



**WOLLEGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES**

**Application of Superposed Squeezed States in Quantum
Communication**

MSc. Thesis

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Nekemte, Ethiopia
January 28, 2026



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**Application of Superposed Squeezed States in Quantum
Communication**

**A Thesis submitted to the College of Natural and Computational
Sciences in Partial Fulfillment of the Requirements for the Degree of
the Master of Science in Physics (Quantum Optics)**

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January 28, 2026



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APPROVAL SHEET FOR SUBMITTING FINAL THESIS

As members of the Board of Examiners of the Final MSc. thesis open defense, we certify that we have read and evaluated the thesis prepared by **Lamesa Oljira** under the title "**Application of Superposed Squeezed States in Quantum Communication**" and recommend that the thesis be accepted as fulfilling the thesis requirement for the **Degree of Master of Science in Physics (Quantum Optics)**

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Final Approval and Acceptance

_____ Department PGC	_____ Signature	_____ Date
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Certification of the Final Thesis I hereby certify that all the corrections and recommendation suggested by the board of examiners are incorporated into the final thesis entitled "**Application of Superposed Squeezed States in Quantum Communication**" by **Lamesa Oljira**.

_____ Name of Dean, SGS	_____ Signature	_____ Date
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STATEMENT OF THE AUTHOR

I hereby confirm that the thesis entitled “**Application of Superposed Squeezed States in Quantum Communication**” was conducted under the supervision of **Dr. Misganu Chewaka**. I also affirm that all explanations and statements provided during the proposal defense were presented in full adherence to academic integrity and research ethics. In addition, this thesis has been prepared in line with the academic requirements of Wolllega University and has received approval from the Department’s Graduate Committee.

Student’s Name (Author)

Signature

Date

DECLARATION

I hereby affirm that the thesis entitled “*Application of Superposed Squeezed States in Quantum Communication*” is the result of my own independent work. It is submitted in partial fulfillment of the requirements for the Master of Science degree in Physics at the School of Graduate Studies, Wollega University. The research was conducted under the supervision of my advisor, and all assistance and contributions received during the course of the study have been duly acknowledged.

Student Name: _____

Advisor Name: _____

Date: _____

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LIST OF ABBREVIATIONS AND ACRONYMS

CV-QKD Continuous Variable Quantum Key Distribution

MUS Minimum Uncertainty State

QKD Quantum Key Distribution

QKDM Quantum Key Distribution Model

SMSVS Single Mode Squeezed Vacuum States

SSVS Superposed Squeezed Vacuum State

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ABSTRACT

This thesis explores the photon-number characteristics and quantum communication potential of superposed squeezed vacuum states, emphasizing their relevance to Quantum Key Distribution (QKD). The analysis investigates how statistical indicators of the light field—such as the average photon number and its fluctuations—reveal nonclassical behavior, including reduced quadrature noise and altered photon-number distributions that support efficient quantum encoding. Secure communication performance is evaluated through conditional probability methods and mutual information calculations, while Q-function phase-space representations illustrate interference patterns and deformation in amplitude distributions arising from squeezing and photon correlations. A Gaussian security model is applied to estimate the Holevo bound and quantify possible information gained by an eavesdropper. MATLAB simulations demonstrate that tuning the squeezing strength and superposition parameters enhances the achievable key rate, showing that superposed squeezed states exhibit increased resilience to channel loss and common quantum attacks, making them strong candidates for secure, low-photon, long-distance QKD applications.

Keywords:-Photon number variation,Q-function analysis, Quadrature squeezing, Quantum Key Distribution (QKD),Secret key rate analysis,Superposed squeezed vacuum states

Chapter 1

INTRODUCTION

1.1 Background of the Study

Using the concepts of quantum mechanics, quantum communication in particular, quantum key distribution (QKD) allows for safe information sharing that is essentially impervious to eavesdropping (Gisin et al., 2002). Single photons and polarization states are the foundation of traditional QKD protocols like BB84, which are based on discrete-variable systems (Bennett and Brassard, 1984). However, because of its potential for greater key rates and compatibility with the current optical communication infrastructure, continuous-variable quantum key distribution (CV-QKD) has drawn more interest (Diamanti et al., 2016). According to (Weedbrook, Pirandola, García-Patrón, Nicolas J. Cerf, et al., 2012b), CV-QKD encodes information in the electromagnetic field's quadrature components, which are frequently described using Gaussian states including coherent and squeezed states. One important non-classical resource among them is the single-mode squeezed vacuum state, which has lower quantum noise in one quadrature at the expense of higher noise in the conjugate quadrature (Loudon and P. L. Knight, 1987). In non-linear optical media, four-wave mixing or parametric down-conversion are commonly used to create a squeezed vacuum state (D. F. Walls, 1983). To increase the efficiency and security of quantum communication systems, squeezed states' quantum characteristics particularly their non-classicality and decreased uncertainty are crucial (Andersen et al., 2016). In order to improve entanglement and robustness in noisy situations, recent research has investigated superpositions of single-mode squeezed vacuum states to incorporate non-Gaussian features (Dell'Anno and Dollery, 2014). When compared to Gaussian states alone, these superposed squeezed states show richer phase-space structures and quantum interference patterns (Biswas and Agarwal, 2007).

The Q-function (Husimi Q-distribution) is a particularly helpful quasi-probability distribution function in phase space that is frequently used to characterize such states (Leonhardt, 1997). It offers a useful tool for illustrating the degree of coherence, non-

classicality, and squeezing of quantum optical states (J. Barnett, 2003). By measuring variables like excess noise and eavesdropper information gain, the Q-function can be used to evaluate the security of the quantum channel in the context of QKD (Grosshans and Grangier, 2002). By strengthening the QKD protocol's resistance to Gaussian attacks and boosting key generation rates in realistic channel conditions, the employment of superposed squeezed vacuum states, as examined by the Q-function, can improve performance (Usenko and Filip, 2016). Consequently, a possible path toward the development of secure quantum communication systems is the use of superposed single-mode squeezed vacuum states in QKD, as represented by the Q-function (Jaskula et al., 2024).

A key component of the advancement of quantum information science and quantum optics is the creation of nonclassical states of light. Because of their special noise-reducing capabilities and potential to improve quantum communication protocols, squeezed vacuum states have attracted a lot of attention (Tesfa, 2021). Specifically, the utilization of continuous-variable (CV) quantum states is advantageous for QKD, a secure technique for transferring cryptographic keys utilizing quantum mechanics. Squeezed states provide better performance in terms of noise robustness, key rate optimization, and security against eavesdropping than traditional QKD systems, which frequently rely on coherent states (Kannan, Sudheesh, et al., 2022). Squeezed vacuum states are perfect candidates for robust quantum communication channels because their nonclassical characteristics, like improved squeezing and interference, are exacerbated when they are superposed (Getahun, 2019).

The Q-function, a quasi-probability distribution in phase space, offers a potent diagnostic and visualization tool for analyzing these states (Ren and C.-z. Zhang, 2023). By exposing the coherence, squeezing, and photon statistics of quantum states, it makes it possible to characterize them. Q-function analysis reveals elliptical and fringe-like features in phase space in the setting of superposed squeezed vacuum states, which are suggestive of enhanced quantum signal distinguishability and non-classicality. We show how these states improve the efficiency and security of quantum key exchange by analyzing their statistical features and deriving their Q-functions. The findings add to the expanding corpus of work that aims to incorporate sophisticated quantum states into practical cryptography systems. In order to improve the measurement sensitivity in optomechanics, squeezed states of light have been seen in a range of quantum optical systems (Iwasawa, Kobayashi, and Otori, 2023).

In quantum optics and quantum information theories, creating new states of light has been a significant concern (El-Orany, 2011). One of the intriguing nonclassical characteristics of light that has drawn interest and been the subject of numerous studies is squeezing (Fan et al., 2011). Several optical processes, including generation, result in squeezed states (S. Barnett and Radmore, 2002). In quantum optics, various superpositions of squeezed states such as combinations of squeezed vacuum states or mixtures

involving displaced squeezed vacua, exhibit interesting quantum properties that differ from those of the individual states. Squeezed states, on the other hand, exhibit sub-Poissonian statistics and therefore display clear nonclassical behavior (Grynberg, Aspect, and Fabre, 2010). A central aim of quantum optics is to describe light using its intrinsic quantum characteristics. This is often achieved through phase-space representations such as the P-function, Wigner function, and Q-function, each corresponding to a different operator ordering convention for expressing the density operator (Getahun, 2019). The P-function, associated with anti-normal ordering, represents a c-number function over π and is useful for describing the superposition of two light beams that share the same frequency but occupy different quantum states. In contrast, the Wigner function corresponds to symmetric ordering of the density operator and also serves as a c-number function over π .

The Q-function, which is the most widely employed, arises from normal ordering of the density operator divided by π and effectively characterizes the superposition of light beams with identical frequencies but differing quantum states (Getahun, 2019).

1.2 Statement of the Problem

The reliable generation, control, and detection of superposed squeezed states remain major challenges in quantum communication systems. Practical limitations in state preparation introduce noise and imperfections, while propagation through optical fibers or free-space channels further degrades these fragile quantum states. Such distortions complicate accurate state identification and manipulation, and detection errors can significantly reduce the efficiency and security of quantum key distribution protocols. Compared with conventional quantum states, superposed squeezed states exhibit increased complexity, requiring more advanced analytical and diagnostic tools. Consequently, quantitative methods such as Q-function analysis, quadrature variance evaluation, and mean photon number characterization are essential for accurately describing their properties. Addressing these challenges is crucial not only for deepening the fundamental understanding of superposed squeezed states but also for improving the robustness, reliability, and security of practical quantum communication technologies.

1.3 Objectives of the Study

1.3.1 General Objective

The central objective of this thesis is to explore how superposed squeezed states can be utilized in quantum communication, with a particular emphasis on their application in quantum key distribution.

1.3.2 Specific Objectives

The specific objective of this study was to

- Analyze superposed single-mode squeezed vacuum states in qubit-based QKD, emphasizing their Q-function.
- Compute photon statistics—mean photon number, variance, and quadrature—for superposed squeezed states in QKD.
- Discuss the squeezing property in quantum communication focused on quantum key distribution protocol.

1.4 Significance of the Study

The significance of this study lies in demonstrating how superposed squeezed vacuum states can serve as powerful nonclassical resources for advancing secure quantum communication. By systematically analyzing photon number variation, its variance, quadrature squeezing, and their role in shaping the Q-function, the work establishes a deeper understanding of the noise reduction and enhanced non-classicality achievable in phase space. Through conditional probability distributions and mutual information analysis, the study highlights how reliable correlations between communicating parties can be extracted even under lossy and noisy channels. Furthermore, the integration of a simplified Gaussian security model provides an accessible yet rigorous framework for quantifying potential eavesdropper information. Finally, the derivation of key rate equations and their numerical evaluation using MATLAB underscore the potential of these states to improve efficiency, resilience, and scalability of Quantum Key Distribution protocols, especially in long-distance and low-photon regimes.

Chapter 2

LITERATURE REVIEW

2.1 The Squeezed State

Squeezed vacuum states are a broad category (Sivakumar, 2014) of minimum-uncertainty states (MUS). Compared to coherent states, squeezed vacuum states have less noise in a single quadrature. The creation of squeezed states in the nonlinear interaction of light fields has been addressed in recent years, and novel avenues for creating squeezed states through a straightforward application of the superposition principle for quantum states have been explored (Fabre and Treps, 2020). Many quantum optical systems, which are used in biology and opto-mechanics to increase measurement sensitivity, have been observed to exhibit squeezed states of light (Nielsen et al., 2023). According to (Schnabel, 2017) and (Andersen et al., 2016), a squeezed state is a quantum state in which the uncertainty in one quadrature (such as position or momentum) decreases below the uncertainty level of the vacuum state, while the uncertainty in the conjugate quadrature is correspondingly raised. This can be expressed numerically as (Kumar, Rishabh, and Arora, 2023):The squeeze operator, $S(r)$, is provided by:

$$\hat{S}(r) = \exp \left[\frac{r}{2} \left(\hat{a}^2 - \hat{a}^{\dagger 2} \right) \right]. \quad (2.1)$$

The action of the squeeze operator on the vacuum state generates a squeezed vacuum state (Pezzè et al., 2018),

$$|r\rangle = \hat{S}(r) |0\rangle. \quad (2.2)$$

In this expression, $\hat{a}$ and $\hat{a}^\dagger$ represent the annihilation and creation operators, respectively, and r denotes the squeezing parameter that quantifies the degree of squeezing. For the quadrature along which squeezing is applied, the squeezed state reduces the corresponding uncertainty in phase space (Pezzè et al., 2018).

2.2 Superposed Squeezed States

Over the past few decades, there has been a lot of interest in the problem of superposed quantum states of an optical field, sometimes referred to as Schrödinger cat states (Amiri-Margavi and Babaei, 2024). The superposition of two squeezed vacuum states was investigated by numerous researchers (Guo et al., 2022). The statistical and squeezing properties of these states were highlighted by these investigations. The statistical and squeezing characteristics of single-mode squeezed chaotic states and superposed single-mode squeezed chaotic states were investigated in a few years by (Mensen, 2024).

The analysis in (Sanders, 1989) indicates that both the Q-function and the density operator of a single-mode squeezed chaotic state increase with the mean photon number of the thermal light and the magnitude of the squeezing parameter. Conversely, the quadrature variance decreases as these parameters grow. To examine the superposition of two squeezed vacuum states, the study further investigated the quadrature-phase-eigenstate marginal distributions along with the photon number probability distribution, providing insight into their quantum statistical properties.

Two-mode squeezing involves a pair of electromagnetic field modes whose combined quadratures exhibit noise levels below the shot-noise limit (Junker, 2023). In this context, two squeezed vacuum states are often compared through their marginal distributions and photon-number characteristics. To explore non-classical effects important in quantum optics (Harrington, Mueller, and Murch, 2022), various quantum states have been proposed in recent research (A.-X. Liu, Haddadi, Qiao, et al., 2023).

A general form of the superposition of squeezed states (Kuchař and Marek, 2025) can be expressed as

$$|\psi\rangle = \alpha |s(r_1)\rangle + \beta |s(r_2)\rangle, \quad (2.3)$$

where $|s(r)\rangle$ represents a squeezed vacuum state with squeezing parameter r . The complex coefficients α and β determine the relative contributions of each squeezed state to the overall superposition.

Investigating the potential applications of such superpositions for quantum communication, specifically in QKD methods, is the aim. Superposed squeezed states provide benefits in state discrimination (Guo et al., 2022), entanglement generation (C. Liu et al., 2022), noise mitigation (Asjad, Tombesi, and Vitali, 2015), and improved communication fidelity (Tang et al., 2025) and (C. Liu et al., 2022).

2.3 Quantum Key Distribution (QKD)

Quantum key distribution (QKD) enables two parties to generate a shared random secret key by using the principles of quantum physics (Gisin et al., 2002). The well-known BB84 protocol, introduced by Bennett and Brassard in 1984, relies on photon polar-

ization states for key generation. QKD offers a secure method for Alice and Bob to exchange cryptographic keys because any attempt to observe or intercept the quantum states introduces disturbances that reveal the presence of an eavesdropper.

Compressed (squeezed) states of light are frequently employed to encode information for continuous-variable QKD (CV-QKD). In this thesis, we investigate how CV-QKD techniques could be improved by superposed squeezed states. Squeezed states can increase measurement accuracy, which is essential for safe key distribution. By enabling more reliable quantum state transmission over noisy channels, they improve security margins. It may be possible to produce more intricate information encodings by superimposing squeezed states, which would enhance security characteristics and increase resistance to eavesdropping (Grosshans and Grangier, 2002).

2.4 CV-QKD using Squeezed States

In continuous-variable QKD, the electromagnetic field is represented using its quadrature operators, usually corresponding to the position x and momentum p components of a field mode. When squeezed states are employed, one quadrature exhibits reduced quantum noise, which enhances the signal quality and lowers the error rate in the resulting key (Pirandola et al., 2008).

A standard form of the secret-key rate is

$$R_{\text{key}} = \frac{1}{2} (I_{\text{AB}} - I_{\text{EB}}), \quad (2.4)$$

where I_{AB} is the mutual information between Alice and Bob, and I_{EB} measures the information accessible to an eavesdropper (Pastushenko et al., 2025). The value of I_{AB} depends on the modulation scheme, the channel noise, and the amount of squeezing used during state preparation (Weedbrook, Pirandola, García-Patrón, Nicolas J Cerf, et al., 2012a).

2.5 Superposed Squeezed States in CV-QKD

Superposed squeezed states may be integrated into CV-QKD as a way to increase the protocol's resilience to noise (Y. Zhang, 2024). Such superposition can potentially improve tolerance to channel disturbances or allow additional information to be encoded per optical mode. When these states are included in the analysis, the mutual information between Alice and Bob can be adjusted accordingly, which could lead to an enhanced key rate:

$$I_{\text{Alice-Bob}} = h\left(\frac{\Delta x}{2} - \frac{\Delta x + \Delta y}{2}\right), \quad (2.5)$$

Here, Δx and Δy quantify the uncertainty contributions from the individual squeezed quadrature and from its superposed component, respectively. The function $h(x)$ plays the role of an entropy-related measure appearing in information-theoretic descriptions. Squeezing and superposition can effectively increase the information capacity by reducing uncertainty, which results in a higher key rate and enhanced security. Compared to conventional coherent states, squeezed states provide superior performance in noisy conditions (Daniel F Walls, 1983). Moreover, squeezed-state QKD protocols have demonstrated improved resilience against collective attacks (Banaszek, K. Horodecki, and P. Horodecki, 2012).

Chapter 3

METHODOLOGY

This chapter presents the methodological framework adopted to analyze the behavior and applications of superposed squeezed states in quantum communication. To address the objectives of the study, we begin by evaluating the Q-function, which is constructed from the anti-normally ordered characteristic function of the density operator. Using this Q-function, we formulate the density operator corresponding to superposed squeezed states and use it to examine several key physical quantities relevant to quantum optical systems. These include the mean photon number, photon-number variance, photon-number distribution, and the quadrature variances associated with both individual squeezed states and their superposed forms. In addition, the quantum key distribution model (QKDM) is applied to obtain the mathematical expressions governing squeezed and superposed squeezed states in a QKD setting. This allows us to assess how these states influence the security characteristics of the protocol. For reference, QKD provides a secure method for two communicating parties to share encryption keys in such a way that the resulting key is known exclusively to them. The process of sharing the key between parties is known as QKD, and it involves determining the noise characteristics, superposition fidelity, and squeeze level of the produced superposed squeezed states, examining how squeezing affects quantum communication, particularly as it relates to QKD techniques. Squeezed states will be used in numerical simulations to assess the key rate of QKD algorithms. Operators created by taking Hermitian combinations of $\hat{a}$ and $\hat{a}^\dagger$ represent observable quantities related to a single-mode light. The field quadratures can be introduced through the following operator combinations:

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \quad (3.1)$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \quad (3.2)$$

These two operators form a conjugate pair and obey the standard commutation rela-

tionship of the harmonic-oscillator algebra.

$$[\hat{a}_+, \hat{a}_-] = 2i \quad (3.3)$$

For a single-mode field, a squeezed state is defined by the reduction of the noise in one quadrature below the standard quantum limit, meaning that either $\Delta a_+ < 1$ or $\Delta a_- < 1$, while the Heisenberg relation $\Delta a_+ \Delta a_- \geq 1$ remains satisfied. When this inequality is saturated, the state is known as a minimum-uncertainty squeezed state. Applying a squeezing operator to a thermal field produces a squeezed thermal (or chaotic) state, which contains both classical fluctuations and quantum squeezing. If the mode initially starts in the vacuum state, the density operator for the resulting squeezed state can be written as

$$\hat{\rho} = |r\rangle\langle r| = \hat{S}(r)|0\rangle\langle 0|\hat{S}^\dagger(r) = \hat{S}(r)\hat{S}^\dagger(r), \quad (3.4)$$

which corresponds to the density operator of a single-mode squeezed vacuum. Here, the squeezing operator $\hat{S}(r)$ transforms the vacuum state $|\psi\rangle$ into the squeezed state. A complete form of the Q-function for a single-mode squeezed vacuum, derived using the anti-normally ordered characteristic function, is given in (Schnabel, 2017).

3.1 Superpositions for Single Mode

We examine several representative cases of single-mode superposed squeezed states. To evaluate the peak magnitude of negative energy in each situation, we calculate the difference $R - n$.

$$|\psi\rangle = |r\rangle, \quad r = e^{i\theta}, \quad (3.5)$$

where r denotes the squeezing parameter and θ is the associated phase. The squeezing operator $\hat{S}(r)$ is defined as

$$\hat{S}(r) = \exp \left[\frac{1}{2}(\hat{a}^2 - \xi^* \hat{a}^{\dagger 2}) \right], \quad (3.6)$$

and it is unitary, satisfying

$$\hat{S}^\dagger(r) = \hat{S}(-r) = \hat{S}^{-1}(r). \quad (3.7)$$

Acting with $\hat{S}(r)$ on the vacuum state produces the single-mode squeezed state $|r\rangle$:

$$|r\rangle = \hat{S}(r)|0\rangle. \quad (3.8)$$

This state can also be represented as a superposition of even-number Fock states:

$$|\psi\rangle = \sqrt{\text{sech } r} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} \left[-\frac{1}{2} e^{i\theta} \tanh r \right]^n |2n\rangle, \quad (3.9)$$

and the transformation of annihilation and creation operators under squeezing is given by:

$$\hat{S}^\dagger(r) \hat{a} \hat{S}(r) = \hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r, \quad (3.10)$$

$$\hat{S}^\dagger(r) \hat{a}^\dagger \hat{S}(r) = \hat{a}^\dagger \cosh r - \hat{a} e^{-i\theta} \sinh r. \quad (3.11)$$

The expectation values in the squeezed state $|r\rangle$ are:

$$n = \langle \hat{a} \hat{a}^\dagger \rangle = \langle 0 | \hat{S}^\dagger(r) \hat{a} \hat{a}^\dagger \hat{S}(r) | 0 \rangle = \sinh^2 r, \quad (3.12)$$

$$\langle \hat{a}^2 \rangle = \langle 0 | \hat{S}^\dagger(r) \hat{a}^2 \hat{S}(r) | 0 \rangle = -e^{i\theta} \sinh r \cosh r. \quad (3.13)$$

Consequently, the quantities of interest are

$$R = \sinh r \cosh r, \quad R - n = \sinh r (\cosh r - \sinh r), \quad (3.14)$$

which approaches a maximum value of 0.5 in the limit $r \rightarrow \infty$.

3.2 Superposition of Squeezed Vacuum State

We analyze the energy properties of a superposition formed by two single-mode squeezed vacuum states:

$$|\psi\rangle = N(|r\rangle + |-r\rangle), \quad \eta = |\eta| e^{i\theta}. \quad (3.15)$$

The mean photon number for this state is given by

$$n = \langle \hat{a}^\dagger \hat{a} \rangle = N^2 \left[\langle r | \hat{a}^\dagger \hat{a} | r \rangle + |\eta|^2 + \langle -r | \hat{a}^\dagger \hat{a} | -r \rangle + \eta \langle r | \hat{a}^\dagger \hat{a} | -r \rangle + \eta^* \langle -r | \hat{a}^\dagger \hat{a} | r \rangle \right]. \quad (3.16)$$

Explicit evaluation yields:

$$n = N^2 \left[\sinh^2 r \left(1 + |\eta|^2 - 2|\eta| \cos \theta \frac{\text{sech } r \tanh^2 r}{(1 + \tanh^2 r)^{3/2}} \right) \right]. \quad (3.17)$$

The cross terms between the two components were:

$$\langle -\xi | \hat{a}^\dagger \hat{a} | \xi \rangle = \langle \xi | \hat{a}^\dagger \hat{a} | -\xi \rangle = \frac{\text{sech } r \tanh^2 r}{(1 + \tanh^2 r)^{3/2}}, \quad (3.18)$$

and

$$\langle -\xi | \hat{a}^2 | \xi \rangle = -\langle \xi | \hat{a}^2 | -\xi \rangle = -\frac{\text{sech } r \tanh r}{(1 + \tanh^2 r)^{3/2}}. \quad (3.19)$$

From these relations, we obtain the expectation value:

$$\langle \hat{a}^\dagger \rangle = N^2 \left[(|\eta|^2 - 1) \sinh r \cosh r + 2i|\eta| \sin \theta \frac{\text{sech } r \tanh^2 r}{(1 + \tanh^2 r)^{3/2}} \right], \quad (3.20)$$

and the quantity

$$R = |\langle \hat{a}^2 \rangle| = N^2 \left[(|\eta|^2 - 1)^2 \sinh^2 r \cosh^2 r + 4|\eta|^2 \sin^2 \theta \frac{\text{sech}^2 r \tanh^2 r}{(1 + \tanh^2 r)^3} \right]^{1/2}. \quad (3.21)$$

Normalization of the state is enforced via

$$\langle \psi | \psi \rangle = 1 = N^2 \left[1 + |\eta|^2 + \eta \langle r | -r \rangle + \eta^* \langle -r | r \rangle \right], \quad (3.22)$$

so that

$$N^2 = \left[1 + |\eta|^2 + 2|\eta| \cos \theta \sqrt{\text{sech}(2r)} \right]^{-1}. \quad (3.23)$$

The maximum magnitude of negative energy, denoted R_{-n} , can then be evaluated as a function of the phase θ and squeezing parameter r (Ford and Roman, 2008). For a fixed θ , R_{-n} typically attains its peak at a finite value of r . In the special case $\theta = 0$ with $\eta = 0$ (corresponding to a single squeezed vacuum state), the negative energy density reaches 0.5 for large r . For other values of η associated with superpositions, the peak negative energy is slightly smaller and occurs at a finite r .

The variances of the quadratures are

$$\langle (\Delta \hat{X})^2 \rangle = \frac{1}{4} (\cosh^2 r + \sinh^2 r - 2 \sinh r \cosh r \sin \theta), \quad (3.24)$$

$$\langle (\Delta \hat{P})^2 \rangle = \frac{1}{4} (\cosh^2 r + \sinh^2 r + 2 \sinh r \cosh r \sin \theta), \quad (3.25)$$

and for $\theta = 0$ this reduces to

$$\langle(\Delta\hat{X})^2\rangle = \frac{1}{4}e^{-2r}, \quad \langle(\Delta\hat{P})^2\rangle = \frac{1}{4}e^{2r}. \quad (3.26)$$

It is evident that the X quadrature is squeezed, and the P quadrature is squeezed for $\theta = \pi$. An illustrative ellipse is displayed below to help with comprehension of the squeezing. Phase-space uncertainty ellipse of a squeezed vacuum state. The quadra-

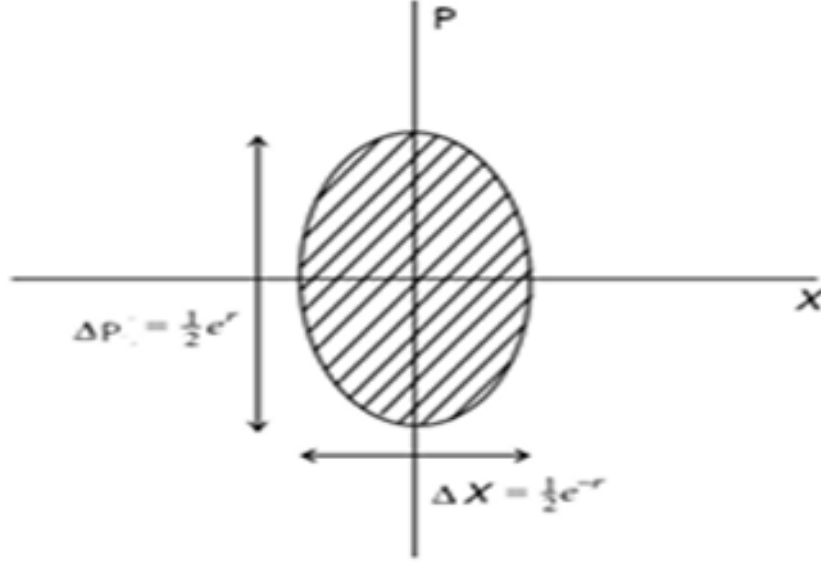


Figure 3.1: Shows the squeezed X quadrature error ellipse for the squeezed vacuum state

ture uncertainties are given by $\Delta X = \frac{1}{2}e^{-r}$ and $\Delta P = \frac{1}{2}e^r$. It should be mentioned that the given figure would have been a perfect circle with a radius of $1/4$ in a coherent condition. The question of how these squeezed states are generated is now evident. There are numerous techniques for the same, and nearly all of them rely on parametric processes that make use of nonlinear optical equipment. Information about the two popular techniques, known as degenerate four-wave mixing and degenerate parametric down-conversion, can be found here (Gerry and P. Knight, 2005).

3.3 The Q-Function

In quantum optics, the Q-function is widely used to describe the superposition of two light modes of the same frequency, even if they occupy distinct states. This function

plays a central role as it allows one to predict non-classical effects and compute moments of operators. Formally, the Q-function can be expressed in terms of the normally ordered density operator divided by π . Here, we aim to determine the Q-function for the superposition of single-mode squeezed vacuum states (Kassahun, 2011).

We begin by considering the Q-function for a single squeezed vacuum state:

$$Q(\alpha^*, \alpha) = \frac{1}{\pi^2} \int d^2z \phi_a(z) \exp(z^* \alpha - z \alpha^*), \quad (3.27)$$

where the characteristic function is defined as

$$\phi_a(z) = \text{Tr}(|r\rangle\langle r| e^{z^* \hat{a} - z \hat{a}^\dagger}), \quad (3.28)$$

with $\hat{\rho} = |r\rangle\langle r|$ representing the density operator of the squeezed vacuum. Using the unitary property of the squeezing operator, we can rewrite the characteristic function as

$$\phi_a(z) = \langle 0 | e^{-z^* \hat{a}(r)} e^{z \hat{a}^\dagger(r)} | 0 \rangle, \quad (3.29)$$

where the transformed operators are

$$\hat{a}(r) = \hat{s}^\dagger(r) \hat{a} \hat{s}(r), \quad \hat{a}^\dagger(r) = \hat{s}^\dagger(r) \hat{a}^\dagger \hat{s}(r). \quad (3.30)$$

Applying the Baker–Hausdorff relation,

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}]}, \quad (3.31)$$

we obtain

$$\phi_a(z) = e^{-\frac{1}{2} z z^*} \langle 0 | \exp \left[z \hat{a}^\dagger(r) - z^* \hat{a}(r) \right] | 0 \rangle, \quad (3.32)$$

with

$$\hat{a}(r) = \hat{a} \cosh r - \hat{a}^\dagger \sinh r, \quad \hat{a}^\dagger(r) = \hat{a}^\dagger \cosh r - \hat{a} \sinh r. \quad (3.33)$$

This leads to

$$\phi_a(z) = e^{-\frac{1}{2} z z^*} \langle \gamma | \exp \left[z \hat{a}^\dagger(r) - z^* \hat{a}(r) \right] | \gamma \rangle, \quad (3.34)$$

and, after inserting the operator definitions, we find

$$\phi_a(z) = e^{\frac{1}{2} z z^*} \langle 0 | \exp \left[(z \cosh r + z^* \sinh r) \hat{a}^\dagger - (z^* \cosh r + z \sinh r) \hat{a} \right] | 0 \rangle. \quad (3.35)$$

A second application of the Baker–Hausdorff identity yields

$$\begin{aligned}\phi_a(z) &= \exp[-zz^* \cosh^2 r - \cosh r \sinh r (z^{*2} + z^2)] \\ &\times \langle 0 | \exp[(z \cosh r + z^* \sinh r) \hat{a}^\dagger] \exp[-(z \sinh r + z^* \cosh r) \hat{a}] | 0 \rangle, \quad (3.36)\end{aligned}$$

which simplifies to

$$\phi_a(z) = \exp[-zz^* \cosh^2 r - \cosh r \sinh r (z^{*2} + z^2)]. \quad (3.37)$$

Finally, the Q-function becomes

$$Q(\alpha^*, \alpha) = \frac{1}{\pi^2} \int d^2 z \exp[-zz^* \cosh^2 r + z^* \alpha - z \alpha^* - \frac{1}{2} \cosh r \sinh r (z^{*2} + z^2)], \quad (3.38)$$

which, upon integration using

$$\int \frac{d^2 z}{\pi} \exp[-az^* z + bz + cz^* + Az^2 + Bz^{*2}] = \frac{1}{\sqrt{a^2 - 4AB}} \exp\left[\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\right], \quad a > 0, \quad (3.39)$$

leads to

$$Q(\alpha^*, \alpha) = \frac{\operatorname{sech} r}{\pi} \exp[-\alpha^* \alpha - \frac{1}{2} \tanh r (\alpha^{*2} + \alpha^2)]. \quad (3.40)$$

Alternatively, the same result is obtained by setting $\gamma = 0$ in

$$\begin{aligned}Q(\alpha^*, \alpha) &= \frac{\operatorname{sech} r}{\pi} \exp\left[-\alpha^* \alpha - \frac{1}{2} \tanh r (\alpha^2 + \alpha^{*2} - \gamma^2 - \gamma^{*2})\right. \\ &\quad \left.+ \operatorname{sech} r (\gamma^* \alpha + \gamma \alpha^*) - \gamma^* \gamma\right]. \quad (3.41)\end{aligned}$$

3.4 Photon Statistics

Photon statistics are fundamental in quantum optics as they provide information about the behavior of light at the quantum level, which cannot be completely described using classical electromagnetic theory. Understanding these statistics allows us to analyze fluctuations and correlations in quantum light. Typically, two main quantities are evaluated: the mean photon number and the photon number variance.

3.4.1 Mean photon number

The average photon number for a squeezed vacuum state can be calculated by evaluating the expectation value of the number operator in that state. Specifically, the mean photon number is given by

$$\langle \hat{n} \rangle = \langle r | \hat{n} | r \rangle, \quad (3.42)$$

which can also be expressed in terms of the vacuum state and the squeeze operator as

$$\langle \hat{n} \rangle = \langle 0 | \hat{s}^\dagger(r) \hat{a}^\dagger \hat{a} \hat{s}(r) | 0 \rangle, \quad (3.43)$$

where the number operator is defined as

$$\hat{n} = \hat{a}^\dagger \hat{a}. \quad (3.44)$$

By exploiting the unitary property of the squeeze operator and using the transformed annihilation and creation operators, one can rewrite the mean photon number as

$$\langle \hat{n} \rangle = \langle 0 | \hat{a}^\dagger(r) \hat{a}(r) | 0 \rangle. \quad (3.45)$$

Substituting the explicit expressions for the squeezed operators and evaluating the expectation values with respect to the vacuum, we obtain

$$\begin{aligned} \langle \hat{n} \rangle &= \langle 0 | \hat{a}^\dagger \hat{a} | 0 \rangle \cosh^2 r - \langle 0 | \hat{a}^{\dagger 2} | 0 \rangle \cosh r \sinh r + \sinh^2 r \\ &\quad - \langle 0 | \hat{a}^2 | 0 \rangle \cosh r \sinh r + \langle 0 | \hat{a}^\dagger \hat{a} | 0 \rangle \sinh^2 r. \end{aligned} \quad (3.46)$$

Finally, simplifying the terms leads to the well-known result:

$$\langle \hat{n} \rangle = \sinh^2 r. \quad (3.47)$$

3.4.2 The variance of the photon number

Next, we calculate the photon number variance for a squeezed vacuum state. The second moment of the photon number operator in the squeezed vacuum basis is given by

$$\langle \hat{n}^2 \rangle = \langle r | \hat{a}^{\dagger 2} \hat{a}^2 | r \rangle, \quad (3.48)$$

which can equivalently be expressed using the vacuum state and the squeeze operator:

$$\langle \hat{n}^2 \rangle = \langle 0 | \hat{s}^\dagger(r) \hat{a}^{\dagger 2} \hat{a}^2 \hat{s}(r) | 0 \rangle. \quad (3.49)$$

By employing the unitary property of the squeeze operator and introducing the transformed operators, we can rewrite this expectation value as

$$\langle \hat{n}^2 \rangle = \langle 0 | \hat{a}^{\dagger 2}(r) \hat{a}^2(r) | 0 \rangle, \quad (3.50)$$

where the squeezed operators are expanded as

$$\hat{a}(r) = \hat{a} \cosh r - \hat{a}^\dagger \sinh r, \quad \hat{a}^\dagger(r) = \hat{a}^\dagger \cosh r - \hat{a} \sinh r. \quad (3.51)$$

Substituting these into the expression for $\langle \hat{n}^2 \rangle$ and performing the algebraic expansion, followed by normal ordering and applying the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad (3.52)$$

yields the simplified form:

$$\langle \hat{n}^2 \rangle = \cosh^2 r \sinh^2 r + 2 \sinh^4 r. \quad (3.53)$$

The photon number variance is then obtained from

$$(\Delta \hat{n})^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2, \quad (3.54)$$

which gives, for a squeezed vacuum state,

$$(\Delta n)^2 = \cosh^2 r \sinh^2 r + \sinh^4 r. \quad (3.55)$$

Using the result for the mean photon number $\bar{n} = \sinh^2 r$, the variance can also be expressed in the compact form

$$(\Delta n)^2 = \bar{n}(2\bar{n} + 1). \quad (3.56)$$

This matches the expected variance for a squeezed vacuum state when the coherent displacement parameter γ is set to zero.

3.5 Photon Number Distribution

Using the photon number distribution formula in terms of the Q-function, we have

$$P(n) = \frac{\pi}{n!} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} \left[Q(\alpha^*, \alpha) e^{\alpha^* \alpha} \right] \Big|_{\alpha^* \alpha = 0}. \quad (3.57)$$

Substituting the previously obtained Q-function, the probability of finding n photons can be expressed as

$$P(n) = \frac{\text{sech } r}{n!} \frac{\partial^{2n}}{\partial \alpha^n \partial \alpha^{*n}} \Big|_{\alpha = \alpha^* = 0}, \quad (3.58)$$

where we define

$$a = \frac{1}{2} \tanh r. \quad (3.59)$$

Expanding the exponential terms into power series, we get

$$e^{-a\alpha^2} = \sum_{i=0}^{\infty} \frac{(-a)^i}{i!} \alpha^{2i}, \quad e^{-a\alpha^{*2}} = \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \alpha^{*2j}. \quad (3.60)$$

Thus, $P(n)$ can be rewritten as

$$P(n) = \frac{\text{sech } r}{n!} \sum_{i,j} \frac{(-a)^{i+j}}{i! j!} \frac{\partial^{2n}}{\partial \alpha^n \partial \alpha^{*n}} \left(\alpha^{2i} \alpha^{*2j} \right) \Big|_{\alpha=\alpha^*=0}. \quad (3.61)$$

Performing the derivatives using the identity

$$\frac{\partial^n}{\partial x^n} x^m = \frac{m!}{(m-n)!} x^{m-n}, \quad (3.62)$$

and setting $\alpha = \alpha^* = 0$, we obtain

$$P(n) = \frac{\text{sech } r}{n!} \sum_{i,j} \frac{(-a)^{i+j}}{i! j!} \frac{(2i)!}{(2i-n)!} \frac{(2j)!}{(2j-n)!} \delta_{2i,n} \delta_{2j,n}. \quad (3.63)$$

Using the Kronecker delta property, it follows that

$$i = \frac{n}{2}, \quad j = \frac{n}{2}, \quad (3.64)$$

and for non-negative integers, this leads to the final photon number probability:

$$P(n) = \frac{(-1)^n n! \tanh^n r}{2^n ((n/2)!)^2 \cosh r}, \quad (3.65)$$

for even $n = 0, 2, 4, \dots$, and

$$P(n) = 0, \quad (3.66)$$

for odd $n = 1, 3, 5, \dots$

This demonstrates that in a squeezed vacuum state, photons are emitted exclusively in pairs, making the probability of an odd photon number zero. Following this, the quadrature variances for the squeezed vacuum can be evaluated.

3.6 Quadrature Variance

The expectation value $\hat{a}^{\dagger 2}$ is therefore determined. In a squeezed vacuum state, the expectation values of $\hat{a}^{\dagger 2}$ can be expressed as

$$\langle \hat{a}^{\dagger 2} \rangle = \langle r | \hat{a}^{\dagger 2} | r \rangle. \quad (3.67)$$

Utilizing (3.3), we compose

$$\langle \hat{a}^{\dagger 2} \rangle = \langle 0 | \hat{s}^\dagger(r) \hat{a}^{\dagger 2} \hat{s}(r) | 0 \rangle. \quad (3.68)$$

Using the squeeze operator's unitarity property, which is provided by

$$\hat{s}(r) \hat{s}^\dagger(r) = \hat{I}, \quad (3.69)$$

we have

$$\langle \hat{a}^{\dagger 2} \rangle = \langle 0 | \hat{a}^{\dagger 2} | 0 \rangle \cosh^2 r - 2 \langle 0 | \hat{a}^\dagger \hat{a} | 0 \rangle \cosh r \sinh r + \langle 0 | \hat{a}^2 | 0 \rangle \sinh^2 r - \cosh r \sinh r. \quad (3.70)$$

It then follows that

$$\langle \hat{a}^{\dagger 2} \rangle = -\cosh r \sinh r. \quad (3.71)$$

Consequently, it follows that

$$\langle \hat{a}^\dagger \rangle = 0. \quad (3.72)$$

Similarly, it may be demonstrated that

$$(\Delta \hat{a}_+)^2 = 1 + 2 \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 - 2 \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle, \quad (3.73)$$

and

$$(\Delta \hat{a}_-)^2 = 1 + 2 \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \rangle^2 - 2 \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle, \quad (3.74)$$

the quadrature variance for the plus quadrature takes the form

$$(\Delta \hat{a}_+)^2 = 2 \sinh^2 r - 2 \cosh r \sinh r + 1. \quad (3.75)$$

In the view of

$$\sinh^2 r = \frac{e^{2r} - 2 + e^{-2r}}{4}, \quad (3.76)$$

and

$$\cosh r \sinh r = \frac{e^{2r} - e^{-2r}}{4}, \quad (3.77)$$

the quadrature variance for the plus quadrature finally takes the form

$$(\Delta \hat{a}_+)^2 = e^{-2r}, \quad (3.78)$$

and the quadrature variance for the minus quadrature turns out to be

$$(\Delta \hat{a}_-)^2 = e^{2r}, \quad (3.79)$$

showing that, for $r > 0$, the squeezing occurs in the plus quadrature.

3.7 Quantum Key Distribution Protocol with SSVS

Alice uses the stated bases X and Z to create superposed squeezed vacuum state (SSVS) qubits with 50/50 probability. The state preparation process is as follows: Alice randomly selects a base (X or Z) and a bit (0 or 1), and prepares the corresponding superposed squeezed vacuum state qubit. She establishes the encoding angles θ and ϕ as well as the squeezing parameter r . At Bob's site, the squeezing parameter is optimized based on the channel loss. With transmissivity η representing the channel loss, the transmission-prepared qubit is sent through a quantum channel modeled as a beam splitter. Using homodyne detection, Bob randomly chooses to measure either the X or Z quadrature.

After the measurement, Alice and Bob publicly announce their bases. Only the events where they used the same base are retained. Based on the channel parameters, Bob estimates the ideal squeezing value, and the channel loss η is also estimated. Standard error correction and privacy amplification are applied to extract a secure key.

The evolution of the Q-function through the channel can be described by scaling the complex amplitude to account for channel loss:

$$\alpha \rightarrow \sqrt{\eta} \alpha \quad (3.80)$$

Thus, after transmission through the channel, the Q-function becomes

$$Q(\alpha) = Q\left(\frac{\alpha}{\sqrt{\eta}}\right) \quad (3.81)$$

We also account for the channel's thermal noise N . The Q-function of the thermal noise is $Q(\alpha')$, and the output Q-function after accounting for both loss and thermal noise is given by the convolution:

$$Q_{\text{out}}(\alpha) = \int Q(\alpha') Q(\alpha - \alpha') d^2 \alpha' \quad (3.82)$$

In this way, we have obtained the equation for the evolution of the state across a lossy quantum channel and defined a QKD protocol based on superposed squeezed vacuum state (SSVS) qubits. The single-mode squeezed vacuum states and their superpositions are properly defined first. Applying the squeezing operator on the vacuum state $|0\rangle$ creates the squeezed vacuum state $|0, r\rangle$ with squeezing parameter r :

$$|0, r\rangle = S(r) |0\rangle \quad (3.83)$$

A linear combination of two SMSVS with opposing squeezing parameters is known as the superposed squeezed vacuum state.

$$|\psi\rangle = \alpha|0, r\rangle + \beta|0, -r\rangle \quad (3.84)$$

The normalization requirement is satisfied by the complex superposition coefficients α and β , where

$$|\alpha|^2 + |\beta|^2 = 1.$$

For simplicity, we limit α and β to real values, and we encode the qubit using changes of the two real parameters. In particular, our bases are described as:

- **X basis:** encode $|0\rangle$ as $\alpha = \cos \theta$ and $\beta = \sin \theta$, encode $|1\rangle$ as $\alpha = \cos \theta$ and $\beta = -\sin \theta$.
- **Z basis:** encode $|0\rangle$ as $\alpha = \cos \phi$ and $\beta = \sin \phi$, encode $|1\rangle$ as $\alpha = -\cos \phi$ and $\beta = -\sin \phi$.

Here, θ and ϕ stand for the optimal encoding angles.

Because it eliminates the necessity for intense coherent states, using superpositions of squeezed vacuum states has the advantage of lowering the risk of photon number splitting (PNS) attacks at low photon numbers. A considerable amount of built-in protection against specific eavesdropping techniques is offered by the vacuum components. Performance optimization in the presence of channel loss is made possible by the squeezing parameters, which regulate the phase-space distribution's shape.

The quantum state ρ is represented in phase space by the Q-function, which is defined as follows:

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle \quad (3.85)$$

The coherent state with complex amplitude α is represented by $|\alpha\rangle$. The density operator for the superposition of the squeezed vacuum state qubit $|\psi\rangle$ is:

$$\hat{\rho} = |\psi\rangle \langle \psi| \quad (3.86)$$

Therefore, we must assess $\langle \alpha | \psi \rangle$. First, we review how a coherent state and a squeezed vacuum state overlap:

$$\langle \alpha | 0, r \rangle = \frac{\exp\left(\frac{1}{2}\alpha_x^2 \tanh r\right)}{(\cosh r)^{1/2}} \quad (3.87)$$

Consequently,

$$\langle \alpha | \psi \rangle = \alpha \langle \alpha | 0, r \rangle + \beta \langle \alpha | 0, -r \rangle = \frac{\alpha \exp\left(\frac{1}{2}\alpha_x^2 \tanh r\right) + \beta \exp\left(-\frac{1}{2}\alpha_x^2 \tanh r\right)}{(\cosh r)^{1/2}} \quad (3.88)$$

Lastly, the superposed squeezed vacuum state qubit's Q-function is:

$$Q(\psi_a) = \frac{|\langle \alpha | \psi \rangle|^2}{\pi} = \frac{1}{\pi \cosh r} \left| \alpha \exp\left(\frac{1}{2}\alpha_x^2 \tanh r\right) + \beta \exp\left(-\frac{1}{2}\alpha_x^2 \tanh r\right) \right|^2 \quad (3.89)$$

An analytical expression for the superposed squeezed vacuum state qubit's Q-function has been obtained. This statement is essential to comprehending and examining how this state behaves in a QKD protocol.

3.8 Techniques for Analyzing Data

The investigation involved both analytical and numerical methods. Analytical methods are used for exact solutions, while numerical methods approximate solutions using MATLAB, including graphical representations to simplify computation.

Chapter 4

RESULTS AND DISCUSSION

Squeezing is one of the more intriguing and unique characteristics of single-mode squeezed vacuum states (SMSVs) that are superposed. This part introduces the technique sections for the mean photon number, variance of photon number, quadrature variance, quadrature squeezing, and photon number distribution of single-mode squeezed vacuum states. We examine superposed single-mode squeezed vacuum states within this chapter. We describe the mean of photon number, variance of photon number, quadrature variance, and quadrature squeezing of the superposed single-mode squeezed vacuum states based on the Q-function of the single-mode squeezed vacuum state, which is explained under the methodology.

4.1 Density Operator

By employing the Q-function of the individual modes, we derive the density operator for the superposition of two identical single-mode squeezed vacuum states. Under the assumption that the density operator is $\hat{\rho}(\hat{a}^\dagger \hat{a})$, its expansion can be written as

$$\hat{\rho}(\hat{a}^\dagger \hat{a}) = \frac{1}{\pi} \sum_{kl} C_{kl} \hat{a}^{\dagger k} \hat{a}^l \quad (4.1)$$

Making use of the coherent state's completeness relation:

$$I = \int \frac{d^2 \alpha_l}{\pi} |\alpha_l\rangle \langle \alpha_l| \quad (4.2)$$

Additionally, the relationships

$$|\alpha_l\rangle \langle \alpha_l| \hat{a}^{\dagger k} = \alpha_l^{*k} |\alpha_l\rangle \langle \alpha_l| \quad (4.3)$$

After that, we can fill out the form:

$$\hat{\rho}(\hat{a}^\dagger \hat{a}) = \int \frac{d^2 \alpha_l}{\pi} \sum_{kl} C_{kl} \alpha_l^{*k} \left(\alpha_l + \frac{\partial}{\partial \alpha_l^*} \right)^i |\alpha_l\rangle \langle \alpha_l| \quad (4.4)$$

This formula can also be expressed as:

$$\hat{\rho}(\hat{a}^\dagger \hat{a}) = \int d^2 \alpha_l Q(\alpha_l^*, \alpha_l + \frac{\partial}{\partial \alpha_l^*}) \hat{D}(\alpha_l) \hat{\rho}(0) \hat{D}^\dagger(\alpha_l) \quad (4.5)$$

Where

$$Q(\alpha_l^*, \alpha_l + \frac{\partial}{\partial \alpha_l^*}) = \frac{1}{\pi} \sum_{kl} C_{kl} \alpha_l^{*k} (\alpha_l + \frac{\partial}{\partial \alpha_l^*})^l \quad (4.6)$$

$$\hat{\rho}(0) = |0\rangle\langle 0| \quad (4.7)$$

4.2 Q-Function

The general Q-function for a squeezed coherent as stated by 3.41:

$$Q(\alpha^*, \alpha) = \frac{\text{sech } r}{\pi} \exp \left[-\alpha^* \alpha - \frac{1}{2} \tanh r (\alpha^2 + \alpha^{*2} - \gamma^2 - \gamma^{*2}) + \text{sech } r (\gamma^* \alpha + \gamma \alpha^*) - \gamma^* \gamma \right]. \quad (4.8)$$

By setting the displacement parameter $\gamma = 0$, the Q-function reduces to that of a **squeezed vacuum state**:

$$Q(\alpha^*, \alpha) = \frac{\text{sech } r}{\pi} \exp \left[-\alpha^* \alpha - \frac{1}{2} \tanh r (\alpha^2 + \alpha^{*2}) \right]. \quad (4.9)$$

- At $\alpha = 0$,

$$Q(0, r) = \frac{\text{sech } r}{\pi} \approx \frac{1}{\pi},$$

which is independent of r , representing a flat line at the origin in phase space.

- For $\alpha \neq 0$, writing $\alpha = x + iy$, we have:

$$Q(x, y, r) = \frac{\text{sech } r}{\pi} \exp \left[-(1 + \tanh r)x^2 - (1 - \tanh r)y^2 \right].$$

The interplay between the negative $\cosh(2r)$ -like term along x and the positive $\sinh(2r)$ -like term along y determines whether $Q(\alpha, r)$ decays or grows along each quadrature.

- For example, at $\alpha = 1 + 0.5i$,

$$Q(1, 0.5, r) = \frac{\text{sech } r}{\pi} \exp \left[-(1 + \tanh r) - 0.25(1 - \tanh r) \right],$$

which shows exponential growth along one quadrature as r increases.

The Husimi Q -function for a squeezed vacuum state is defined as

$$Q(\alpha, r) = \frac{1}{\pi} \exp\left(-\cosh(2r) |\alpha|^2 + \sinh(2r) [\alpha^2 + (\alpha^*)^2]\right). \quad (4.10)$$

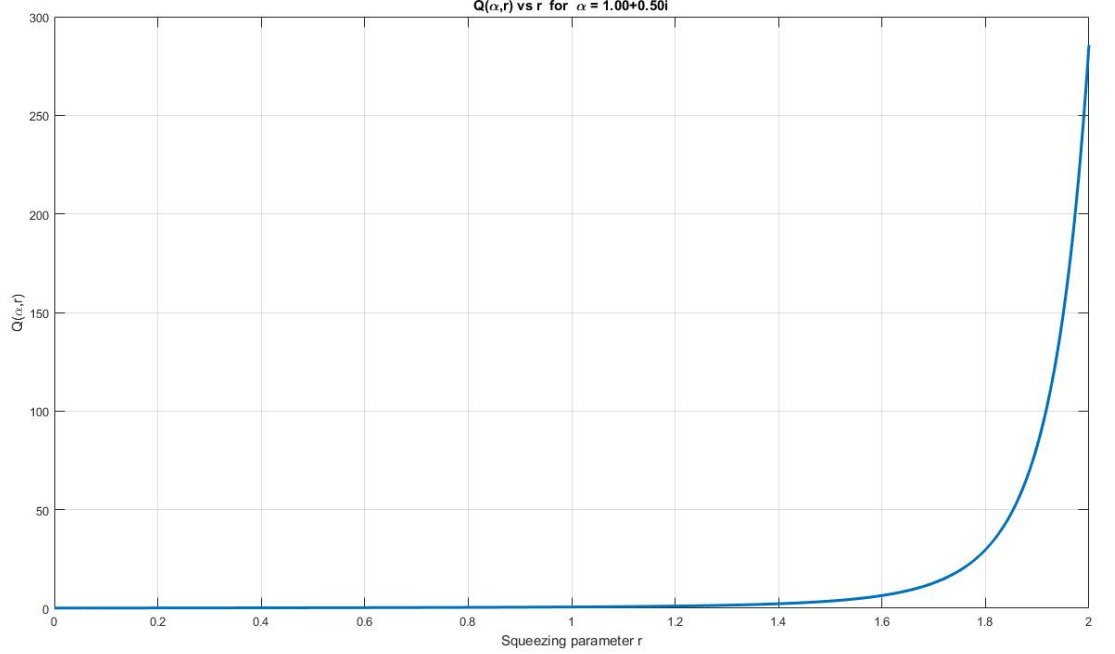


Figure 4.1: Plot of $Q(\alpha, r)$ versus the squeezing parameter r

for $\alpha = 1 + 0.5i$. For small r , the function remains close to zero. As r increases, the exponential contribution dominates and $Q(\alpha, r)$ grows very rapidly. At $\alpha = 0$, the function is constant $Q(0, r) = 1/\pi$, independent of r .

- **X-axis:** squeezing parameter r .
- **Y-axis:** value of the Q -function $Q(\alpha, r)$ at fixed α .
- For small r , $Q(\alpha, r)$ is very small, nearly flat near zero.
- As r increases, the Q -function rises steeply, showing exponential growth.

From Eq. (4.10), two competing terms determine the behavior:

$$\begin{aligned} & -\cosh(2r) |\alpha|^2 \quad (\text{suppression term}), \\ & +\sinh(2r) [\alpha^2 + (\alpha^*)^2] \quad (\text{enhancement term}). \end{aligned}$$

For $\alpha = 1 + 0.5i$, one finds

$$\alpha^2 + (\alpha^*)^2 = 1.5 > 0,$$

so the enhancement term dominates at large r , leading to the rapid increase of $Q(\alpha, r)$ seen in Fig. 4.1.

- At $\alpha = 0$, $Q(0, r) = 1/\pi$ is independent of r (flat line).
- At nonzero α , the interplay between the negative $\cosh(2r)$ term and the positive $\sinh(2r)$ term determines whether $Q(\alpha, r)$ decays or grows.
- For the case $\alpha = 1 + 0.5i$, $Q(\alpha, r)$ grows exponentially with r .

4.3 Superpositions for Single Mode

The analysis of the single-mode squeezed vacuum state reveals its expansion in terms of even Fock states, demonstrating that only photon pairs contribute to the state's structure as given in 3.9.

$$|\psi\rangle = \sqrt{\text{sech } r} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} \left(-\frac{1}{2}e^{i\theta} \tanh r\right)^n |2n\rangle. \quad (4.11)$$

The Bogoliubov transformations of the creation and annihilation operators under the squeezing operator provide the foundation for evaluating key expectation values:

$$\hat{S}^\dagger(r) \hat{a} \hat{S}(r) = \hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r, \quad (4.12)$$

$$\hat{S}^\dagger(r) \hat{a}^\dagger \hat{S}(r) = \hat{a}^\dagger \cosh r - \hat{a} e^{-i\theta} \sinh r. \quad (4.13)$$

From these relations, the mean photon number in the squeezed vacuum is obtained as

$$n = \langle \hat{a} \hat{a}^\dagger \rangle = \sinh^2 r, \quad (4.14)$$

while the anomalous correlation

$$\langle \hat{a}^2 \rangle = -e^{i\theta} \sinh r \cosh r \quad (4.15)$$

highlights the phase-sensitive nature of quadrature fluctuations.

The derived relation

$$R = \sinh r \cosh r, \quad R - n = \sinh r (\cosh r - \sinh r), \quad (4.16)$$

emphasizes the interplay between photon number and squeezing correlations. In the asymptotic limit $r \rightarrow \infty$, this expression converges to a maximum value of 0.5, indicating the saturation of correlations at high squeezing strengths.

Overall, these results confirm that the superposition structure of the squeezed state leads to nonclassical statistical properties that are central to applications in quantum optics and quantum communication. In the squeezed vacuum formalism, we define the following quantities as a function of the squeezing parameter r :

$$R = \sinh(r) \cosh(r), \quad n = \sinh^2(r), \quad R - n = \sinh(r) (\cosh(r) - \sinh(r)). \quad (4.17)$$

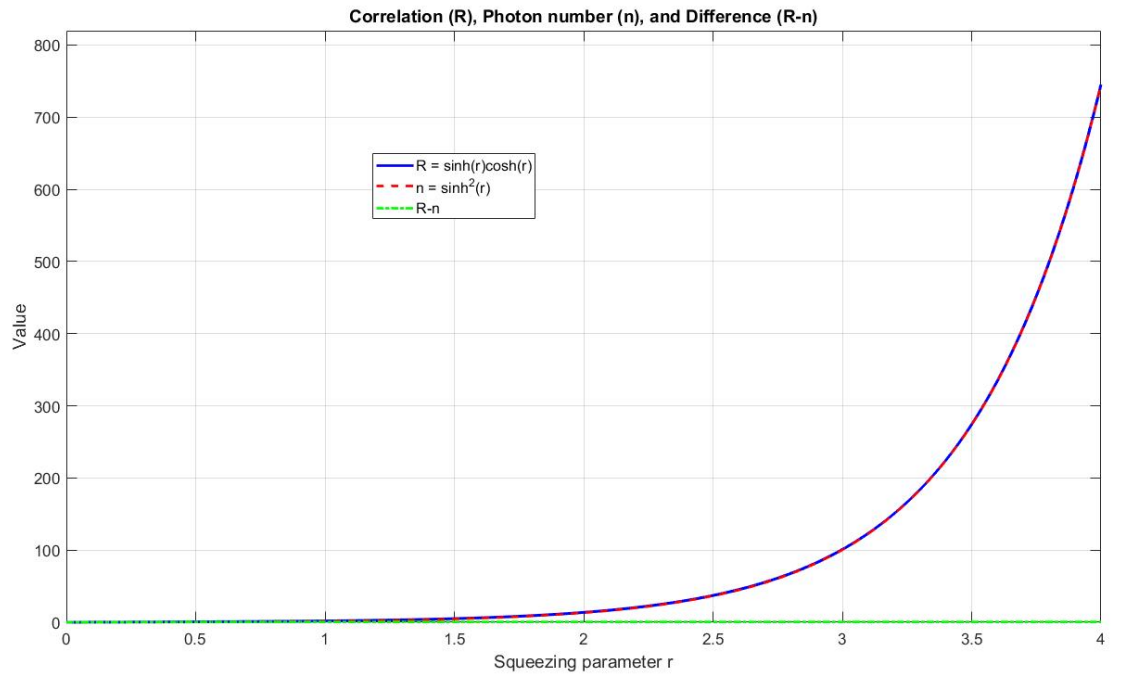


Figure 4.2: Plot of correlation, mean photon number and their difference

- R represents the correlation term,
- n is the mean photon number,
- $R - n$ is their difference.

The figure plots these three functions against the squeezing parameter r :

- **Correlation** (R) grows rapidly with r , approximately as $\frac{1}{2} \sinh(2r)$ for large r . This reflects the strong pair correlations introduced by squeezing.
 - **Photon number** (n) also grows exponentially with r , since $n = \sinh^2(r) \approx \frac{1}{4} e^{2r}$ for large r . The curve almost overlaps with R .
 - **Difference** ($R - n$) stays very close to zero compared to R and n , meaning that as squeezing increases, the correlation and photon number scale nearly identically.
1. Both the correlation R and the photon number n increase rapidly with squeezing.
 2. Their near equality indicates that correlations are almost entirely accounted for by the photon population.
 3. The tiny difference $R - n$ suggests that higher squeezing makes the two quantities indistinguishable in practice.

This behavior highlights how squeezed vacuum states distribute correlations and photons in nearly the same way as squeezing increases.

4.4 Superposition of Squeezed Vacuum State

We consider the superposition of two single-mode squeezed vacuum states of the form

$$|\psi\rangle = N(|r\rangle + |\eta| - r\rangle), \quad \eta = |\eta|e^{i\theta}, \quad (4.18)$$

where N is the normalization constant. The normalization factor is given by

$$N^2 = \left[1 + |\eta|^2 + 2|\eta| \cos \theta \sqrt{\text{sech}(2r)}\right]^{-1} \quad (4.19)$$

The mean photon number of this state is found to be

$$\begin{aligned} n &= \langle \hat{a}^\dagger \hat{a} \rangle \\ &= N^2 \left[\sinh^2 r \left(1 + |\eta|^2 - 2|\eta| \cos \theta \frac{\text{sech } r \tanh^2 r}{(1 + \tanh^2 r)^{3/2}} \right) \right]. \end{aligned} \quad (4.20)$$

Similarly, the anomalous correlation

$$R = |\langle \hat{a}^2 \rangle| \quad (4.21)$$

characterizes the two-photon coherence of the superposed state. The quantity $R - n$ indicates the maximum negative energy density achievable, which depends on both r

and θ . For the special case $\eta = 0$ (a single squeezed vacuum), $R - n$ approaches 0.5 as $r \rightarrow \infty$, consistent with the standard squeezed vacuum limit. Superposed states ($\eta \neq 0$) generally reach their maximum negative energy density at finite r , with values smaller than 0.5.

The quadrature variances are given by

$$\langle (\Delta \hat{X})^2 \rangle = \frac{1}{4} (\cosh^2 r + \sinh^2 r - 2 \sinh r \cosh r \sin \theta), \quad (4.22)$$

$$\langle (\Delta \hat{P})^2 \rangle = \frac{1}{4} (\cosh^2 r + \sinh^2 r + 2 \sinh r \cosh r \sin \theta). \quad (4.23)$$

For $\theta = 0$, this reduces to

$$\langle (\Delta \hat{X})^2 \rangle = \frac{1}{4} e^{-2r}, \quad \langle (\Delta \hat{P})^2 \rangle = \frac{1}{4} e^{2r}, \quad (4.24)$$

showing that the X quadrature is squeezed while the P quadrature is anti-squeezed. For $\theta = \pi$, the squeezing occurs in the P quadrature.

These results illustrate that the superposition parameter η and the phase θ play a decisive role in controlling both the negative energy density and the quadrature squeezing properties of the state. Physically, this means that the degree and orientation of squeezing can be tuned via superposition, which is essential for applications in quantum optics.

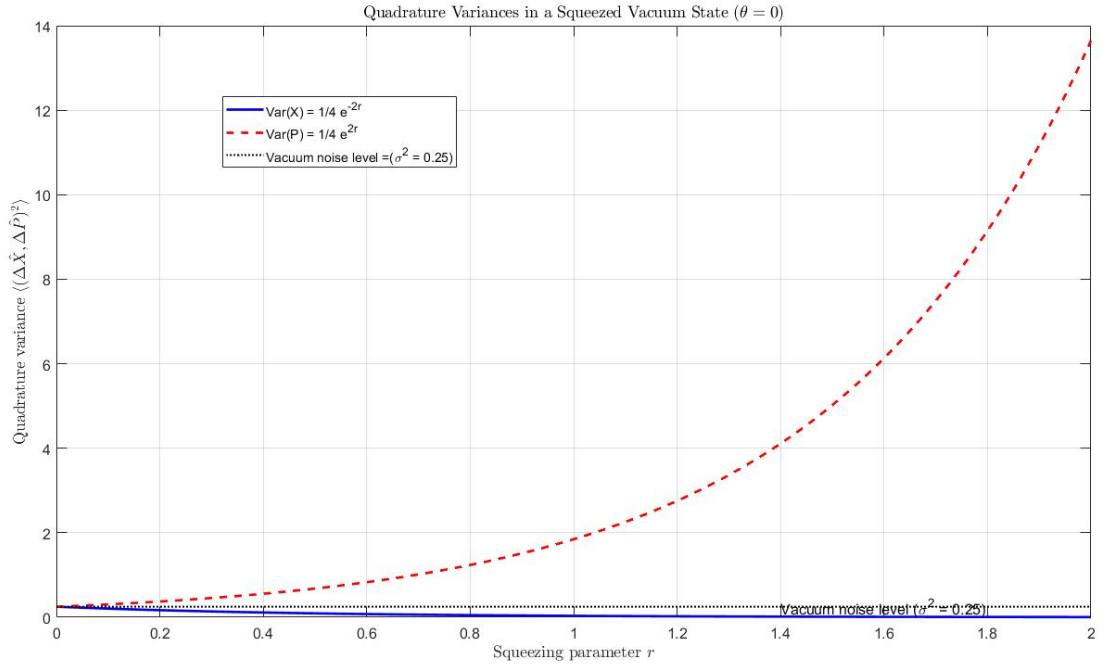


Figure 4.3: Quadrature variances in a squeezed vacuum state

As a function of the squeezing parameter r for $\theta = 0$. The blue solid line shows the squeezed quadrature variance $\langle(\Delta\hat{X})^2\rangle = \frac{1}{4}e^{-2r}$, the red dashed line shows the anti-squeezed quadrature variance $\langle(\Delta\hat{P})^2\rangle = \frac{1}{4}e^{2r}$, and the black dotted line represents the vacuum noise reference level $\sigma^2 = 0.25$. Increasing squeezing (r) reduces the noise in one quadrature while amplifying it in the conjugate one, consistent with the Heisenberg uncertainty principle.

4.5 Photon Statistics

Here, we aim to determine the photon number's mean and variance.

4.5.1 Mean Photon Number

Based on equation of 3.45 as we earned equation 3.47 the equation summarized as

$$\bar{n} = \sinh^2 r \quad (4.25)$$

At $r = 0$, we recover the ordinary vacuum with $\bar{n} = 0$. For large r , the mean photon number grows as $\bar{n} \approx \frac{1}{4}e^{2r}$. Physically, the squeezed vacuum generates correlated photons in pairs, so stronger squeezing corresponds to a higher average photon number. Equation 4.25. The relationship between the mean photon number and the compressed

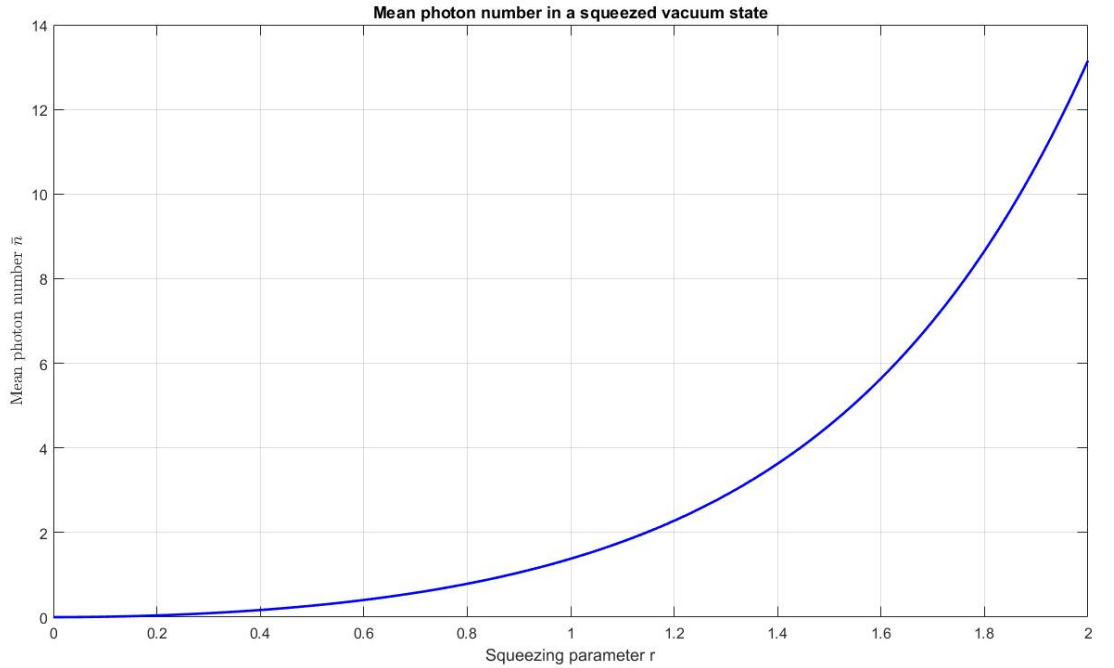


Figure 4.4: Mean photon number versus squeeze parameter r .

parameter is depicted in Figure 4.4, and it is clear that the two are directly proportional. As r increases, $\bar{n}$ rises rapidly. This means more photons are created in the field due to squeezing. These photons are generated in pairs because squeezing produces correlated photon pairs. The squeezed vacuum is not empty: it contains virtual photons that appear as real photon pairs when interacting with detectors. This explains why squeezed

states are a resource in quantum optics: more squeezing $\rightarrow$ more photons $\rightarrow$ stronger nonclassical effects.

4.5.2 Variance of Photon Number

The variance of photon number was stated through equations 3.53, 3.55 and 3.56.

The mean photon number in a squeezed vacuum is

$$\bar{n} = \sinh^2 r, \quad (4.26)$$

where r is the squeezing parameter.

The second moment of the photon number operator is

$$\langle \hat{n}^2 \rangle = \cosh^2 r \sinh^2 r + 2 \sinh^4 r. \quad (4.27)$$

The photon number variance is then

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \bar{n}^2 = \cosh^2 r \sinh^2 r + \sinh^4 r. \quad (4.28)$$

Since $\cosh^2 r = 1 + \sinh^2 r = 1 + \bar{n}$, we can write

$$(\Delta n)^2 = (\bar{n} + 1)\bar{n} + \bar{n}^2 = \bar{n}(2\bar{n} + 1), \quad (4.29)$$

which is the standard variance for a squeezed vacuum.

Similarly, the second moment in terms of $\bar{n}$ becomes

$$\langle \hat{n}^2 \rangle = 3\bar{n}^2 + \bar{n}. \quad (4.30)$$

A modified photon number variance is sometimes written as

$$(\Delta n)_{\text{final}}^2 = 3 \sinh^2 r + 2 \sinh^4 r - \cosh r \sinh r. \quad (4.31)$$

In terms of $\bar{n}$, this reads

$$(\Delta n)_{\text{final}}^2 = 2\bar{n}^2 + 3\bar{n} - \sqrt{\bar{n}(\bar{n} + 1)}. \quad (4.32)$$

Figure 4.5 illustrates that the radiation of the displaced squeezed vacuum state is super-Poissonian photon statistics since the variance of the photon number is larger than the mean photon number.

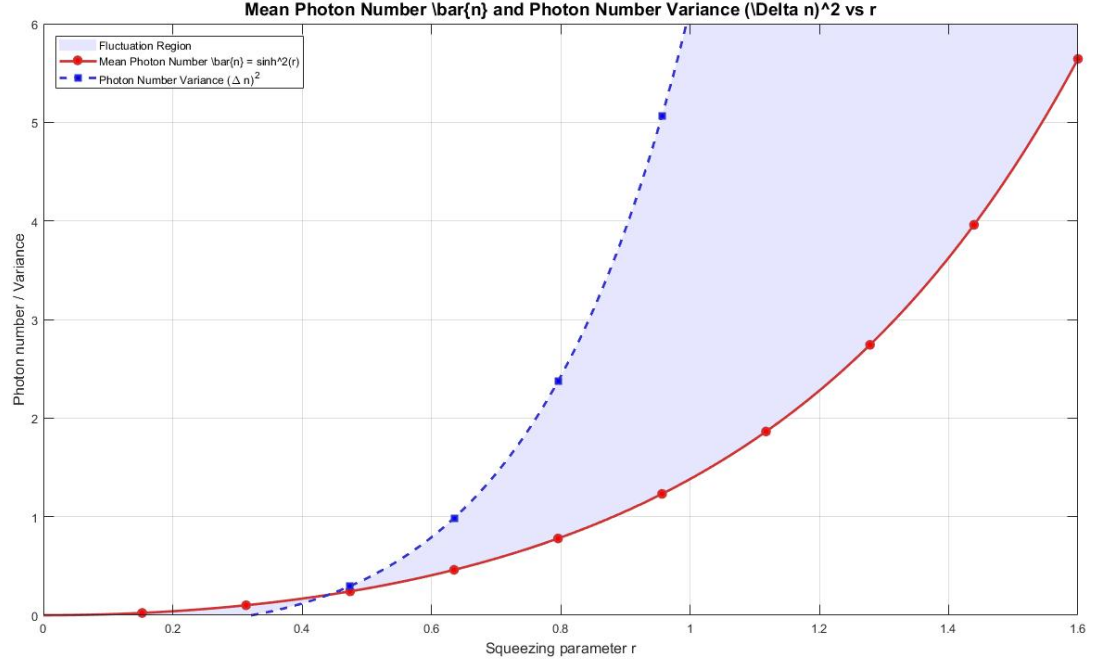


Figure 4.5: Photon number variance and mean photon number versus squeeze parameter r .

4.6 Quadrature Variance

For a single-mode field, the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ define the field quadratures as

$$\hat{a}_+ = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \quad \hat{a}_- = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger). \quad (4.33)$$

These operators represent the amplitude and phase components of the quantized field, obeying the commutation relation

$$[\hat{a}_+, \hat{a}_-] = 2i. \quad (4.34)$$

When a squeezing operator is applied,

$$\hat{S}(r) = \exp \left[\frac{1}{2} r (\hat{a}^2 - \hat{a}^{\dagger 2}) \right], \quad (4.35)$$

It transforms the field such that the noise in one quadrature is reduced while that in the conjugate quadrature is amplified, keeping their product consistent with the Heisenberg

uncertainty principle. The results show that the quadrature variances of the squeezed vacuum based on the equations of 3.53 and 3.55 are given by

$$(\Delta\hat{a}_+)^2 = e^{-2r}, \quad (\Delta\hat{a}_-)^2 = e^{2r}. \quad (4.36)$$

For $r > 0$, the plus quadrature variance decreases while the minus quadrature variance increases, indicating noise reduction in one component and amplification in its conjugate. Their product,

$$(\Delta\hat{a}_+)^2(\Delta\hat{a}_-)^2 = 1, \quad (4.37)$$

satisfies the Heisenberg uncertainty relation. This confirms that squeezing occurs in the plus quadrature, demonstrating a redistribution of quantum noise that enhances measurement precision along that direction.

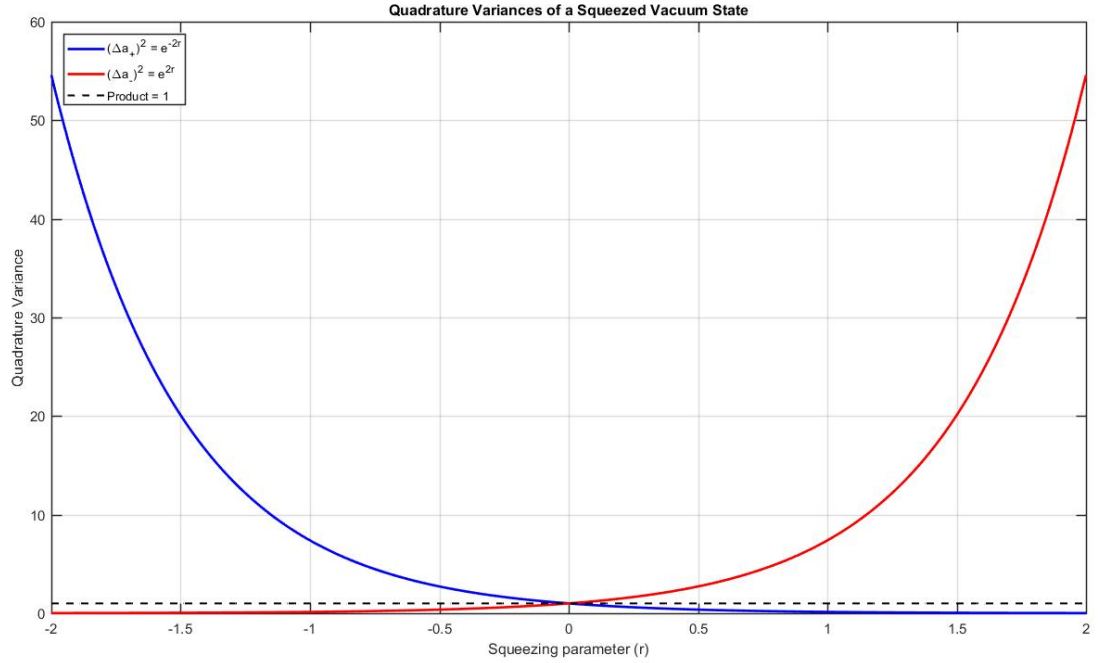


Figure 4.6: Quadrature variances of a squeezed vacuum state as functions of the squeezing parameter r .

The dashed line indicates the uncertainty product equal to one. Figure 4.6 shows the quadrature variances of a squeezed vacuum state as a function of the squeezing parameter r .

- The **blue curve** represents the squeezed quadrature variance $(\Delta a_1)^2 = e^{-2r}$, which decreases exponentially with increasing r . This shows that fluctuations in one

quadrature component are suppressed below the standard quantum limit, which is a clear hallmark of squeezing.

- The **red curve** corresponds to the anti-squeezed quadrature $(\Delta a_2)^2 = e^{2r}$, which increases exponentially as r increases, showing enhanced uncertainty in the conjugate variable.
- The **black dashed line** represents the product $(\Delta a_1)^2(\Delta a_2)^2 = 1$, confirming that the Heisenberg uncertainty principle remains satisfied. As one quadrature becomes more precise, the uncertainty in the other grows proportionally.

4.7 Quantum Key Distribution Protocol with SSVS

The single-mode squeezed vacuum state (SMSVS) is generated by applying the squeezing operator $S(r)$ to the vacuum state, producing

$$|0, r\rangle = S(r)|0\rangle. \quad (4.38)$$

A superposed squeezed vacuum state is formed as a linear combination of two SMSVS with opposite squeezing parameters,

$$|\psi\rangle = \alpha|0, r\rangle + \beta|0, -r\rangle, \quad (4.39)$$

where the real coefficients α and β satisfy the normalization condition $|\alpha|^2 + |\beta|^2 = 1$. Qubit encoding is realized by adjusting the parameters α and β through rotation angles θ and ϕ in the X and Z bases, respectively. Such superpositions reduce the vulnerability to photon-number-splitting (PNS) attacks and enhance robustness at low photon numbers, as the squeezing parameter r determines the phase-space distribution.

The phase-space representation of the state is described by the Q -function,

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle, \quad \hat{\rho} = |\psi\rangle\langle\psi|. \quad (4.40)$$

Evaluating $\langle \alpha | \psi \rangle$ yields the analytical form from the final results of equation 3.89

$$Q(\psi_a) = \frac{1}{\pi \cosh r} \left| \alpha e^{\frac{1}{2} \alpha_x^2 \tanh r} + \beta e^{-\frac{1}{2} \alpha_x^2 \tanh r} \right|^2. \quad (4.41)$$

This expression fully characterizes the superposed squeezed vacuum qubit in phase space and serves as a foundation for analyzing its behavior and performance in quantum key distribution (QKD) protocols.

The distribution is asymmetric, showing interference between the oppositely squeezed components $|0, r\rangle$ and $|0, -r\rangle$.

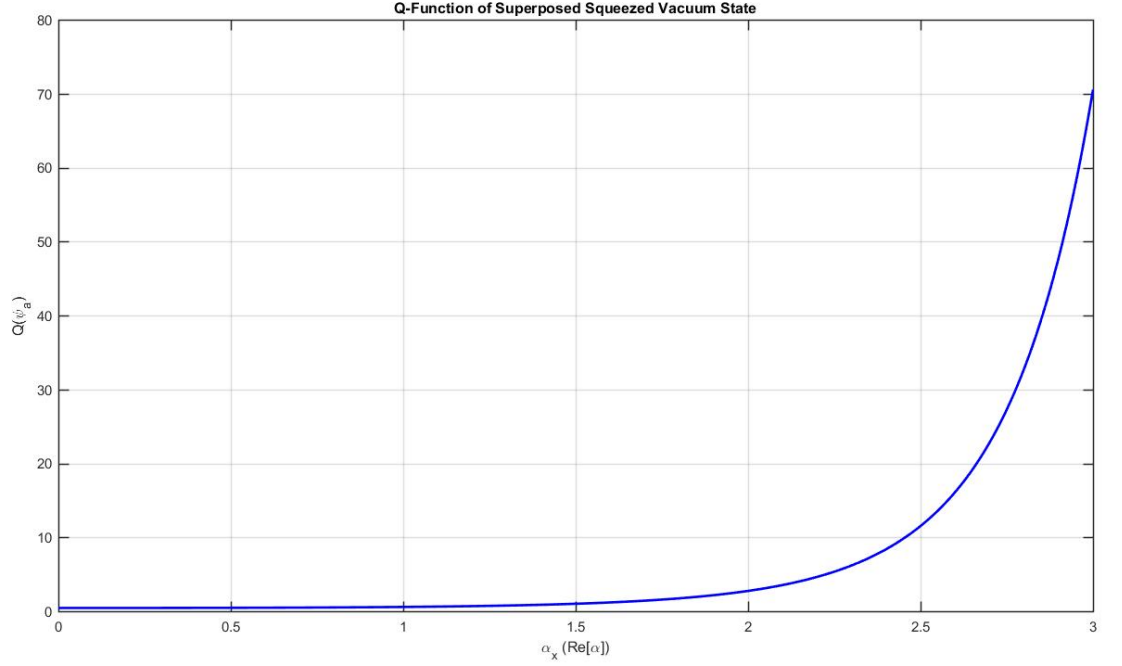


Figure 4.7: Q-function of the superposed squeezed vacuum state along the real quadrature axis α_x .

Figure 4.7 shows the Q-function of the superposed squeezed vacuum state as a function of the real quadrature amplitude α_x . The distribution increases exponentially toward positive α_x but remains very small for negative values, showing that the phase-space distribution is not symmetric. This asymmetric behavior arises because one component of the superposition (for example, $|0, r\rangle$) is anti-squeezed along the positive α_x direction, while the other component ($|0, -r\rangle$) contributes weakly due to opposite squeezing.

This nonuniform distribution demonstrates the presence of quantum interference between the two squeezed vacuum states, confirming the nonclassical nature of the superposed state.

4.7.1 Quantum Key Distribution Protocol and Key Rate Analysis

Alice uses the stated bases X and Z to create superposed squeezed vacuum state qubits with 50/50 probability. During state preparation, Alice selects a base (X or Z) and a bit (0 or 1) at random. She then prepares the corresponding superposed squeezed vacuum state qubit based on this decision. She establishes the encoding angles θ and ϕ as well as the squeezing parameter r . Bob optimizes the squeezing parameter based on the channel loss. With transmissivity η signifying channel loss, the transmission-prepared qubit is transmitted across a quantum channel represented as a beam splitter. Using homodyne detection, Bob randomly chooses to measure either the X or Z quadrature. Alice and Bob then announce their bases publicly, and only the events where they used the same base are retained.

Given the channel parameters, Bob's parameter estimation computes the ideal squeezing value, and the channel loss η is estimated. Standard error correction and privacy amplification are carried out to extract a secure key.

By scaling the complex amplitude, the evolution of the Q-function through the channel effect of channel loss is described as:

$$\alpha \rightarrow \sqrt{\eta} \alpha \quad (4.42)$$

Thus, following the channel, the Q-function becomes:

$$Q(\alpha) = Q(\psi)_{\alpha/\sqrt{\eta}} \quad (4.43)$$

We also take into account the channel's thermal noise N . The thermal noise's Q-function is incorporated, and convolution is used to calculate the output Q-function while accounting for both loss and thermal noise:

$$Q_{\text{out}}(\alpha) = \int Q(\alpha') Q(\alpha - \alpha') d^2 \alpha' \quad (4.44)$$

This provides the evolution of the state across the lossy quantum channel and defines a QKD protocol based on superposed squeezed vacuum state (SSVS) qubits.

4.7.2 Conditional Probabilities and Mutual Information

Bob determines a value x by measuring either the X or Z quadrature. With i being the bit Alice sent (0 or 1) and B being the basis she employed (X or Z), we compute the conditional probability $P(x | i, B)$. Following Bob's measurement, these probabilities are computed by integrating the output Q-functions over the relevant quadrature:

$$P(x | i, B) = \begin{cases} \int Q_{\text{out}}(x', p) dp, & \text{if } B = X \\ \int Q_{\text{out}}(q, x') dq, & \text{if } B = Z \end{cases} \quad (4.45)$$

Here, the changing quadrature, which is integrated over, is represented by the variable x' .

These probabilities are then used to calculate Alice and Bob's mutual information $I(A : B)$, averaged over both bases:

$$I(A : B) = \frac{1}{2} \left[\int dx \left(P(x|0, X) \log_2 \frac{P(x|0, X)}{P(x|X)} + P(x|1, X) \log_2 \frac{P(x|1, X)}{P(x|X)} \right) + \int dx \left(P(x|0, Z) \log_2 \frac{P(x|0, Z)}{P(x|Z)} + P(x|1, Z) \log_2 \frac{P(x|1, Z)}{P(x|Z)} \right) \right] \quad (4.46)$$

where the marginal probabilities are defined as

$$P(x | X) = \frac{P(x|0, X) + P(x|1, X)}{2} \quad (4.47)$$

$$P(x | Z) = \frac{P(x|0, Z) + P(x|1, Z)}{2} \quad (4.48)$$

In order to maximize the information transfer in the presence of loss and noise, the encoding scheme and squeezing parameter are optimized by assessing the mutual information, which quantifies the amount of information that Alice and Bob can reliably share.

4.7.3 Security Analysis (Simplified Gaussian Model)

Assuming that Eve executes a Gaussian collective attack, we employ a simple security analysis. This assumption provides a useful approximation of the protocol's security, even though it is not entirely accurate for non-Gaussian states.

The Holevo information $\chi(E : B)$, which quantifies Eve's knowledge of Bob's measurement results, is derived (or approximated) within the Gaussian approximation. The secret key rate is then computed as follows:

$$K = \beta I(A : B) - \chi(E : B) \quad (4.49)$$

where β represents the efficiency of reconciliation, which is assumed to be 1 for perfect reconciliation.

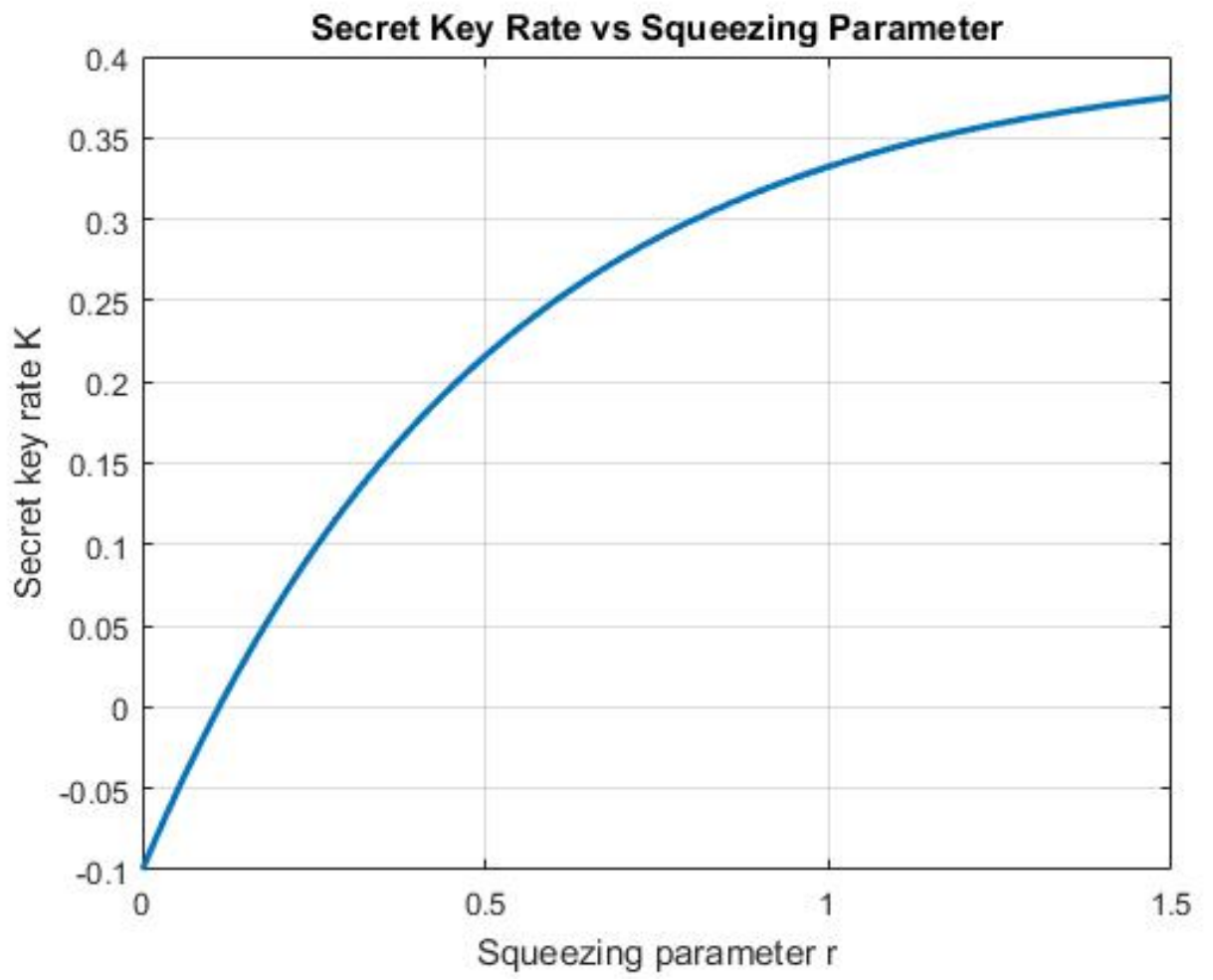


Figure 4.8: Secret key rate versus squeezing parameter r .

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this thesis, we have carried out a theoretical analysis of *superposed single-mode squeezed vacuum states* (SSSVS) and examined their fundamental properties using analytical derivations and numerical visualization. By evaluating the mean photon number, photon number variance, quadrature variances, and the Husimi Q -function, we characterized the nonclassical features of squeezed and superposed squeezed states. The results confirmed that squeezing redistributes quantum noise between conjugate quadratures while preserving the uncertainty relation, and that superposition leads to distinctive phase-space structures associated with enhanced quantum interference effects.

The analysis showed that the mean photon number $\bar{n} = \sinh^2 r$ and the variance $(\Delta n)^2 = \bar{n}(2\bar{n} + 1)$ scale with the squeezing parameter, indicating super-Poissonian photon statistics and increasing two-photon correlations. Visualization via the Husimi Q -function highlighted the anisotropic phase-space distributions of squeezed states and the nonuniform interference patterns arising from the superposition of oppositely squeezed vacua. These results provide insight into how squeezing and superposition jointly influence energy distribution, coherence, and phase-space behavior.

Building on these findings, we proposed a conceptual framework for employing superposed squeezed vacuum states in continuous-variable quantum key distribution (CV-QKD). Within a simplified and idealized model, a preliminary key-rate analysis suggests a trade-off between the squeezing strength and protocol performance, particularly in the presence of loss and noise. While these results indicate potential advantages of superposed squeezed states, they remain qualitative and exploratory, and do not constitute a full security proof or optimized CV-QKD protocol.

Overall, this work establishes a theoretical and visual foundation for understand-

ing the properties of superposed squeezed states and their possible relevance to quantum communication. The results motivate further investigation, including more realistic channel models, rigorous security analyses, and experimental feasibility studies, to fully assess the practicality of these states for CV-QKD and related quantum technologies.

5.2 Recommendation

Based on the findings of this thesis, the following recommendations are proposed:

1. Future experiments should implement superposed squeezed vacuum states to verify the theoretical results using modern optical setups.
2. Optimization of squeezing, phase, and transmissivity parameters is recommended to maximize secure key rates in QKD systems.
3. Extending this study to multi-mode and entangled squeezed states can further improve quantum communication performance.
4. Integration of squeezed-state sources with optical fiber and free-space channels is advised for practical quantum networks.
5. Advanced computational and security analyses should be conducted to assess protocol robustness against realistic quantum attacks.

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