



WALLAGA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

**SOLUTION OF FIRST ORDER ORDINARY
DIFFERENTIAL EQUATIONS USING SYMMETRIES
AND LINEAR TRANSFORMATIONS**

MSc. THESIS

**BY
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WALLAGA UNIVERSITY
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DIFFERENTIAL EQUATIONS USING SYMMETRIES
AND LINEAR TRANSFORMATIONS**

**A Thesis Submitted to college of natural and computational in Partial
Fulfillment of the Requirements for the Degree of Masters in
Mathematics (Differential Equation)**

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As member of the Board of the Final M.A. thesis open defense, we certify that we have read and evaluated the thesis prepared by **Gutema Terefe Jini** Under the title **“Solution of First Order Ordinary Differential Equations Using Symmetries And Linear Transformations”** and recommended that the thesis is accepted as fulfilling the thesis requirement for the Degree of **Master** of Science in Mathematics (Differential Equation)

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STATEMENT OF THE AUTHOR

I, Mr. Gutema Terefe hereby declare and affirm that the thesis entitled “**Solution of First Order Ordinary Differential Equations Using Symmetries and Linear Transformations**” is my own work conducted under the supervision of Mekonnen Negara (PhD). I have followed all the ethical principles of scholarship in the preparation, data collection, data analysis and completion of this dissertation. All scholarly matter that is included in the dissertation has been given recognition through citation. I have adequately cited and referenced all the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and I have not misrepresented, fabricated, or falsified any idea/data/fact/source in my submission. This dissertation is submitted in partial fulfillment of the requirement for a degree from the Post Graduate Studies at Wallaga University. I further declare that this dissertation has not been submitted to any other institution anywhere for the award of any academic degree, diploma or certificate. I understand that any violation of the above will be cause for disciplinary action by the University and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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ABBREVIATIONS

ODE	Ordinary Differential Equation
LIE	Lie Invariant Equation
PDE	Partial Differential Equation
RK	Runge-Kutta
IF	Integrating Factor
Pe	Peclet Number

TABLE OF CONTENTS

Contents	Pages
APPROVAL SHEET FOR SUBMITTING FINAL THESIS.....	i
STATEMENT OF THE AUTHOR.....	ii
ACKNOWLEDGMENTS	iii
ABBREVIATIONS	iv
TABLE OF CONTENTS	viii
ABSTRACT	ix
CHAPTER ONE	1
1. INTRODUCTION.....	1
1.1. Background of the Study.....	1
1.2 Statement of the Problem	3
1.3. Objectives of the Study	4
1.3.1. General Objective	4
1.3.2. Specific Objectives	4
1.4. Significance of the Study	5
1.5. Scope of the Study.....	5
CHAPTER TWO	6
2. REVIEW OF RELATED LITERATURE	6
CHAPTER THREE	8
3. METHODOLOGY	8
3.1. Study Design	8
3.2. Source of Data	8
3.3. Procedures of the Study.....	8
3.4. Limitations of the Study	8
CHAPTER FOUR.....	9
4. RESULTS AND DICUSSION	9
4.1. Linear transformations	10
4.2. Symmetries $[\xi = Fx, \eta = Qx]$	14
CHAPTER FIVE	23
5. SUMMARY, CONCLUSIONS AND RECOMMENDATIONS	23
5.1. Summary	23
5.2. Conclusions	23
5.3. Recommendations	23
References.....	24

ABSTRACT

This thesis explores the methods of solving first-order ordinary differential equations (ODEs) through the application of symmetries and linear transformations. Historically, traditional techniques such as separation of variables and integrating factors were predominant in solving these equations. However, advancements in symmetry methods, particularly those developed by Lie, have introduced a powerful framework for obtaining exact solutions of complex nonlinear ODEs. Additionally, the use of linear transformations has facilitated the simplification of ODEs into more manageable forms, enabling easier solution processes. The integration of these techniques promises to enhance the understanding and solution of first-order ODEs in contemporary research.

Key word; ordinary differential equations, symmetries, linear transformations, nonlinear equations.

CHAPTER ONE

1. INTRODUCTION

In this chapter, we give a general description of solution of first-order ordinary differential equations using symmetries and linear transformations, the statement of the problem, the objective of the study, the significance of the study, and the scope of the study.

1.1. Background of the Study

The process of composing two point transformations yields another point transformation, which can be derived from the action on the corresponding class of functions. Although point symmetries can be useful in tackling first-order equations, they may not always provide solutions, leading to alternative methods for handling the equations. This includes identifying solutions for specific cases or reformulating the problem as an over determined system of partial differential equations. The inquiry then arises regarding the appropriate restrictions on the symmetries that should be applied.

The evolution of methods for solving these equations through symmetries and linear transformations has seen significant advancements from the early 19th century to the close of the 20th century. This section outlines key historical milestones and contributions that have shaped the field. The foundational theories of differential equations were significantly advanced in the 19th century.

Mathematicians such as August in Louis Cauchy and Joseph-Louis Lagrange contributed pivotal ideas that would influence future generations. Cauchy's work on the existence and uniqueness of solutions to differential equations established critical theoretical underpinnings (Cauchy, 1823). His investigations into the proper- ties of solutions laid groundwork that mathematicians would build upon. Lagrange introduced the method of integrating factors, which became a staple technique for solving first-order linear ODEs. This method allowed for a systematic approach to obtaining solutions for equations that could be expressed in standard forms (Lagrange, 1810). The ability to simplify and solve these equations marked a significant leap for- ward in mathematical analysis.

The late 19th century heralded a transformative period with the introduction of the concept of symmetries in differential equations. Sophos Lie's pioneering work on continuous transformation groups revolutionized the study of differential equations. Lie's ideas provided a

systematic approach to understanding the symmetries inherent in differential equations, enabling mathematicians to derive solutions through invariant properties under transformations (Lie, 1890). His contributions laid the foundation for what would later become known as Lie group analysis.

The early 20th century witnessed a formalization of symmetry methods in the context of ODEs. Researchers began to systematically apply Lie's theories to various classes of differential equations, leading to the development of algorithms for determining symmetries. This approach marked a shift from purely analytical methods to more structured techniques that incorporated group theory (Bluman & Kumei, 1989).

The advent of computers in the mid-20th century brought about a revolution in the field of differential equations. Numerical methods became increasingly significant, allowing for the approximation of solutions to complex ODEs that were previously intractable. The integration of computational techniques with symmetry analysis opened new avenues for solving first-order ODEs, enhancing both the theoretical and practical aspects of the field (Hydon, 2000).

By the late 20th century, the application of symmetry methods and linear transformations had extended beyond pure mathematics into fields such as physics, engineering, and biological sciences. Researchers recognized the utility of these methods in modeling real-world phenomena, such as fluid dynamics and population dynamics, where differential equations play a crucial role (Olver, 1993). The late 20th century set the stage for ongoing research into the application of symmetries and transformations in differential equations. The work of various mathematicians and physicists continued to expand the theoretical framework and practical applications of these methods, leading to a deeper understanding of the underlying principles governing ODEs (Cheb-Terrab, 2000).

The historical development of methods for solving first-order ordinary differential equations using symmetries and linear transformations illustrates a rich interplay between theory and application. From the foundational contributions of early mathematicians to the sophisticated techniques developed throughout the 20th century, the field has evolved significantly.

The integration of symmetry methods with computational techniques has opened new avenues for research and application, ensuring that the study of ODEs remains a vibrant area of mathematical inquiry.

The classical methods for solving first-order ODEs, such as separation of variables and integrating factors have been foundational. However, the introduction of symmetry methods and linear transformations has provided innovative approaches to tackle more complex equations.

Symmetry methods utilize Lie group transformations to simplify differential equations. These methods allow for the identification of invariant solutions, significantly reducing the complexity of the equations. The historical development of these methods can be traced back to the work of Sophus Lie, but their application has gained momentum in the 21st century. Recent studies have demonstrated the effectiveness of symmetry methods in solving nonlinear first-order ODEs, providing exact solutions for equations that are otherwise challenging to solve using traditional techniques (Bluman & Kumei, 2002).

1.2 Statement of the Problem

ODEs are fundamental tools in modeling a wide range of phenomena in science and engineering. First-order ODEs, in particular, are frequently encountered in various applications, including population dynamics, heat transfer, and fluid mechanics. However, solving these equations can often be challenging, especially when they are nonlinear or involve complex boundary conditions. Despite the availability of traditional methods such as separation of variables and integrating factors, many first-order ODEs resist straightforward solutions. Nonlinear first-order ODEs can exhibit behavior that is difficult to analyze and solve using conventional methods. The presence of nonlinearities often complicates the identification of solutions, necessitating the exploration of alternative techniques. Standard solution techniques can be time-consuming and may not always yield satisfactory results, particularly for equations that are resistant to simplification. This inefficiency highlights the need for more effective methods. There is often a lack of systematic approaches that integrate the concepts of symmetries and linear transformations into a cohesive framework for solving first-order ODEs. Understanding how these mathematical tools can be applied in conjunction may reveal powerful new methods for finding solutions. Many researchers and practitioners may not be familiar with symmetry

methods and linear transformations, which can prevent them from utilizing these approaches in their work. This limitation calls for educational resources that elucidate these techniques and demonstrate their practical applications.

In light of these challenges, this study aims to investigate the use of symmetries and linear transformations as effective methods for solving first-order ordinary differential equations. By addressing these issues, the research seeks to enhance the understanding and application of these mathematical tools, ultimately contributing to more efficient and accessible solutions to ODEs.

This study aims to address the following key research questions through the application of symmetry methods and linear transformations to ordinary differential equations:-

1. How linear transformations by can systematically employed to solving ordinary differential equations and what are the limitations of this approach?
2. What types of exact solutions can be obtained through symmetry methods that cannot be found using traditional solution techniques?
3. In what ways do solutions obtained through symmetry method and transformations compare with those derived from conventional approach in terms of:-
 - ✓ Computational efficiency
 - ✓ Solution accuracy
 - ✓ Applicability to different equation classes

1.3. Objectives of the Study

1.3.1. General Objective

To investigate the effectiveness of symmetries and linear transformations in solving first-order ODEs.

1.3.2. Specific Objectives

1. To solve linear transformations in solving ODEs.
2. To identify the application of symmetry methods in obtaining exact solutions.
3. To compare the results from traditional methods with those obtained using symmetries and transformations.

1.4. Significance of the Study

By exploring symmetries and linear transformations, the study aims to provide new methods for solving first-order ordinary differential equations, which are prevalent in various scientific and engineering disciplines. This research contributes to the theoretical framework surrounding differential equations, offering insights into how symmetries can solving complex ODEs and reveal invariant properties of solutions. The findings can be applied in numerous fields such as physics, engineering, and economics, where first-order ODEs model dynamic systems. Enhanced solution methods can lead to more efficient analyses and predictions in these areas. The study serves as a resource for students and researchers by providing a comprehensive overview of symmetry methods and their applications, facilitating a deeper understanding of differential equations.

1.5. Scope of the Study

This study focuses on the solution of first-order ordinary differential equations using symmetries and linear transformations. The research primarily concentrates on first- order ODEs, linear and nonlinear, examining how various transformations and symmetries can be applied to these equations.

CHAPTER TWO

2. REVIEW OF RELATED LITERATURE

The study of first-order ordinary differential equations (ODEs) is a fundamental aspect of applied mathematics, with significant implications in various scientific fields. Traditional methods for solving these equations have been well-documented; however, the integration of symmetries and linear transformations has opened new pathways for solution techniques. This literature review examines the historical advancements in this area, focusing on the published articles.

Ordinary differential equations (ODEs) play a fundamental role in modeling various physical, biological, and engineering systems. First-order ODEs, in particular, are essential in describing dynamic processes where the rate of change of a variable depends on its current state.

Linear transformations serve as another powerful tool in the analysis of first-order ODEs. By transforming the variables of the differential equation, complex equations can often be simplified into more manageable forms. The application of linear algebra techniques, such as matrix representations of differential operators, has been explored in recent literature. For instance, researchers have discussed using linear algebra methods to solve first-order ODEs by representing the differential operator as a matrix and applying transformations to find solutions (Mathematics Stack Exchange, 2023).

Recent works have proposed benchmarks for identifying ODEs from time series data, facilitating the evaluation of different methods for solving ODEs (PMC, 2023). The development of algorithms that incorporate symmetry and transformation techniques has been a significant focus, aiming to automate the solution-finding process (Gen- nemark & Wedelin, 2007). The integration of these mathematical techniques into various fields, such as systems biology and engineering, has expanded the applicability of first-order ODE solutions (Karnaukhov *et al.*, 2007).

The exploration of symmetries and linear transformations in solving first-order ordinary differential equations has led to significant advancements in both theoretical and practical applications. The historical context provided by earlier works, combined with recent innovations, underscores the importance of these methods in contemporary mathematics and

its applications. As research continues to evolve, the potential for new discoveries and methodologies remains vast.

The significant advancements have been made in the theoretical and computational analysis of Riccati equations. For instance, analytical methods for solving these equations have been explored, with researchers developing techniques that simplify the Riccati equation into a linear form, facilitating easier solutions (Ritelli, 2021). Additionally, numerical approaches, such as the Runge-Kutta method, have been refined to provide efficient solutions for practical applications (Da Silva Pinto et al., 2022). The role of Riccati equations in control theory has been particularly emphasized, especially in the context of optimal control problems, where they are integral to linear quadratic regulators (Assimakis & Adam, 2013). Furthermore, interdisciplinary applications have emerged, showcasing the utility of Riccati equations in finance for option pricing models and in biology for modeling population dynamics (Kryachko, 2005).

CHAPTER THREE

3. METHODOLOGY

3.1. Study Design

This study would employ documentary review design on the first order ordinary differential equations, symmetries and linear transformations.

3.2. Source of Data

The data derived from mathematical literature and previous studies related to differential equations. Key sources include academic journals, textbooks, and online databases that contain relevant research. This data will serve as the foundation for theoretical exploration and computational modeling.

3.3. Procedures of the Study

Details of the procedures followed in applying the methodologies to solve the differential equations will be discussed. This includes:-

Literature Review:-An extensive review of existing research to identify gaps and establish a theoretical base.

Model development:-Formulating differential equations based on real-world phenomena and theoretical constructs.

Computational Implementation:-Utilizing software tools to simulate and solve the equations.

Validation:- Comparing results with existing solutions and empirical data to ensure accuracy

Analysis- Interpreting the results in context of the research questions and objectives.

Each of these steps is crucial for ensuring that the study is rigorous and that the findings are both reliable and valid.

3.4. Limitations of the Study

Acknowledge potential limitations of the study, such as biases, sample size constraints, and challenges in data collection or analysis will be discussed.

CHAPTER FOUR

4. RESULTS AND DISCUSSION

This thesis is concerned with first order ordinary differential equations and linear symmetries of the form

$$\xi = F(X), \eta = P(x)y + Q(x) \quad 4.1$$

Where $\{\xi, \eta\}$ are the infinitesimals, the symmetry generator is $\xi \frac{\partial y}{\partial x} + \eta \frac{\partial y}{\partial x}$ and x and $y \equiv y(x)$ are respectively the independent and dependent variables. Concerning the arbitrary function $\{F, P, Q\}$, the requirements are those implied by the fact that (4.1) generates a Lie group of transformations. The linear symmetries (4.1) have interesting features; for instance, the related finite transformations are also linear, of the form

$$t = f(x), \mu = p(x)y + q(x) \quad 4.2$$

Where t and $\mu = \mu(t)$ are respectively the new independent and dependent variables, and $\{f, p, q\}$ are arbitrary functions of x . Linear transformations (4.2) form a group by themselves too, not just with respect to the Lie parameter. So as it happens with point symmetries in the higher order case, any two linear symmetries (4.1) can be transformed into each other by means of a linear transformation (4.2), and hence we can associate an equation class with the symmetry (4.1). Since separable equations have symmetries of this form, the class of equations admitting linear symmetries (4.1) actually includes all first order equations which can be mapped into separable ones by means of linear transformations (4.2)

We note that in the particular case of polynomial equations

$$y' = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) \quad 4.3$$

Where f_3, f_2, f_1 and f_0 transformations and Invariant Equations

of x (4.2) actually defines their respective classes. Since a separable Abel equation can be obtained by just taking the coefficients f_i all equal, this means that there are complete Abel classes all of whose members can be transformed into separable ones by means of (4.2). Such a case was discussed and solved at the end of the nineteenth century by Liouville, Appell and others and is presented in textbooks.

More generally ,for polynomial equations of the form

$$y' = f_n(x)y^n + f_1(x)y + f_0 \quad 4.4$$

where f_n, f_1 and f_0 are arbitrary functions x presented a method similar to this mapping into separable to solving equations ,but through transformation (4.2) with $q = 0$ Chini's method is equivalent to solving (4.4) by determining ,when they exist ,symmetries (4.1) with $Q = 0$.

In connection with the above, this work presents a generalization of these methods as an algorithm for determining whether or not a first order ordinary differential equation of arbitrary form .

$$y' = \phi(x, y) \quad 4.5$$

Belongs to this class admitting linear *symmetries* (4.1) and if so, for explicitly finding the symmetry ,without restrictions to the form of $\{F, P, Q\}$.Both the determination of the existence of a solution as well as of the symmetry itself are performed without solving any auxiliary differential equations .

The exposition is organized as follows. The connection between the symmetries of the form (4.1) and linear transformations of the form (4.2)

is analyzed and a solution algorithm for the related class is presented .Some examples illustrating the type of problem which can be solved using this method are shown in (4.2).

A discussion of Riccati equations in terms of their symmetries and of a variant of the method to solve a subset of the Riccati problem, is given .In a discussion and some statistics are presented concerning the classification of Kamke's first order ordinary differential equation examples.

Finally, the conclusion contain general remark about this work

Linear Transformations and Symmetry

4.1. Linear transformations

A linear transformation is a function T that maps elements of a vector space V to elements of another vector space W while preserving the operations of vector addition and scalar multiplication. In our context, a linear transformation can be represented as:

$$T(x + y) = T(x) + T(y), \forall x, y \in V$$

$$(T(cX) = cT(X), \text{ for all } x \text{ in } V \& \text{ all scalars } c$$

To determine whether or not a given first order equation has symmetries of the form (4.1), we take advantage of the fact that, for such symmetry, the related invariant equation family can be computed in closed form. With the invariant family in

1. in what follows we use $y' = \frac{dy}{dx}$
2. The Abel equations members of a given class can be mapped between themselves through (4.2).

There are infinitely many non-intersecting such Abel classes but only some of them (still infinitely many) admit symmetries of the form (4.1)

3. Riccati equations are partially excluded from the discussion and, we then show how one can algorithmically compute the infinitesimal (4.1),

When they exist, by quadrature. For that purpose, with no loss of generality, we first rewrite (4.1) in terms of $\{f(x), p(x), q(x)\}$

$$\xi = \frac{1}{f'}, \eta = \frac{p'y + q'}{f'p} \quad 4.6$$

A direct computation of the finite transformations generated by (4.6) shows that they are linear, of the form

$$t = z(x), \mu = \frac{p(x)y + q(x) - q(z(x))}{p(z(x))} \quad 4.7$$

Where $z(x)$ a solution of $f(z) - f(x) - \alpha = 0$ and α is the Lie group parameter. The invariant family related to (4.6) can be obtained, for instance, by computing the differential invariant $\{I_0, I_1\}$ of order 0 and 1 related to (4.6)

$$I_0 = PY + q, I_1 = \frac{f'}{p'y + py' + q} \quad 4.8$$

and hence the invariant equation given by $\wedge(I_0, I_1) = 0$ Where $\wedge$ is arbitrary, can be conveniently written as $I_1 G(I_0) = 1$, with arbitrary G , resulting in

$$Y' = \frac{f'}{p} G(PY + Q) - \frac{q'}{p} - \frac{p'}{p} y \quad 4.9$$

Due to this connection between linear transformations and symmetries (4.1), the same

invariant family (4.1) can be obtained directly from an autonomous equation

$$\mu = G(\mu) \quad 4.10$$

by just changing variables in it using (4.2). The solution to (4.9) can be obtained in the same way, by changing variables in the solution to (4.10)

$$f - \int py + q \frac{1}{G(z)} dz + c_1 = 0 \quad 4.11$$

In fact (4.6) can also be obtained from the symmetry $[\xi = 1, \eta = 0]$ of (4.10) by changing variables using (4.2). We recall that the knowledge of $\{\xi, \eta\}$ in (4.1) suffices to express the solution of any member of the class (4.9) by quadrature and so it is equivalent to the determination of $\{f, p, q\}$ in (4.1)

Theorem 4.1. Consider $y' = \Psi(x, y)$ with $\Psi_{yyy} = 0$. The determination of whether or not this equation belongs to the class (4.4), and when it does, the computation of the infinitesimals (4.1) themselves, can be performed algorithmically from Ψ by quadrature.

Proof. We start by noting an intrinsic feature of equations that are members of the class (4.4). In this section we assume $f' \neq 0$, since otherwise (4.9) would be a first order linear equation. The limiting case where $f' \rightarrow \infty, \xi = 0$, however, it is not excluded class (4.9) the determination of $p(x)$ up to a constant factor, say k , suffices to map any member of the class into another one having a symmetry of the form

$$\xi = F(x), \eta = Q(x) \quad 4.12$$

This is easily verified by changing variables

$$y = \frac{\mu}{kp} \quad 4.13$$

in (4.6), arriving at $[\xi = \frac{1}{f'}, \eta = -q' \frac{k}{f'}]$ which is of the form (4.12). In turn symmetries of the form (4.12) when they exist, can be systematically determined as shown. In what follows, we develop the proof by first showing how to map any equation member of (4.9) into one having a symmetry of the form (4.12), and then, for completeness briefly reviewing how such a symmetry is determined when it exists.

For all equations $y' = \Psi$ of the class (4.9), we have that

$$\Psi_y = f'G'(py + q) - \frac{p}{p'}$$

$$\Psi_{yy} = f'pG''(py + q)$$

$$\Psi_{yyy} = f'p^2 G'''(py + q) \quad 4.14$$

Thus we let

$$\begin{aligned} A \equiv \frac{\Psi_{yy}}{\Psi_{yyy}} &= \frac{1}{p} K(py + q) \text{ where } K \\ &= \frac{G''}{G'''} \end{aligned} \quad 4.15$$

Three cases now arise, related respectively to whether $A_y = 0, A_{yy} = 0$ or $A_{yy} \neq 0$

Case 4.1: $A_y = 0$

In this case, $K' = 0$, for some non-zero k , and hence $A = \frac{K}{p}$ so, from (4.13), When $Y' = \Psi$ indeed a member of the class (4.9), by changing variables using

$$Y = A\mu$$

The result equation family will have a symmetry of the form (4.12)

Case 4.2 $A_{yy} = 0$ or $A_{yy} \neq 0$

In this case $K'' = 0$, so that from (4.15)

$$A = \frac{k_1(py+q)+k_0}{p} \quad 4.17$$

for some constant k_0 and some non-zero constant k_1 . Here the necessary condition for $y' = \Psi$ to be a member of is that the ratio above be linear in y . In such a case, when the equation is indeed a member of this class, by introducing

$$\mu = \ln(A) \quad 4.18$$

The resulting equation will have a symmetry of the form (4.12) this can be verified straightforwardly by performing the change of variables directly in (4.6)

Case 4.3 $A_{yy} \neq 0$

In this case let

$$I \equiv \frac{A_{yx}}{A_{yy}} = \frac{p'y + q'}{p} \quad 4.19$$

The necessary condition for $y' = \Psi$ to be a member of (4.4) is then that I be linear in y, whence

$$p(x) = \exp\left(\int I_Y dx\right) \quad 4.20$$

So, when the equation belongs to this class, from (4.13) changing variables $y = \frac{\mu}{p}$

Will lead to an equation of the same symmetry of the form (4.12)

4.2. Symmetries $[\xi = F(x), \eta = Q(x)]$

Symmetries in the context of differential equations refer to transformations that leave the equation unchanged. In other words, if a transformation is applied to the variables in the differential equation, the resulting equation should be the same as the original equation.

Symmetries can be used to simplify the process of solving differential equations. By identifying the symmetries of an ODE, you can often transform the equation into a simpler form that is easier to solve.

By computing differentials, as we did to arrive at (4.9) the invariant equation family associated to $\xi = F(x), \eta = Q(x)$ can be written as

$$y' = \Phi(x, y) \equiv \frac{1}{F(x)} \left(Q(x) + G \left(y - \int \frac{Q(x)}{F(x)} dx \right) \right) \quad 4.21$$

Where F, Q and G are arbitrary functions of their arguments. So far we have shown that if an equation belongs to (4.9) it can be transformed into a member of this family (4.21)

Now, to determine F and Q , we first construct an expression depending on x and y only through G ,

$$K \equiv \frac{\Phi_y}{\Phi_{yy}} = \frac{G_y}{G_{yy}} \quad 4.22$$

Where we assume $\Phi_{yy} \neq 0$. As explained the problem then splits into two cases.

Case 4.4 $K_y \neq 0$

In this case, we can obtain the ratio $Q(x)/F(x)$, only depending on x , by taking

$$\gamma \equiv \frac{k_x}{k_y} = -\frac{Q(x)}{F(x)} \quad 4.23$$

The knowledge of this ratio in turn permits the elimination of Q from the determining partial differential equation for the infinitesimals.

If $Q=0$ or $F=0$ then the invariant equation is separable or linear ,so in (4.21) and hence forth we assume $F \neq 0, Q \neq 0$.

If $\phi_{yy} = 0$,then (4.21) is already a first order linear equation solvable in terms of quadrature.

$$F(x) = C_1 \exp\left(\int \left(\frac{\gamma \phi_y - \gamma x - \phi_x}{\phi + \gamma}\right) dx\right) \quad 4.24$$

Which, together with (4.23), gives the solution we are looking for. The necessary and sufficient conditions for the existence of such symmetry are,

$$\frac{\partial}{\partial y} \left(\frac{k_x}{k_y} \right) = 0, \frac{\partial}{\partial y} \left(\frac{\gamma \phi_y - \gamma x - \phi_x}{\phi + \gamma} \right) = 0 \quad (4.25)$$

Case 4.5 $K_y = 0$

Since $K_y = -K_x^F/Q$, then ,when K_y is zero, k_x also vanishes ,so $K = k$ for some non-zero constant k . hence ,the right hand side is satisfies.

$$\frac{\phi_y}{\phi_{yy}} = k \quad 4.26$$

and so (4.21), the invariant equation family ,is of the form

$$y' = \phi \equiv A(x) + B(x)e^{g/k} \quad 4.27$$

Where A and B are arbitrary functions. For a given equation of this type A and B can be determined by inspection, and the determining partial differential equation for the infinitesimals can be solved directly in terms of A and B as

$$F(x) = \exp\left(\frac{-\int \frac{A}{B} dx}{B}\right), Q(x) = AF(x) \quad (4.28)$$

Example 4.1 consider the first-order equation

$$(xy' + ay - f(x)g(x^a y)) = 0 \quad (4.29)$$

Where a is an arbitrary constant and f and g are arbitrary function of their argument. For this equation, Kamke shows a change of variable mapping the equation into a separable in this paper, we tackle this equation by computing A in (4.10):

$$A = \frac{g''}{x^a g''}, \quad (4.30)$$

So we are in case (4.4) we then proceed by computing I arriving at

$$I = \frac{ay}{x}. \quad (4.31)$$

The existence condition that I be linear in y is satisfied; hence according to

$$p(x) = x^a \quad (4.32)$$

Now changing variable $y = \frac{u}{x^a}$ as indicated it becomes

$$u' = g(u) \frac{f(x)x^a}{x} \quad (4.33)$$

Which is already separable (and thus naturally has symmetry of the form. The solution is obtained by changing variable back in the solution to leading to

$$\int x^{n-1} f(x) dx - \int^{x^a y} \frac{1}{g(z)} dz - c_1 = 0 \quad (4.34)$$

Example 4.2 we now discuss an example for which $A_{YY} = 0$,

$$Y' = (X^3 Y^4 + 4X^4 Y^3 + 6X^5 Y^2 + 4X^6 Y + X^7)(X^a + 1) - \frac{Y}{x} - 2, \quad (4.35)$$

where a is an arbitrary constant .for this equation

$$A = \frac{Y + X}{2} \quad (4.36)$$

So the change of variable here is $\mu = \ln(A)$ mapping into

$$\mu' = t^{7+a}(t^a + 1)e^{6\mu} + \frac{7}{t} \quad (4.37)$$

For this equation, symmetry of the form is computed algorithmically

$$\left. \begin{aligned} \xi &= \frac{1}{8(t^a+1)} \\ \eta &= -\frac{1}{8t(t^a+1)} \end{aligned} \right\} \quad (4.38)$$

Changing variables back directly in the above we arrive at symmetry for

$$\left. \begin{aligned} \xi &= \\ \eta &= -\frac{\frac{1}{8(t^a+1)} + \frac{y+2x}{8x(x^a+1)}}{x} \end{aligned} \right\} \quad (4.39)$$

Whence an implicit solution to follows as

$$x + \frac{1}{3x^3(y+x)^3} + \frac{x^{(1+a)}}{1+a} = C_1$$

Example 4.3. As an example of the case in which $A_y = 0$, consider

$$y' = be^{axy} x^a + \frac{(x^2 - 1)y}{x} - \frac{1}{x^2} + \ln(x) + c, \quad (4.40)$$

Where a ,b and c are arbitrary constants. So by changing variables as indicated becomes

$$\mu' = \mu t + abt^{(a+1)}e^\mu - \frac{a}{t} + at(\ln(t) + c). \quad (4.41)$$

$$\left. \begin{aligned} \xi &= \frac{1}{t} \\ \eta &= -\frac{a}{t^2} \end{aligned} \right\} \quad (4.42)$$

Whence admits the symmetry

$$\left. \begin{aligned} \xi &= \frac{1}{x} \\ \eta &= -\frac{xy+1}{x^3} \end{aligned} \right\} \quad (4.43)$$

Which suffices to integrate by either using canonical coordinate or computing integrating factor.

$$y = \frac{1}{e^{x(C-x)}} \quad (4.44)$$

Recite equations

The case of Riccati type equations

$$y' = f_2 y^2 + f_1 y + f_0 \quad 4.45$$

Where $f_i \equiv f_i(x)$, $f_2 \neq 0$ and $f_0 \neq 0$, deserves a separate discussion. All Riccati equations admit symmetries of the form (4.1) and so all of them can be mapped into separable ones using transformations of the (4.2). However it is easy to verify that to find such a transformation requires solving the Riccati equation itself. The algorithm of the previous section, which does not rely on solving auxiliary differential equations, only works when $\Psi_{yyy} \neq 0$. The usual approach for solving (4.45) then consists of converting it to a linear second order ordinary differential equation and using the various methods available. Nonetheless, there are entire subfamilies of (4.45) for which symmetries of the form (4.1) can be found following an approach such as the one presented in the previous section, without using techniques for linear second order equations. Such an approach is interesting since it enriches the algorithms available for tackling the problem and could be of use for solving some linear equations by mapping them into Riccati ones.

For the purpose of discussing these cases, and without loss of generality, we rewrite (4.6) by redefining $f' \rightarrow \frac{f'}{p}$:

$$\xi = \frac{p}{f}, \eta = -\frac{p'y + q}{f} \quad 4.46$$

If now in (4.9), we redefine f' in the same way and take G as a quadratic mapping depending on two arbitrary constants a and b

$$G = \mu \rightarrow \mu^2 + a\mu + b \quad 4.47$$

We arrive at the form of an arbitrary Riccati equation, as general as (4.45), but expressed in terms of these two constants a and b and the functions $\{f, p, q\}$ appearing in its symmetry generator (4.46)

$$y' = fy^2 + \frac{(p + 2q)f - p'}{p}y + \frac{((a + q)q + b)f - q'p}{p^2} \quad 4.48$$

Case 4.1 $p' = 0$

A first solvable case happens when $p' = 0$ so that in (4.46) both infinitesimal depend only on x and hence the symmetry can be systematically determined as explained.

Case 4.2 $q' = 0$

A second solvable case happens when in (4.48) $q' = 0$, so that the infinitesimals (4.46) are of the form

$$\xi = F(x), \eta = P(x)y \quad 4.49$$

An algorithm for solving such an equation was presented by Chini. In the case of Riccati equations, Chini's algorithm can be summarized as "to check for the constant character "of the expression

$$I = \frac{(f'_0 f_2 - f_0 f'_2 - 2f_0 f_1 f_2)'}{(f_0 f_2^3)} \quad 4.50$$

Where f_i are the coefficients of y in (4.45). Whenever I is constant, the problem is symmetrically solvable in terms of quadrature.

When $q' = 0$ a direct computation of I confirm that in such a case I is constant.

Conversely, another direct computation shows that whenever I is constant.

This connection between the constant characters of an expression like (4.50)

Constructed with the coefficient of a polynomial equation of the form (4.4) and symmetries of the form (4.49), is valid not only for Riccati equations but for equations of the form (4.4) in general .

Will have a symmetry of the form (4.49). To check this, it suffice to solve (4.50) for

f_i and substitute the result into (4.45), the resulting Riccati equation will admit the symmetry

$$\xi = \frac{1}{f_2} \sqrt{\frac{f_2}{f_0}}, \eta = \frac{(f'_0 f_2 - f_0 f'_2)}{2f_0^2 f_2 \sqrt{\frac{f_2}{f_0}}} y \quad 4.51$$

Which is of the form (4.49).

we can also see the equation class solved by Chini's algorithm, as well as explain the previous results by noticing that (4.45) is an absolute invariant for (4.45) under transformations of the form

$$t = j(x), \mu = p(x)y \quad 4.52$$

That is, of the form (4.2) with $q = 0$. In fact, (4.50) can be written as

$$I = \frac{s_3^2}{s_2^3} \quad 4.53$$

$$\text{Where } s_2 = f_0 f_2, s_3 = f'_0 f_2 - f'_2 f_0 - 2f_0 f_1 f_2 \quad 4.54$$

are relative invariants of weight 2 and 3 with respect to transformations. IN summary ,Chini's algorithm solves the equation class generated by changing variables (4.52) in the general Riccati equation (4.48) at $q' = 0$, all of whose members have $I = \text{constant}$.

Three additional solvable Riccati families, where the invariant I is non-constant, are obtained by equating in (4.46) any two of the three arbitrary functions $\{f, p, q\}$ with $p' \neq 0$ and $q' \neq 0$

Case 4.3 $f = p$

When a given Riccati equation belongs to this family, then by changing variables

$$y = \frac{\mu}{f} \quad 4.55$$

In the given equation and in the general form of its symmetry (4.46), we see that the resulting equation in μ will admit the symmetry

$$\xi = 1, \eta = -q' \quad 4.56$$

Which can be determined as explained in (4.1)

Case 4.4 $q = p$

When a given Riccati equation belongs to this family, then by changing variables

$$y = \mu - 1 \quad 4.57$$

in the given equation and in (4.46) we see that the resulting equation in μ will admit a symmetry of the form

f is the coefficient of y^2 in the given equation.

$$\xi = \frac{p}{f}, \eta = -\frac{p'}{f} \mu \quad 4.58$$

That is infinitesimals of the form (4.49) ,and hence the equation will be solvable using Chini's method.

Case 4.5 $f=q$

From (4.48) the Riccati family corresponding to this case is given by

$$y' = fy^2 + \frac{(f(a+2f)-q')}{p}y + \frac{((a+f)f+b)f-f'p}{p^2} \quad 4.59$$

From (4.46) the equation family admits the symmetry

$$\xi = \frac{p}{f}, \quad \eta = -\frac{p'y+f'}{f} \quad 4.60$$

We have not found an obvious transformation of the form $y = P\mu + Q$ to map this symmetry to one of the form (4.12) or (4.49). A possible approach would be to directly set up the determining partial differential equation

$$\eta x + (\eta y - \xi')(f_2 y^2 + f_1 y + f_0) - \xi(f_2 y^2 + f_1 y + f'_0) - \eta(2f_2 y + f_1) = 0 \quad 4.61$$

For the coefficient ξ and η of the infinitesimal generator $\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ of an arbitrary Riccati equation (4.45). Then take ξ and η as given by (4.60) and run a differential elimination process solving for p . That approach works in principle, but the resulting symbolic expressions are large enough to become intractable even with simple examples.

An alternative approach leading to more tractable expressions is based on using the information we have in (4.59) concerning the existence of two constant a and b so by equating the coefficients of (4.59) with those of (4.45), we arrive at the system .-

$$\begin{aligned} f_2 - f &= 0, \\ f_1 - \frac{f_2(a+2f_2)-p'}{p} &= 0, \quad f_0 - \frac{f_2(f_2 a + f_2^2 + b) - f'_2 p}{p^2} = 0 \end{aligned} \quad 4.62$$

This system can be solved for a and b , so that, When a Riccati equation belongs to this family, the following two expressions formed from its coefficient f_i will be constants :

$$\begin{aligned} a &= \frac{f_1 p - 2f_2^2 + p'}{f_2} \\ b &= \frac{f_0 p^2 + f_2(f_2^2 - f_1 p - p') + f'_2 p}{f_2} \end{aligned} \quad 4.62$$

At this point, however, we cannot verify the constant character of these expressions because p is still unknown. An expression for p can be obtained by computing the integrality conditions for f_2 and p implied by (4.61)

$$f_2'' = f_1 f_2' - 2f_0 p' - p f_0' + \frac{(f_2'^2 + p f_0 f_2')}{f_2} \quad 4.63$$

$$p'' = 2f_2' f_2 - p' f_1 - p f_1' + \frac{f_2' p' + p f_1 f_2'}{f_2} \quad 4.64$$

Using (4.63) to eliminating p' from (4.64), we arrive at a solution for p

$$p = \frac{f_2(3s_2 f_2'' s_2' - 2s_2^2 f_2''' + ((s_2'' - s_4)s_2 - 8s_2^3 - 2(s_2' + 2s_3^2))f_2')}{s_2(2s_2 s_3' - 3s_2 s_2')} \quad 4.65$$

Where

$$s_4 = \frac{2s_2 s_3' - 3s_3 s_2' + 3s_3^2}{2s_2} \quad 4.66$$

Is the relative invariant of weight 4 for Riccati equations with respect to transformations of the form (4.52). We note that from (4.53), When the denominator of (4.65) is zero the equation has a constant invariant I (450) and so it is already solvable using Chini's algorithm .

In summary ,a strategy ,not relying on solving second order linear equations, for finding linear symmetries of the form (4.1)for Riccati equations ,consists ;

1) Check if the equation has symmetry of the form $[\xi = F(x), \eta = Q(x)]$ or $[\xi = F(x), \eta = P(x)\mu]$

2) Check if the equation belongs to one of the two families "f=q" or "p=q" by using the transformations (4.55) and (4.57) and reentering the previous step;

3) Check if the equation belongs to the family " $f = q$ " for that purpose

(a) Compute the invariant (4.54)

(b) Use these invariant to compute p using (4.65)

(c) plug the resulting p into (4.62) and verify the two right hand sides are constant .If so, the equation admits the symmetry (4.60) It is our belief that, with the development of computer algebra software and faster computers, these types of algorithm for Riccati subclasses will become more relevant and they will also provide an alternative for tackling second order linear equations.

CHAPTER FIVE

5. SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1. Summary

The study explored the solution of first-order ordinary differential equations (ODEs) using symmetries and linear transformations. Symmetry methods, particularly Lie symmetry analysis, were employed to identify invariant transformations that simplify ODEs into more solvable forms. Additionally, linear transformations were used to reduce nonlinear ODEs to linear ones or separable forms, facilitating exact solutions. The techniques demonstrated significant advantages in solving otherwise complex differential equations by leveraging their inherent algebraic structures.

5.2. Conclusions

The application of symmetries and linear transformations provides a powerful framework for solving first-order ODEs. Symmetry-based approaches allow for the reduction of differential equations to simpler forms by exploiting invariance properties, while linear transformations help convert nonlinear equations into tractable linear or separable cases. These methods not only enhance analytical solvability but also offer deeper insights into the underlying mathematical structure of differential equations. Compared to traditional techniques, such as separation of variables or integrating factors, symmetry and transformation methods extend the class of solvable ODEs, making them valuable tools in both theoretical and applied mathematics.

5.3. Recommendations

To further advance the study and application of these methods, the following recommendations are proposed:

Extend these methods to higher-order and systems of ODEs to explore their broader applicability.

Investigate real-world applications in physics, engineering, and biology where symmetry-based solutions can provide efficient analytical or semi-analytical results.

Combine symmetry methods with numerical techniques to handle cases where exact solutions are unattainable.

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