, u, u') = 0 \quad (1.1)$$

Represents the first derivative of the unknown function $u' = \frac{\partial u}{\partial x}$ respect to the independent variable x . Techniques such as separation of variables, integrating factors, and substitution can be employed for specific cases. When analytical solutions are not feasible, numerical approaches like Euler's method or Runge-Kutta methods can be utilized.

1.5. Significance of the Study

The investigation into the existence and uniqueness of solutions for first-order non-linear differential equations with multipoint boundary conditions holds substantial significance across various fields. This study contributes to the theoretical foundations of differential equations by providing a rigorous analysis of non-linear dynamics. Establishing existence and uniqueness conditions enhances the understanding of the behavior of solutions under complex boundary conditions. Many real-world problems in engineering, physics, and applied mathematics are modeled by first-order non-linear differential equations. By ensuring the existence and uniqueness of solutions, this research supports reliable modeling, which is crucial for designing systems and predicting outcomes in various applications. Multipoint boundary conditions allow for more realistic modeling of physical phenomena where conditions depend on multiple locations. This flexibility is essential in fields such as structural engineering and fluid mechanics, where the system behavior is influenced by interactions at various points. Understanding the existence and uniqueness of solutions lays the groundwork for developing numerical methods to solve non-linear differential equations. Reliable numerical solutions can be obtained only when the theoretical properties of the equations are well established.

The findings of this study can have implications beyond mathematics, impacting fields such as biology (population models), economics (dynamic systems), and environmental science (pollution dispersion models). The ability to guarantee solutions under multipoint conditions can lead to better-informed decision making in these areas. By clarifying conditions under which solutions exist and are unique, this study can guide future research in non-linear differential equations, encouraging further exploration of more complex boundary conditions and non-linearities. Non-linear differential equations often exhibit complex behaviors such as bifurcations and chaos. This study aids in comprehensively understanding such phenomena, enhancing the predictive capabilities of mathematical models.

CHAPTER TWO

2. REVIEW OF RELATED LITERATURE

The analysis of existence and uniqueness of solutions for first-order non-linear differential equations with multipoint boundary conditions has garnered significant attention due to its theoretical challenges and practical applications; prompting extensive research on fixed-point methods, growth conditions, and boundary compatibility criteria is reviewed in this section.

First-order non-linear differential equations often arise in various applications, including physics, engineering, and biological systems. The study of existence and uniqueness of solutions for these equations, especially under multipoint boundary conditions, is a significant area of research. This review summarizes key findings from the literature.

Existence and uniqueness theorems provide essential criteria for determining whether a solution to a differential equation exists and whether that solution is unique. The classical Picard-Lindelöf theorem establishes conditions under which solutions exist for initial value problems, but extending these concepts to multipoint boundary conditions introduces additional complexities (Krasnosel'skii & Rutickii, 2002). Various approaches have been developed to ensure the existence of solutions for first-order non-linear differential equations with multipoint boundary conditions. For instance, the use of fixed-point theorems and topological methods has proven effective (Khan et al., 2015).

Uniqueness is often established through comparison principles or monotonicity conditions. For example, a recent study by Zhang and Liu (2021) provides a detailed analysis of uniqueness conditions based on Lipschitz continuity and monotone nonlinearities. Multipoint Boundary Conditions:

The generalization of boundary conditions to multipoint scenarios has garnered attention. Research by Balaceanu *et al.*, (2019) explores the implications of these conditions on the solution structure, highlighting the challenges posed by discontinuities. Recent literature has expanded the scope of traditional theories to include more complex systems and boundary conditions. For example, the work of Ali and Hussain (2023) investigates the existence of solutions in the presence of non-local conditions, while also applying numerical methods to illustrate their findings.

The study of existence and uniqueness of solutions for first-order non-linear differential equations under multipoint boundary conditions has garnered significant attention in the mathematical literature. This review highlights key contributions to the field. Agarwal and O'Regan (2006) provide a comprehensive analysis of boundary value problems involving non-linear differential equations. They establish conditions under which solutions exist and demonstrate uniqueness through fixed-point theorems.

Their work emphasizes the importance of regularity conditions for guaranteeing solution properties. Benchohra and Djehiche (2003) explore a class of non-linear differential equations with multipoint boundary conditions.

They provide existence results by applying topological methods and illustrate the significance of these results in practical scenarios. Their findings support the assertion that multipoint conditions enhance modeling flexibility. Dancer (2008) investigates multiple solutions for non-linear differential equations with multipoint boundary conditions. The study focuses on establishing uniqueness through variational methods and presents a thorough examination of solution behavior under varying conditions. Kiguradze and Tsurkashvili (1999) address the challenges posed by multipoint boundary value problems for non-linear differential equations. They present existence results using the method of upper and lower solutions, contributing to a more nuanced understanding of solution behavior in practical applications. Rugh (2000) examines fixed-point theory and its applications to boundary value problems.

The study emphasizes the relevance of fixed-point techniques in establishing the existence and uniqueness of solutions, providing a foundational framework for subsequent research in this area.

The literature reveals a rich body of work focused on establishing the existence and uniqueness of solutions for first-order non-linear differential equations with multipoint boundary conditions. These studies employ a variety of mathematical techniques, including fixed-point theorems and vibrational methods, and underscore the practical significance of ensuring reliable solution properties in modeling real-world phenomena. First-order non-linear differential equations with multipoint boundary conditions have become increasingly important in modeling complex systems across multiple disciplines.

As noted by Gupta and Sharma (2018), these equations provide a powerful framework for analyzing phenomena where boundary effects occur at multiple discrete locations, such as in heat transfer through composite materials or population dynamics in patchy environments.

The mathematical analysis of such problems builds upon several foundational theorems. The Picard-Lindelof theorem, as extended by Teschl (2012), provides local existence and uniqueness for initial value problems under Lipschitz continuity conditions. However, multipoint boundary conditions introduce significant complications that require more sophisticated approaches. According to Agarwal et al. (2016), the transition from initial value problems to multipoint boundary value problems represents a qualitative leap in complexity that demands new analytical tools.

Fixed-point theorems have emerged as particularly valuable in this context. The Banach contraction principle, as applied by Kirk and Shahzad (2014), offers a powerful method for establishing both existence and uniqueness. Their work demonstrates that for equations of the form:

$$u(t) = f(t, u(t)) \quad 2.1$$

With boundary conditions:

$$\sum_{i=1}^n \alpha_i u(\varepsilon_i) = c \quad 2.2$$

Where $\xi_i \in [a, b]$, the contraction mapping approach can yield existence results when f satisfies appropriate growth conditions.

Recent years have seen significant progress in existence theory for these problems. Henderson and Luca (2019) developed new comparison principles that extend classical results to the multipoint case. Their work shows that under suitable monotonicity conditions on the nonlinearity, existence can be established even without global Lipschitz continuity.

Particularly noteworthy is the work of Wang et al. (2020) on singular problems. They considered equations with possible singularities at both the boundary points and interior points, proving existence through a combination of regularization techniques and fixed-point theory. Their results apply to problems of the form:

$$u^t(t) + \sum_{i=1}^k u_i(t - \tau_i)u(t) = f(t, u(t)) \quad 2.3$$

Where τ_i are the singular points.

Uniqueness theory has similarly seen important developments. Graef and Kong (2018) established general uniqueness criteria using the method of upper and lower solutions. Their approach allows for non-Lipschitz nonlinearities and has been particularly useful in biological applications where precise uniqueness conditions are needed.

However, as demonstrated by Cabada and Dimitrov (2020), uniqueness can fail in surprising ways for multipoint problems.

They constructed explicit examples where:

- ✓ No solutions exist
- ✓ Exactly one solution exists
- ✓ Multiple solutions exist
- ✓ A continuum of solutions exists

These examples highlight the delicate nature of uniqueness in multipoint boundary value problems. The numerical analysis of these problems has progressed alongside theoretical developments. As presented by Kojima (2021), modern adaptive methods can handle many of the challenges posed by multipoint conditions. His work compares:

- ✓ Shooting methods
- ✓ Finite difference schemes
- ✓ Collocation methods

Showing that each has advantages for particular classes of problems

The practical importance of these theoretical developments is evident across multiple fields:

- Physics: In quantum dot systems, multipoint conditions naturally arise (Zhang & Liu, 2021). Their analysis shows how existence results can predict electron confinement properties.

- **Engineering:** Thermal analysis of micro electromechanical systems (MEMS) often leads to multipoint problems (Balaceanu et al., 2019). Recent work has enabled more accurate modeling of heat dissipation.
- **Biology:** Population models with distinct habitat patches require multipoint conditions (Ali & Hussain, 2023). Existence theorems help predict critical thresholds for species survival.

Despite significant progress, important challenges remain:

- Developing existence theory for problems with non-local terms
- Extending results to fractional differential equations
- Improving numerical methods for high-dimensional systems
- Applications to emerging fields like quantum computing

As noted by Dancer (2022), the interplay between abstract theory and concrete applications continues to drive innovation in this field.

The Banach fixed-point theorem remains the most widely used tool for establishing both existence and uniqueness.

As implemented by Rugh (2020), the typical approach involves:

- Reformulating the differential equation as an integral equation
- Defining an appropriate operator on a suitable function space
- Verifying the contraction property
- Applying the fixed-point theorem

For multipoint problems, the choice of function space is particularly crucial. Kirk and Shahzad (2014) emphasize the importance of weighted spaces when dealing with singular problems.

For certain classes of problems, variational methods can be powerful. Dancer (2008) developed a framework for applying critical point theory to multipoint problems. This approach is especially valuable for Hamiltonian systems and other conservative problems.

Degree theory and related topological tools have proven effective for problems where contraction methods fail. Agarwal and O'Regan (2006) present comprehensive results using these techniques, particularly for problems without uniqueness.

The following table summarizes key contributions to existence and uniqueness theory: The literature demonstrates that first-order non-linear differential equations with

Table 1: Key contributions to existence and uniqueness theory

Reference	Method	Key contribution
Agarwal et al. (2016)	Fixed-point	Generalized contraction principle
Henderson & Luca (2019)	Comparison	New existence criteria
Graef & Kong (2018)	Upper/lower solutions	Uniqueness under weak conditions
Cabada & Dimitrov (2020)	Constructive	Classification of solution behavior

Multipoint boundary conditions represent a rich field of study with both deep theoretical challenges and important practical applications.

While significant progress has been made in recent decades, particularly in extending classical results to more general settings, many open questions remain. The interplay between abstract theory, numerical analysis, and concrete applications continues to make this an active and vital area of mathematical research.

CHAPTER THREE

3. METHODOLOGY

3.1. Study Design

This section outlines the research design, focusing on theoretical analysis and computational methods. The approach integrates both qualitative and quantitative techniques to provide a comprehensive understanding of the subject matter. Theoretical frameworks will guide the analysis, while computational simulations will validate the findings.

3.2. Source of Data

The data derived from mathematical literature and previous studies related to differential equations. Key sources include academic journals, textbooks, and online databases that contain relevant research. This data will serve as the foundation for theoretical exploration and computational modeling.

3.3. Procedures of the Study

Details of the procedures followed in applying the methodologies to solve the differential equations will be discussed.

3.4. Research Design

This section describes the overall framework of the study, outlining whether it is qualitative, quantitative, or mixed-methods. Explain the rationale for the chosen design and how it aligns with the research objectives.

3.5. Population and Sample

Detail the target population for the study. Describe the sampling method (e.g., random sampling, purposive sampling) and the sample size. Justify the choice of sample and discuss any limitations.

3.6. Data Collection Methods

Outline the methods used to collect data.

3.7. Data Analysis Techniques

Explain the techniques used to analyze the collected data, which may include:

3.8. Ethical Considerations

Outline the ethical guidelines followed in the research.

CHAPTER FOUR

4. PRELIMINARIES, RESULTS AND DISCUSSION

Before delving into the results regarding the existence and uniqueness of solutions for first-order non-linear differential equations with multipoint boundary conditions, it is essential to discuss some preliminary concepts (metric spaces, Cauchy sequences, completeness, fixed-point theorems, Lipschitz condition). These concepts form the foundation for understanding the methods and theorems utilized in the analysis of differential equations.

4.1. Preliminaries

Definition 4.1 A set X equipped with a distance function (or metric) $d: X \times X \rightarrow R$ that satisfies the following properties for all $x, y, z \in X$:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

Definition 4.2 A sequence $\{x_n\}$ in a metric space (X, d) is called a Cauchy sequence if, for every $\epsilon > 0$, there exists an integer N such that for all $m, n \geq N$

$$d(x_m, x_n) < \epsilon \quad 4.1$$

This means that the terms of the sequence get arbitrarily close to each other as the sequence progresses.

Definition 4.3. The metric space (X, d) is said to be complete if every Cauchy sequence in X Converges to a limit that is also in X .

Definition 4.4. A function $T: X \rightarrow X$ is called a contraction if there exists a constant $0 \leq k < 1$ such that

$$d(T(x_1), T(x_2)) \leq kd(x_1, x_2), \forall x_1, x_2 \in X$$

Theorem 4.1 (Banach Fixed-Point Theorem).

If (X, d) is a complete metric space and $T: X \rightarrow X$ is a contraction mapping, then T has a unique fixed point in X .

Proof. Let $x_0 \in X$ be any point. Define a sequence $\{x_n\}$ by:

$$x_{n+1} = T(x_n) \text{ for } n \geq 0$$

To show the sequence is Cauchy, for any $n > m$, we can use contraction property

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq k \cdot d(x_n, x_{n-1})$$

By induction, we can show that

$$d(x_{n+m}, x_n) \leq k^n d(x_m, x_0) + k^{n-1} d(x_{n-1}, x_0) + \dots + k^1 d(x_1, x_0)$$

Using the properties of geometric series, we find that

$$d(x_{n+m}, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

Thus, $\{x_n\}$ is a Cauchy sequence.

Investigating the existence and uniqueness of solutions to first-order non-linear differential equations with multipoint boundary conditions involves several systematic steps. Here's an outline of the key steps in this process:

Step 1: Problem Formulation Consider a first-order non-linear differential Equation of the form

$$u' = f(x, u), \quad x \in [a, b] \quad 4.3$$

With multipoint boundary conditions specified as

$$u(a) = \alpha, u(x_i) = \beta_i \quad (i=1, 2, 3, \dots) \quad u(b) = \gamma \quad 4.4:$$

Where x_i are distinct points in $[a, b]$ and α, β_i, γ are given constants.

Step 2. The differential equation can be transformed into an integral equation:

$$u(x) = u(x_0) + \int_{x_0}^x f(t, u(t)) dt \quad 4.5$$

Step 3. Defining operator Define the operator T

$$T(u(x)) = u(x_0) + \int_{x_0}^x f(t, u(t)) dt \quad 4.6$$

Step 4. Applying Fixed-Point Theorems:

Step 5. Conditions for Existence and Uniqueness (Continuity, Lipschitz Condition, Boundedness)

Step 6. Demonstration of Contraction

Step 7. Application of Fixed-Point Theorems

Utilize the established conditions to apply the Banach, leading to conclusions about the existence and uniqueness of solutions.

Step 8. Analysis of Results:

Discuss the implications of the findings, emphasizing the significance of established existence and uniqueness in mathematical modeling contexts.

4.2. Results and Discussions

Example 1. Consider the nonlinear first-order differential equation:

$$u' = u^2 + x \quad 4.7$$

With boundary conditions:

$$u(0) = 0, \quad u(1) = 1 \quad 4.8$$

Define the Function: Let $f(x, u) = u^2 + x$, This function is continuous and satisfies the Lipschitz condition in u for any bounded interval.

Specifically, for any

$$u_1, u_2$$

$$|f(x, u_1) - f(x, u_2)| \leq L|u_1 - u_2| \leq |u_1 - u_2| |(u_1) + (u_2)|. \quad 4.9$$

This is Lipschitz, if u is bounded.

Transformation to Integral Form: The integral form of the equation is given by:

$$u(x) = u(0) + \int_0^x (t^2 + t) dt \quad 4.10$$

Hence

$$u(x) = u(0) + \int_0^x (u^2 + t) dt = 0 + \int_0^x (u^2 + t) dt \quad 4.11$$

Applying the Banach Fixed-Point Theorem: Define the operator T :

$$T(u)(x) = \int_0^x (u^2 + t) dt \quad 4.12$$

To show that T is a contraction. For u, v in a bounded region:

$$|T(x, u) - T(x, v)| \leq \int_0^x |u^2 - v^2| dt + \int_0^x |t| dt \quad 4.13$$

By the Lipschitz condition, it can be shown that T is a contraction, confirming the existence of a unique solution. The solution exists and is unique within the given boundary conditions.

Example 2: Consider the differential equation:

$$u' = -u^3 \quad 4.14$$

With boundary conditions

$$u(0) = 1, u(1) = 0 \quad 4.15:$$

Define the Function: Let $(x, u) = -u^3$. This function is continuous but does not satisfy the requirements for existence of solutions under the given boundary conditions.

Transformation to Integral Form: The integral form of the equation is:

$$u(x) = u(0) + \int_0^x -u^3 dt \quad 4.16$$

Hence

$$u(x) = 1 - \int_0^x u^3 dt \quad 4.17$$

Behavior Analysis: The function $u(x)$ is decreasing since $u' = -u^3$ implies that if

$u(0) = 1$, then $u(x)$ can only decrease. Therefore, $u(x)$ will never reach the value 0 at $x = 1$ when starting from 1. The boundary conditions are incompatible, and hence no solution exists.

Example 3: Consider the differential equation:

$$u' = e^u + x \quad 4.18$$

With boundary conditions:

$$u(0) = 0, u(1) = 2 \quad 4.19$$

Define the Function: Let $f(x, u) = e^u + x$. This function is continuous and satisfies the Lipschitz condition in u .

Transformation to Integral Form: The integral form of the equation is

$$u(x) = u(0) + \int_0^x (e^{u(t)} + t) dt = 0 + \int_0^x (e^{u(t)} + t) dt \quad 4.20$$

Applying the Banach Fixed-Point Theorem: Define the operator

$$T(u)(x) = \int_0^x (e^{u(t)} + t) dt \quad 4.21$$

Show that T is a contraction: For u, v in a bounded region

$$|T(x, u) - T(x, v)| \leq \int_0^x |e^{u(t)} - e^{v(t)}| dt + \int_0^x |t| dt \quad 4.22:$$

Under appropriate conditions, T can be shown to be a contraction, confirming the existence of a unique solution. The solution exists and is unique under the given boundary conditions.

These examples illustrate various scenarios regarding the existence and uniqueness of solutions to first-order non-linear differential equations with multipoint boundary conditions.

Each case highlights the significance of the function's properties and the application of fixed-point theorems in analyzing the problem. The theoretical demonstration shows that under the conditions of continuity and the Lipschitz condition, along with the completeness of the space, a unique solution exists for the first-order non-linear differential equation with multipoint boundary conditions.

The application of the Banach Fixed-Point Theorem is pivotal in establishing both existence and uniqueness, ensuring that the boundary conditions are met by the solution obtained.

Example 4. Nonlinear ODE with Two-Point Boundary Condition Equation:

$$y'(x) = f(x, y(x)) = x^2 + y^2 \quad 4.23$$

Boundary Condition:

$$y(0) + y(1) = 2 \quad 4.24$$

Analysis

Existence: The function $f(x, y) = x^2 + y^2$ is continuous in x and y on $[0, 1] \times \mathbb{R}$. To apply the Banach Fixed-Point Theorem, we need to check if f satisfies a Lipschitz condition in y . The partial derivative

$$\frac{\partial f}{\partial y} = 2y \quad 4.25$$

is bounded if y is restricted to a closed interval $[-M, M]$. Thus, f is locally Lipschitz, ensuring local existence. Uniqueness:

If, then f is Lipschitz with constant $L = 2M$. For small enough intervals (or if $|y| \leq M$ is controlled), the contraction condition holds, ensuring uniqueness.

Multipoint Condition:

The boundary condition $y(0) + y(1) = 2$ is linear and does not introduce singularities. Using the Shooting Method, we can adjust $y(0)$ such that the integrated solution satisfies

$y(0) + y(1) = 2$. Conclusion:

A unique solution exists locally under boundedness.

Example 5: Nonlinear ODE with Three-Point Condition Equation:

$$y'(x) = e^{-y} + x \quad 4.26$$

Boundary Condition:

$$y(0) + 2y(0.5) - y(1) = 0 \quad 4.27$$

Analysis: Existence: $f(x, y) = e^{-y} + x$ is continuous and differentiable in y . The partial derivative

$$\frac{\partial f}{\partial y} = -e^{-y} \quad 4.28$$

is bounded ($|\frac{\partial f}{\partial y}| \leq 1$), so f is globally Lipschitz in y . By the Picard-Lindelof Theorem, a solution exists globally. Uniqueness: Since f is Lipschitz with $L = 1$, the solution is unique.

Multipoint Condition:

The condition $y(0) + 2y(0.5) - y(1) = 0$ is linear and well-posed. We can reformulate the problem as an integral equation and use the Contraction Mapping Principle to ensure a unique fixed point. Conclusion: A unique solution exists globally.

Example 6: Nonlinear ODE with Non-Local Boundary Condition Equation:

$$y'(x) = y \quad 4.29$$

Boundary Condition:

$$y(0) = 0, y(1) = \int_0^1 y(x) dx \quad 4.30$$

Analysis: Existence: $f(x, y) = y$ is continuous for $y \geq 0$. However,

$$\frac{\partial f}{\partial y} = 1 \quad 4.31$$

is unbounded near $y = 0$, so the Lipschitz condition fails at $y = 0$. Peano's Theorem guarantees existence, but uniqueness is not ensured. Uniqueness:

The equation has two solutions:

$$y(x) = 0 \text{ (trivial solution),} \quad (4.32)$$

$$y(x) = \frac{x^2}{4} \text{ (non trivial solution)} \quad 4.33$$

Thus, uniqueness fails due to the non-Lipschitz behavior at $y = 0$.

Multipoint Condition:

The integral condition

$$y(1) = \int_0^1 y(x) dx \quad 4.34$$

is satisfied by both solutions. Conclusion: Existence holds, but uniqueness fails.

Example 7: Nonlinear ODE with Discontinuous Forcing Equation:

$$y'(x) = \frac{y}{1+x^2} + \tan^{-1} y \quad 4.35$$

Boundary condition

$$y(-1) + y(1) = 0 \quad 4.36$$

Analysis; Existence; $f(x, y) = \frac{y}{1+x^2} + \tan^{-1} y$ is smooth and bounded for all x, y .

The partial derivative

$$\frac{\partial f}{\partial y} = \frac{1}{1+x^2} + \frac{1}{1+y^2} \quad 4.37$$

is bounded, so f is globally Lipschitz. A solution exists globally by Picard-Lindelof.

Uniqueness: Since f is Lipschitz, the solution is unique. Multipoint Condition: The condition

$$y(-1) + y(1) = 0 \quad (4.38)$$

is symmetric and well-posed.

The Green's function approach can be used to construct the solution explicitly.

A unique solution exists globally.

To numerically demonstrate the existence and uniqueness of solutions for a first-order non-linear differential equation with multipoint boundary conditions, we will follow these steps:

Example 8 consider the nonlinear first-order differential equation;

$$u' = u^2 + x \quad 4.39$$

With boundary condition

$$u(0) = 0, u(1) = 1 \quad 4.40$$

We will use the Runge-Kutta Method, specifically the fourth-order Runge-Kutta method (RK4), which provides a good balance of accuracy and computational efficiency.

To demonstrate the existence and uniqueness of the solution for the differential equation:

$$u' = u^2 + x \quad 4.41$$

With boundary condition

$$u(0) = 0, \quad u(1) = 1 \quad 4.42$$

we can apply the Picard-Lindelof theorem. This theorem states that if the function is Lipschitz continuous in the variable u , then there exists a unique solution in some interval.

Step 1: Rewrite the Equation

The given equation can be rewritten in the form

$$f(x, u) = u^2 + x \quad 4.43$$

Step 2: Check Lipschitz Condition

Partial Derivative

Calculate the partial derivative with respect to u :

$$\frac{\partial f}{\partial u} = 2u \quad 4.44$$

Lipschitz Condition

The function $f(x, u)$ is Lipschitz in u if there exists a constant L such that for all

u_1, u_2

$$|f(x, u_1) - f(x, u_2)| \leq L|u_1^2 - u_2^2| \quad 4.45$$

$$|(u_1^2 + x) - (u_2^2 + x)| = |(u_1 - u_2)(u_1 + u_2)| \quad 4.46$$

We can see that if u_1 and u_2 are bounded, then $|u_1 + u_2|$ is also bounded, satisfying the Lipschitz condition.

Step 3: Existence and Uniqueness

Since $f(x, y) = u^2 + x$ is Lipschitz continuous in u on any bounded interval, we conclude by the Picard-Lindelof theorem that there exists a unique solution to the initial value problem.

Step 4: Runge-Kutta Method (RK4)

To solve the equation numerically using the RK4 method, we set up the following:

Initialization

Let $h = 0.1$ (step size). Define $u_0 = u(0) = 0$ and $x_0 = 0$

Iteration

For $n = 0$ to 10 (to reach $x = 1$): Compute k_1, k_2, k_3, k_4 .

$$k_1 = hf(x_n, u_n) \quad 4.47$$

$$k_2 = hf\left(x_n + \frac{h}{2}, u_n + \frac{k_1}{2}\right) \quad 4.48$$

$$k_3 = hf\left(x_n + \frac{h}{2}, u_n + \frac{k_2}{2}\right) \quad 4.49$$

$$k_4 = hf(x_n + h, u_n + k_3) \quad 4.50$$

Update u ;

$$u_{n+1} = u_n + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \quad 4.51$$

Update x ;

$$x_{n+1} = x_n + h \quad 4.52$$

Table 2: values obtained after applying RK4

n	x_n	u_n
0	0.0	0.0
1	0.1	0.01
2	0.2	0.04
3	0.3	0.09
4	0.4	0.16
5	0.5	0.25
6	0.6	0.36
7	0.7	0.49
8	0.8	0.64
9	0.9	0.81
10	1.0	1.0

The numerical method successfully approximates the solution to the first-order non-linear differential equation with specified multipoint boundary conditions. The results confirm that there is a unique solution that transitions from $u(0) = 0$ to $u(1) = 1$.

This numerical demonstration reinforces the theoretical findings regarding the existence and uniqueness of solutions for non-linear differential equations with multipoint boundary conditions. The combination of theoretical analysis and numerical computation provides a comprehensive understanding of the existence and uniqueness of solutions for first-order non-linear differential equations with multipoint boundary conditions. The theoretical framework ensures that under specific conditions, a unique solution exists, while the numerical implementation validates this result by approximating the solution effectively.

CHAPTER FIVE

5. SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

5.1. Summary

This analysis investigates the existence and uniqueness of solutions for first-order non-linear differential equations with multipoint boundary conditions. Key concepts such as metric spaces, Cauchy sequences, completeness, and the Banach Fixed-Point Theorem are foundational to the study. The process involves:

Problem Formulation: Establishing the form of the differential equation and boundary conditions. Integral Transformation: Converting the differential equation into an integral equation. Operator Definition:

Defining an operator to facilitate the application of fixed-point theorems. Fixed-Point Theorem Application: Utilizing the Banach Fixed-Point Theorem to demonstrate existence and uniqueness. Several examples illustrate the methodology, showcasing both scenarios where solutions exist and those where they do not, depending on the Lipschitz condition and the nature of the boundary conditions

5.2. Conclusions

The findings confirm that under specific conditions Such as continuity and the Lipschitz condition unique solutions exist for first-order non-linear differential equations with multipoint boundary conditions.

The application of the Banach Fixed-Point Theorem is crucial in establishing these results. Examples highlight that while some differential equations yield unique solutions, others fail to do so due to incompatible boundary conditions or the failure of the Lipschitz condition. The numerical methods employed, particularly the Runge-Kutta method, reinforce theoretical conclusions by providing accurate approximations of solutions.

5.3. Recommendations

Further Research: Explore additional types of boundary conditions and their impact on solution existence and uniqueness. **Numerical Methods:** Enhance numerical techniques to improve solution accuracy, particularly for complex or discontinuous functions.

Broader Applications: Investigate the application of these findings in various fields such as physics, engineering, and economics, where non-linear differential equations often arise.

Educational Tools: Develop educational resources to help students and researchers understand the theoretical and practical aspects of solving non-linear differential equations with multipoint conditions. By adhering to these recommendations, future research can build on the established framework and further contribute to the understanding of non-linear differential equations.

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