



WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF PHYSICS

THREE-LEVEL ATOM INTERACTING WITH THERMAL LIGHT

MSc Thesis

Name: RABUMA GAMACHU INJIFETA

Advisor: Misganu Chewaka (PhD)

January 8, 2026

Nekemte, Ethiopia



WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF PHYSICS

THREE-LEVEL ATOM INTERACTING WITH THERMAL LIGHT
A Thesis Submitted to Department of Physics, Wallaga University, in
Partial Fulfillment of the Requirements for the Masters of science Degree
in Physics (Quantum Optics)

By: RABUMA GAMACHU INJIFETA

Advisor: Misganu Chewaka (PhD)

January 8, 2026
Nekemte, Ethiopia

DECLARATION

This is to certify that the work related to the thesis report entitled “Three-Level Atom Interacting with Thermal Light” was carried out by Mr. Rabuma Gamachu under my supervision for the partial fulfillment of the requirements for the award of the degree of Master of Science in Quantum Optics”.

Misganu Chewaka (PhD)

Major Advisor

Signature

Date



WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

P.O. Box: 395, Nekemte, Ethiopia.

APPROVAL SHEET FOR SUBMITTING FINAL THESIS

As member of the Board of Examining of the Final MSc. thesis open defense, we certify that we have read and evaluated the thesis prepared by Rabuma Gamachu Injifeta under the title **“Three-Level Atom Interacting with Thermal Light”** and recommended that the thesis is accepted as fulfilling the thesis requirement for the Degree of Master in Physics.

_____	_____	_____
ChairPerson	Signature	Date

_____	_____	_____
Name of Internal Examiner	Signature	Date

_____	_____	_____
Name of External examiner	Signature	Date

_____	_____	_____
Department PGC	Signature	Date

_____	_____	_____
Dean of College	Signature	Date

Certification of the Final Thesis I hereby that all the correction and recommendation suggested by the board of examiners are incorporated in to the final thesis entitled “**Three-Level Atom Interacting with Thermal Light**” by Rabuma Gemechu Injifeta

Dean, SGS

Signature

Date

BIBLIOGRAPHICAL SKETCH

The author was born on February 1985EC in Oromia Regional State, East Wollega Zone, Ebantu district, Burka Gudina Kebele. He attended his education at Hinde Elementary School in 1993-2001. He continued his high school at Hinde Secondary School in 2002-2003 and he train Teachers Training College (TTC) from 2004-2006 at Nekemte Teachers Training College. After graduation, he was employed in East Shoa, Gumbichu district Elementary School. After a year, again he joined the first-degree program in Wallaga University in the field of Physics in 2007 E.C. soon after he again joined he again joined Wallaga University for his further education in M.Sc. in Physics in 203 E.C.

STATEMENT OF THE AUTHOR

I declare that this thesis Research entitled “Three-Level Atom Interacting with Thermal Light” It is my work and that all sources of materials used for this thesis have been appropriately acknowledged. This thesis is submitted in partial fulfillment of the requirements for master’s degree in Physics at Wallaga University. I seriously declare that this thesis is not submitted to any other institution any for the award of any degree or diploma. The thesis deposited at Wallaga University Library to make available to borrowers under rule of the Wallaga University Library. Brief quotations from this thesis are allowable without special permission in accurate acknowledgment of the source is made. However, requests for permission for extended quotations from or reproduction in part of this manuscript may be granted by Wallaga University, the school of graduate studies in all other instances, permission must be obtained from the author.

Name of Student

Signature

Date

Department: Physics

College: Natural and Computational Sciences

ACKNOWLEDGEMENT

First of all, thanks to all powerful almighty God for these illustrations of blessing throughout my work successfully.

Secondly, I would like to thank Dr. Misganu Chewaka for his kindly approach. No more word can be possible enough to express my heartfelt about the constructive advice and the research methodology concepts.

Lastly, I would like to express my sincere love and appreciation to all my family members for their moral support and encouragement and also their efforts to overcome various challenges that might have affected my work.

ABSTRACT

We investigate the steady-state properties and dynamics of a three-level cascade atom interacting with a thermal light field, focusing on how thermal photon occupation affects population distributions, coherence decay, and fluorescence spectra. Using a rate-equation model and analytical expressions for the steady-state populations, we examine the time evolution of the atomic populations under low, intermediate, and high thermal photon numbers $\bar{n}$. At intermediate $\bar{n}$, a balance between spontaneous emission and thermal absorption emerges, yielding nonzero stable populations at all levels; conversely, for low $\bar{n}$, spontaneous emission dominates, and the system rapidly decays to the ground state. We also investigate the degradation of coherence $|\rho_{fe}(t)|$ between the upper and intermediate levels. At high thermal photon numbers, thermalization drives the system toward equipartition, such that the populations of all levels become nearly equal. We show that increasing thermal occupation shortens the lifetime of quantum superpositions and accelerates decoherence, as evidenced by the loss of $|\rho_{fe}(t)|$. In the steady-state fluorescence spectrum of the $|e\rangle \rightarrow |g\rangle$ transition, Lorentzian broadening, proportional to the effective decoherence rate, highlights the influence of thermal photons on phase coherence and spectral features. These findings have direct implications for quantum information processing: they demonstrate how thermal noise markedly degrades coherence, gate fidelity, and photon emission quality.

Keywords: *population distribution; three-level cascade atom; thermal light field; steady state population; rate-equation model; coherence decay.*

ACRONYMS

EIT:	Electromagnetically Induced Transparency
FWHM:	Full Width at Half Maximum
GKSL:	Gorini–Kossakowski–Sudarshan–Lindblad
HWHM:	Half Width at Half Maximum
QED:	Quantum Electrodynamics
QRT:	Quantum Regression Theorem
RWA:	Rotating Wave Approximation

List of Figures

3.1	Three-level atom in a cascade configuration	19
4.1	Time evolution of the level populations $\rho_{ff}(t)$, $\rho_{ee}(t)$, and $\rho_{gg}(t)$ for a three-level cascade atom with a mean photon number $\bar{n} = 0.01$, coupled to a thermal bath. The parameters are $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0.5$	47
4.2	Time evolution of populations at intermediate thermal photon number ($\bar{n} = 1$) parameters $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0.5$	49
4.3	Time evolution of populations at high thermal photon number ($\bar{n} = 10$) parameters $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0.5$. . .	51
4.4	Steady-state populations ρ_{gg}^{ss} , ρ_{ee}^{ss} , ρ_{ff}^{ss} as a function of mean photon number $\bar{n}$ Parameters $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0$	52
4.5	Time evolution of the magnitude of the coherence $ \rho_{fe}(t) $ the upper level $ f\rangle$ and intermediate level $ e\rangle$ in a three-level cascade atom for different thermal photon numbers $\bar{n}$	54
4.6	Steady-state fluorescence spectrum $S(\omega)$ of the $ e\rangle \rightarrow g\rangle$ transition in a three-level cascade atom under thermal light.	56

List of Tables

Contents

BIBLIOGRAPHICAL SKETCH	v
STATEMENT OF THE AUTHOR	vi
ACKNOWLEDGEMENT	vii
ABSTRACT	ix
ACRONYMS	ix
1 INTRODUCTION	1
1.1 Background of the study	1
1.2 Problem Statement	4
1.3 Research Questions	5
1.4 Objectives of the Study	6
1.4.1 General Objective	6
1.4.2 Specific Objectives	6
1.5 Significance of the Study	6
1.6 Delimitation of the Study	7
1.7 Limitation of the Study	7
2 CHAPTER TWO: REVIEW OF RELATED LITERATURE	8

2.1	Three-Level Cascade Systems and Thermal Light Interactions	8
2.2	Hamiltonian of Three-Level Cascade Atom Interacting with thermal Light	9
2.2.1	Matrix Form (Atomic Part Only)	10
2.3	Master Equation of Three-Level Cascade Atom	11
2.4	Population Dynamics and Steady-State Behavior	11
2.5	Coherence and Decoherence Effects	12
2.6	Thermodynamic Implications	13
2.6.1	Thermodynamic Properties	13
2.7	Applications in Quantum Technologies	14
2.8	Collapse and Revival of Three-Level Cascade System	15
2.8.1	Collapse	15
2.8.2	Revival	15
2.9	Input-Output formalism and Photon Flux	16
2.10	Quantum Regression Theorem and Fluorescence Spectrum	16
2.11	Photon Statistics: Thermal versus Coherent Light	17
3	METHODOLOGY	18
3.1	Theoretical Model and Tools	18
3.2	Physical System: Cascade Configuration	18
3.3	Hamiltonian of Cascade Configuration.	19
3.3.1	Atomic Hamiltonian ($\hat{H}_{\text{atom}}$)	20
3.3.2	Field Hamiltonian ($\hat{H}_{\text{field}}$)	20
3.3.3	Interaction Hamiltonian ($\hat{H}_{\text{int}}$)	21
3.3.4	Total Hamiltonian	21

3.4	Thermal (Chaotic) Light	22
3.5	Density Matrix	24
3.6	Density Matrix Time Evolution	25
3.7	Open Quantum Systems (Lindblad Master Equation) . .	26
3.8	Population (Rate) Equations	31
3.8.1	Rates Matrix Form	32
3.8.2	Steady State Population	33
3.9	Coherence (Rate) Equations	35
3.9.1	Rate Matrix Form	37
3.9.2	Steady States Coherence	38
3.10	Input–Output Theorem	39
3.11	Photon Flux	40
3.12	Steady-State Photon Flux	41
3.13	Fluorescence Power Spectrum	42
3.14	Quantum Regression Theorem (QRT)	43
3.15	Analytic Form (Lorentzian Limit)	43
3.16	Photon Statistics	45
3.17	Photon Number Statistics versus $g^{(2)}(\tau)$	45
4	CHAPTER FOUR: RESULT AND DISCUSSION	47
4.1	Results	47
4.2	Rate Equation Model for Population Decay Atom under Weak Thermal Excitation	47
4.3	Intermediate Temperature Population Dynamics in a Three- Level Cascade Atom	49

4.4	High Temperature Population Dynamics in a Three Level Cascade Atom	50
4.5	Steady State Population Distribution vs. Thermal Photon Number in a Three-Level Cascade Atom	51
4.6	Thermal Photon Induced Coherence Decay between Upper and Intermediate Levels in a Cascade Atom	54
4.7	Fluorescence Spectrum under Thermal Bath in a Three-Level Cascade System	55
5	CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATION	58
5.1	Summary	58
5.2	Conclusions	59
5.3	Recommendation	60

1 INTRODUCTION

1.1 Background of the study

The interaction of atoms with lasers and other coherent light sources has been the focus of previous research. On the other hand, thermal light, which is characterized by oscillations in intensity and incoherence due to the unpredictable phases of its constituent modes, is commonly encountered in practical environments. Blackbody radiation and other forms of thermal light were modeled as statistical mixtures using Bose-Einstein statistics. Temperature-dependent probabilities were used to realize each combination of photon counts (Goodman, 1985).

Researchers used the density matrix formalism to model such systems, allowing them to represent both mixed and pure quantum states. The development of the isolated system was then expanded to incorporate environmental interactions using the Lindblad master equation, which was used under the rotating wave approximation. By maintaining essential characteristics like positivity and the normalisation of probabilities, this formulation made sure that the system's evolution remained physically consistent (Schaller & Brandes, 2008).

The interactions between two-level atoms and thermal fields have been the main focus of earlier research; these studies showed limited ability to control quantum states and rapid decoherence. Nevertheless, little is known about how thermal light affects three-level systems, particularly in cascade configurations. A deeper understanding of phenomena such

as decoherence, thermalization, and population inversion under realistic, imperfect lighting conditions could be obtained by examining these interactions (Macovei, Ficek, & Keitel, 2006).

The interaction between light and matter in situations where temperature-defined photon populations dictate transitions in a cascade atomic system has been described via theoretical modeling that links thermodynamics with quantum statistical mechanics. Understanding phenomena such as laser cooling and quantum information processing in open systems requires using this ensemble-based methodology. By offering a thorough framework for simulating the evolution of such systems under realistic circumstances, this work aims to close the knowledge gap regarding the dynamics of a three-level cascade atomic system interacting with thermal light. Wang et al. (2017) established a closely related theoretical framework by showing that laser-cooled atoms can induce photon thermalization into a grand canonical ensemble, thereby unifying thermodynamics and light-matter interactions.

A thermally driven cascade atom emits a fluorescence spectrum composed of a coherent delta peak at the driving frequency and inelastic Lorentzian lines at each allowed atomic transition, the latter arising from decaying dipole correlations whose dynamics follow the same master equation via the quantum regression theorem. Each transition contributes one Lorentzian line whose linewidth is set by its coherence decay rate and whose intensity tracks the steady-state population of the emit-

ting level (Dalton & Garraway, 2007).

A nonclassical light, especially antibunched photons, serves as a robust single-photon resource in quantum communication, computing, and sensing, it is crucial to understand both the spectral decomposition and photon statistics (e.g., antibunched vs. Poissonian) in such cascade emission.

1.2 Problem Statement

Understanding how different kinds of light fields interact with atomic systems has been essential to the development of quantum technologies in quantum optics. The three-level atom in a cascade configuration, one of several atomic configurations, offers a rich platform for researching quantum coherence, photon statistics, and population dynamics, because its energy levels are arranged so that an atom transitions from the highest to the lowest energy state via an intermediate level. While considerable research has been devoted to interactions with coherent and squeezed light, relatively less attention has been paid to the effects of thermal light on such three-level systems, despite its relevance in realistic and noisy environments (Zhang, Xie, & Zhou, 2025).

The unpredictable phase and intensity fluctuations of thermal light present special difficulties for manipulating atomic populations and maintaining coherence. Earlier research, which focused mainly on two-level atoms interacting with thermal fields, revealed rapid decoherence and very limited control over quantum states. However, how these decoherence effects manifest in a more intricate three-level cascade structure remained largely unexplored, especially with respect to emission spectra, transition probabilities, and population trapping (Macovei, Ficek, & Keitel, 2023). Furthermore, previously unexplored nonlinearities arise from the interaction between thermal photon statistics and the cascade system topology, in which spontaneous emission from the upper level influences

the dynamics of the lower transition. Even under ordinary thermal excitation, these nonlinearities could give rise to nonclassical emission patterns or unexpected steady-state behaviors (Obada, Eied, & Abd Al Kader, 2024).

Thus, the main problem this study seeks to address is the lack of theoretical understanding regarding the interaction of thermal light with a three-level atom arranged in a cascade configuration. Specifically, the study aims to examine the effects of thermal field characteristics on the system's emission behavior, coherence properties, and population dynamics. By developing a master equation approach and solving for steady-state populations and spectra, this work intends to close a gap in the literature and advance our knowledge of open quantum systems driven by classical noisy fields (Belay, 2023).

1.3 Research Questions

1. Do analytical expressions from a rate-equation model correctly predict steady-state populations for low, intermediate, and high thermal photon numbers?
2. How do atomic-level population trajectories differ under low, intermediate, and high thermal photon numbers over time?
3. How does the number of thermal photons affect the rate of decay of coherence between the upper and intermediate levels?
4. How are the transition rates influenced by thermal photons?

1.4 Objectives of the Study

1.4.1 General Objective

The main objective of this study was to investigate the quantum dynamics of a three-level atom interacting with thermal light.

1.4.2 Specific Objectives

More specifically, the study aimed to:

1. To derive analytical steady-state populations using a rate equation model and validate them for low, medium, and high thermal photon numbers.
2. To explore how atomic-level populations evolve over time under different thermal photon regimes (low, moderate, high).
3. To study how increasing thermal photon number degrades coherence between the upper and intermediate levels.
4. To determine transition rates influenced by thermal photons.

1.5 Significance of the Study

This study deepened our understanding of how thermal light interacts with a three-level cascade atomic system, knowledge that is essential for developing quantum technologies capable of operating in noisy, realistic environments. It demonstrated how thermal photon statistics affect atomic population dynamics, coherence, and photon correlations, a finding with direct relevance for designing and improving quantum

information processors and quantum sensors that make use of thermal light. Furthermore, this study enhanced the relevance of applications in quantum optics, laser physics, astrophysics, and quantum thermodynamics, where controlling decoherence and leveraging atomic transitions are crucial. It achieved this by emulating naturally occurring radiation such as thermal light rather than relying on idealized coherent fields.

1.6 Delimitation of the Study

The investigation focused solely on the atomic population dynamics, coherence, and photon statistics within a designated three-level cascade system interacting with thermal light. It did not consider other configurations (such as Λ type or V type atoms), the influence of coherent driving fields, multi-atom interactions, or higher-order nonlinear effects. Moreover, no other quantum or classical light sources were included in the analysis; only thermal light was treated as the external field, omitting other types of quantum or classical light sources.

1.7 Limitation of the Study

This exploration expanded our understanding of the effects of thermal light, ubiquitous and inherently noisy, on the dynamics of a three-level cascade atomic system, particularly concerning population behavior, coherence production, and photon correlations. These insights are crucial for designing quantum devices, such as sensors and information processors that must operate reliably in realistic, noisy environments.

2 CHAPTER TWO: REVIEW OF RELATED LITERATURE

A major area of research in quantum optics is the study of three-level atomic systems interacting with thermal light, which bridges the gap between fundamental quantum mechanics and real-world applications in quantum technologies. This review examines important advancements and discoveries in this field, with particular emphasis on the dynamics of such systems under the influences of a thermal environment.

2.1 Three-Level Cascade Systems and Thermal Light Interactions

An atom in a three-level system configured in a cascade arrangement typically has three energy levels: an upper excited state, an intermediate state, and a lower (ground) state. Transitions occur sequentially, first from the highest to the intermediate level, then from the intermediate to the ground state. This configuration enables a rich variety of quantum optical phenomena, such as spontaneous emission, absorption, and stimulated emission across two different transitions, making it very significant (Ferguson, Ficek, & Dalton, 1996).

Adding to the complexity, interesting consequences arise because of the incoherent nature of thermal light. Light that follows statistical photon distributions governed by Bose–Einstein statistics when such a system interacts with it. Unlike coherent laser light, thermal light exhibits random intensity variations, giving rise to super-Poissonian photon statis-

tics, in which the variance of photon number exceeds the mean. These stochastic fluctuations greatly affect the population dynamics of each atomic level, alter coherence properties, and broaden the emission spectra. Transitions induced by thermal light reduce the degree of quantum coherence normally maintained in coherently driven optical systems and smear out the sharpness of resonance (Finn, Greenlees, Kumar, & Lewis, 1988).

2.2 Hamiltonian of Three-Level Cascade Atom Interacting with thermal Light

We can use the following compact, conventional Hamiltonian. Let ω_{eg} be the transition frequency for $|e\rangle \leftrightarrow |g\rangle$ and ω_{fe} the frequency for $|f\rangle \leftrightarrow |e\rangle$. We assume a cascade (ladder) level ordering: $|g\rangle$ (ground), $|e\rangle$ (intermediate), and $|f\rangle$ (top). "Thermal light" means that the initial field states and baths are thermal, and the Hamiltonian is the same as that for a quantized electromagnetic field.

The most general full multimode Hamiltonian can be written as

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{field} + \hat{H}_{int} \quad (2.1)$$

with

$$\hat{H}_{atom} = \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_{fg}|f\rangle\langle f| \quad (2.2)$$

For the field and interaction terms, we write

$$\hat{H}_{field} = \sum_{k,\lambda} \hbar\omega_k a_{k\lambda}^\dagger a_{k\lambda} e^{i\mathbf{k}\cdot\mathbf{r}_0}, \quad (2.3)$$

$$\hat{H}_{int} = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}(\mathbf{r}_0) = \sum_{k,\lambda} \sum_{i>j} \left(g_{k\lambda}^{(ij)} a_{k\lambda} |j\rangle\langle i| + \text{h.c.} \right) \quad (2.4)$$

where the allowed dipole transitions are $i > j$, specifically $|f\rangle \leftrightarrow |e\rangle$ and $|e\rangle \leftrightarrow |g\rangle$. The coupling constants are

$$g_{k\lambda}^{(ij)} = -i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} (\mathbf{d}_{ij} \cdot \hat{\mathbf{e}}_{k\lambda}) e^{i\mathbf{k} \cdot \mathbf{r}_0}, \quad (2.5)$$

Utilizing

$$\mathbf{d}_{ij} = \langle i | \hat{\mathbf{d}} | j \rangle \quad (2.6)$$

this Hamiltonian is written before the rotating wave approximation (RWA). Only energy conserving terms are kept in the interaction Hamiltonian that is shown.

2.2.1 Matrix Form (Atomic Part Only)

We take the basis $\{|g\rangle, |e\rangle, |f\rangle\}$. Then

$$\hat{H}_{\text{atom}} = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega_{eg} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.7)$$

with the relation $\omega_{fg} = \omega_{fe} + \omega_{eg}$.

In the same basis, the interaction (atomic transition) operators appear as matrices. For example:

- The operator $|e\rangle\langle f|$ is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.8)$$

- The operator $|g\rangle\langle e|$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.9)$$

2.3 Master Equation of Three-Level Cascade Atom

The theory of open quantum systems, in which the atom (as the quantum system) interacts with an external environment such as vacuum modes or thermal radiation, which causes dissipation and degrades coherence, serves as the foundation for the master equation of a three-level atom. This equation, which couples coherent dynamics (unitary evolution) with incoherent processes like spontaneous emission and thermal absorption, describes the time evolution of the atom's density matrix $\rho(t)$ (Breuer & Petruccione, 2002). The Lindblad form, the most commonly used structure in the Markovian open-system framework, is generally used to express the master equation. The von Neumann (or Liouville–von Neumann) equation is generalized by the Lindblad master equation, which adds dissipative terms to the unitary evolution while preserving positivity and the trace of the density matrix. It can be written as

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}_{atom}, \rho] + \sum_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right) \quad (2.10)$$

where $\hat{H}$ is the system Hamiltonian governing coherent dynamics, and the set $\{L_i\}$ are the Lindblad (quantum jump) operators responsible for dissipative processes such as decay and dephasing.

2.4 Population Dynamics and Steady-State Behavior

The population dynamics of three-level atomic systems under thermal light are studied. The derived rate equations capture the excitation and decay dynamics for the populations of the atomic levels. Steady-state solutions reveal the system's thermalization behavior, showing how the

population distribution depends on transition rates and thermal photon occupation numbers. At low temperatures, the system favors the ground state, consistent with the Boltzmann distribution; at high temperatures, the populations tend to equalize, reflecting a thermal distribution (Koyu & Tscherbul, 2021).

2.5 Coherence and Decoherence Effects

The off-diagonal (coherence) components of the atomic density matrix are disrupted by interactions with thermal light, which introduce decoherence. Thermal photons degrade quantum phenomena such as coherent population trapping and superposition-based processes by reducing coherence between energy levels (for example, ρ_{ge} , ρ_{ef} , ρ_{gf}), owing to their randomly fluctuating phases and intensities. Since even classical thermal excitation can occasionally produce nonlinear steady-state effects or non-classical emission features, it is crucial to understand how thermal photon statistics interact with the level structure of a cascade (three-level) system (Ayehu, 2021).

In a three-level atom, off-diagonal density-matrix elements

$$\rho_{ij} = \langle i | \rho | j \rangle \quad \text{for } i \neq j. \quad (2.11)$$

represent quantum coherences between states. For example, ρ_{ge} links the ground $|g\rangle$ and excited $|e\rangle$ states; ρ_{ef} connects the excited $|e\rangle$ and higher $|f\rangle$ states; and ρ_{gf} links the ground and upper levels. These terms indicate superpositions between energy levels. Over time, these off-diagonal

elements decay due to decoherence, often following an exponential law:

$$\frac{d\rho_{ij}}{dt} = -\Gamma_{ij}\rho_{ij} \quad (2.12)$$

where Γ_{ij} is the decoherence rate between states $|i\rangle$ and $|j\rangle$. This behavior illustrates the progressive loss of coherence in the system (Zhong & Robicheaux, 2016).

2.6 Thermodynamic Implications

Through processes such as spontaneous emission, absorption, and stimulated emission, fundamental thermodynamic concepts can be applied to a three level cascade atomic system interacting with thermal light. Each of these processes contributes to entropy production as well as energy exchange. When connected to both a hot and a cold reservoir, the system may transport energy between them, effectively functioning as a quantum heat engine. Each transition's mean photon occupancy follows the Bose–Einstein distribution:

$$\bar{n}(\omega_{ij}) = \frac{1}{e^{\frac{\hbar\omega_{ij}}{k_B T}} - 1} \quad (2.13)$$

This occupancy affects transition rates and, consequently, the overall population dynamics. Thus, thermodynamic properties such as heat flows, work extraction, and entropy production are embodied within the quantum system (Laskar, 2024).

2.6.1 Thermodynamic Properties

$$Z = \sum_i e^{-E_i/(k_B T)},$$

summing over the atom's three energy levels, is the key quantity in the canonical ensemble (fixed temperature, volume, and atom number). Essential thermodynamic quantities can be obtained from Z . The Helmholtz free energy is

$$F = -k_B T \ln Z, \quad (2.14)$$

the internal energy

$$U = k_B T^2 \frac{\partial \ln Z}{\partial T}, \quad (2.15)$$

and the entropy

$$S = -k_B T \sum_i p_i \ln p_i, \quad (2.16)$$

with

$$P_i = \frac{e^{\left(-\frac{E_i}{k_B T}\right)}}{Z} \quad (2.17)$$

being the thermal occupation probability. These relationships provide a clear connection between macroscopic thermodynamic behavior and the atom's microscopic energy levels.

2.7 Applications in Quantum Technologies

Numerous quantum technologies benefit from studying three-level atomic system-level interacting with thermal light (Younes, Putnam, Hamilton, & Campbell, 2025). For example, laser cooling methods, which cool atoms via interactions with laser beams, require an understanding of thermalization dynamics. Furthermore, understanding population dynamics and coherence features is essential for quantum information processing, where controlling population distributions and preserving coherence are critical for reliable operation. Moreover, investigating nonlin-

earities and steady-state behavior under thermal light may lead to new quantum devices and protocols.

2.8 Collapse and Revival of Three-Level Cascade System

2.8.1 Collapse

In a three-level cascade atom, thermal light induces incoherent transitions through the states $|f\rangle \rightarrow |e\rangle \rightarrow |g\rangle$ governed by a master equation with Lindblad terms. The average photon number $\langle n \rangle$, from the Bose–Einstein distribution, drives both stimulated absorption and emission, leading to rapid decoherence and population redistribution. Unlike coherent fields that allow collapse–revival dynamics, thermal light suppresses off-diagonal density matrix elements, eliminating superpositions and preventing revival events (Rau & Zhao, 2003).

2.8.2 Revival

When the thermal field has partial quantization or structured mode coupling, residual atom–field correlations can restore coherence and populations in a cascade three-level atom ($|g\rangle \rightarrow |e\rangle \rightarrow |f\rangle$) interacting with thermal light. Usually, incoherence suppresses revival events. To model this, one uses a Lindblad master equation for dissipation along with a three-level extension of the Jaynes–Cummings-type Hamiltonian in the rotating wave approximation. Although thermal noise tends to wash out coherence, numerical studies (Dalton & Garraway, 2003) in organized reservoirs show that, when the field retains some structure or coherence, revival-like oscillations may appear in the intermediate and upper-level

populations.

2.9 Input-Output formalism and Photon Flux

Internal quantum operators and observable output fields are connected by input-output theory, which is widely used in QED, open systems, and quantum networks. A typical relation is

$$a_{\text{out}}(t) = a_{\text{in}}(t) + \sqrt{\gamma_i} \sigma_i(t) \quad (2.18)$$

where γ_i is the spontaneous emission rate. The photon flux in the output mode is then

$$\Phi_i(t) = \langle a_{\text{out}}^\dagger a_{\text{out}} \rangle = \gamma_i \langle \sigma_i^\dagger \sigma_i \rangle_t + \langle a_{\text{in}}^\dagger a_{\text{in}} \rangle \quad (2.19)$$

2.10 Quantum Regression Theorem and Fluorescence Spectrum

A common method in quantum optics for relating two time correlation functions to the dynamics of single time expectation values (via the master equation) is the quantum regression theorem (QRT). For example, the power spectrum for resonance fluorescence can be written as:

$$S_i(\omega) = \frac{1}{2\pi R} \int_{-\infty}^{\infty} \langle a_{\text{out}}(t + \tau) a_{\text{out}}^\dagger(t) \rangle_{\text{ss}} e^{i(\omega - \omega_L)\tau} d\tau \quad (2.20)$$

In this expression, the correlation $a_{\text{out}}(t + \tau) a_{\text{out}}^\dagger(t)$ evolves by the same Liouvillian (i.e. the same master equation dynamics) that governs $\sigma_i^\dagger(t)$ QRT; one propagates the operator $\sigma_i^\dagger(t)$ forward in the interval τ under the Liouvillian superoperator, using the steady state as the starting condition (Swain, 1981).

2.11 Photon Statistics: Thermal versus Coherent Light

The fluctuations in photon counts are characterized by photon statistics. Thermal (or chaotic) light exhibits super Poissonian behavior and follows Bose–Einstein statistics, meaning

$$(\Delta n)^2 = \langle n \rangle + \langle n \rangle^2 \quad (2.21)$$

Conversely, coherent light (for example, from an ideal laser) has Poissonian statistics, for which

$$(\Delta n)^2 = \langle n \rangle \quad (2.22)$$

A useful metric is the Mandel Q parameter. For Poissonian statistics,

$$Q = \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle} \quad (2.23)$$

which is zero for poissonian, positive for supe-Poissonian,, and negative for sub-poissonian statistics.

3 METHODOLOGY

3.1 Theoretical Model and Tools

In this chapter, we employ the density matrix formalism via QuTiP (Quantum Toolbox in Python) to describe a cascade three-level atom interacting with thermal light. Using a three-level extension of a Jaynes-Cummings-type Hamiltonian under the rotating wave approximation, we couple the atomic transitions to thermal modes and construct a Lindblad master equation that includes spontaneous emission, absorption, and stimulated emission by the thermal reservoir. We then obtain the time evolution of populations and coherence through numerical solution of the master equation, showing how partial coherence or field structure can permit revival-like oscillations even in the presence of thermal excitation.

3.2 Physical System: Cascade Configuration

In a cascade (or ladder, Ξ type) three-level atom, three energy states are arranged in ascending order: the ground state $|g\rangle$, the intermediate state $|e\rangle$, and the excited state $|f\rangle$. The allowed dipole transitions occur sequentially: first from $|g\rangle$ to $|e\rangle$, then from $|e\rangle$ to $|f\rangle$. Importantly, direct transitions between the ground and excited states ($|g\rangle \leftrightarrow |f\rangle$) are forbidden. Each allowed transition couples to its own thermal field (or reservoir), which interacts exclusively with that specific transition. These are often described as thermal reservoirs or blackbody radiation fields (Dalton & Garraway, 2003).

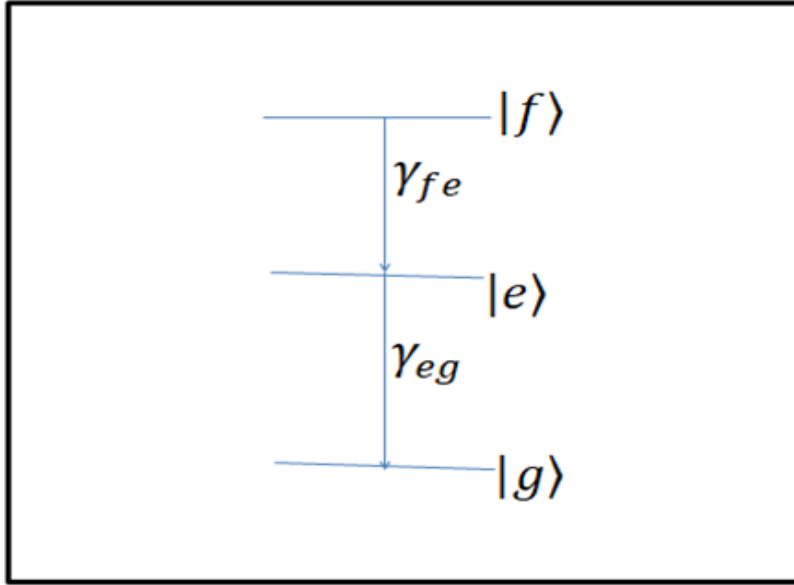


Figure 3.1: Three-level atom in a cascade configuration

3.3 Hamiltonian of Cascade Configuration.

Under thermal illumination, the Hamiltonian of a cascade-configured three-level atom can be divided into three main parts: the atomic Hamiltonian, the field Hamiltonian, and the interaction Hamiltonian. The atomic Hamiltonian encodes the ladder type transitions ($|g\rangle \rightarrow |e\rangle \rightarrow |f\rangle$) and the intrinsic energy structure of the isolated atom. The field Hamiltonian models the electromagnetic environment by treating the thermal radiation modes as quantized harmonic oscillators. The interaction Hamiltonian couples atomic transitions to the field and mediates energy exchange between the atom and the thermal bath via absorption and emission. Together, these three components provide a complete open system description of the cascade atom interacting with a thermal light field (Shrestha et al., 2004).

3.3.1 Atomic Hamiltonian ($\hat{H}_{\text{atom}}$)

We consider a three-level atom with energy eigenstates $|g\rangle$, $|e\rangle$, and $|f\rangle$, having corresponding energies $\hbar\omega_g$, $\hbar\omega_e$, and $\hbar\omega_f$. The atomic Hamiltonian can be written as

$$\hat{H}_{\text{atom}} = \hbar\omega_g|g\rangle\langle g| + \hbar\omega_e|e\rangle\langle e| + \hbar\omega_f|f\rangle\langle f|. \quad (3.1)$$

We then take the ground state energy $E_g = 0$ as reference and define the transition energies

$$E_e = \hbar\omega_{eg}, \quad E_f = \hbar\omega_{fg}. \quad (3.2)$$

Under this convention, the Hamiltonian becomes

$$\hat{H}_{\text{atom}} = \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_{fg}|f\rangle\langle f|. \quad (3.3)$$

It is often convenient to define the transition frequencies

$$\omega_{eg} = \omega_e - \omega_g, \quad \omega_{fe} = \omega_f - \omega_e, \quad (3.4)$$

so that all energies are expressed relative to the ground state $|g\rangle$.

3.3.2 Field Hamiltonian ($\hat{H}_{\text{field}}$)

We model the thermal light as a reservoir of quantized modes. The free field Hamiltonian is given by

$$\hat{H}_{\text{field}} = \sum_k \hbar\omega_k a_k^\dagger a_k \quad (3.5)$$

In a two-mode representation, this becomes

$$\hat{H}_{\text{field}} = \hbar\omega_1 a_1^\dagger a_1 + \hbar\omega_2 a_2^\dagger a_2 \quad (3.6)$$

Here, $a_2^\dagger, a_2$ act on the mode associated with the $|e\rangle \leftrightarrow |f\rangle$ transition, while $a_1^\dagger, a_1$ are the creation and annihilation operators for the mode coupled to the $|g\rangle \leftrightarrow |e\rangle$ transition. The frequencies of the modes are given by ω_k .

3.3.3 Interaction Hamiltonian ($\hat{H}_{\text{int}}$)

The interaction between the atom and the field is described using the rotating wave approximation (RWA) and the dipole approximation:

$$\hat{H}_{\text{int}} = -\mathbf{d} \cdot \mathbf{E} \quad (3.7)$$

where $\mathbf{E}$ is the quantized electric field and $\mathbf{d}$ is the atomic dipole operator. For the two allowed transitions $|g\rangle \leftrightarrow |e\rangle$ and $|e\rangle \leftrightarrow |f\rangle$, the interaction Hamiltonian becomes

$$\hat{H}_{\text{int}} = \hbar g_{eg} (\sigma_{eg} a_{eg} + \sigma_{ge} a_{eg}^\dagger) + \hbar g_{fe} (\sigma_{fe} a_{fe} + \sigma_{ef} a_{fe}^\dagger) \quad (3.8)$$

where:

- $\sigma_{ij} = |i\rangle\langle j|$ are the atomic transition operators,
- g_{eg} and g_{fe} are coupling constants corresponding to each transition,
- a_{eg}, a_{fe} (and their Hermitian conjugates) are the annihilation and creation operators of the field modes at the respective transition frequencies.

3.3.4 Total Hamiltonian

The full Hamiltonian for the cascade three-level atom interacting with thermal light is obtained by summing all contributions:

$$\hat{H} = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}} + \hat{H}_{\text{int}} \quad (3.9)$$

Explicitly, using the forms from earlier:

$$\begin{aligned}
\hat{H} = & \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_{fg}|f\rangle\langle f| \\
& + \hbar\omega_1 a_1^\dagger a_1 + \hbar\omega_2 a_2^\dagger a_2 \\
& + \hbar g_{eg} (\sigma_{eg} a_{eg} + \sigma_{ge} a_{eg}^\dagger) + \hbar g_{fe} (\sigma_{fe} a_{fe} + \sigma_{ef} a_{fe}^\dagger)
\end{aligned} \tag{3.10}$$

Here, the first two terms come from $\hat{H}_{\text{atom}}$, the next two from $\hat{H}_{\text{field}}$, and the last two from $\hat{H}_{\text{int}}$.

3.4 Thermal (Chaotic) Light

A thermal source in equilibrium at a temperature T emits thermal light, also referred to as chaotic or blackbody radiation. The photons in such light follow Bose–Einstein statistics. Its spectral properties are described by Planck’s law, which dictates how the radiation intensity varies with frequency and temperature (Mandel & Wolf, 1995). Thermal light differs from coherent sources such as lasers in that it exhibits characteristic intensity fluctuations, arising from the random phases among the various light modes. The thermal density matrix ρ_{th} is used to characterize the state of thermal light in quantum optics and thermodynamics. This matrix, weighted by probabilities determined by the average photon number $\bar{n}$, represents a statistical mixture of photon number states $|n\rangle$.

Mathematically,

$$\rho_{th} = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}} |n\rangle\langle n| \tag{3.11}$$

encodes the probability that the field contains $\mathbf{n}$ photons. Each mode $\mathbf{k}$ of the radiation field is described by the Hermitian Hamiltonian of a

quantum harmonic oscillator:

$$\hat{H} = \hbar\omega \left(a_k^\dagger a_k + \frac{1}{2} \right) \quad (3.12)$$

where $\hbar\omega$ is the photon energy in that mode, and $a_k^\dagger, a_k$ are the creation and annihilation operators for photons in mode k . At thermal equilibrium, the density matrix is given by the Boltzmann factor:

$$\rho_{\text{th}} = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\beta E_n} |n\rangle \langle n| \quad (3.13)$$

where Z is the partition function, $E_n = \hbar\omega \left(n + \frac{1}{2} \right)$ is the energy of the n -photon state, and $\beta = \frac{1}{k_B T}$ is the inverse temperature. The partition function is

$$Z = \sum_{n=0}^{\infty} e^{-\beta E_n} \quad (3.14)$$

Since $E_n = \hbar\omega \left(n + \frac{1}{2} \right)$, one finds

$$Z = e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} (e^{-\beta\hbar\omega})^n = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} \quad (3.15)$$

The thermal density matrix can then be written as

$$\rho_{\text{th}} = (1 - e^{-\beta\hbar\omega}) \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} |n\rangle \langle n| \quad (3.16)$$

This expression can also be restated using the Bose–Einstein definition of the average photon number $\bar{n}$:

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1} \quad (3.17)$$

From this we find

$$e^{-\beta\hbar\omega} = \frac{\bar{n}}{1 + \bar{n}} \quad (3.18)$$

which then leads back to the original expression for ρ_{th} in terms of $\bar{n}$. Physically, the mean number of photons occupying energy level $\bar{n}$ at

a temperature T is denoted by $\bar{n}$. The mean photon number for each atomic transition in a three-level system is associated with its own thermal reservoir. For instance, the transitions $|f\rangle \leftrightarrow |e\rangle$ and $|e\rangle \leftrightarrow |g\rangle$ are coupled to thermal reservoirs with mean photon numbers $\bar{n}_{fe}$ and $\bar{n}_{eg}$, respectively:

$$\bar{n}_{fe} = \frac{1}{e^{\hbar\omega_{fe}/k_B T} - 1}, \quad \bar{n}_{eg} = \frac{1}{e^{\hbar\omega_{eg}/k_B T} - 1} \quad (3.19)$$

With photon populations controlled by these temperature-dependent distributions, this approach allows us to describe how thermal light drives transitions in quantum systems.

3.5 Density Matrix

We can choose a basis $\{|i\rangle\}_{i=1}^N$ that spans an N -dimensional Hilbert space. On that basis, for a three-level system:

$$|g\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |e\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |f\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (3.20)$$

Then the density matrix ρ may be written as

$$\rho = \begin{bmatrix} \rho_{gg} & \rho_{ge} & \rho_{gf} \\ \rho_{eg} & \rho_{ee} & \rho_{ef} \\ \rho_{fg} & \rho_{fe} & \rho_{ff} \end{bmatrix} \quad (3.21)$$

The populations are the diagonal elements ρ_{ii} (which satisfy $\rho_{ii} \geq 0$) and they must obey

$$\sum_{i=1}^N \rho_{ii} = 1 \quad (3.22)$$

Because the density matrix is Hermitian, the off-diagonal elements (called coherences) satisfy

$$\rho_{ij} = \rho_{ji}^* \quad (3.23)$$

3.6 Density Matrix Time Evolution

The Schrödinger equation, which governs how the state vector changes over time, describes the time evolution of a pure state in quantum mechanics. For a pure state, the Schrödinger equation is:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \quad (3.24)$$

where $\hat{H}(t)$ is the Hamiltonian operator representing the total energy of the system. The corresponding density operator for a pure state is

$$\hat{\rho}(t) = |\psi(t)\rangle \langle \psi(t)| \quad (3.25)$$

Differentiating $\hat{\rho}(t)$ with respect to time gives

$$\frac{d\hat{\rho}}{dt} = \frac{d}{dt} (|\psi(t)\rangle \langle \psi(t)|) = \left(\frac{d}{dt} |\psi(t)\rangle \right) \langle \psi(t)| + |\psi(t)\rangle \left(\frac{d}{dt} \langle \psi(t)| \right) \quad (3.26)$$

Using the Schrödinger equation and its Hermitian conjugate:

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} \hat{H}(t) |\psi(t)\rangle \quad (3.27)$$

$$\frac{d}{dt} \langle \psi(t)| = \frac{i}{\hbar} \langle \psi(t)| \hat{H}(t) \quad (3.28)$$

Substituting these into Eq. (3.26) yields

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} \hat{H}(t) |\psi(t)\rangle \langle \psi(t)| + \frac{i}{\hbar} |\psi(t)\rangle \langle \psi(t)| \hat{H}(t) \quad (3.29)$$

This simplifies to the familiar von Neumann (Liouville–von Neumann) equation:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}(t), \hat{\rho}(t)] \quad (3.30)$$

Here $[A, B] = AB - BA$ denotes the commutator of operators A and B .

This equation characterizes the time evolution of the density operator

for a pure state and provides a more general framework that also applies to mixed states. For a statistical mixture of pure states, the density operator is

$$\hat{\rho} = \sum_j \rho_j |\psi_j\rangle\langle\psi_j| \quad (3.31)$$

where ρ_j the probabilities are associated with each pure state $|\psi_j\rangle$. Since the Schrödinger evolution is linear in each component, the von Neumann equation still applies to statistical mixtures and is valid in this more general situation as well (Von Neumann, 1932).

3.7 Open Quantum Systems (Lindblad Master Equation)

In an open quantum system, the system of interest is coupled to an external environment, or bath. The master equation in Lindblad, form also known as the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation or simply the Lindbladian, is a powerful result that follows from certain simplifying assumptions. Generally, the bath has an infinite number of degrees of freedom and is too complex to describe or simulate exactly. Below is a somewhat simplified form of the Lindblad master equation for a weakly interacting system and bath (Brasil, Fanchini, & Napolitano, 2013). Tracing over the field and using the Born–Markov approximations yields a reduced master equation for the atom:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}_{\text{atom}}, \rho] + \mathcal{L}_{\text{diss}}(\rho) \quad (3.32)$$

Here we set $\omega_g = 0$, and

$$\hat{H}_{\text{atom}} = \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_{fg}|f\rangle\langle f| \quad (3.33)$$

The superoperator $\mathcal{L}_{\text{diss}}$ includes dissipators corresponding to every allowed transition, induced by coupling to the thermal baths. To account for a finite-temperature environment, we introduce four collapse (jump) operators: two for emission and two for absorption.

First, we define the atomic transition operators as follows:

$$\sigma_{fe} = |e\rangle\langle f|, \quad \sigma_{ef} = |f\rangle\langle e| \text{ for the } |f\rangle \leftrightarrow |e\rangle \text{ transitions,}$$

$$\sigma_{eg} = |g\rangle\langle e|, \quad \sigma_{ge} = |e\rangle\langle g| \text{ for the } |e\rangle \leftrightarrow |g\rangle \text{ transitions.}$$

Then we construct the Lindblad collapse operators as:

$$L_{fe}^- = \sqrt{\gamma_{fe}(\bar{n}_{fe} + 1)} \sigma_{fe}, \quad L_{fe}^+ = \sqrt{\gamma_{fe}\bar{n}_{fe}} \sigma_{ef} \quad (3.34)$$

$$L_{eg}^- = \sqrt{\gamma_{eg}(\bar{n}_{eg} + 1)} \sigma_{eg}, \quad L_{eg}^+ = \sqrt{\gamma_{eg}\bar{n}_{eg}} \sigma_{ge} \quad (3.35)$$

In order to account for both spontaneous and stimulated emission, each collapse operator includes a factor combining a decay rate γ_{ij} with either $\bar{n}_{ij} + 1$ or $\bar{n}$ to represent emission or absorption from the thermal reservoir. For a system coupled to a thermal bath, this structure conforms to standard Lindblad modeling: one jump operator represents loss (emission) and another represents environmental gain (absorption). The Lindblad dissipator is defined by

$$\mathcal{D}[L]\rho = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\} \quad (3.36)$$

Thus, the master equation becomes

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}_{\text{atom}}, \rho] + \sum_{x \in \{fe, eg\}} (\mathcal{D}[L_x^-]\rho + \mathcal{D}[L_x^+]\rho) \quad (3.37)$$

Equivalently, one might write

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}_{\text{atom}}, \rho] + \mathcal{L}_{fe}[\rho] + \mathcal{L}_{eg}[\rho] + \mathcal{L}_{fg}[\rho] \quad (3.38)$$

If the direct $|f\rangle \leftrightarrow |g\rangle$ transition is forbidden, as in a cascade configuration, we can drop $\mathcal{L}_{fg}$. The master equation then becomes

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}_{\text{atom}}, \rho] + \mathcal{L}_{fe}[\rho] + \mathcal{L}_{eg}[\rho] \quad (3.39)$$

This master equation accounts for both thermal excitation and spontaneous emission for each allowed transition. For each transition, the Liouvillian superoperator $\mathcal{L}[\rho]$ gives the Lindblad form of the master equation in a thermal bath:

$$L[\rho] = \gamma_{ij}(\bar{n}_{ij}+1) \left(\sigma_{ij}\rho\sigma_{ij}^\dagger - \frac{1}{2}\{\sigma_{ij}^\dagger\sigma_{ij}, \rho\} \right) + \gamma_{ij}\bar{n}_{ij} \left(\sigma_{ji}\rho\sigma_{ji}^\dagger - \frac{1}{2}\{\sigma_{ji}^\dagger\sigma_{ji}, \rho\} \right) \quad (3.40)$$

This form is applied to each transition. For example, the upper transition $|f\rangle \leftrightarrow |e\rangle$ is treated this way:

$$\mathcal{L}_{fe}[\rho] = \gamma_{fe}(\bar{n}_{fe} + 1) \left(\sigma_{fe}\rho\sigma_{fe}^\dagger - \frac{1}{2}\{\sigma_{fe}^\dagger\sigma_{fe}, \rho\} \right) \quad (3.41)$$

The first dissipator describes the decay (downward transition) from $|f\rangle$ to $|e\rangle$. Its rate is proportional to $\gamma_{fe}(\bar{n}_{fe}+1)$, capturing both spontaneous emission ($\bar{n}_{fe} = 0$) and stimulated emission.

$$\mathcal{D}[\sigma_{fe}]\rho = \gamma_{fe}(\bar{n}_{fe} + 1) \left(\sigma_{fe}\rho\sigma_{fe}^\dagger - \frac{1}{2}\{\sigma_{fe}^\dagger\sigma_{fe}, \rho\} \right) \quad (3.42)$$

By contrast, thermal excitation (the upward transition) from $|e\rangle$ to $|f\rangle$ is described by the second dissipator. Its rate, proportional to $\gamma_{fe}\bar{n}_{fe}$, represents the probability that the system absorbs energy from the environment. It is written as:

$$\mathcal{D}[\sigma_{ef}]\rho = \gamma_{fe}\bar{n}_{fe} \left(\sigma_{ef}\rho\sigma_{ef}^\dagger - \frac{1}{2}\{\sigma_{ef}^\dagger\sigma_{ef}, \rho\} \right) \quad (3.43)$$

Here, $\sigma_{ef} = |e\rangle\langle f|$ acts as the lowering operator (transitioning from $|f\rangle$ to $|e\rangle$), and $\sigma_{fe} = |f\rangle\langle e|$ is the raising operator (transitioning from $|e\rangle$ to $|f\rangle$).

$|f\rangle\rangle$). Meanwhile, the population projectors, $\sigma_{ff} = |f\rangle\langle f|$ and $\sigma_{ee} = |e\rangle\langle e|$ monitor the occupancy of their respective states. When combined, these terms give a Lindblad-form master equation that balances thermally induced transitions with spontaneous decay, thereby ensuring a physically accurate description of the dynamics of the two-level system interacting with a thermal bath (Breuer & Petruccione, 2002).

$$\mathcal{L}_{fe}[\rho] = \gamma_{fe}(\bar{n}_{fe}+1) \left(\sigma_{fe}\rho\sigma_{ef} - \frac{1}{2}\{\sigma_{ef}\sigma_{fe}, \rho\} \right) + \gamma_{fe}\bar{n}_{fe} \left(\sigma_{fe}\rho\sigma_{ef} - \frac{1}{2}\{\sigma_{ef}\sigma_{fe}, \rho\} \right) \quad (3.44)$$

We use the same Lindblad-superoperator structure as above to characterize the lower transition between $|e\rangle$ and $|g\rangle$. Under the rotating wave approximation in the interaction picture, the relevant part of the Liouvillian becomes:

$$\mathcal{L}_{eg}[\rho] = \gamma_{eg}(\bar{n}_{eg} + 1) \left(\sigma_{eg}\rho\sigma_{eg}^\dagger - \frac{1}{2}\{\sigma_{eg}^\dagger\sigma_{eg}, \rho\} \right) \quad (3.45)$$

With a rate that scales as $\gamma_{eg}(\bar{n}_{eg} + 1)$, the first term models decay from the excited state to the ground state $|e\rangle \rightarrow |g\rangle$. It reflects both spontaneous emission (which occurs even at zero thermal occupancy) and stimulated emission (which increases with thermal excitation). The corresponding dissipator is:

$$\mathcal{D}[\sigma_{eg}]\rho = \gamma_{eg}(\bar{n}_{eg} + 1) \left(\sigma_{eg}\rho\sigma_{ge} - \frac{1}{2}\{\sigma_{ge}\sigma_{eg}, \rho\} \right) \quad (3.46)$$

Thermal excitation (the upward transition $|g\rangle \rightarrow |e\rangle$) is captured by the second term. Its rate, proportional to $\gamma_{eg}\bar{n}_{eg}$, represents the probability of absorbing energy from the bath. Its dissipative form is:

$$\mathcal{D}[\sigma_{ge}]\rho = \gamma_{eg}\bar{n}_{eg} \left(\sigma_{ge}\rho\sigma_{eg} - \frac{1}{2}\{\sigma_{eg}\sigma_{ge}, \rho\} \right) \quad (3.47)$$

The lowering operator in this case is $\sigma_{ge} = |g\rangle\langle e|$, which effects a transition from $|e\rangle$ down to $|g\rangle$. The raising operator is $\sigma_{eg} = |e\rangle\langle g|$, which enables excitation from $|g\rangle$ to $|e\rangle$. The population projectors $\sigma_{ee} = |e\rangle\langle e|$ and $\sigma_{gg} = |g\rangle\langle g|$ monitor the occupancy of each level.

The combination of these Lindblad type terms describes the two level system's evolution interacting with a thermal environment in a balanced and thermodynamically consistent manner (Manzano, 2019). A compact form for the full Lindblad superoperator combining both effects is:

$$\mathcal{L}_{eg}[\rho] = (\bar{n}_{eg}+1) \left(\sigma_{eg}\rho\sigma_{ge} - \frac{1}{2}\{\sigma_{ge}\sigma_{eg}, \rho\} \right) + \gamma_{eg}\bar{n}_{eg} \left(\sigma_{ge}\rho\sigma_{eg} - \frac{1}{2}\{\sigma_{eg}\sigma_{ge}, \rho\} \right) \quad (3.48)$$

The loss of excitation (decay) is represented in the first term by $\sigma_{ge}\rho\sigma_{eg}$, together with the anti-commutator term to preserve trace and positivity in the dynamics. Thermal (upward) excitation from the ground state is handled by the second term. This structure mirrors the general Lindblad form for bosonic modes, where annihilation and creation operators play analogous roles in modeling thermalization.

The full Lindblad master equation can be written as

$$\frac{d\rho}{dt} = -i[\hat{H}_{\text{atom}}, \rho] + \gamma_{fe}(\bar{n}_{fe}+1) \mathcal{D}[\sigma_{fe}]\rho + \gamma_{fe}\bar{n}_{fe} \mathcal{D}[\sigma_{fe}^\dagger]\rho + \gamma_{eg}(\bar{n}_{eg}+1) \mathcal{D}[\sigma_{eg}]\rho + \gamma_{eg}\bar{n}_{eg} \mathcal{D}[\sigma_{eg}^\dagger]\rho \quad (3.49)$$

Emission (including spontaneous emission) is represented by terms proportional to $\gamma_{fe}(\bar{n}_{fe}+1)$. Thermal absorption (i.e., excitation by the bath) is represented by terms proportional to $\gamma_{eg}\bar{n}_{eg}$.

3.8 Population (Rate) Equations

Since the Lamb shift contributes only to coherent energy shifts and does not change the populations, we use only the dissipator terms to derive the rate equations for the level populations

$$P_i = \rho_{ii} = \langle i | \rho | i \rangle \quad (3.50)$$

We now focus on the $|f\rangle \leftrightarrow |e\rangle$ transition and its two dissipators. The relevant term is

$$\mathcal{L}_{fe}[\rho] = \gamma_{fe}(\bar{n}_{fe} + 1) \mathcal{D}[\sigma_{fe}]\rho + \gamma_{fe}\bar{n}_{fe} \mathcal{D}[\sigma_{fe}^\dagger]\rho \quad (3.51)$$

We calculate the contribution to the population ρ_{ff} from the first dissipator. For the first term, $\sigma_{fe}\rho\sigma_{ef}$ has zero ρ_{ff} components. But from the anticommutator part $\{\sigma_{ef}\sigma_{fe}, \rho\}$, we get a contribution $2\rho_{ff}$. Thus, the matrix element is

$$\langle f | \mathcal{D}[\sigma_{fe}]\rho | f \rangle = -\rho_{ff}, \quad (3.52)$$

and so multiplying by $\gamma_{fe}(\bar{n}_{fe} + 1)$ gives the rate

$$-\gamma_{fe}(\bar{n}_{fe} + 1)\rho_{ff}. \quad (3.53)$$

For the second term $\mathcal{D}[\sigma_{fe}^\dagger]\rho$, the anticommutator term vanishes. Thus,

$$\langle f | \sigma_{ef}\rho\sigma_{fe} | f \rangle = \rho_{ee}, \quad (3.54)$$

and when multiplied by $\gamma_{fe}\bar{n}_{fe}$, this yields the positive contribution

$$+\gamma_{fe}\bar{n}_{fe}\rho_{ee}. \quad (3.55)$$

Putting these results together, we obtain

$$\frac{d\rho_{ff}}{dt} = -\gamma_{fe}(\bar{n}_{fe} + 1)\rho_{ff} + \gamma_{fe}\bar{n}_{fe}\rho_{ee}. \quad (3.56)$$

Similarly, one shows that the $|e\rangle \leftrightarrow |g\rangle$ channel affects only ρ_{ee} , ρ_{gg} , ρ_{ff} . By applying the same method to each channel, one arrives at the full system of rate equations for all level populations.

The intermediate level equation is written as:

$$\frac{d\rho_{ee}}{dt} = \gamma_{fe}(\bar{n}_{fe} + 1)\rho_{ff} - \gamma_{fe}\bar{n}_{fe}\rho_{ee} - \gamma_{eg}(\bar{n}_{eg} + 1)\rho_{ee} + \gamma_{eg}\bar{n}_{eg}\rho_{gg}, \quad (3.57)$$

and the fundamental formula for the ground level is:

$$\frac{d\rho_{gg}}{dt} = \gamma_{eg}(\bar{n}_{eg} + 1)\rho_{ee} - \gamma_{eg}\bar{n}_{eg}\rho_{gg}. \quad (3.58)$$

By ensuring that the sum of populations remains constant over time, these rate equations preserve $\text{Tr}(\rho)$ (Carmichael, 1993).

3.8.1 Rates Matrix Form

A compact matrix (master equation) form provides a concise expression of the rate equations for a three-level cascade atom interacting with thermal light. We define the population vector:

$$P(t) = \begin{pmatrix} \rho_{ff} \\ \rho_{ee} \\ \rho_{gg} \end{pmatrix} \quad (3.59)$$

and write its time evolution as

$$\frac{d}{dt}P(t) = M P(t), \quad (3.60)$$

where M is the matrix of transition rates, given by

$$M = \begin{pmatrix} -\gamma_{fe}(\bar{n}_{fe} + 1) & \gamma_{fe}\bar{n}_{fe} & 0 \\ \gamma_{fe}(\bar{n}_{fe} + 1) & -[\gamma_{fe}\bar{n}_{fe} + \gamma_{eg}(\bar{n}_{eg} + 1)] & \gamma_{eg}\bar{n}_{eg} \\ 0 & \gamma_{eg}(\bar{n}_{eg} + 1) & -\gamma_{eg}\bar{n}_{eg} \end{pmatrix} \quad (3.61)$$

In explicit component form, the population changes satisfy:

$$\frac{d\rho_{ff}}{dt} = -\gamma_{fe}(\bar{n}_{fe} + 1)\rho_{ff} + \gamma_{fe}\bar{n}_{fe}\rho_{ee}, \quad (3.62)$$

$$\frac{d\rho_{ee}}{dt} = \gamma_{fe}(\bar{n}_{fe} + 1)\rho_{ff} - \gamma_{fe}\bar{n}_{fe}\rho_{ee} - \gamma_{eg}(\bar{n}_{eg} + 1)\rho_{ee} + \gamma_{eg}\bar{n}_{eg}\rho_{gg}, \quad (3.63)$$

$$\frac{d\rho_{gg}}{dt} = \gamma_{eg}(\bar{n}_{eg} + 1)\rho_{ee} - \gamma_{eg}\bar{n}_{eg}\rho_{gg}. \quad (3.64)$$

The population vector evolves linearly under the influence of the matrix M , whose off-diagonal elements capture absorption and stimulated emission transitions, while its diagonal entries ensure correct outgoing rates and conservation of total probability (i.e., normalization). This formulation matches the canonical structure of a Markovian master equation. In standard Markovian master equation frameworks, the matrix from

$$\frac{d}{dt}P(t) = M P(t) \quad (3.65)$$

Thus accurately encapsulates the dynamical behavior of the system (Budini, 2006).

3.8.2 Steady State Population

We solve the matrix equation

$$M \vec{P}_{ss} = 0 \quad (3.66)$$

together with the normalization condition. Imposing the steady-state condition $M \vec{P}_{ss} = 0$ yields three linear equations (one per row). The populations are represented by the vector $(\rho_{ff}, \rho_{ee}, \rho_{gg})^T$. From the third row and Eq. (3.61), we deduce:

$$\gamma_{eg}(\bar{n}_{eg} + 1) \rho_{ee}^{ss} = \gamma_{eg}\bar{n}_{eg} \rho_{gg}^{ss}, \quad (3.67)$$

hence we find that

$$\rho_{ee}^{ss} = \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1} \rho_{gg}^{ss}. \quad (3.68)$$

Considering Eq. (3.61), from the first row we obtain:

$$\gamma_{fe}(\bar{n}_{fe} + 1) \rho_{ff}^{ss} = \gamma_{fe} \bar{n}_{fe} \rho_{ee}^{ss} \quad \Rightarrow \quad \rho_{ff}^{ss} = \frac{\bar{n}_{fe}}{\bar{n}_{fe} + 1} \rho_{ee}^{ss}. \quad (3.69)$$

We then express all populations in terms of ρ_{gg}^{ss} . Substitute Eq. (3.68) into Eq. (3.69): since

$$\rho_{ee}^{ss} = \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1} \rho_{gg}^{ss}, \quad \rho_{ff}^{ss} = \frac{\bar{n}_{fe}}{\bar{n}_{fe} + 1} \times \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1} \rho_{gg}^{ss}. \quad (3.70)$$

Using the normalization condition

$$\rho_{ff}^{ss} + \rho_{ee}^{ss} + \rho_{gg}^{ss} = 1. \quad (3.71)$$

We introduce the shorthand

$$a = \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1}, \quad b = \frac{\bar{n}_{fe}}{\bar{n}_{fe} + 1}. \quad (3.72)$$

Then

$$\rho_{gg}^{ss}(1 + a + ab) = 1 \quad \Rightarrow \quad \rho_{gg}^{ss} = \frac{1}{1 + a + ab}. \quad (3.73)$$

Hence

$$\rho_{ee}^{ss} = \frac{a}{1 + a + ab}, \quad \rho_{ff}^{ss} = \frac{ab}{1 + a + ab}. \quad (3.74)$$

Returning to the original variables:

$$\rho_{gg}^{ss} = \frac{1}{1 + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} \times \frac{\bar{n}_{fe}}{\bar{n}_{fe}+1}}, \quad (3.75)$$

$$\rho_{ee}^{ss} = \frac{\frac{\bar{n}_{eg}}{\bar{n}_{eg}+1}}{1 + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} \times \frac{\bar{n}_{fe}}{\bar{n}_{fe}+1}}, \quad (3.76)$$

$$\rho_{ff}^{ss} = \frac{\frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} \times \frac{\bar{n}_{fe}}{\bar{n}_{fe}+1}}{1 + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} \times \frac{\bar{n}_{fe}}{\bar{n}_{fe}+1}}. \quad (3.77)$$

If we set $r_1 = a$ and $r_3 = b$, then we can write:

$$\rho_{gg}^{ss} = \frac{1}{1 + r_1 + r_3}, \quad \rho_{ee}^{ss} = \frac{r_1}{1 + r_1 + r_3}, \quad \rho_{ff}^{ss} = \frac{r_1 r_3}{1 + r_1 + r_3}. \quad (3.78)$$

Where r_1, r_3 normalized fractions between zero and one

In the high temperature limit ($\bar{n}_{eg}, \bar{n}_{fe} \gg 1$), both ratios r_1 and r_3 approach 1, so the populations become nearly equal:

$$\rho_{ff} \approx \frac{1}{3} \approx \rho_{gg} \approx \rho_{ee}. \quad (3.79)$$

In the low-temperature limit (i.e. $\bar{n}_{eg}, \bar{n}_{fe} \rightarrow 0$, $r_1 \rightarrow 0$ and $r_3 \rightarrow 0$), then $\rho_{gg} \rightarrow 1$ and $\rho_{ee}, \rho_{ff} \rightarrow 0$, reflecting that spontaneous emission drives the system into the ground state. Consequently, the system spontaneously thermalized to equilibrium according to the bath driven transition rates, and the effective photon occupation numbers and transition rates provided a compact representation of the steady state population distribution (Sen, Dey, Nath, & Gangopadhyay, 2015).

3.9 Coherence (Rate) Equations

We work on the basis $\{|f\rangle, |e\rangle, |g\rangle\}$ and derive the rate equations for the off-diagonal (coherence) elements of the density matrix for the cascade atom, taking into account both unitary evolution and dissipation. The free (unperturbed) Hamiltonian is

$$\hat{H} = \hbar\omega_g|g\rangle\langle g| + \hbar\omega_e|e\rangle\langle e| + \hbar\omega_f|f\rangle\langle f|, \quad (3.80)$$

and we set $\hbar = 1$. For off-diagonal elements ρ_{mn} (with $m \neq n$), we define $\omega_{mn} = \omega_m - \omega_n$, which gives rise to the usual phase rotation term $-i\omega_{mn}\rho_{mn}$. Next, we include dissipation via the Lindblad master

equation:

$$\begin{aligned}
\frac{d\rho}{dt} = & -i[\hat{H}_{\text{atom}}, \rho] \\
& + \gamma_{fe}(\bar{n}_{fe} + 1)D[\sigma_{fe}]\rho + \gamma_{fe}\bar{n}_{fe}D[\sigma_{fe}^+]\rho \\
& + \gamma_{eg}(\bar{n}_{eg} + 1)D[\sigma_{eg}]\rho + \gamma_{eg}\bar{n}_{eg}D[\sigma_{eg}^+]\rho.
\end{aligned} \tag{3.81}$$

Here,

$$\sigma_{fe} = |f\rangle\langle e|, \quad \sigma_{ef} = |e\rangle\langle f|, \quad \sigma_{eg} = |e\rangle\langle g|, \quad \sigma_{ge} = |g\rangle\langle e|.$$

The thermal occupation numbers are $\bar{n}_{fe}$ and $\bar{n}_{eg}$. From this master equation, one can extract differential equations for each coherence ρ_{mn} . For example, for ρ_{fe} , the coherent part contributes $-i\omega_{fe}\rho_{fe}$, and the dissipative terms introduce decay (dephasing) at rates involving $\gamma_{fe}, \gamma_{eg}, \bar{n}_{fe}, \bar{n}_{eg}$, etc. For each of the three coherences ρ_{fe}, ρ_{eg} , and ρ_{fg} , we compute the off-diagonal density matrix elements; for example,

$$\rho_{fe} = |e\rangle\rho\langle f|. \tag{3.82}$$

The dissipative contributions are evaluated via the Lindblad dissipators $D[\sigma_{fe}]$ and $D[\sigma_{eg}]$, while the unitary evolution contributes a term $-i\omega_{fe}\rho_{fe}$. For the coherence ρ_{fe} , the unitary term is $-i\omega_{fe}\rho_{fe}$. Taking into account the thermal occupation number, the fe bath contribution gives $-\frac{1}{2}\gamma_{fe}(2\bar{n}_{fe} + 1)\rho_{fe}$, and the dissipator. $D[\sigma_{fe}]$ contributes an additional $-\frac{1}{2}\gamma_{fe}$. The $D[\sigma_{eg}]$ term yields the eg bath contribution $-\frac{1}{2}\gamma_{eg}(2\bar{n}_{eg} + 1)\rho_{fe}$. Summing these yields the rate equation:

$$\frac{d\rho_{fe}}{dt} = -i\omega_{fe}\rho_{fe} - \frac{1}{2}[\gamma_{fe}(2\bar{n}_{fe} + 1) + \gamma_{eg}\bar{n}_{eg}]\rho_{fe}. \tag{3.83}$$

The unitary term contributes $-i\omega_{eg}\rho_{eg}$ to the coherence ρ_{eg} . Each of the dissipators $D[\sigma_{eg}]$ and $D[\sigma_{eg}^+]$ contributes $-\frac{1}{2}\gamma_{eg}(\bar{n}_{eg} + 1)\rho_{eg}$. From

the $D[\sigma_{eg}^+]$ term, one obtains an additional $-\frac{1}{2}\gamma_{eg}\bar{n}_{eg}\rho_{eg}$. Altogether, combining these contributions gives:

$$\frac{d\rho_{eg}}{dt} = -i\omega_{eg}\rho_{eg} - \frac{1}{2}[\gamma_{eg}(2\bar{n}_{eg} + 1) + \gamma_{fe}\bar{n}_{eg}]\rho_{eg}. \quad (3.84)$$

For the coherence ρ_{fg} , the unitary term contributes $-i\omega_{fg}\rho_{fg}$. The corresponding dissipative terms lead to the rate equation:

$$\frac{d\rho_{fg}}{dt} = -i\omega_{fg}\rho_{fg} - \frac{1}{2}[\gamma_{fg}(\bar{n}_{fg} + 1) + \gamma_{eg}\bar{n}_{eg}]\rho_{fg}. \quad (3.85)$$

These rate equations include both unitary and dissipative dynamics, and they characterize the time evolution of the coherences in the cascade atomic system. The general form of the Lindblad master equation, which governs the evolution of open quantum systems by accounting for both coherent and incoherent processes, is consistent with the derived equations (Breuer & Petruccione, 2002).

3.9.1 Rate Matrix Form

A coherence vector is defined as follows:

$$C(t) = \begin{pmatrix} \rho_{ge} \\ \rho_{ef} \\ \rho_{gf} \end{pmatrix} \quad (3.86)$$

Additionally, we express the temporal evolution of the coherences in the compact matrix form:

$$\frac{d\mathbf{c}}{dt} = M \cdot \mathbf{c} \quad (3.87)$$

The matrix M is given by

$$\begin{bmatrix} -i\omega_{ge} - \frac{1}{2}[\gamma_{eg}(2\bar{n}_{eg} + 1) + \gamma_{fe}\bar{n}_{eg}] & 0 & 0 \\ 0 & -i\omega_{ef} - \frac{1}{2}[\gamma_{fe}(2\bar{n}_{fe} + 1) + \gamma_{eg}(\bar{n}_{eg} + 1)] & 0 \\ 0 & -i\omega_{gf} - \frac{1}{2}[\gamma_{eg}\bar{n}_{fe} + \gamma_{fe}(\bar{n}_{fe} + 1)] & 0 \end{bmatrix} \quad (3.88)$$

This formulation clearly demonstrates that each term comprises both an incoherent decay rate and a coherent frequency component, and that each coherence decays independently (i.e. without cross coupling).

3.9.2 Steady States Coherence

Since there was no coherent driving present, and only incoherent (thermal) light interacted with the atom, the density matrix in the steady state took a fully diagonal form. All off-diagonal coherence terms vanished, and the populations resided solely in the diagonal elements. Specifically, only three elements remained nonzero and followed the thermal distribution ratios: The steady-state populations were given by ρ_{gg}^{ss} , ρ_{ee}^{ss} , and ρ_{ff}^{ss} . The coherences ρ_{ge}^{ss} , ρ_{ef}^{ss} , ρ_{gf}^{ss} , and their complex conjugates vanished entirely. This resulted in the following steady-state density matrix:

$$\rho_{ss} = \begin{pmatrix} \rho_{gg}^{ss} & 0 & 0 \\ 0 & \rho_{ee}^{ss} & 0 \\ 0 & 0 & \rho_{ff}^{ss} \end{pmatrix}. \quad (3.89)$$

The diagonal entries were obtained using the previously derived expressions for the populations in terms of:

$$r_1 = \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1}, \quad r_3 = \frac{\bar{n}_{fe}}{\bar{n}_{fe} + 1}. \quad (3.90)$$

$$\rho_{gg}^{ss} = \frac{1}{1 + r_1 + r_3}, \quad \rho_{ee}^{ss} = \frac{r_1}{1 + r_1 + r_3}, \quad \rho_{ff}^{ss} = \frac{r_1 r_3}{1 + r_1 + r_3}. \quad (3.91)$$

Thus, the explicit steady-state density matrix becomes:

$$\rho_{ss} = \begin{pmatrix} \frac{1}{1 + r_1 + r_3} & 0 & 0 \\ 0 & \frac{r_1}{1 + r_1 + r_3} & 0 \\ 0 & 0 & \frac{r_1 r_3}{1 + r_1 + r_3} \end{pmatrix}. \quad (3.92)$$

The steady-state density matrix has zero entries for all off-diagonal elements. This demonstrates that, under purely thermal (incoherent) forcing, only populations persist and no coherences remain (i.e., $\rho_{ij} = 0$ for $i \neq j$). As shown earlier via the master equation analysis, the diagonal entries exactly correspond to the steady-state thermal distribution over the atomic levels. In fact, the populations settle into the familiar Boltzmann (thermal) distribution, while all coherence terms (e.g., ρ_{eg}) decay to zero in the steady state when the master equation is solved via the Einstein rate equation formalism. Under the high-temperature limit, the steady-state density matrix becomes:

$$\rho_{gg}^{(\text{high-T})} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}. \quad (3.93)$$

In contrast, in the low temperature limit, the steady-state density matrix becomes:

$$\rho_{gg}^{(\text{low-T})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.94)$$

With dominant spontaneous emission and minimal thermal excitation, this matrix indicates that the ground state dominates the system.

3.10 Input–Output Theorem

The input–output formalism links the output field operator $a_{\text{out}}(t)$ to the atomic lowering operator $\sigma_i(t)$ and the input field $a_{\text{in}}(t)$. Following Gardiner and Collett (1985), we have

$$a_{\text{out}}(t) = a_{\text{in}}(t) + \sqrt{\gamma_i} \sigma_i(t). \quad (3.95)$$

Here γ_i is the spontaneous-emission rate for that transition, and $\sigma_i(t)$ is the atomic lowering operator for that transition. For the incident

(thermal) input field operator $a_{\text{in}}(t)$, the following relation applies:

$$\langle a_{\text{out}}^\dagger(t) a_{\text{in}}(t') \rangle = \bar{n}_i \delta(t - t'), \quad (3.96)$$

where $\bar{n}_i$ is the average thermal photon number in the input channel. This input–output relation thus links the internal atomic dynamics (via σ_i) with the observable fluorescence field (via a_{out}).

3.11 Photon Flux

The photon flux is defined as the number of photons an emitter emits into a particular mode (or radiation channel) per unit time. It accounts for both spontaneous emission (driven by vacuum fluctuations) and stimulated processes (driven by thermal photons or by coherent driving) in a quantum emitter. According to (Fan, Kocabaş, & Shen, 2010), this quantity indicates how many photons are escaping the atom into a specified electromagnetic mode. The photon flux in output mode i is defined as

$$\Phi_i(t) = \langle a_{\text{out}}^\dagger(t) a_{\text{out}}(t) \rangle. \quad (3.97)$$

Using (3.95), one obtains

$$\Phi_i(t) = \gamma_i \langle \sigma_i^\dagger \sigma_i \rangle_t + \langle a_{\text{in}}^\dagger a_{\text{in}} \rangle + \sqrt{\gamma_i} \left(\langle a_{\text{in}}^\dagger \sigma_i \rangle + \langle \sigma_i^\dagger a_{\text{in}} \rangle \right). \quad (3.98)$$

If one assumes that there is no correlation between the atom and the input field (i.e., $\langle a_{\text{in}}^\dagger \sigma_i \rangle = 0$), and similarly $\langle \sigma_i^\dagger a_{\text{in}} \rangle = 0$, this reduces to

$$\Phi_i(t) = \gamma_i \langle \sigma_i^\dagger \sigma_i \rangle_t + \langle a_{\text{in}}^\dagger a_{\text{in}} \rangle. \quad (3.99)$$

For a thermal bath input field one has $\langle a_{\text{in}}^\dagger a_{\text{in}} \rangle = n_i$, where n_i is the thermal occupation number. Hence, in steady state one finds

$$\Phi_i^{\text{ss}} = \gamma_i \langle \sigma_i^\dagger \sigma_i \rangle_{\text{ss}} + n_i. \quad (3.100)$$

In this expression, the second term represents the background (thermal) flux, while the first term accounts for spontaneous and stimulated emissions. For a cascade atom with three levels and thermal occupancies: the lower transition $|e\rangle \rightarrow |g\rangle$ has thermal occupancy $\bar{n}_{eg}$; the upper transition $|f\rangle \rightarrow |e\rangle$ has thermal occupancy $\bar{n}_{fe}$.

Accordingly, the photon flux for the lower transition is

$$\Phi_{eg}(t) = \gamma_{eg} [(\bar{n}_{eg} + 1) \rho_{ee}(t) - \bar{n}_{eg} \rho_{gg}(t)]. \quad (3.101)$$

Here $\bar{n}_{eg}\rho_{ee}$ corresponds to stimulated emission plus spontaneous emission proportional to the excited-state population, while $(\bar{n}_{eg} + 1)\rho_{ee}$ reflects absorption triggered by thermal photons. Likewise, for the upper transition one writes

$$\Phi_{fe}(t) = -\gamma_{fe} [(\bar{n}_{fe} + 1) \rho_{ff}(t) + \bar{n}_{fe} \rho_{ee}(t)]. \quad (3.102)$$

3.12 Steady-State Photon Flux

At steady state we replace the time-dependent populations $\rho_{ii}(t)$ by their constant steady-state values ρ_{ii}^{ss} . In particular, one assumes

$$\rho_{ee}^{\text{ss}} = \rho_{gg}^{\text{ss}} = \rho_{ff}^{\text{ss}} \quad (3.103)$$

Then the steady-state flux for the lower transition becomes

$$\Phi_{eg}^{\text{ss}} = \gamma_{eg} [(\bar{n}_{eg} + 1) \rho_{ee}^{\text{ss}} - \bar{n}_{eg} \rho_{ee}^{\text{ss}}]. \quad (3.104)$$

Inserting the steady-state populations

$$\rho_{ee}^{\text{ss}} = \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1}, \quad \rho_{gg}^{\text{ss}} = \frac{1}{\bar{n}_{eg} + 1}, \quad (3.105)$$

one finds

$$\Phi_{eg}^{\text{ss}} = \gamma_{eg} \left[(\bar{n}_{eg} + 1) \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1} - \bar{n}_{eg} \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1} \right] = 0. \quad (3.106)$$

Similarly, for the upper transition one obtains

$$\Phi_{fe}^{\text{ss}} = -\gamma_{fe} [\bar{n}_{fe} \rho_{ee}^{\text{ss}} - \bar{n}_{fe} \rho_{ee}^{\text{ss}}] = 0. \quad (3.107)$$

Thus, in steady state, neither transition produces a net photon flux.

3.13 Fluorescence Power Spectrum

We apply the quantum version of the Wiener–Khinchin theorem to represent the power spectrum of the emitted light (Chen & Chen, 2022). Define the output field operator $a_{\text{out}}(t)$ for transition i with transition frequency ω_i . The power spectrum of the emitted light in steady state is

$$S_i(\omega) = \frac{1}{2\pi} \Re \int_{-\infty}^{+\infty} d\tau e^{i(\omega - \omega_i)\tau} \langle a_{\text{out}}^\dagger(t) a_{\text{out}}(t + \tau) \rangle_{\text{ss}}. \quad (3.108)$$

If we neglect correlations between the input and the atom, and substitute the input–output relation, one obtains

$$S_i(\omega) \approx \frac{\gamma_i}{2\pi} \Re \int_{-\infty}^{+\infty} d\tau e^{i(\omega - \omega_i)\tau} \left[\langle a_{\text{in}}^\dagger(t) a_{\text{in}}(t + \tau) \rangle + \gamma_i \langle \sigma_i^\dagger(t) \sigma_i(t + \tau) \rangle_{\text{ss}} \right]. \quad (3.109)$$

Denoting the first bracket as the background spectrum $S_{\text{in}}(\omega)$, one writes

$$\begin{aligned} S_i(\omega) \approx & \frac{\gamma_i}{2\pi} \Re \int_{-\infty}^{+\infty} d\tau e^{i(\omega - \omega_i)\tau} S_{\text{in}}(\omega) \\ & + \frac{\gamma_i^2}{2\pi} \Re \int_{-\infty}^{+\infty} d\tau e^{i(\omega - \omega_i)\tau} \langle \sigma_i^\dagger(t) \sigma_i(t + \tau) \rangle_{\text{ss}}. \end{aligned} \quad (3.110)$$

3.14 Quantum Regression Theorem (QRT)

We define the atomic two-time correlation

$$C_{eg}(\tau) = \langle \sigma_{ge}^\dagger(t) \sigma_{eg}(t + \tau) \rangle_{\text{ss}}. \quad (3.111)$$

In steady state this depends only on the time difference τ . According to the QRT, the operator $\sigma_{ge}^\dagger \rho_{\text{ss}}$ evolves under the same Liouvillian $\mathcal{L}$ that governs the density matrix. Thus,

$$C_{eg}(\tau) = \text{Tr} [\sigma_{eg} e^{\mathcal{L}\tau} (\sigma_{ge}^\dagger \rho_{\text{ss}})] . \quad (3.112)$$

Hence the spectrum representation becomes

$$S_{eg}(\omega) = \frac{1}{2\pi} \Re \left\{ \int_{-\infty}^{+\infty} d\tau e^{i(\omega - \omega_{eg})\tau} \text{Tr} [\sigma_{eg} e^{\mathcal{L}\tau} (\sigma_{ge}^\dagger \rho_{\text{ss}})] \right\}. \quad (3.113)$$

For the cascade atom the total fluorescence spectrum is

$$S(\omega) = S_{fe}(\omega) + S_{eg}(\omega). \quad (3.114)$$

3.15 Analytic Form (Lorentzian Limit)

In many simple cases (weak driving, no strong coherent coupling, a single transition, nearly single exponential decay) one obtains

$$\begin{aligned} C_{eg}(\tau) &= \langle \sigma_{ge}^\dagger(t) \sigma_{eg}(t + \tau) \rangle_{\text{ss}} \\ &= A_{eg} e^{-(\Gamma_{ge} - i\omega_{eg})\tau}, \quad \tau \geq 0. \end{aligned} \quad (3.115)$$

where Γ_{ge} is the coherence-decay rate, and A_{eg} is an amplitude (often $\langle \sigma_{ge}^\dagger \sigma_{eg} \rangle_{\text{ss}}$). Then, restricting the integral to $\tau \geq 0$, one finds.

Then the spectrum integral (taking only $\tau \geq 0$ for stationarity) becomes:

$$\begin{aligned}
S_{eg}(\omega) &\propto \Re \int_0^{+\infty} d\tau e^{i(\omega - \omega_{eg})\tau} A_{eg} e^{-(\Gamma_{ge} - i\omega_{eg})\tau} \\
&= A_{eg} \int_0^{+\infty} d\tau e^{-\Gamma_{ge}\tau} e^{i(\omega - \omega_{eg} + \omega_{eg})\tau} \\
&= A_{eg} \int_0^{+\infty} d\tau e^{-(\Gamma_{ge} - i\Delta)\tau},
\end{aligned} \tag{3.116}$$

where $\Delta = \omega - \omega_{eg}$. Compute the integral:

$$\int_0^{+\infty} d\tau e^{-(\Gamma_{ge} - i\Delta)\tau} = \frac{1}{\Gamma_{ge} - i\Delta}, \quad \Gamma_{ge} > 0. \tag{3.117}$$

Then:

$$\Re \left\{ \frac{1}{\Gamma_{ge} - i\Delta} \right\} = \frac{\Gamma_{ge}}{\Gamma_{ge}^2 + \Delta^2}. \tag{3.118}$$

Taking the real part yields a Lorentzian line shape:

$$S_{eg}(\omega) \propto A_{eg} \frac{\Gamma_{ge}}{(\omega - \omega_{eg})^2 + \Gamma_{ge}^2}. \tag{3.119}$$

Thus the full width at half maximum (FWHM) is $2\Gamma_{ge}$, and the half width at half maximum (HWHM) is Γ_{ge} . Frequently one writes

$$S_{eg}(\omega) \approx \frac{\gamma_{eg}(\bar{n}_{eg} + 1) \langle \sigma_{eg}^\dagger \sigma_{ge} \rangle_{ss}}{\pi} \frac{\left(\frac{1}{2}\gamma_{eg}\right)}{(\omega - \omega_{eg})^2 + \left(\frac{1}{2}\gamma_{eg}\right)^2}. \tag{3.120}$$

which implies the identity $\Gamma_{ge} = \frac{1}{2}\gamma_{eg}$. The prefactor includes the steady-state population, the factor $(\bar{n}_{eg} + 1)$ (for stimulated emission in a thermal bath), and the spontaneous emission rate γ_i . Hence the cascade atom's fluorescence spectrum shows two Lorentzian peaks at ω_{fe} and ω_{eg} , with linewidths γ_{fe} and γ_{eg} , respectively.

3.16 Photon Statistics

The second-order correlation function (photon statistics) for channel i is defined as

$$g_i^{(2)}(\tau) = \frac{\langle a_{\text{out}}^\dagger(t) a_{\text{out}}^\dagger(t+\tau) a_{\text{out}}(t+\tau) a_{\text{out}}(t) \rangle}{\left(\langle a_{\text{out}}^\dagger a_{\text{out}} \rangle \right)^2}. \quad (3.121)$$

Using the input–output relation and the QRT one obtains

$$g^{(2)}(\tau) = \frac{\text{Tr}[\sigma_{eg}^\dagger \sigma_{ge} e^{\mathcal{L}\tau} (\sigma_{eg} \rho_{\text{ss}} \sigma_{ge}^\dagger)]}{\left(\langle \sigma_{eg}^\dagger \sigma_{ge} \rangle_{\text{ss}} \right)^2}. \quad (3.122)$$

Cross-correlations between the upper and lower transitions in the cascade process can also be investigated. For example:

$$g_{(fe, eg)}^{(2)}(\tau) = \frac{\langle a_{\text{out}, fe}^\dagger(t) a_{\text{out}, eg}^\dagger(t+\tau) a_{\text{out}, eg}(t+\tau) a_{\text{out}, fe}(t) \rangle}{\langle a_{\text{out}, fe}^\dagger a_{\text{out}, fe} \rangle \langle a_{\text{out}, eg}^\dagger a_{\text{out}, eg} \rangle}. \quad (3.123)$$

This captures the time-ordering inherent in cascade emission: typically the photon from $|f\rangle \rightarrow |e\rangle$ is emitted before the one from $|e\rangle \rightarrow |g\rangle$. As a reminder: the annihilation operator is $a_{\text{out}}(t)$, and the corresponding creation operator is $a_{\text{out}}^\dagger(t)$. The zero-delay correlation value $g^{(2)}(0)$ provides key information:

$$\begin{aligned} g^{(2)}(0) &= 1 && \text{for a coherent (Poissonian) light state,} \\ g^{(2)}(0) &< 1 && \text{indicates photon antibunching (non-classical light),} \\ g^{(2)}(0) &> 1 && \text{indicates photon bunching (thermal or chaotic light).} \end{aligned}$$

3.17 Photon Number Statistics versus $g^{(2)}(\tau)$

The probability of finding a particular number of photons in a given mode is described by photon statistics. According to Bose–Einstein

statistics, thermal light exhibits super-Poissonian behavior, meaning its variance is greater than its mean (Mandel & Wolf, 1995). The formula for the probability $P(n)$ of finding n photons in a single thermal light mode is

$$P(n) = \frac{1}{\langle n \rangle + 1} \left(\frac{\langle n \rangle}{\langle n \rangle + 1} \right)^n. \quad (3.124)$$

where $\langle n \rangle$ is the mean photon number in that mode. The variance of thermal light exceeds $\langle n \rangle$,

$$\Delta n^2 = \langle n \rangle^2 + \langle n \rangle. \quad (3.125)$$

In contrast, Poissonian statistics apply to coherent light, such as that from an ideal laser, for which the variance equals the mean. The probability of finding n photons is given by

$$P(n) = \frac{\bar{n}^n e^{-\bar{n}}}{n!}, \quad (3.126)$$

and its variance equals the mean:

$$\Delta n^2 = \bar{n}. \quad (3.127)$$

These statistical distributions complement the correlation-function approach: a sub-Poissonian source ($\Delta n^2 < \bar{n}$) corresponds to $g^{(2)}(0) < 1$, a signature of quantum light.

4 CHAPTER FOUR: RESULT AND DISCUSSION

4.1 Results

This chapter analyzes and discusses results from numerical simulations of a three-level atom interacting with thermal light in a cascade configuration. The time evolution of population inversion and coherence is examined across different values of the thermal photon number parameter

4.2 Rate Equation Model for Population Decay Atom under Weak Thermal Excitation

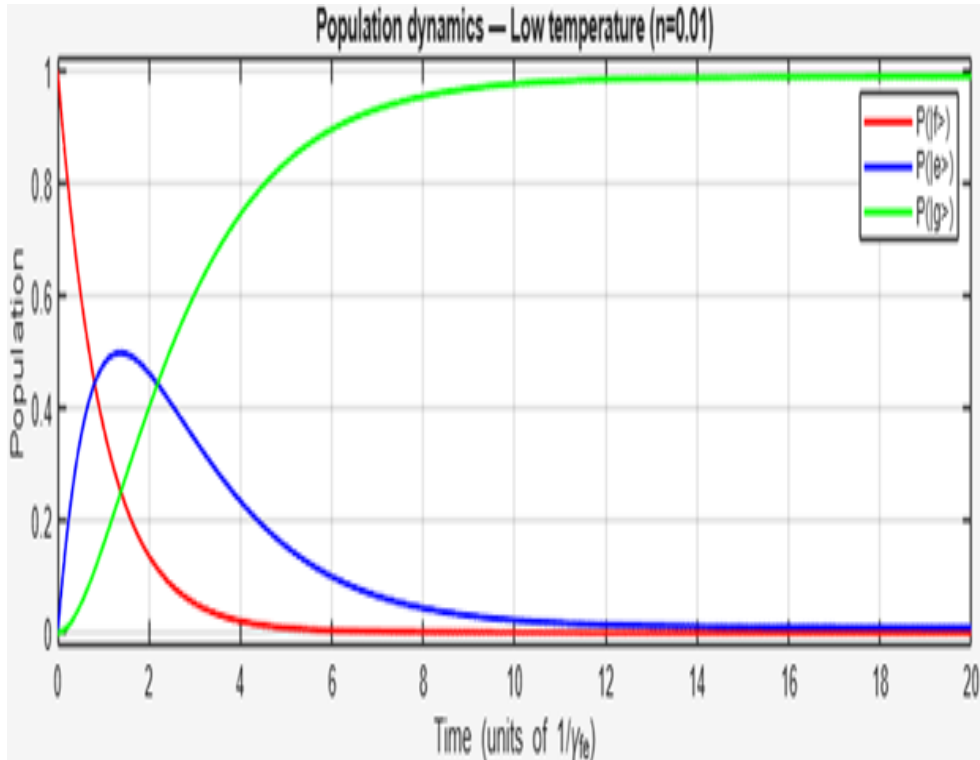


Figure 4.1: Time evolution of the level populations $\rho_{ff}(t)$, $\rho_{ee}(t)$, and $\rho_{gg}(t)$ for a three-level cascade atom with a mean photon number $\bar{n} = 0.01$, coupled to a thermal bath. The parameters are $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0.5$.

Figure 4.1 shows the time evolution of the level populations $\rho_{ff}(t)$, $\rho_{ee}(t)$, and $\rho_{gg}(t)$ for a three-level cascade atom with a mean photon number $\bar{n} = 0.01$, coupled to a thermal bath. The parameters are $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0.5$. The initial state is the excited atomic level $|f\rangle$ (i.e. $\rho_{ff}(0) = 1$). Because the thermal excitation is minimal, the figure shows a rapid decay from $|f\rangle$ to $|g\rangle$, eventually giving ground-state dominance. The transitions $|f\rangle \rightarrow |e\rangle$ (rate γ_{fe}) and $|e\rangle \rightarrow |g\rangle$ (rate γ_{eg}) occur, with upward (absorption) rates γ_n and downward (spontaneous + stimulated emission) rates $\gamma(1 + \bar{n})$. The system begins entirely in the excited state $|f\rangle$ in Figure 4.1, corresponding to the low-temperature regime ($\bar{n} = 0.01$). Because there are very few thermal photons, upward transitions are suppressed and spontaneous emission dominates. According to the rate-equation model,

$$M = \begin{pmatrix} -\gamma_{fe}(\bar{n}_{fe} + 1) & \gamma_{fe}\bar{n}_{fe} & 0 \\ \gamma_{fe}(\bar{n}_{fe} + 1) & -[\gamma_{fe}\bar{n}_{fe} + \gamma_{eg}(\bar{n}_{eg} + 1)] & \gamma_{eg}\bar{n}_{eg} \\ 0 & \gamma_{eg}(\bar{n}_{eg} + 1) & -\gamma_{eg}\bar{n}_{eg} \end{pmatrix}. \quad (4.1)$$

The system decays quickly to the ground state. As a result, the population in $|f\rangle$ decays rapidly, $|e\rangle$ is only briefly occupied, and nearly all of the population eventually accumulates in the ground state $|g\rangle$. In the absence of substantial thermal excitation, this behavior matches the expected spontaneous decay dynamics.

4.3 Intermediate Temperature Population Dynamics in a Three-Level Cascade Atom

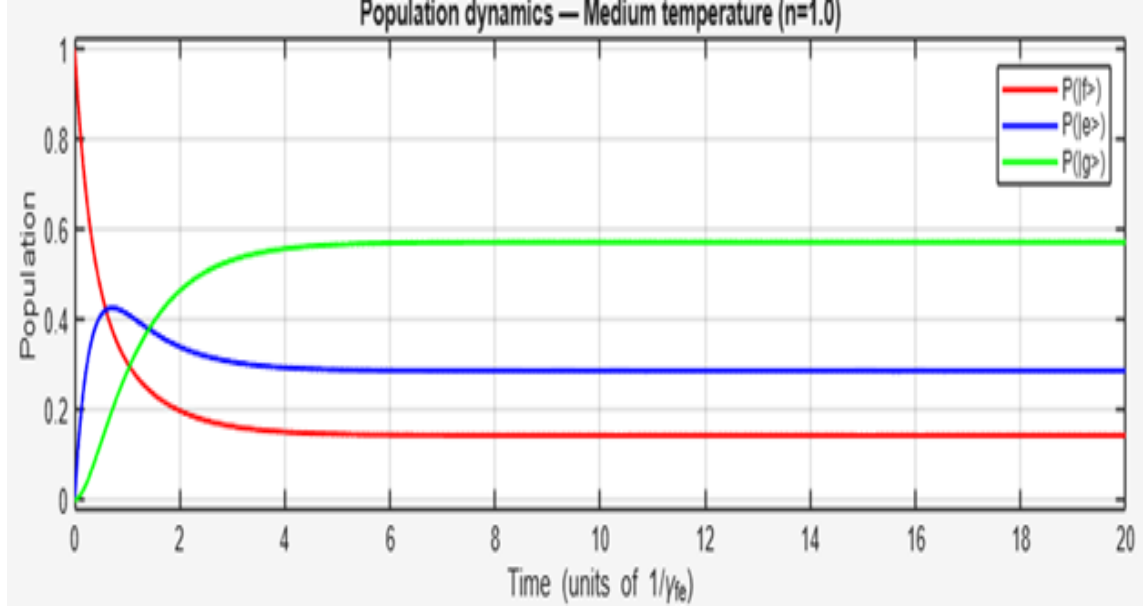


Figure 4.2: Time evolution of populations at intermediate thermal photon number ($\bar{n} = 1$) parameters $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0.5$

Figure 4.2 shows the population time evolution at an intermediate thermal photon number ($\bar{n} = 1$) with parameters $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0.5$. The initial condition is $\rho_{ff}(0) = 1$. The figure illustrates the competition between thermal absorption and downward decay. The ground state no longer fully dominates, and the intermediate level $|e\rangle$ attains a substantial steady-state population. The behavior is notably different in the medium-temperature regime ($\bar{n} = 1$), as shown in Figure 4.2. In this case, the populations do not simply decay to the ground state, as both stimulated emission and absorption processes occur at rates comparable to spontaneous emission. Instead, although the ground state remains the most populated, the intermediate level $|e\rangle$ and the excited state $|f\rangle$

attain significant steady-state populations. The relations

$$\gamma_{eg}(\bar{n}_{eg} + 1)\rho_{ee}^{\text{ss}} = \gamma_{eg}\bar{n}_{eg}\rho_{gg}^{\text{ss}}, \quad (4.2)$$

$$\rho_{ee}^{\text{ss}} = \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1}\rho_{gg}^{\text{ss}}, \quad (4.3)$$

$$\gamma_{fe}(\bar{n}_{fe} + 1)\rho_{ff}^{\text{ss}} = \gamma_{fe}\bar{n}_{fe}\rho_{ee}^{\text{ss}}, \quad (4.4)$$

$$\rho_{ff}^{\text{ss}} = \frac{\bar{n}_{fe}}{\bar{n}_{fe} + 1}\rho_{ee}^{\text{ss}}, \quad (4.5)$$

which combine into

$$\rho_{ee}^{\text{ss}} = \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1}\rho_{gg}^{\text{ss}}, \quad \rho_{ff}^{\text{ss}} = \frac{\bar{n}_{fe}}{\bar{n}_{fe} + 1} \cdot \frac{\bar{n}_{eg}}{\bar{n}_{eg} + 1}\rho_{gg}^{\text{ss}}, \quad (4.6)$$

capture the balance between upward stimulated transitions and downward spontaneous emission. This leads to thermalization toward a Boltzmann-like distribution, consistent with the detailed balance of transition rates.

4.4 High Temperature Population Dynamics in a Three Level Cascade Atom

Figure 4.3 shows the population time evolution at a high thermal photon number ($\bar{n} = 10$) with parameters $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0.5$. The initial condition is $\rho_{ff}(0) = 1$. High-temperature thermalization is evident in the steady state, where strong thermal excitations drive continuous population exchange among all three levels, resulting in approximately equal populations. In the high temperature regime ($\bar{n} = 10$) shown in Figure 4.3, the populations gradually approach nearly equal values in all three states. Both upward and downward transition rates become large and nearly balanced, and strong thermal excitation dominates. As a result, the system tends toward equipartition, with approximately one

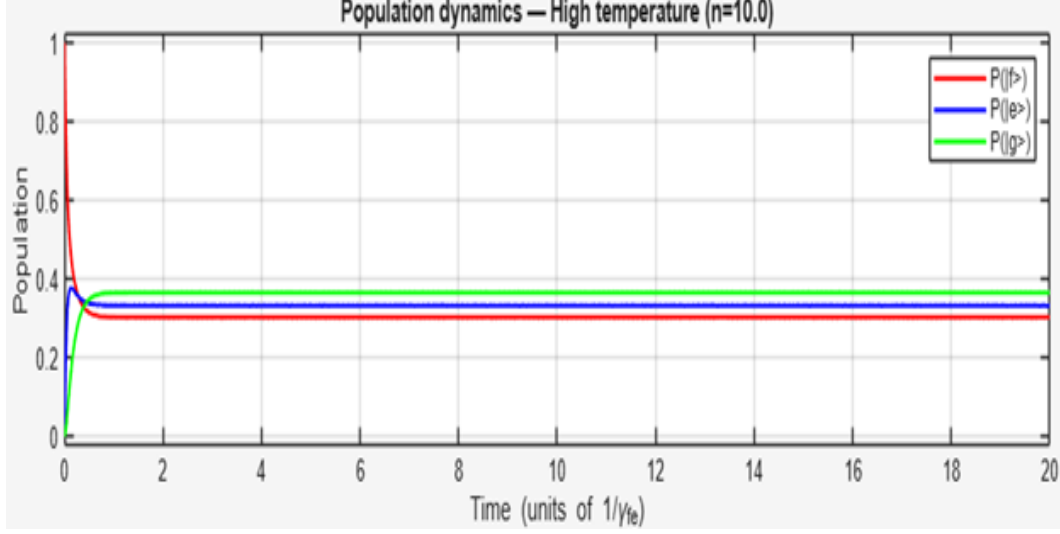


Figure 4.3: Time evolution of populations at high thermal photon number ($\bar{n} = 10$) parameters $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0.5$

third of the population residing in each level. In accordance with the high temperature limit of

$$\rho_{gg}^{ss} = \frac{1}{1 + r_1 + r_3}, \quad \rho_{ee}^{ss} = \frac{r_1}{1 + r_1 + r_3}, \quad \rho_{ff}^{ss} = \frac{r_1 r_3}{1 + r_1 + r_3}, \quad (4.7)$$

Thermal excitations strongly drive population redistribution among all three levels, resulting in nearly identical steady-state values. The figure illustrates how, in the high-temperature limit, the cascade system loses its preference for the ground state, as thermal light erases population differences.

4.5 Steady State Population Distribution vs. Thermal Photon Number in a Three-Level Cascade Atom

Figure 4.4 shows the steady-state populations ρ_{gg}^{ss} , ρ_{ee}^{ss} , and ρ_{ff}^{ss} of a three-level cascade atom as a function of the mean thermal photon number $\bar{n}$. The parameters are $\gamma_{eg} = 0.5$ and $\gamma_{fe} = 1.0$. The populations are

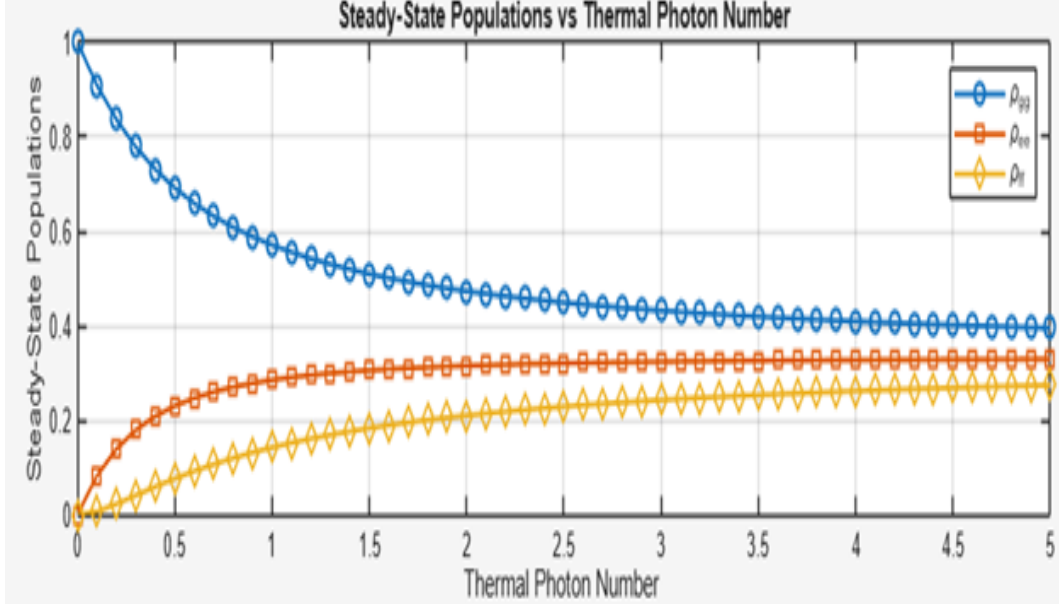


Figure 4.4: Steady-state populations ρ_{gg}^{ss} , ρ_{ee}^{ss} , ρ_{ff}^{ss} as a function of mean photon number $\bar{n}$ Parameters $\gamma_{fe} = 1.0$ and $\gamma_{eg} = 0$

calculated as:

$$\rho_{gg}^{ss} = \frac{1}{1 + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} \cdot \frac{\bar{n}_{fe}}{\bar{n}_{fe}+1}}, \quad (4.8)$$

$$\rho_{ee}^{ss} = \frac{\frac{\bar{n}_{eg}}{\bar{n}_{eg}+1}}{1 + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} \cdot \frac{\bar{n}_{fe}}{\bar{n}_{fe}+1}}, \quad (4.9)$$

$$\rho_{ff}^{ss} = \frac{\frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} \cdot \frac{\bar{n}_{fe}}{\bar{n}_{fe}+1}}{1 + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} + \frac{\bar{n}_{eg}}{\bar{n}_{eg}+1} \cdot \frac{\bar{n}_{fe}}{\bar{n}_{fe}+1}}. \quad (4.10)$$

At very low thermal photon numbers ($\bar{n} \approx 0$), the atom predominantly occupies the ground state $|g\rangle$, with ρ_{gg}^{ss} approaching unity, while the excited states $|e\rangle$ and $|f\rangle$ remain nearly unpopulated. This behavior reflects the dominance of spontaneous emission when thermal contributions are negligible. As $\bar{n}$ increases, thermal absorption and stimulated emission become more significant, redistributing the populations among all three levels. The intermediate level $|e\rangle$, acting as a bridge between the top and bottom states, acquires a non-zero steady-state population.

Eventually, when the populations converge toward equal values:

$$\rho_{gg}^{ss} \approx \rho_{ee}^{ss} \approx \rho_{ff}^{ss} = \frac{1}{3}, \quad (4.11)$$

consistent with thermal equilibrium (i.e., equipartition among degenerate levels in the high-temperature limit). The results demonstrate how temperature governs the distribution of atomic populations in a thermal light environment. Consequently, the combined figures highlight the interplay between spontaneous emission and thermally driven transitions during the thermalization of a three-level cascade atom exposed to thermal light. At low thermal photon numbers, the atom initially prepared in the higher excited state $|f\rangle$ rapidly decays to the ground state, as shown by the time-evolution plots. As thermal occupation increases, this decay is slowed by thermal absorption and stimulated emission, which populate the intermediate and upper levels. This dynamic behavior is then summarized in the steady-state plot, which reveals a temperature-dependent shift from ground-state dominance to population equalization, characteristic of a thermal distribution governed by Bose–Einstein statistics. Altogether, these results suggest that temperature not only determines the atom’s final population distribution but also significantly affects the relaxation dynamics. This insight is important for applications in thermal control of atomic systems, quantum technologies operating in noisy environments, and quantum optics.

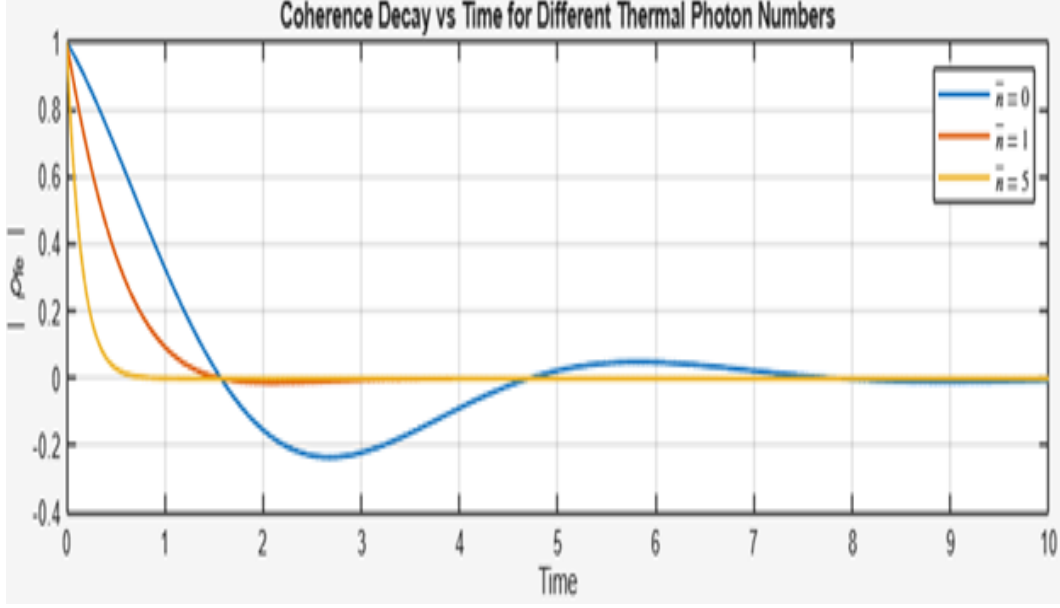


Figure 4.5: Time evolution of the magnitude of the coherence $|\rho_{fe}(t)|$ the upper level $|f\rangle$ and intermediate level $|e\rangle$ in a three-level cascade atom for different thermal photon numbers $\bar{n}$

4.6 Thermal Photon Induced Coherence Decay between Upper and Intermediate Levels in a Cascade Atom

Figure 4.5 shows the time development of the coherence magnitude $|\rho_{fe}(t)|$ between the upper level $|f\rangle$ and the intermediate level $|e\rangle$ for different thermal photon numbers $\bar{n}$. The solid curves correspond to $\bar{n} = 0$ (blue), $\bar{n} = 1$ (orange), and $\bar{n} = 5$ (green). The coherence decay is given by:

$$|\rho_{fe}(t)| = |\rho_{fe}(0)| e^{-\frac{1}{2}(\gamma_{fe} + \gamma_{th})t} e^{-i\omega_{fe}t}, \quad (4.12)$$

where γ_{th} is the additional decoherence rate induced by thermal photons (which depends on $\bar{n}$), and γ_{fe} is the spontaneous-emission decay rate of the $|f\rangle \leftrightarrow |e\rangle$ transition. The exponential envelope indicates decoher-

ence arising from spontaneous emission (with $\gamma_{fe} = 1$, $\gamma_{eg} = 0.5$) and thermal excitations, whereas the oscillations represent the unitary evolution at the transition frequency $\omega_{fe} = 1$. In a three-level cascade atom, the plot of $|\rho_{fe}(t)|$ shows how coherence decays under the influence of thermal photons between the upper level $|f\rangle$ and the intermediate level $|e\rangle$. For example, with $\bar{n} = 0$, the decay is slow and driven mainly by spontaneous emission, but for higher values, such as $\bar{n} = 5$, the coherence is almost immediately damped. This behavior illustrates how thermal photons induce random phase fluctuations that destroy quantum superpositions by driving upward and downward transitions. Greater thermal photon occupation thus accelerates decoherence, making it more difficult to preserve quantum coherence, a crucial property for quantum control applications like precise photon emission, entanglement generation, and manipulation of superposition states. Therefore, to achieve effective quantum control and maintain coherence, low-temperature conditions with small $\bar{n}$ are preferable.

4.7 Fluorescence Spectrum under Thermal Bath in a Three-Level Cascade System

Figure 4.6: Under thermal illumination, the steady-state fluorescence spectrum $S(\omega)$ of the $|e\rangle \rightarrow |g\rangle$ transition in a three-level cascade atom is Lorentzian in shape and centered at the transition frequency $\omega_{fe} = 1$. The linewidth is given by the effective decoherence rate

$$\Gamma_{ge} = \frac{1}{2} \left(\gamma_{eg}(2\bar{n} + 1) + \gamma_{fe}\bar{n} \right), \quad (4.13)$$

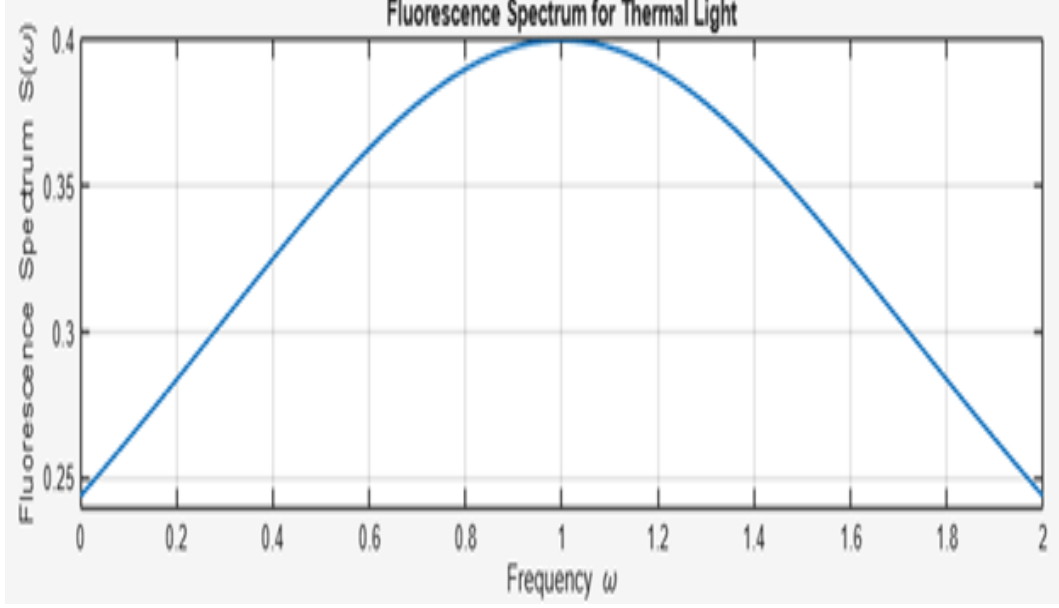


Figure 4.6: Steady-state fluorescence spectrum $S(\omega)$ of the $|e\rangle \rightarrow |g\rangle$ transition in a three-level cascade atom under thermal light.

with $\gamma_{eg} = 0.5$, $\gamma_{fe} = 1$, and $\bar{n} = 1$. The steady-state excited population $\rho_{ee}^{ss} = 0.5$ determines the peak amplitude of the spectrum. The broadening illustrates how thermal photons influence the spectral shape and coherence of atomic emission by combining the effects of spontaneous emission and thermal photon-induced decoherence. The number of atoms available to emit photons is reflected in the peak amplitude, which is proportional to the steady-state excited population $\rho_{ee}^{ss} = 0.5$. Interaction with the thermal bath broadens the emission line an effect more pronounced at higher thermal photon numbers $\bar{n}$, which increase Γ_{ge} . This spectrum illustrates how thermal photons reduce the temporal coherence of the emitted light by contributing both to population redistribution and phase decoherence. Understanding this behavior is crucial for analyzing photon statistics and designing improved quantum control protocols in thermal environments. Additionally, the fluorescence and

coherence decay spectra of a three-level cascade atom under thermal light carry important implications for quantum information processing. Thermal noise is a primary source of decoherence, limiting the lifetime of quantum superpositions and degrading the fidelity of quantum operations. This is evident in the rapid decay of coherences $|\rho_{fe}|$ and $|\rho_{eg}|$, as the thermal photon number $\bar{n}$ increases. Maintaining coherence is vital in quantum information tasks such as photon-based quantum communication, entanglement generation, and quantum gate operations; thus, large thermal occupation numbers degrade performance and reduce the efficacy of quantum protocols. Likewise, the Lorentzian broadening of the fluorescence spectrum indicates decreased temporal coherence, implying that thermal baths impair the quality and indistinguishability of single-photon sources by randomizing photon phases in addition to redistributing populations. To preserve coherence, enhance gate fidelity, and enable reliable quantum control, these results underscore the importance of low-temperature or otherwise engineered environments to suppress thermal photons.

5 CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATION

This chapter deals with summary, conclusions and recommendations. In this section, the summary, conclusions and fundamental findings of the study are drawn. Finally, some possible recommendations are forwarded based on the findings of the study.

5.1 Summary

The dynamics and steady state behavior of a three level cascade atom interacting with a thermal light field are examined in this work. By analyzing fluorescence spectra, coherence decay, and time dependent populations, we show how thermal photons influence the atom's behavior. At low thermal photon numbers, spontaneous emission dominates, causing slow decoherence and a rapid collapse to the ground state. For moderate photon numbers, absorption and stimulated emission compete with spontaneous decay, producing significant stable populations in the intermediate and excited states. At high photon numbers, the populations equalize at all levels, indicating thermalization.

As the thermal photon occupation increases, the fluorescence spectrum broadens, indicating reduced temporal coherence, while coherence between atomic levels decays more rapidly. These effects highlight the significance of thermal noise in quantum systems. The findings have direct implications for quantum information processing, stressing the importance of low temperature environments to maintain high-fidelity

operations, preserve quantum coherence, and maximize the performance of photon-based quantum technologies.

5.2 Conclusions

In this study, we examine the fluorescence spectra, coherence properties, steady state populations, and dynamics of a three level cascade atom interacting with a thermal light field. We demonstrate that the population distribution and the temporal evolution of atomic coherences depend strongly on the thermal photon occupancy.

At low thermal photon numbers, spontaneous emission dominates, resulting in slow decoherence and the system collapsing to the ground state. As the thermal photon number increases, absorption and stimulated emission compete with spontaneous decay, causing more rapid coherence decay, broader fluorescence spectra, and nonzero stable populations in the intermediate and excited states. In the high temperature limit, populations tend toward equipartition, indicating complete thermalization.

These findings emphasize the importance of low temperature or otherwise optimized conditions for preserving quantum superpositions, by demonstrating the substantial impact of thermal noise on quantum coherence and the spectral features of atomic transitions. The study shows that controlling thermal excitations is vital for maintaining coherence, enhancing gate fidelity, and ensuring reliable quantum operations, offering valuable insights for quantum information processing.

5.3 Recommendation

1. Low-Temperature Operation: Experiments and quantum devices should operate in low-temperature environments with reduced thermal photon occupation to maintain quantum coherence and optimize control over atomic states.

2. Thermal Noise Mitigation: To ensure higher fidelity in quantum gates and photon-based protocols, thermal photon-induced decoherence must be suppressed through engineered environments or shielding techniques in quantum information applications.

3. Optimized Atomic Systems: By carefully considering the effects of thermal noise on population dynamics and coherence, three-level cascade atoms and similar systems can be tailored for quantum control, communication, and sensing.

4. Future Research Directions: Further studies could explore cavity-coupled or multi-atom systems under thermal noise, as well as advanced decoherence mitigation strategies such as reservoir engineering, squeezing, or dynamical decoupling.

References

- [1] Ayehu,D. (2021). Squeezing and entanglement properties of the cavity light with decoherence in a cascade three level laser. *Journal of Russian Laser Research*, 42(2), 136–145.
- [2] Belay,W.A. (2023). Access to land for women in Ethiopia. *African Journal of Land Policy and Geospatial Sciences*, 6(5), 961–977.
- [3] Brasil,C.A., Fanchini,F.F., & Napolitano,R.deJ. (2013). A simple derivation of the Lindblad equation. *Revista Brasileira de Ensino de Física*, 35(1), 1303 1–1303 9.
- [4] Breuer,H. P., & Petruccione,F. (2002). *The theory of open quantum systems* (pp.1–648). Oxford University Press.
- [5] Budini,A.A. (2006). Lindblad rate equations. *Physical Review A*, 74(5), Article053815.
- [6] Carmichael,H.J. (1993). *An open systems approach to quantum optics: Lectures presented at the UniversitéLibredeBruxelles, October28 to November4,1991* (Lecture Notes in Physics Monographs, Vol.18, pp.i xxiv,1 179). Springer Verlag.
- [7] Chen,Y., & Chen,L. (2022). Quantum Wiener–Khinchin theorem for spectral domain optical coherence tomography. *Physical Review Applied*, 18(1), 014077.

- [8] Dalton,B.J., & Garraway,B.M. (2003). Non Markovian decay of a three level cascade atom in a structured reservoir. *Physical Review A*, 68(3), 033809.
- [9] Dalton,B.J., & Garraway,B.M. (2007). Quantum master equations for atoms in thermal fields. *Journal of Modern Optics*, 54(3), 413–423.
- [10] Fan,S., Kocabaş,Ş.E., & Shen,J. T. (2010). Input output formalism for few photon transport in one dimensional nanophotonic waveguides coupled to a qubit. *Physical Review A*, 82(6), 063821.
- [11] Ferguson,M.R., Ficek,Z., & Dalton,B.J. (1996). Resonance fluorescence spectra of three level atoms in a squeezed vacuum. *Physical Review A*, 54(3), 2379–2390.
- [12] Finn,M.A., Greenlees,G.W., Kumar,J., & Lewis,D.A. (1988). Title of the paper. *Physical Review A*, 38(2), 733–739.
- [13] Gao,W. C., Wang,T. J., Cao,C., & Wang,C. (2017). Efficient purification and concentration for Λ type three level entangled quantum dots using non reciprocal microresonators. *Quantum Information Processing*, 16(7), 182.
- [14] Gardiner,C.W., & Collett,M.J. (2025). Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation. *Physical Review A*, 31(6), 3761–3774.
- [15] Goodman,J.W. (1985). Statistical properties of laser light. In *The Fokker–Planck equation* (pp.374–413). Springer.

- [16] Koyu,S., & Tscherbul,T.V. (2021). Long lived quantum coherent dynamics of a Λ system driven by a thermal environment. *The Journal of Chemical Physics*, 157(12), 124302.
- [17] Laskar,R. (2024). Proposal for Composite Quantum Electromagnetically Induced Transparency Heat Engine Coupled by a Nanomechanical Mirror [Preprint].
- [18] Macovei,M., Ficek,Z., & Keitel,C.H. (2025). Collective coherent population trapping in a thermal field. *Physical Review A*, 73(6), 063818.
- [19] Mandel,L., & Wolf,E. (1995). *Optical coherence and quantum optics*. Cambridge University Press. (pp.i xxiv + 1 1166)
- [20] Manzano,D. (2019). A short introduction to the Lindblad master equation [Preprint].
- [21] Obada,A. S.F., Eied,A.A., & Abd Al Kader,G.M. (2024). Treatment of the emission and absorption spectra of a general formalism λ type three level atom driven by a two mode field with nonlinearities. *Journal of Russian Laser Research*, 29(5), 398–407.
- [22] Rau,A.R.P., & Zhao,W. (2003). Decoherence in a driven three level system. *Physical Review A*, 68(5), 052102–052106.
- [23] Schaller,G., & Brandes,T. (2008). Quantum and information thermodynamics: A unifying framework based on repeated interactions. *Physical Review X*, 7(2), 021003.

- [24] Sen,S., Dey,T.K., Nath,M.R., & Gangopadhyay,G. (2015). Comparison of electromagnetically induced transparency in lambda, cascade and vee three level systems. *Journal of Modern Optics*, 62(3), 166–174.
- [25] Shrestha,M., etal. (2004). Decoherence and population inversion in a three level atom interacting with thermal light. *Physical Review A*, 70(4), 043804.
- [26] Swain,M. (1981). Bilingual education for majority and minority language children. *Studia Linguistica*, 35(1–2), 15–32.
- [27] Von Neumann,J. (1932). *Mathematical foundations of quantum mechanics*. Princeton University Press. (460 pp)
- [28] Wang,C.H., Gullans,M.J., Porto,J.V., Phillips,W.D., & Taylor,J.M. (2017). Photon thermalization via laser cooling of atoms. *Physical Review A*, 98, 013834.
- [29] Younes,A., Putnam,R., Hamilton,P., & Campbell,W.C. (2025). Internal state cooling of an atom with thermal light. *Entropy*, 27(3), 222.
- [30] Zhang,Z. M., Xie,S. W., & Zhou,S. K. (2025). Quantum theory of the micromaser with ultracold cascade three level atoms. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 32(16), 4013–4023.
- [31] Zhong,C., & Robicheaux,F. (2016). Coherence and quasistable states in a strong infrared field. *Physical Review A*, 93(3), 033410.