



Wollega University

**College of Natural and Computational Sciences
Department of Mathematics**

**Analysis of Dengue Epidemic Model with Optimal Control
Strategies**

Master of Science

in

Mathematical Modeling

by

Fikadu Dibisa Olike

**November 2025
Nekemte, Ethiopia**

Analysis of Dengue Epidemic Model with Optimal Control Strategies

**College of Natural and Computational Sciences
Wollega University**

**Fikadu Dibisa Olika
ID.No: WU1309042**

Advisor:

Dr. Lemesa Bedjisa

A Thesis Submitted to the Department of Mathematics for the Requirements of Partial Fulfillment of the degree of Master of Science in Applied Mathematics (Mathematical Modeling).

**November 2025
Nekemte, Ethiopia**



Wollega University
School of Graduate Studies
Approval Sheet

Approval Sheet for submitting final thesis

As members of the Board of Examining of the Final Msc. thesis open defense, we certify that we have read and evaluated the thesis prepared by Fikadu Dibisa Olika under the title "Analysis of dengue epidemic model with optimal control strategies" and recommend that the thesis be accepted as Fulfilling the thesis requirement for the degree of Master of Science (Msc) in Mathematics (Mathematical Modeling).

1. _____	_____	_____
Chairperson	Signature	Date
_____	_____	_____
I/Examiner	Signature	Date
_____	_____	_____
E/Examiner	Signature	Date
2. _____	_____	_____
DGC	Signature	Date
3. _____	_____	_____
Dean of College	Signature	Date
4. _____	_____	_____
Postgraduate Dean	Signature	Date

Declaration

We, undersigned, declare that this thesis is original work and prepared by Fikadu Dibisa Olika under our guidance. All sources of materials used for this thesis have been indicated and acknowledged. We further confirm that the thesis has not been submitted to any other higher learning institution for the purpose of learning any degree.

Advisor: Lemesa Bedjisa (PhD)

Signature: _____

Date: _____

Statement of the Author

I, Mr. Fikadu Dibisa Oliko, hereby declare and affirm that the thesis entitled Analysis of dengue epidemic model with optimal control strategies is my own work conducted under the supervision of my advisor Lemesa Bedjisa Dano (PhD). I have followed all the ethical principles of scholarship in the preparation and analysis and completion of this thesis. All scholarly matter that is included in the thesis has been given recognition through citation. I have adequately cited and referenced all the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and I have not misrepresented, fabricated, or falsified any idea /data/fact/ source in my submission. This thesis is submitted in partial fulfillment of the requirement for a degree from the Post Graduate Studies at Wollega University. I further declare that this thesis has not been submitted to any other institution anywhere for the award of any academic degree, diploma or certificate. I understand that any violation of the above will be cause for disciplinary action by the University and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Name: Fikadu Dibisa Oliko

Signature: _____

Department: Mathematics

College: Natural and Computational Sciences

List of Acronyms

ACER	Average Cost-Effectiveness Ratio
DFE	Disease-Free Equilibrium
EE	Endemic Equilibrium
FBSM	Forward Backward Sweep Method
ICER	Incremental Cost-Effectiveness Ratio
IVP	Initial Value Problem

Contents

Approval Sheet	ii
Declaration	ii
Statement of the Author	iii
List of Acronyms	iv
Table of Contents	v
List of Tables	viii
List of Figures	ix
Acknowledgments	x
Abstract	xi
1 Introduction	1
1.1 Background	1
1.1.1 Overview of Dengue epidemics	1
1.1.2 Causes of dengue fever	1
1.1.3 Transmission mode	2
1.1.4 Prevention and control	2
1.1.5 Optimal control theory	3
1.2 Statement of the problems	4
1.3 Research Questions	4
1.4 Research Objectives	5
1.4.1 General Objective	5

1.4.2	Specific objectives	5
1.5	Significance of the study	5
1.6	Mathematical Preliminaries	7
1.6.1	Basics on Stability Analysis	11
1.6.2	Routh-Hurwitz Conditions	13
1.6.3	Pontryagin's Maximum Principle	14
1.6.4	Forward Backward Sweep Method	15
1.7	Epidemiological Background	15
1.7.1	Basic Terminologies	15
1.7.2	The SIR model	16
1.7.3	Infectious Diseases	18
2	Review of Related Literature	19
3	Research Methodology	26
3.1	Study area and period	26
3.2	Study design	26
3.3	Numerical simulation	26
3.4	Study Procedure	27
3.5	Source of Information	27
4	Modeling the Transmission Dynamics of Dengue Fever with Optimal Control Strategies	28
4.1	Introduction	28
4.2	Dengue Model Formulation	28
4.2.1	Mathematical Model	28
4.2.2	Model Assumption	29
4.2.3	Flow chart diagram	30
4.2.4	Model Equations	31
4.3	Qualitative analysis	32
4.3.1	Existence of solution	32
4.3.2	Positivity and boundedness of solution	34
4.4	Disease free-equilibrium and Basic Reproduction Number	38
4.4.1	Disease free equilibrium	38

4.4.2	Basic reproductin number (R_0)	39
4.4.3	Local stability of DFE	41
4.4.4	Global Stability of DFE	42
4.5	Existence of Endemic Equilibrium	44
4.5.1	Local stability of EE	45
4.5.2	Global stability of EE	48
4.6	Sensitivity analysis	49
4.7	Optimal Control Analysis	52
4.7.1	Existence of Optimal Control	53
4.7.2	Characterization of optimality	55
4.8	Numerical simulation and discussion	58
4.8.1	Use of personal preventive control measure only	60
4.8.2	Use of insecticide control measure only	60
4.8.3	Use of personal preventive and treatment control measures only	61
4.8.4	Use of personal preventive, treatment and insecticide spray only	62
4.9	Local and Globa Stality of Numerical Simulation	64
4.10	Cost-Effectiveness Analysis	65
5	Summary, Conclusions and Recommendations	69
5.1	Summary	69
5.2	Conclusions	70
5.3	Recommendations	71

List of Tables

1.1	Various kinds of stability or instability correspond to the eigenvalues of the Jacobian matrix A	12
4.1	Sensitivity indices of the basic reproduction number associated with basic model parameter.	51
4.2	Infection averted and the total cost	66
4.3	Infection averted and the total cost	66
4.4	Infection averted and the total cost	67
4.5	Infection averted and the total cost	67
4.6	Infection averted and the total cost	68

List of Figures

1.1	A: <i>Aedes aegypti</i> biting to human, and B: Life cycle of mosquito passes from different.	2
4.1	Flowchart diagram for dengue fever transmission.	31
4.2	The effect of parameters δ and d_m on the basic reproduction number.	51
4.3	Dynamic behavior of (a) human mobility, (b) susceptible human, (c) infected human, (d) recovered human, (e) susceptible mosquito and (f) infected mosquito for the endemic case for when $R_0 = 1.2086 > 1$ using parameter values in Table 4.2.	59
4.4	The impact of personal prevention control measure on the dynamic behavior of (a) infected human, (b) infected mosquito and its control profile.	60
4.5	The impact of insecticide chemical spray control measure on the dynamic behavior of (a) infected human and (b) infected mosquito with its (c) control profile.	61
4.6	The Impact of personal preventive and treatment control measures on the dynamic behavior of (a) infected humans, (b) infected mosquitoes, and (c) the corresponding control profile.	62
4.7	The Impact of personal preventive, treatment and insecticide control measures on the dynamic behavior of (a) infected humans, (b) infected mosquitoes, and (c) the corresponding control profile.	63
4.8	The local stability of the disease at DFE (a) infected humans, (b) The global stability at DFE infected mosquitoes, and (c) The Global stability of the infected host vrs infected mosquito.	64

Acknowledgments

First and foremost, I would like to thank my Almighty God who gave the strength to work on this seminar on specialization.

Next, I would like to thank my advisor Dr. Lemesa Bedjisa who helped me to do every activity at the same time. My special and heart full gratitude for his great assistance, contribution, useful suggestion and constructive idea on this excellence advising and limitless effort in encouraging me in my work, correcting and giving comments by devoting his time. I greatly appreciate him who gave me valuable suggestions and comments during the preparation of this proposal.

Thirdly I would like to thank my family for being tired with me until now and supporting me financially and encouraging me on every day and I would also like to thank my greatest thanks and appreciation to my wife Obse mideksa for her permanent love and support during my graduate thesis work and beyond.

Finally, I would like to thanks my colleagues who helping me in all ways as well as whom shared different ideas and supporting me by their own material needed to complete this proposal.

Abstract

Dengue is virus which is caused by aedes Agypti mosquito the can transmitted by this vectors. The global dynamics and control strategies of an epidemic model system for dengue and we analyze the model considering the controls as constants. The biological feasibility of the model system is examined. The existence and stability of the equilibrium states are studied concerning the basic reproduction number. Both the local and the global asymptotic stability of the system's equilibrium states are considered. Next, we introduce the time-dependent controls, namely, personal prevention for dengue infection, treatment for the infected persons, and environmental hygiene against Dengue into the model. A control problem is proposed and solved analytically. The necessary conditions for the effective control of the disease are derived by applying Pontryagin's maximum principle. Moreover, several possible combinations of the controls are considered, and also their productiveness is compared through the several numerical illustrations. It is suggested from the simulation works that the best strategy to minimize the transmission of dengue virus is to adopt all the three controls optimally among the considered strategies. Findings: Results indicate that reducing mosquito biting rates and increasing human recovery rates are effective strategies for controlling dengue transmission. The model exhibits backward bifurcation, suggesting that even when R_0 is less than one, the disease can persist in the population under certain conditions. Novelty: This study presents a novel modeling framework that can inform the development of targeted prevention and management strategies for dengue.

Keywords: Dengue fever, Mathematical Modeling, Ordinary differential equations, Basic reproduction number, Sensitivity analysis, Optimal control.

Chapter 1

Introduction

1.1 Background

Dengue fever is a rapidly spreading mosquito-borne disease caused by four serotypes of the Flavivirus, transmitted mainly by *Aedes aegypti* and *Aedes albopictus*. Once limited to sporadic outbreaks, it has become a major global health threat due to urbanization, climate change, and increased travel.([Sheet, 2024](#)) WHO estimates about 390 million infections annually, with epidemics often driven by serotype shifts and secondary infections that heighten the risk of severe disease.

Dengue fever is a mosquito-borne infectious disease which is transmitted through *Aedes aegypti* mosquitoes.

1.1.1 Overview of Dengue epidemics

Due to absence of proper medicine or vaccination, mathematical modeling plays an important role in understanding the disease dynamics and in designing strategies to control the spread of dengue virus. It is one of the major global health issues in the tropical and subtropical regions of the world. Its dynamics are very complicated owing to the coupling among multiple transmission pathways and various components in pathogen. Dengue fever is a series bacterial infection caused by the by Mosquito. People who are infected with infected with Dengue

1.1.2 Causes of dengue fever

Dengue is an infectious disease that is caused by the dengue virus (DEN virus or DENV). This virus is transmitted through the bite of an infected female *Aedes aegypti* or *Aedes*

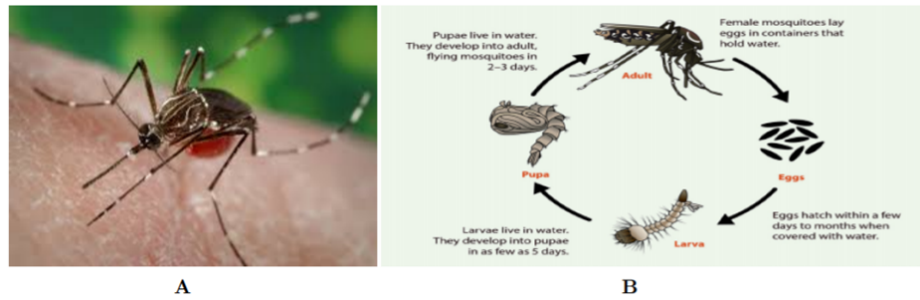


Figure 1.1: A: *Aedes aegypti* biting to human, and B: Life cycle of mosquito passes from different.

bopictus. (Sheet, 2024) it is estimated that of the world population is at risk of dengue every year. Dengue has been the subject of main concern in many tropical and subtropical countries including Indonesia. Since the first case of dengue in Indonesia was reported in Jakarta and Surabaya way back in 1968 the incidence of dengue in Indonesia continues to spread to this day and some symptoms of Dengue virus, severe headache, nausea, high fever, vomiting and joint pain. Generally people who have suffered from a different virus type are earlier are highly likely affected by these diseases. It is observed that the children below the age of 15 and some senior adults are likely the most victim of Dengue.

1.1.3 Transmission mode

Dengue viruses are transmitted through people the bites of infected female *Aedes* mosquitoes to human and Human are the primary host and source of virus for female mosquito to acquire Dengue through blood meal from viremic human. The pass way of Dengue shown below.

1.1.4 Prevention and control

To prevent Dengue we focus on the eliminating mosquito breeding ground, protecting yourself from bites, and seeking early medical attention if you suspect an infection. (CDC, 2020) in addition to vector people can prevent mosquito bites by taking personal protection measures: Wearing clothes that cover the body and minimize exposure to mosquito bites; Mosquito repellent Mosquitoes installing net screens on doors and Windows. Optimal control helps identify the best timing and intensity of interventions. Optimal control strategies are useful mathematical tools to help find an effective mitigation strategy for some diseases, such as

vector-borne diseases, mycobacterium tuberculosis, and optimal control is used to identify the most effective and cost-efficient strategies for disease prevention and control, focusing on minimizing the number of infected individual and cost of intervention. Therefore, the author used as optimal control personal prevention, environmental hygiene and treatment.

Maintaining community awareness by reducing the ignorance of dengue is essential to guarantee intervention success in controlling the spread of dengue. Mathematical models have been used widely by researchers to understand how vector-borne diseases spread. Dengue fever is not only causing devastating economic losses but also significantly affects human health in terms of morbidity and mortality.([Draidi Areed et al., 2024](#)) It is recognized as the supreme threatening infection disease in the world with the highest ratio of cases per year. The symptoms of dengue infection include mild or high fever, headache, mild foot swelling, skin rash, internal haemorrhaging, joint pain, circulatory failure, and death. The main vector, *Aedes aegypti* mosquito, transmits different serotypes of dengue to the host. The mosquitoes gain the dengue virus when they bite an infected human host. The infected mosquitoes then having the capacity of spreading dengue virus if they bite another host and remain infected for the rest of their lives. It is reported in other works that vertical transmission of virus in both humans and mosquitoes are very rare. A partially effective vaccine has been introduced for dengue infection and is available in the market for use in several countries. Mathematical models are used to understand the dynamic behaviour of infection disease and are recognized as the best tools for addressing public health questions. To be more specific it is used to recognize the dominant factors that greatly affect the transmission of infection diseases.

1.1.5 Optimal control theory

Optimal control helps identify the best timing and intensity of interventions. Optimal control strategies are useful mathematical tools to help find an effective mitigation strategy for some diseases, such as vector-borne diseases, mycobacterium tuberculosis, and optimal control is used to identify the most effective and cost efficient strategies for disease prevention and control ,focusing on minimizing the number of infected individual and cost of intervention. Therefore the author used as optimal control personal prevention, environmental hygiene and treatment.

1.2 Statement of the problems

Mathematical models for the transmission of dengue fever with mass action, standard and saturated incidence have been developed by several authors in the last few years back in order to investigate the dynamic behavior of this arbovirus disease (Esteva and Vargas, 1998; Cai et al., 2009; Khatua and Kar, 2020). Very recently, several authors have developed compartmental mathematical modeling of dengue fever in order to study the dynamic behavior and its global stability (Din et al., 2021; Aldila et al., 2023).

In 1998, Esteva and Vargas (1998) formulated a simple compartmental deterministic dengue fever epidemic model. They authors suggested the control measure for vector population. Additionally, Cai et al. (2009) developed a simple deterministic mathematical modeling of dengue fever to investigate the global stability of a dengue epidemic model with saturation incidence for the interaction between susceptible human and infectious mosquito, and bilinear incidence for the interaction between susceptible mosquito and infectious human population. Based on this model, Khatua and Kar (2020) extended the dynamic behavior of dengue fever to a problem of optimal control by incorporating two time dependent control measures. Aldila et al. (2023) improved the work of Khatua and Kar (2020) by considering the impact of community ignorance on the dynamics of dengue fever. Many studies showed that the human mobility has effect on the transmission dynamics of dengue fever.

In this thesis, we considered a MSIR-SI model for the dynamics of dengue disease with standard and bilinear incidence by incorporating three time dependent control measures. Despite the availability of mathematical models to simulate dengue transmission dynamics, there is a critical gap in integrating optimal control theory with cost-effectiveness analysis to guide policy decisions. Specifically, there is a need to (i) identify the most effective combination of control strategies that minimize the number of infections, (ii) quantify the economic trade-offs between different interventions, and (iii) provide actionable insights for resource allocation in dengue-endemic regions.

1.3 Research Questions

This research answer the question the following:

- How to develop a nonlinear deterministic mathematical model to study the dynamic transmission of dengue fever?

- How to incorporate three control measures in order to reduce the transmission of dengue disease in the region?
- How to examine the effectiveness of the intervention strategies by determining the cost associated with each strategy?

1.4 Research Objectives

This section presents the research objective of the study, mainly the general and specific objectives of the study.

1.4.1 General Objective

The main objective of this thesis was to analyse the optimal control and cost-effectiveness analyses of dengue epidemic model using mathematical modeling approach.

1.4.2 Specific objectives

In this subsection, we are desired to list all possible specific objectives we are going to achieve from this study. The specific objectives of this thesis were:

- To develop a nonlinear deterministic mathematical model to study the dynamic transmission of dengue fever,
- To incorporate three control measures in order to reduce the transmission of dengue disease in the region, and
- To examine the effectiveness of the intervention strategies by determining the cost associated with each strategy,

In order to completely address the problems around the dynamic transmission of dengue in the region, we stick to the aforementioned objectives of the study.

1.5 Significance of the study

Mathematical modelling is an instrument that is capable of saving millions of live, assisting in policy designing and decision-making. It is broad and has wide range applications in such

a way that controlling the spread of diseases and optimizing economic growth of a certain country are among few applications to list. It can also be exploited to help us understand the universe and the conditions needed to sustain life.

In this study, we employ mathematical modelling of Dengue epidemic model and extension of the deterministic model into optimal control and cost effectiveness Analysis. In previous research, there is no mathematical study considering the asymptomatic carriers of dengue infection therefore, the purpose of this study is to incorporate asymptomatic and symptomatic infected classes to investigate how they influence dynamic behaviour of dengue fever, and further examine the optimal control measures to lessen the level of infection and costs. Therefore, the significance are.

1. Advances Mathematical Epidemiology. This includes

- By integrating optimal control theory with dengue transmission dynamics, the study contributes to the growing body of literature on disease modeling and control.
- Bridges Theory and Practice: It provides a framework that links mathematical rigor with real-world decision-making, offering tools that health authorities can use to simulate and evaluate intervention strategies.

2. Public Health Impact: This includes

- Improves Disease Management: The model identifies the most effective and economically viable strategies to reduce dengue infections, hospitalizations, and deaths.
- Supports Policy Formulation: Results from the cost-effectiveness analysis can guide governments and NGOs in allocating limited resources toward interventions that yield the highest public health benefit.

3. Socioeconomic Relevance: This includes

- Reduces Economic Burden: Dengue outbreaks impose significant costs on health-care systems and affected communities. This research helps minimize these costs through optimized intervention planning.

- Empowers Local Communities: By tailoring strategies to specific regions or resource constraints, the study promotes sustainable and community-driven approaches to dengue control.

4. Long-Term Vision: This includes

- Scalable Framework: The methodology can be adapted to other vector-borne diseases such as malaria or Zika, making it a versatile tool for epidemic preparedness.
- Promotes Interdisciplinary Collaboration: The research encourages collaboration between mathematicians, epidemiologists, economists, and public health officials

1.6 Mathematical Preliminaries

This section is devoted to presenting definitions, lemmas, theorems and theories that are linked to the study in thesis. We provide detail source for each definition, lemma, theorem and theory or principles used hereunder.

Definition 1.6.1. (*Boyce et al., 2017*) *A differential equation an equation involving a equality and its derivatives in relation to one or more independent variables is known as a differential equation.*

Definition 1.6.2. *Let z represent a dynamical systems state Next, given z a developed deterministic model is given by:*

$$\frac{d_x}{d_t} = f(t, x, \lambda) \quad \text{where } \lambda \in \mathbb{R}^{\kappa} \quad \text{donatetheparameterthatdetermineshowthesystemevolve } Z \in \mathbb{R}^{\kappa} \text{ and } t \in \mathbb{R}$$

Ordinary and partial differential equations are the two categories of differential equations. An ODE is a quantity description where the quantity depends only on one variable. Comparatively, a partial differential equation involves a quantity that has multiple independent variables. This thesis focuses primarily on deterministic models, which are represented as ordinary differential equation

In this subsection, we provide basic definitions commonly used in classical ordinary differential equation. It has a wide range of applications in epidemiology to model the dynamics of any infectious disease.

Definition 1.6.3. Suppose z is an unknown state variable in t . Thus,

$$\frac{dz}{dt} = f(t, z(t)). \quad (1.1)$$

is a system of ordinary differential equations (ODEs) where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ for all $n \in \mathbb{N}$.

Suppose that equation (4.41) is subjected to the initial conditions (ICs)

$$z(0) = z_0. \quad (1.2)$$

Thus, equation (4.41) along with equation (1.2) is termed as initial value problem. Note that equation (4.41) can also be written in a column vector form as follows:

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \vdots \\ \dot{z}_{n-1}(t) \\ \dot{z}_n(t) \end{bmatrix} = \begin{bmatrix} f_1(z_1(t), z_2(t), \dots, z_{n-1}(t), z_n(t)) \\ f_2(z_1(t), z_2(t), \dots, z_{n-1}(t), z_n(t)), \\ \vdots \\ f_{n-1}(z_1(t), z_2(t), \dots, z_{n-1}(t), z_n(t)), \\ f_n(z_1(t), z_2(t), \dots, z_{n-1}(t), z_n(t)). \end{bmatrix}$$

In the next discussions, we call equation (4.41) as a system of ODEs, an integer order system and/or a dynamical system, and apply freely in epidemiological models (Martcheva, 2015). System (4.41) can be either

- linear or nonlinear: The mathematical model will be linear if the operators are linear and it is nonlinear otherwise. In different contexts, linearity and nonlinearity are defined differently, and linear models can include nonlinear expressions.
- static or dynamic: A dynamic model accounts for changes in the system's status over time. As opposed to this, a static (or steady-state) model, which is time-invariant, estimates the system in equilibrium. Dynamic models are typically represented using ordinary differential differential equations, fractional differential equations, or difference equations.
- explicit or implicit: If all of the input parameters for the whole model are known, and the output parameters can be calculated using a finite number of computations, the model is said to be explicit. However, there are times when only the output parameters are known, necessitating the use of an iterative strategy such as Newton's technique or Broyden's method to solve the corresponding inputs.

- autonomous or non-autonomous: A dynamical system can be either autonomous or non-autonomous system of differential equations depending on the independent variable involved. It is said to be autonomous if f does not explicitly depend on the independent variable (t) and non-autonomous otherwise.
- discrete or continuous: In contrast to a continuous model, which depicts objects as continuous, a discrete model views objects as discrete, such as chemical model particle states or statistical model states.

In this thesis, however, we are interested in the autonomous nonlinear classical and fractional order differential equations to develop a mathematical model for HBV infection and its progression to liver cirrhosis.

Theorem 1.6.1 (Differential Inequality). *Let $z(t)$ satisfy*

$$\frac{dz}{dt} \leq dz + b, \quad z(0) = z_0. \quad (1.3)$$

for $b, d \in \mathbb{R}$ are constants. Then the solution $z(t)$ of equation (1.3)

$$z(t) \leq z_0 e^{dt} + \frac{b}{d} (e^{dt} - 1), \quad b \neq 0,$$

and

$$z(t) \leq z_0 + bt$$

Basically, the calculation of equilibrium points is common sense in any dynamic system, and it is a value of the state variables where the state variables do not change and remain constant.

Definition 1.6.4. *An equilibrium point of a dynamical system (4.41) is a point $z^* \in \mathbb{R}^n$ that satisfies $f_i(z^*) = 0$ for all $1 \leq i \leq n$.*

The term equilibrium point, critical point or fixed point are used interchangeable in any dynamical system. Analyzing the stability of an equilibrium point for a given dynamical system is perhaps the most fundamental problem in a qualitative study of biological systems.

Definition 1.6.5 (Equilibrium point). *An equilibrium point (z^*) of a dynamical system (4.41) is said to be locally stable if all eigenvalues of the Jacobian matrix evaluated at z are negative.*

Theorem 1.6.2 (Linearisation). *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuously differentiable function and z^* be the equilibrium point of f in equation (4.41). Let $A = \frac{\partial f}{\partial z} \Big|_{z=z^*}$ be the linearization of f , and $\sigma(A) \subset \{Re(\lambda) < 0\}$.*

The Jacobian matrix denoted by A or $Df(z)$ of the dynamical system (4.41) evaluated around its equilibrium point is given by

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \cdots & \frac{\partial f_1}{\partial z_{n-1}} & \frac{\partial f_1}{\partial z_n}, \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \cdots & \frac{\partial f_2}{\partial z_{n-1}} & \frac{\partial f_2}{\partial z_n}, \\ \vdots & \vdots & \ddots & \vdots & \vdots, \\ \frac{\partial f_{n-1}}{\partial z_1} & \frac{\partial f_{n-1}}{\partial z_2} & \cdots & \frac{\partial f_{n-1}}{\partial z_{n-1}} & \frac{\partial f_{n-1}}{\partial z_n}, \\ \frac{\partial f_n}{\partial z_1} & \frac{\partial f_n}{\partial z_2} & \cdots & \frac{\partial f_n}{\partial z_{n-1}} & \frac{\partial f_n}{\partial z_n}, \end{pmatrix} \quad (1.4)$$

A system of nonlinear ODEs (4.41) can be converted into system of linear ODEs by simply linearizing the system of ODEs about an equilibrium point which will take the form

$$\frac{dz}{dt} = Az(t). \quad (1.5)$$

where $A \in \mathbb{R}^n$ is the Jacobian matrix evaluated at the equilibrium point (z^*). The solution of this system of linear ODEs (1.5) is given by

$$z(t) = z_0 e^{\lambda t}. \quad (1.6)$$

where z_0 is the initial condition and λ is an eigenvalue that is the root of the characteristic equation

$$|A - \lambda I| = 0. \quad (1.7)$$

where $I \in \mathbb{R}^n$ is an identity matrix. It is to be noted that the solution from equation (1.7) is stable if all the eigenvalues of equation (1.7) have negative real parts and unstable if at least one eigenvalue has a positive real part.

Definition 1.6.6. *An equilibrium point z^* is said to be*

- *hyperbolic if none of the eigenvalues of A evaluated at z^* has a zero real part and*
- *non-hyperbolic if at least one of the eigenvalues of the Jacobian matrix A evaluated at z^* has a non-zero real part.*

We note that the stability of a hyperbolic equilibrium point of any dynamical system with an integer-order derivative is determined by the signs of the real parts of the eigenvalues of its Jacobian matrix. If all the eigenvalues of the Jacobian matrix have negative real parts then this hyperbolic equilibrium point is asymptotically stable.

Definition 1.6.7. *Lipschitz condition* Let $(\mathbb{R}, \|\cdot\|)$ be a normed linear space. A function $f: D \rightarrow \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is said to be Lipschitz in z if there exists a Lipschitz constant M such that

$$\|f(z_1) - f(z_2)\| \leq M\|z_1 - z_2\|.$$

is satisfied for all $z_1, z_2 \in D$ and $z_1 \neq z_2$. We note that f is globally Lipschitz if it is Lipschitz on every bounded subset of $\mathbb{R}^n$. The norm $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}$.

1.6.1 Basics on Stability Analysis

For the stability analysis of the equilibrium point of an integer order system, we employ the Jacobian matrix evaluated at the equilibrium point, Routh-Hurwitz criteria for the local asymptotic stability and Lyapunov function, and LaSalle's invariance principle for the global stability analysis of the system.

Definition 1.6.8. A solution $z(t)$ of an integer order system (4.41) along the initial condition in (1.2) with respect to the open set U is said to be

- *stable* if for each $\varepsilon > 0$; there exists $\delta(\varepsilon) > 0$ such that if $\tilde{z}(t)$ is another solution of (4.41) with $\tilde{z}(0) = \tilde{z}_0$ and $\|\tilde{z}_0 - z_0\| < \delta(\varepsilon)$, then $\|\tilde{z}(t) - z(t)\| < \varepsilon$ for all $t \geq 0$.
- *asymptotically stable* if it is stable, and there exists $\gamma > 0$ such that if $\|\tilde{z}_0 - z_0\| \leq \gamma$, then $\lim_{t \rightarrow \infty} \|\tilde{z}(t) - z(t)\| = 0$.
- *unstable* if given any $\varepsilon > 0$, there always exists an initial condition $z(0)$ with $\|z(0) - z_\varepsilon\| < \varepsilon$ such that $z(t)$ is arbitrarily large as $t \rightarrow \infty$.
- *globally stable* if for any solution $z(t)$ with an initial condition z_0 , then there exists $\varepsilon > 0$ such that $\|z(t) - z_0\| < \varepsilon$ for all $t \geq 0$.

Moreover, the stability of the solution of the integer order system (4.41) depends on the eigenvalues of the Jacobian matrix A , and the possible behaviors of the solutions depend on

Table 1.1: Various kinds of stability or instability correspond to the eigenvalues of the Jacobian matrix A

Eigenvalue	Equilibrium Point	Stability
$\lambda_1 > \lambda_2 > 0$	node	unsable
$\lambda_1 < \lambda_2 < 0$	node	asymptotically sable
$\lambda_2 < 0 < \lambda_1$	saddle point	unsable
$\lambda_1 = \lambda_2 < 0$	degenerate node	asymptotically sable
$\lambda_1 = \lambda_2 > 0$	degenerate node	unsable
$\lambda_1, \lambda_2 = a \pm ib$	spiral (focus) point	
$a > 0$		unstable
$a < 0$		asymptotically stable
$\lambda_i = \pm ib$	central	stable

the signs of the eigenvalue with negative or positive real parts. Table 1.1 summarizes all the possible solution behaviors

We note that nodes, saddles, and spirals are hyperbolic equilibria while centers are non-hyperbolic equilibria of the dynamical system (4.41). Figure indicates the various behavior of the equilibrium point given in Table above.

Definition 1.6.9. Let D be an open subset of $\mathbb{R}^n$ for all $n \in \mathbb{R}$. A real-valued $C^1(D)$ function $V : D \rightarrow \mathbb{R}$ is said to be positive definite if the following conditions hold

- $V(z) = 0$ for all $z = [0, 0, \dots, 0]$,
- $V(z) > 0$ for all $z = (z_1, z_2, \dots, z_n) \in D$ and $z \neq 0$,
- $V(z) \rightarrow \infty$ as $z \rightarrow \infty$

It is to be noted that V is said to be negative definite if $-V$ is positive definite.

Definition 1.6.10 (Lyapunov function, ()). Suppose U is an open subset of $\mathbb{R}^n$. A continuous differential function $V : U \rightarrow \mathbb{R}$ is a Lyapunov function for a nonlinear system (4.41) at equilibrium point (z^*) if

- $V(z)$ is positive definite at z^* , and
- $\frac{dV(z)}{dt} \leq 0$ for all $z \in U$

Basically, there are no concrete procedures for constructing Lyapunov functions. However, there are some functional forms that serve as a baseline that satisfy the two conditions of the Lyapunov function.

Definition 1.6.11. Let $z^* = (z_1^*, z_2^*, \dots, z_n^*)$ be an equilibrium point of the initial value problem in equations (4.41) and (1.2). Let $V: D \rightarrow \mathbb{R}$ be a Lyapunov function. Then z^* is said to be asymptotically stable if it satisfies

- $V(0) = 0$,
- $V(z) > 0$, in $D \setminus \{0\}$
- $\dot{V}(x) < 0$, in $D \setminus \{0\}$

Here the fourth condition in the globally asymptotically stable is termed as radially unbounded where it refers to the more and more $z(t)$ go away from the equilibrium point $V(z)$ will increase more and more.

The stability of an equilibrium point in Lyapunov's method particularly focuses on the LaSalle invariance principle that gives conditions that describe the behavior as $t \rightarrow \infty$ of all solutions of an autonomous ODE

Definition 1.6.12 (LaSalle Invariant Principle, ()). Let $D \subset \mathbb{R}$ be a compact positively invariant set with respect to the dynamical system (4.41). Let $V: D \rightarrow \mathbb{R}$ be a continuously differentiable function such that $\frac{dV}{dt} \leq 0$ in D . Let $E \subset D$ be the set of all points in D where $\frac{dV}{dt} = 0$. Let $M \subset E$ be the largest invariant set in E .

$$\lim_{t \rightarrow \infty} \left(\inf_{x \in M} \|z(t) - x\| \right) = 0$$

1.6.2 Routh-Hurwitz Conditions

In this section, we present the Routh-Hurwitz criteria to ensure the stability of the integer order system (4.41) depending on the coefficient of the characteristic equation in case some eigenvalues of the system are difficult to find out. One method we will consider uses the Routh-Hurwitz conditions. Let the characteristic equation in equation (1.7) be given by

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_0 = 0, \quad (1.8)$$

where a_i ($i = 1, 2, \dots, n$) are real constant. The necessary and sufficient conditions on the a_i in equation (1.8) such that the roots of $p(\lambda)$ have $Re(\lambda) < 0$ are the Routh-Hurwitz conditions,

$$D_1 = a_1 > 0, \quad D_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0, \quad D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0, \quad (1.9)$$

$$D_k = \begin{vmatrix} a_1 & a_3 & \cdot & \cdot & \cdot & \cdot \\ 1 & a_2 & a_4 & \cdot & \cdot & \cdot \\ 0 & a_1 & a_3 & \cdot & \cdot & \cdot \\ 0 & 1 & a_2 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & a_k \end{vmatrix} > 0, \quad (k = 4, 5, \dots, n)$$

where $a_i = 0$ for $i \geq n$. Note that the discriminant D of the characteristic equation (1.9) is defined by $D = (-1)^{\frac{n(n-1)}{2}} R(p, p)$ where p is the derivative of p , and for $q(z) = z^i + b_1 z^{i-1} + b_2 z^{i-2} + \dots + b_i$, $R(f, q)$ is the determinant of the corresponding Sylvester $(n+l) \times (n+l)$ matrix.

1.6.3 Pontryagin's Maximum Principle

The Pontryagin's Maximum Principle (PMP) provides the necessary conditions that an optimal control must satisfy. This principle converts the fractional order system (4.5) into minimization problem point-wise Hamiltonian ($\mathcal{H}$) with respect to controls $u_i(t)$ ($i = 1, \dots, m$). It determines how the state variables $z(t)$, controls $u(t)$, and the co-state variables $\lambda(t)$ change over time through the equation of motion for z and λ .

Definition 1.6.13 (Hamiltonian function). *In FOCP in (4.42), the function*

$$\mathcal{H}(t, z(t), u(t), \lambda(t)) = f(t, z(t), u(t)) + \lambda(t)g(t, z(t), u(t)) \quad (1.10)$$

is called the Hamiltonian function and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ is adjoint variable.

Theorem 1.6.3 (Pontryagin's Maximal Principle, (Pontryagin, 2018)). *Let $u^*(t)$ and $z^*(t)$ be optimal for fractional optimal control problem (1.1), then there exists a function $\lambda(t) : [0, t_f] \rightarrow \mathbb{R}^n$ such that;*

$$\mathcal{H}(t, z^*(t), \lambda(t), u(t)) \leq \mathcal{H}(t, z^*(t), \lambda(t), u^*(t)) \quad (1.11)$$

for all controls u at each time t , and

$$\begin{cases} \frac{dz}{dt} = \frac{\partial H}{\partial \lambda}(z, \lambda, u), & z(0) = z_0, \\ \frac{d\lambda}{dt} = -\frac{\partial H}{\partial z}(z, \lambda, u), \\ \lambda(t_f) = 0 \\ \frac{\partial H}{\partial u} = 0. \end{cases} \quad (1.12)$$

where z, u and $\lambda(t_f)$ are the state equation, adjoint equation, and the transversality condition respectively.

1.6.4 Forward Backward Sweep Method

In this subsection, we present the forward backward sweep method to solve optimality system with forward and backward sweeps. Forward Backward Sweep Method (FBSM) is an effective iterative approach to solving the optimality systems. We write the MATLAB algorithm for FBSM to solve any fractional optimal control problem using the following sequences of steps.

1.7 Epidemiological Background

The compartmental modeling approach is one of the oldest approaches used to describe interactions between physiological variables in the history of infectious diseases.

1.7.1 Basic Terminologies

Definition 1.7.1. (*Martcheva, 2015*) A sub-population is said to be in the

- susceptible class (S) - is individuals who are vulnerable to disease at any instant of time t , The susceptible population acquires HBV infection due to effective contact with HBV infected individuals
- infected class (I) - individuals who got infected through either contact or blood-borne and can transmit the disease but who can able to clear the virus from their body within less than 6 months,
- recovered class (R) - individuals who are removed either due to death or get cured.

Many more compartments such as latent class, treatment class, vaccinated class, and hospitalized class can be included as per the assumptions.

Definition 1.7.2. (*Martcheva, 2015*) *In an epidemiological model,*

- an incidence rate is the number of new cases per unit of time. There are many kinds of incidence rates. Among these, the bilinear incidence rate and the standard incidence rate are the most commonly used incidences in the mathematical modeling of infectious diseases. The bilinear incidence rate (also known as mass action) is used in diseases for which disease-relevant contact increases with an increase in the population size. While the standard incidence is used for diseases for which the contact rate cannot increase indefinitely and is limited even if the population size increases. Moreover, the mass action incidence and the standard incidence agree when the total population size is constant.
- prevalence measures the number of individuals who have the disease at a specific time.
- epidemic is a sudden outbreak of a disease that infects a substantial portion of the population in a small-scale region before it disappears.
- endemic is a constant occurrence of disease.
- transmission rate is the probability of infection given contact between a susceptible and infected individual. It is determined by three distinct factors: those include the prevalence of infectives, the underlying population contact structure, and the probability of transmission given contact.
- A transition rate is a transfer between compartments.

We note that a contact in a population is said to be homogeneous mixing if all susceptible individuals are equally likely infected.

1.7.2 The SIR model

The first SIR epidemic model of an infectious disease was formulated by Kermack-McKendrick in 1927. The model assumes that the population size is fixed (i.e., no births, deaths due to disease, or deaths by natural causes), incubation period of the infectious agent is instantaneous, and the duration of infectivity is same as length of the disease. It also assumes a

completely homogeneous population with no age, spatial, or social structure. The model has the form

$$\begin{cases} \dot{S}(t) = -\beta SI, \\ \dot{I}(t) = \beta SI - \sigma I, \\ \dot{R}(t) = \sigma I, \end{cases} \quad (1.13)$$

The term βSI quantifies the rate at which individuals are being infected, which describes that the number of individuals being infected per unit of time t is proportional to both the number of S available to be infected and the number of I that are around to infect them. The parameter β is the proportionality constant, it represents the transmission rate per capita, which is dependent on the infectivity of the disease in question as well as characteristics of the population we are dealing with. The model (1.13) is subject to initial conditions.

$$S(0) = S_0 > 0, I(0) = I_0 > 0, \text{ and } R(0) = R_0 = 0$$

The state variable R does not appear in either of the equations in the model (1.13) indicate that removed individuals do not affect the disease spread at all. Since the total population is constant, the sum of all rates is zero.

$$\frac{dN}{dt} = 0 = \frac{d(S+I+R)}{dt} \quad (1.14)$$

From the first equation of the model, we obtain

$$\frac{dS}{dt} = -\beta SI < 0 \quad (1.15)$$

Applying equation (1.3) on equation we obtain the solution $S(t)$ as follows:

$$S(t) \leq S_0 \exp \left\{ -\beta \int_0^t I(x) dx \right\}$$

Hence, S decreases from its initial value.

From the second equation of model (1.13), we have

$$\frac{dI}{dt} = \begin{cases} I(\beta S - \sigma) > 0 & \text{if } S > \frac{\sigma}{\beta} \\ I(\beta S - \sigma) < 0 & \text{if } S < \frac{\sigma}{\beta} \end{cases} \quad (1.16)$$

From equation (1.16), we notice that if $S_0 < \frac{\sigma}{\beta}$, then $I(t)$ will decrease for all time and $I(t) \rightarrow 0$ as $t \rightarrow \infty$. However, if $S_0 > \frac{\sigma}{\beta}$, then $I(t)$ will initially increase, and then later, when S has reduced to $\frac{\sigma}{\beta}$, will start to decrease.

Furthermore, the maximum infected population ($I_{\max}$) in the SI -plane can be obtained by combining the first and second equations of the model. That is,

$$\frac{dI}{dS} = \frac{\beta SI - \sigma I}{-\beta SI} = -1 + \frac{\sigma}{\beta S} \quad (1.17)$$

Integrating both sides of the above equation and letting $S = S_0$ when $I = I_0$ yields

$$I = -S + \left(\frac{\sigma}{\beta}\right) \ln S + I_0 + S_0 - \left(\frac{\sigma}{\beta}\right) \ln S_0 \quad (1.18)$$

If $S_0 \geq \frac{\sigma}{\beta}$, then there is $I_{\max}$ for $t > 0$ that occurs at $S = \frac{\sigma}{\beta}$. Therefore, we have

$$I_{\max} = \begin{cases} \left(\frac{\sigma}{\beta}\right) \ln S + I_0 + S_0 - \left(\frac{\sigma}{\beta}\right) \ln S_0, & \text{if } S_0 \geq \frac{\sigma}{\beta} \\ I_0, & \text{if } S_0 \leq \frac{\sigma}{\beta} \end{cases}$$

1.7.3 Infectious Diseases

Definition 1.7.3. *An infectious disease is a clinically evident illness resulting from the presence of a pathogenic microbial agent. The microbial agent causing the disease can be bacterial, viral, fungal, and any parasitic. include tuberculosis and pneumonia; viral diseases include HIV and influenza; the most widespread fungal diseases are dermatomycoses, parasitic infections are caused by macro parasites such as protozoa, helminths, trematodes, and cestodes.*

Definition 1.7.4 (Martcheva (2015)). *A basic reproduction number is the average number of secondary infectious caused by a single infective in a totally susceptible population.*

Chapter 2

Review of Related Literature

In this chapter, we reviewed some related literature on the transmission dynamics of dengue fever, and its control mechanisms. Mathematical models with optimal control strategies are used in investigating and exploring the dynamic behavior of dengue fever for making decisions about intervention strategies to combat it. For this purpose, we selected publications on dengue virus infection mainly from the last five years, although did not.

[Barmak et al. \(2016\)](#) explored the effect of human mobility on the spatio-temporal dynamics of dengue with a stochastic model that takes into account the epidemiological dynamics of the infected mosquitoes and humans, with different mobility patterns of the human population. The authors observed that human mobility strongly affects the spread of infection by increasing the final size and by changing the morphology of the epidemic outbreaks. Finally, the findings of the proposed work indicated that human mobility could be the main driving force in the dynamics of vector epidemics.

[O'Neill et al. \(2019\)](#) suggested an scale deployment wolbachia to protect community from Dengue he use optimal control strategies of dengue dynamics using two controls variables, insecticide spray and Elimination of breeding site of dengue. The authors showed that all solutions of the model are positive and bounded with initial conditions in a certain meaningful set. Besides, the existence of disease free equilibrium and endemic equilibrium points is investigated and the basic reproduction number is computed. Then, the authors studied the local stability is deetermined by using jacobian matrix. In addition, they showed that the system has a locally asymptotically stable at disease-free equilibrium point when the reproduction number is less than one and locally asymptotically stable endemic equilibrium point when the reproduction number is greater than one. Furthermore, using Pontryagins minimum principle, the necessary condition of an optimal control of Dengue disease dynamics is determined. Finally, the author concluded from the numerical simulation that the

combination of control strategy is the best way to minimize the impact of disease.

[Jan and Xiao \(2019\)](#) extended dengue disease transmission models by including transmit vaccinated class. The authors considered that a portion of recovered individual loses immunity and moves to the susceptibles with limited immunity and hence a less transmission probability. They obtained and verified the fundamental properties of the proposed model. Additionally, they computed the sensitivity analysis in order to identify the key factors that greatly affect the dengue infection. Moreover, they proposed an optimal control problem by formulating the control objective associated with the low infection and costs. Furthermore, they carried out a large scale numerical simulations to show the effectiveness of control strategies. Finally, they recommended that control measures effectively control and eradicate the dengue infections during the whole epidemic for the simulation results.

[Abidemi et al. \(2020\)](#) presented a compartmental deterministic model including human prevention and vector control interventions for the dynamics of dengue fever spread. The authors conducted the theoretical analysis of the model to obtain the associated dengue-free equilibrium. They considered numerical simulations of different strategies of control combination. Additionally, they studied the impact of the dynamical behaviour of the system. Finally, they found that dengue prevalence can be reduced in a community by implementing any control intervention which combines human prevention and vector control measures.

[Xue et al. \(2021\)](#) developed an ordinary differential equation model taking into account multiple strains of dengue virus. The authors assessed the effectiveness of human vaccination considering its waning and failure using the model. Additionally, they derived the lower bound and upper bound for the final size of the epidemic. From the sensitivity analysis, they found that the cumulative number of infected humans is small when the vaccination rate is high or the waning rate is low for previously infected humans. Moreover, they analyzed the necessary conditions for implementing optimal control and derived the corresponding optimal solutions for mitigation dengue virus transmission by applying Pontryagin's Maximum Principle. Finally, they suggest that their findings may provide guidance for the public health authorities to implement human vaccination and other mitigation strategies.

[Dwivedi et al. \(2022\)](#) proposed and studied a nonlinear vector-host model to investigate the transmission dynamics of dengue virus which can be controlled by vaccination as well as treatment. The authors also presented the analysis of model start with the basic reproduction number. They calibrated their proposed model corresponding to transmission of dengue

virus using real data for six Indian states by using the least square method. Moreover, the authors performed normalized forward sensitivity analysis of the basic reproduction number and obtained that mitigating the disease transmission rate of mosquito and human population is the most important factor in achieving dengue control. Furthermore, they developed an objective functional to minimize the cost of the vaccination and solved with the aid of the Pontryagin's Maximum Principle. They suggested that results from the implementation of the optimal treatment policy showed a significant reduction of the hospitalized individuals as well as infected individuals. Finally, they performed extensive numerical simulations to validate their theoretical analysis with aid of the estimated model parameters.

Furthermore, [Gyasi et al. \(2023\)](#) developed analysis of Dengue fever in west Africa. a nonlinear deterministic Dengue transmission dynamic using optimal control analysis and cost-effective strategies. First, the authors showed that the model solution is non-negative and bounded. Next, they calculated a basic reproductive number using the Disease-free equilibrium point via the next generation matrix. The researcher also described that linearization and the Lyapunov function are then used to demonstrate how Disease-free equilibrium is both locally and globally stable. The Disease-free equilibrium point is asymptotically stable both locally and globally if the basic reproduction number is less than one; otherwise, an endemic disease equilibrium emerges. Furthermore, the models parameters were analyzed for sensitivity. Moreover, applying the Pontryagin minimum principle, the optimal control minimization interventions are determined using thee control strategies, namely personal prevention prevention, treatment and personal hygiene. Lastly, based up on numerical simulation of systems of optimality, they suggested that personal prevention is the most optimal and least expensive Dengue eradication strategy.

[Yoda et al. \(2024\)](#) developed an SEIR-SI type model for dengue disease in order to better observe the dynamics of infection in human beings. The authors calculated the basic reproduction number R_0 and determined the equilibrium points. Additionally, they presented the numerical simulations obtained using Python to support the theoretical work. Moreover, they studied the sensitivity of the parameters included in the expression of R_0 with the aim of identifying the most influential parameters in the dynamics of dengue disease spread. Finally, they introduced two functions u and v , respectively indicating the treatment of the infected people and any prevention system minimizing contact between humans and the disease causing vectors.

[Kumar et al. \(2024\)](#) developed an optimal control problem for dengue fever dynamics based on re-infection and treatment. The author proposed a multi compartmental model with reinfection and treatment analyses stability reproduction number, sensitivity and devise optimal control strategies through pontryagon minimum principle. The researcher also described that linearization and the Lyapunov function are then used to demonstrate how Disease-free equilibrium is both locally and globally stable. The Disease-free equilibrium point is asymptotically stable both locally and globally if the basic reproduction number is less than one; otherwise, an endemic Disease equilibrium emerges. Furthermore, the model parameters were analysed for sensitivity. Moreover, applying the Pontryagin minimum principle, the optimal infection of Dengue minimization interventions is determined using two control strategies, namely personal prevention and treatment.

On other hand, [Sow et al. \(2024\)](#) develops compartmental model incorporating Vaccination and treatment interplay as optimal control. The authors incorporating vaccination and treatment stability analysis shows combined strategies outperform single approaches an optimal control frame work confirm. Moreover, the authors showed that the system has a locally asymptotically stable at disease free equilibrium point when the reproduction number is less than one and locally asymptotically stable endemic equilibrium point when the reproduction number is greater than one. In order to optimize the best strategy for reducing Dengue virus in society, the author incorporates variable controls into the model. Finally, the numerical results show that using all controls simultaneously is the best strategy for reducing the spread of Dengue virus.

For instance, [Yoda et al. \(2024\)](#) develops SEIR-SI with optimal control via pontryagon principles. The Authors tries to differentiate SEIR with SI model which host model and vector model. from this he determines positivity, basic reproduction number and the model are local stability occur at disease free equilibrium . The basic reproductive number of the Dengue epidemic model model was also determined using the next generation matrix technique at the Dengue free equilibrium points. In addition, the authors showed that the reproductive number R_0 Disease free and endemic equilibrium points are locally and globally asymptotically stable. The analysis and numerical simulations showed that as the effective contact rate depends control variables, prevention and treatment. Lastly, the results showed that the two strategies, prevention and treatment are the most effective to reduce the spread of Dengue in the population

[Hamdan and Kiliçman \(2021\)](#) deterministic mathematical model of the dengue transmission is developed by considering the effect of temperature on the transmission dynamics. It has a locally asymptotically stable disease-free equilibrium (DFE) point whenever the basic reproduction number (R_0) is less than unity. This model has shown a possibility of backward bifurcation, where the stable DFE coexists with a stable endemic equilibrium point when $R_0 < 1$. Using the entomological data of the Aedes mosquito population and the experimental data of the dengue transmission in Malaysia, R_0 is evaluated at different.

[Aldila et al. \(2023\)](#) recently investigated the SIR model of Dengue dynamics. The researchers showed mathematical model analysis including non-negativity and boundedness of the model, Disease free equilibrium point, and local stability. In order to reduce the number of diseases they extended SIR model into optimal control model using three controls strategies, namely: personal preventive, environmental hygiene and treatment. Furthermore, it is investigated how control strategies should be applied to prevent Dengue prevention. optimally. They solved optimal control problem using Pontryagin's Minimum Principle and numerical simulation with Matlab. Their simulation results showed that the first scenario give the optimal solution in reducing the number of Infected sub populations with minimum controls.

[Harjule et al. \(2024\)](#) proposed a simple mathematical model that examines the dynamics of dengue illness transmission. The authors calculate the positivity and boundedness of the solutions of the model, and used next-generation matrix to calculate the basic reproduction number. They computed the sensitivity analysis of R_0 using the normalized forward sensitivity index. They found that the basic reproduction number primarily depends on factors such as the transmission rate of disease and the rate at which mosquitoes become infected by infectious human beings and depends inversely on factors such as the recovery rate of humans and the rate at which mosquitoes transmit the virus to humans. The authors concluded that the study has demonstrated the importance of various parameters that contribute to the spreading of the disease such as the transmission rate of disease, the rate at which mosquitoes become infected from infectious human beings, and the rate at which mosquitoes transmit the virus to humans. Finally, they recommended that the study have essential ramifications for public health policy and may contribute to a future reduction in the incidence of dengue.

For the same times [Sk et al. \(2024\)](#) proposes a fractional order model. the authors want to transmission and recovery process. He introduces a fractional order Dengue model conduct.

Stability analysis via PRCC and formulate a fractional order in optimal control. Finally, The analysis shows that Disease free equilibrium is locally and globally asymptotically stable whenever $R_0 < 1$. Also, the endemic equilibrium point is locally and globally asymptotically stable whenever $R_0 > 1$. Then, the model was extended to optimal control, and some numerical simulations with and without optimal control are also performed to verify the theoretical analysis using MATLAB. Numerical simulation of optimal control model shows that the personal prevention and treatment strategy is the most effective strategy to reduce the dynamic transmission of Dengue virus.

[Ahman et al. \(2025\)](#) They develops a comprehensive mathematical model that integrates multiple transmission routes to improve understanding of dengue dynamics and inform effective control strategies. We develop a compartmental SEIR-based model that captures dengue virus transmission through mosquito vectors, vertical (transovarial), and sexual routes. The model undergoes rigorous mathematical analysis to derive equilibrium points and assess their stability. Both local and global sensitivity analyses are performed to identify key drivers of disease dynamics. Additionally, the model is calibrated using weekly dengue incidence data from Delhi, India, to validate its predictive capacity. The sensitivity analysis identifies the most influential parameters driving transmission. Although the human-to-human contact rate (sexual transmission) has a high sensitivity index, the actual contribution of sexual transmission to the basic reproduction number. finally the researcher concluded that the numerical results show that using all controls simultaneously is the best strategy for reducing dengue epidemic disease.

[Eegunjobi et al. \(2023\)](#) proposes the idea of Modelling the super-infection of two strains of dengue virus. he proposes the model of Dengue infection analysis. The authors showed mathematical model analysis including non-negativity and bound edness of the model, disease free equilibrium point, and local stability. In order to reduce the number of infected cases, they extended SIR model into optimal control model using three controls strategies, namely personal preventive for dengue by the bednet vaccination, body locion and long sleeve clothes it is investigated how control strategies should be applied to prevent Dengue cases optimally. They solved optimal control problem using Pontryagin's Minimum Principle and Runge Kutta order fourth method for numerical simulation with Matlab. Their simulation results showed that the first scenario give the optimal solution in reducing the number of Infected sub populations with minimum controls effort and short time period. Based on their

results, the authors concluded that in order to prevent and reduce the number of Dengue infection, the control strategy giving preventive for the infected individuals must be also followed by prevention infection in society. where all of these control strategies must be carried out together to give the optimal results.

[Islam et al. \(2024\)](#) Mathematical analysis and prediction of future outbreak of dengue on time-varying contact rate using machine learning approach Most of these mathematical models, however, did not account for the dengue dynamics with optimal control analysis and cost effectiveness analysis. In this research the authors work, the Dengue dynamics model in is extended by introducing the recovered compartment that becomes susceptible and an optimal control model using three controls strategies, namely: personal prevention, treatment and insecticide spray to minimize the Dengue dynamics. Moreover, using cost-effective analysis, we identified an optima and least cost strategies to compact the dynamics of Dengue from the community.

[Draidi Areed et al. \(2024\)](#) proposes the Model to Enhance Dengue Prediction and Control for the Mekong Delta Region of Vietnam. Frist the researcher try to identify the positivity and boundedness characteristic equation and Eigen values of the dengue epidemic model. The basic reproductive number of the model was also determined using the next generation operator technique at the endemic equilibrium points. In addition the authors showed that the reproductive number R_0 , local stability and global stability are locally and globally asymptotically stable. The analysis and numerical simulations showed that as the effective contact rate decreases, infected individuals decrease. In their study, they extended the model to include three time-dependent control variables, prevention and treatment, insecticide spray. Lastly, the results showed that the two strategies, prevention and treatment, are the most effective to reduce the spread of disease in the population.

Chapter 3

Research Methodology

This chapter is dedicated to learning about the study period, study design, information sources, model analysis techniques, and mathematical procedures that will be used in the study.

3.1 Study area and period

The research was carried out at Wollega University, Mathematics Department from June 2024 to December 2025.

3.2 Study design

This study used analytical and numerical design to achieve the stated objectives of the research work.

3.3 Numerical simulation

Numerical method is used to approximate numerical solution by mathematical model. we utilized the MATLAB software is used for all simulations. To obtain the optimal solution to the dynamic system with control in equation we apply the forwardbackward sweep method. This iterative method consists in solving the system with the guess for the controls over the time interval using a forward fourth-order Runge Kutta scheme and the transversality conditions. Then, the adjoint system is solved by applying a backward fourth-order Runge Kutta scheme to the current iteratin solution of the system. The controls are updated by means of a convex combination of the previous controls and the values in the characterization process. To repeat the solutions of state variables in the system and adjoint systems the

updated controls are used. This situation is continued until the values of unknowns at the previous iteration and at the present iteration are close enough.

3.4 Study Procedure

Mathematical models describe the behaviour of systems using mathematical concepts. It starts with a clear description of the processes based on the system understanding. The translation into mathematical equations should then be done with a specific goal in mind. The study involved both analytical and numerical methods. That means analytical method is used to find exact solution. At the beginning, we considered the Dengue model called susceptible -infected- Infected recovered that uses a system of non-linear ordinary differential equations to describe the Disease dynamics. We calculated the basic reproduction number by using next generation method. Sensitivity analysis of model parameters for their effects on the outcome of the model results is performed. Local and global stability for Disease free equilibrium point and Disease equilibrium is analyzed using linearization techniques and Lyapunov function theory respectively. Study of dynamical behaviour in terms of threshold parameters and numerical simulations. Besides, a disease dynamics model is also extended to the optimal control problem of disease dynamics model. It is stated that optimal controls exist. Then, using the Pontryagin minimum principle, the necessary conditions for optimal control were derived.

3.5 Source of Information

The study involved gathering data input and information obtained from internet-based sources, some books available in libraries, and recent publication researches.

Chapter 4

Modeling the Transmission Dynamics of Dengue Fever with Optimal Control Strategies

4.1 Introduction

In this chapter, we present the proposed dengue transmission model and analyze its dynamics using both theoretical and numerical approaches. The theoretical analysis provides insights into the qualitative behavior of the model, while the numerical simulations are conducted to verify and illustrate the theoretical results. Together, these complementary methods offer a comprehensive understanding of the proposed model behavior and its implications for dengue transmission.

4.2 Dengue Model Formulation

This section is devoted to present the mathematical model, model assumptions, flow chart diagram and model equations for studying the dynamic behavior of dengue fever.

4.2.1 Mathematical Model

In this subsection, we provide mathematical model for the transmission dynamics of dengue fever epidemic model. We develop the dengue model from the works of many scholars in the field. We construct the mathematical model for dengue fever based on the works of (Cai et al., 2009; Khatua and Kar, 2020; Din et al., 2021; Aldila et al., 2023). In particular, Cai et al. (2009) formulated a five compartmental model for studying the dynamic transmission of dengue fever by assuming saturate incidence type on the human population, and bilinear incidence type on the mosquito population. Based on this work, we constructed a simple

deterministic *MSIR* – *SI* model, which comprises of four human sub-population and two mosquito subpopulations.

Human mobility plays significant role in the dynamic transmission of vector borne diseases (Barmak et al., 2016; Adegbite et al., 2023). It is obvious that some people mobilize in search of shelter because of instability reason, and some other people move from place to place for running their businesses.

In the next subsection, we considered the following basic assumptions and important features to enhance the dengue mathematical model formulation.

4.2.2 Model Assumption

In this subsection, we present some basic assumptions that enhance the mathematical model for the dengue fever epidemic by incorporating some important features.

- All parameters and state variables of the proposed model are assumed to be non-negative real numbers.
- For host population, we divide the population under consideration into four classes: M_h , S_h , I_h and R_h represent, respectively, the number of traveler (human mobility), susceptible, infective and recovered hosts at time t . The total number

$$N_h = M_h + S_h + I_h + R_h.$$

- For vector population, we denote S_m and I_m the number of susceptible and infective mosquito at time t in the vector population, respectively.
- Due to its short life, a mosquito never recovers from the infection and we do not consider the recovered class in this population. The vector total population sizes.

$$N_m = S_m + I_m.$$

- The recruitment rate to the susceptible human and mosquito population is b_h and b_m , respectively.
- We assume that β stands for the number of bites per mosquito per host in unit time

- The infection transmission rate for the human population and the mosquito population are assumed as α_h and α_m , respectively.
- The incidence function for the human population is considered as the standard type and that of for the mosquito population is taken as bilinear type.
- For the human population, the natural mortality rate is taken as d_h . The death rate of the mosquito population is considered as d_m .
- The human population recovers at a rate δ . However, the recovered individual will have permanent immunity to the disease.
- Homogeneous mixing is considered.

The description of all other system parameters is provided in Table 4.1.

On the other hand, the mosquito population is subdivided into two compartments: susceptible mosquito and infected mosquito.

4.2.3 Flow chart diagram

In this subsection, we provide a schematic diagram for the transmission dynamics of dengue fever. It is a graphical representation of how individuals move between different disease states. In addition, it is a fundamental tool for visualizing the structure of a mathematical model of infectious disease transmission.

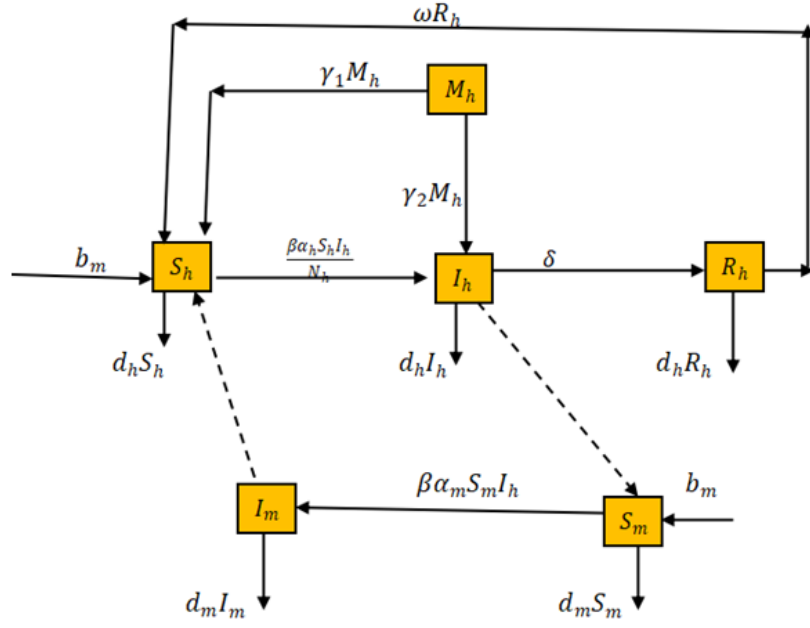


Figure 4.1: Flowchart diagram for dengue fever transmission.

In Figure 4.1, the solid arrow represents the transition of individuals between compartments, while the broken arrow represents the interaction between human and mosquito populations.

4.2.4 Model Equations

In this subsection, we present the governing equation of the proposed model for dengue fever. We construct the model equations based on the assumptions above and the flow chart diagram presented in Figure 4.1. From the flowchart diagram in Figure 4.1, the dynamics of the interaction between humans and mosquitoes is given by the following system of nonlinear ordinary differential equations.

$$\begin{cases} \frac{dM_h}{dt} = -(\gamma_1 + \gamma_2)M_h, \\ \frac{dS_h}{dt} = b_h + \gamma_1 M_h - \frac{\beta \alpha_h S_h I_m}{N_h} - d_h S_h + \omega R_h, \\ \frac{dI_h}{dt} = \frac{\beta \alpha_h S_h I_m}{N_h} - (d_h + \delta)I_h + \gamma_2 M_h, \\ \frac{dR_h}{dt} = \delta I_h - (d_h + \omega)R_h, \\ \frac{dS_m}{dt} = b_m - \beta \alpha_m S_m I_h - d_m S_m, \\ \frac{dI_m}{dt} = \beta \alpha_m S_m I_h - d_m I_m. \end{cases} \quad (4.1)$$

subject to the following initial condition

$$M_h(0) \geq 0, S_h(0) > 0, I_h(0) \geq 0, R_h(0) \geq 0, S_m(0) > 0, I_m(0) \geq 0. \quad (4.2)$$

The initial point are simple the starting point of the population with in certain rate.but parameter value is the constant value.

4.3 Qualitative analysis

In this section, we present the existence, uniqueness, positivity and boundedness of the solutions of the model (4.1). To perform a qualitative analysis of the given system of differential equations, we look at general dynamical behavior stability, equilibrium points, and the biological or epidemiological interpretations of the parameters and equations. This is common in mathematical epidemiology, where we model diseases spreading through human and mosquito populations.

4.3.1 Existence of solution

This subsection is devoted to establishing the existence and uniqueness of the solution of the model under consideration.

Theorem 4.3.1. *The solution $(M_h, S_h, I_h, R_h, S_m, I_m)$ of model (4.1) exists and unique.*

Proof. To establish uniqueness of the solution, we employ the Lipschitz condition. Let

$$N_h = M_h + S_h + I_h + R_h. \quad (4.3)$$

$$N_m = S_m + I_m. \quad (4.4)$$

and assume all parameters are non-negative.

Define

$$X(t) = \begin{bmatrix} M_h(t) \\ S_h(t) \\ I_h(t) \\ R_h(t) \\ S_m(t) \\ I_m(t) \end{bmatrix}$$

and let the system be written as

$$\frac{dX}{dt} = F(X).$$

Our goal is to show that $F(X)$ is Lipschitz continuous on a domain $D \subset \mathbb{R}^6$ where $N_h > 0$ and all state variables are non-negative.

We now check each component of $F(X)$ for the Lipschitz property according to Definition (4.1). Let

$$X = (M_h, S_h, I_h, R_h, S_m, I_m), \quad \tilde{X} = (\tilde{M}_h, \tilde{S}_h, \tilde{I}_h, \tilde{R}_h, \tilde{S}_m, \tilde{I}_m).$$

Firstly, we show the following component:

$$f_1(X) = \frac{dM_h}{dt} = -(\gamma_1 + \gamma_2)M_h.$$

This term is linear in M_h , and we have

$$|f_1(X) - f_1(\tilde{X})| = (\gamma_1 + \gamma_2) |M_h - \tilde{M}_h|.$$

Thus, f_1 is Lipschitz with constant $L = \gamma_1 + \gamma_2$.

Now, we show the second component:

$$f_2(X) = \frac{dS_h}{dt} = b_h - \frac{\beta \alpha_h S_h I_m}{N_h} + \omega R_h - d_h S_h + \gamma_1 M_h.$$

is the only only nonlinear term

$$\frac{\beta \alpha_h S_h I_m}{N_h}, \quad N_h = S_h + I_h + R_h.$$

Consider the following expressions

$$\left| \frac{\beta \alpha_h S_h I_m}{N_h} - \frac{\beta \alpha_h \tilde{S}_h \tilde{I}_m}{\tilde{N}_h} \right| \leq \beta \alpha_h \left| \frac{S_h I_m}{N_h} - \frac{\tilde{S}_h \tilde{I}_m}{\tilde{N}_h} \right|.$$

To ensure boundedness, define a compact domain $D \subset \mathbb{R}^6$ such that

$$0 \leq x_i \leq K, \quad \text{for all components } x_i \text{ of } X.$$

On this domain, $F(X)$ is Lipschitz continuous.

$$f_3(x) = \frac{dI_h}{dt} = \frac{\beta \alpha_h S_h I_h}{dt} - (d_h + \delta) + \gamma_2 M_h.$$

again the linear term and rational term are Lipschitz.

$$f_4(x) = \frac{dR_h}{dt} = \delta I_h - (d_h + \omega) R_h.$$

linear with constant are lipschitz on its domain therefore f_4 are lipschitz.

$$f_5(x) = \frac{dS_m}{dt} = b_m - \beta \alpha S_m I_h - S - m I_h.$$

is constant $b_m - \beta \alpha S_m I_h - S - m I_h$ is linear term, we have

$$f_6(x) = \frac{dI_m}{dt} = \beta \alpha I_h - d_m I_m.$$

all are linear form it is lipschitz in its domain.

$$0 \leq x_i \leq k \quad \text{and} \quad N_h = S_h + I_h + R_h \geq \varepsilon > 0.$$

□

4.3.2 Positivity and boundedness of solution

In compartmental epidemic models such as dengue models involving humans and mosquito vectors. These must remain non-negative for all time. Positive invariant means that if you start with non-negative initial conditions, the solution trajectories stay in the non-negative region. This ensures your model is biologically realistic and mathematically well-posed refers to population survive at any time.

In the next theorem, we prove that that solution of system (4.1) is always positive, which refers that all subpopulation survive at any time t .

Theorem 4.3.2. *If the initial population size is non-negative, then the solution*

$$(M_h(t), S_h(t), I_h(t), R_h(t), S_m(t), I_m(t))$$

is non negative solution for $t > 0$.

Proof. Using the fourth equation from equation (4.1), we have

$$\frac{dR_h}{dt} = \delta I_h - (d_h + \omega) R_h.$$

$$\frac{dR_h}{dt} \geq -(d_h + \omega) R_h.$$

By separable variable then integrate

$$\int_0^t \frac{dR_h}{R_h} \geq - \int_0^t (d_h + \omega) dt$$

$$R_{h(t)} \geq R_{h(0)} e^{-(d_h + \omega)t} \geq 0$$

From the last equation

$$\frac{dI_m}{dt} = \beta \alpha I_h - S_m d_m I_m.$$

$$\frac{dI_m}{dt} \geq -S_m d_m I_m.$$

using separable integral

$$\int_0^t \frac{dI_m}{I_m} = \int_0^t -S_m d_m dt.$$

$$I_{m(t)} \geq I_{m(0)} e^{m t} \geq 0.$$

from equation(3)

$$\frac{dI_h}{dt} = \frac{\beta \alpha_h S_h I_H}{N_h} - (d_h + \delta) I_h + \gamma_2 M_h.$$

$$\frac{dI_h}{dt} \geq -(d_h + \delta) I_h.$$

by separable variable integral

$$\int_0^t \frac{dI_h}{I_h} \geq \int_0^t -(d_h + \delta) dt.$$

$$I_h(t) \geq I_h(0)e^{-(d_h+\delta)t} \geq 0.$$

$$\frac{dS_m}{dt} = -\beta\alpha S_h I_h b_m - d_m S_m.$$

from the the second equation the positivity can shown as

$$\frac{dS_m}{dt} = -\beta\alpha S_h I_h b_m - d_m S_m.$$

$$\frac{dS_m}{dt} \geq -d_m S_m.$$

$$\int_0^t \frac{dS_m}{S_m} \geq \int_0^t -d_m dt.$$

$$S_m \geq S_m(0)e^{-d_m t} \geq 0.$$

Therefore the solution of model (4.1) is positive. $\square$

Next, we prove the boundedness of the solution system (4.1). An invariant region is a subset of the phase space such that if the system's state starts in this region, it remains in the region for all future times under the evolution defined by the system

Theorem 4.3.3. *Suppose the closed region $\Omega = \{M_h(t), S_h(t), I_h(t), R_h(t), S_m(t), I_m(t)\} \in \mathbb{R}^6$. such that*

$$\begin{aligned} 0 \leq M_h(t) + S_h(t) + I_h(t) + R_h(t) &\leq \frac{b_h}{d_h} \quad \text{and} \\ 0 \leq S_m(t) + I_m(t) &\leq \frac{b_m}{d_m}. \end{aligned}$$

Proof. We first take the total human population, which is given by

$$N_h(t) = M_h(t) + S_h(t) + I_h(t) + R_h(t). \quad (4.5)$$

By differentiating equation (4.5), we have

$$\frac{dN_h(t)}{dt} = \frac{d}{dt} (M_h(t) + S_h(t) + I_h(t) + R_h(t)).$$

Further, it implies that

$$\frac{dN_h(t)}{dt} = \frac{dM_h(t)}{dt} + \frac{dS_h(t)}{dt} + \frac{dI_h(t)}{dt} + \frac{dR_h(t)}{dt}. \quad (4.6)$$

Substituting the state variables for human population from equation (4.1) into equation (4.6), we have

$$\frac{dN_h(t)}{dt} = b_h - \frac{\beta \alpha_h S_h I_m}{N_h} - \omega R_h - d_h S_h + \gamma_1 M_h + \frac{\beta \alpha_h S_h I_h}{N_h} - (d_h + \delta) I_h + \gamma_2 M_h + \delta I_h.$$

$$\frac{dN(t)}{dt} = b_h - d_h(S_h + I_m + R_h) - \omega R_h + \gamma_1 M_h + \gamma_2 M_h.$$

$$= b_h - d_h N_h - \omega R_h + (\gamma_1 + \gamma_2) M_h.$$

also recall that

$$\frac{dM_h}{dt} = -(\gamma_1 + \gamma_2) M_h. \quad (4.7)$$

$$(4.8)$$

From this $M_h \rightarrow 0$ then $t \rightarrow \infty$.

$$\frac{dN_h(t)}{dt} \leq b_h - d_h N_h.$$

which implies that

$$N_h(t) \leq \frac{b_h}{d_h}.$$

Hence, the invariant region for the human population becomes

$$\Omega_1 = \{(M_h, S_h, I_h, R_h) \in \mathbb{R}^4 : 0 < M_h + S_h + I_h + R_h \leq \frac{b_h}{d_h}\}. \quad (4.9)$$

Now, the total population for the vector (mosquito) population is given by

$$N_m(t) = S_m(t) + I_m(t). \quad (4.10)$$

Differentiating equation (4.10) from the vector population, we have

$$\frac{dN_m}{dt} = \frac{dS_m(t)}{dt} + \frac{dI_m(t)}{dt}. \quad (4.11)$$

Substituting the state variables for the vector population from equation (4.1) in equation (4.11), we get

$$\frac{dN_m}{dt} = b_m - \beta \alpha S_m I_h - d_m S_m + \beta \alpha S_m I_h - d_m I_m. \quad (4.12)$$

Simplification from equation (4.12) gives

$$\frac{dN_m}{dt} = b_m - d_m(S_m + I_m).$$

Further, it is simplified to the following equation

$$N_m(t) \leq \frac{b_m}{d_m}.$$

Hence, the invariant region for the vector population becomes

$$\Omega_2 = \{(S_m, I_m) \in \mathbb{R}^2 : 0 < S_m + I_m \leq \frac{b_m}{d_m}\}. \quad (4.13)$$

Using equations (4.9) and (4.13), we obtain that

$$\Omega_1 \times \Omega_2 \subseteq \mathbb{R}^4 \times \mathbb{R}^2 = \Omega.$$

where Ω is defined as:

$$\Omega = \left\{ (M_h, S_h, I_h, R_h, S_m, I_m) \in \mathbb{R}^6 \mid M_h + S_h + I_h + R_h \leq \frac{b_h}{d_h} \quad S_m + I_m \leq \frac{b_m}{d_m} \right\}. \quad (4.14)$$

This region is positive invariant if all the trajectories starting in Ω remain in Ω . This indicates that the right hand side of the system are non negative on the boundaries or point inward Ω . $\square$

4.4 Disease free-equilibrium and Basic Reproduction Number

This section presents the disease free-equilibrium point and the basic reproduction number. Firstly, we determine the disease free-equilibrium point of the model under consideration. Secondly, we calculate the basic reproduction number at the disease free-equilibrium (DFE) point using the next-generation matrix.

4.4.1 Disease free equilibrium

The disease-free equilibrium refers to a steady state of an epidemiological model where no individuals are infected, that is, all infected compartments are zero, and the population remains susceptible (plus possibly other non-infected states). To determine the disease-free equilibrium (DFE) point of system (4.1), we set all differential equations equivalent to zero with $M_h = I_h = R_h = I_m = 0$.

From the second equation of system (4.1), we have

$$\begin{aligned}
\frac{dS_h}{dt} &= b_h - \frac{\beta \alpha_h S_h I_m}{N_h} - \omega R_h - d_h S_h + \gamma_1 M_h = 0. \\
\implies b_h - \frac{\beta \alpha_h S_h I_m}{N_h} - \omega R_h - d_h S_h + \gamma_1 M_h &= 0. \\
\implies S_h &= \frac{b_h}{d_h}.
\end{aligned} \tag{4.15}$$

Similarly, from the fifth equation of the model under consideration we get

$$\begin{aligned}
\frac{dS_m}{dt} &= b_m - \frac{\beta \alpha_m S_m I_h}{N_m} - d_m S_m = 0. \\
\implies b_m - \frac{\beta \alpha_m S_m I_h}{N_m} - d_m S_m &= 0. \\
\implies S_m &= \frac{b_m}{d_m}.
\end{aligned}$$

Therefore, DFE of model (4.1), denoted by $E^0 = (M_h^0, S_h^0, I_h^0, R_h^0, S_m^0, I_m^0)$ is given by

$$E^0 = \left\{ 0, \frac{b_h}{d_h}, 0, 0, \frac{b_m}{d_m}, 0 \right\}. \tag{4.16}$$

4.4.2 Basic reproduction number (R_0)

The basic reproduction number can be computed from the system of equations, usually by considering the next generation matrix method (or by linearizing the system near disease-free equilibrium). This is essentially the average number of secondary infections produced by a single infected individual in a completely susceptible population.

Let F represent the new infection terms and V represent the transition terms (transfer in or out of the infected compartments). Define the state vector as:

$$X = \begin{pmatrix} I_h \\ I_m \end{pmatrix}$$

where I_h is the number of infected humans and I_m is the number of infected mosquitoes.

The Jacobian of the new infection terms (F) with respect to infected compartments is:

$$F = \begin{pmatrix} \frac{\partial}{\partial I_h} \left(\frac{\beta \alpha_h S_h I_m}{N_h} \right) & \frac{\partial}{\partial I_m} \left(\frac{\beta \alpha_h S_h I_m}{N_h} \right) \\ \frac{\partial}{\partial I_h} (\beta \alpha_m S_m I_h) & \frac{\partial}{\partial I_m} (\beta \alpha_m S_m I_h) \end{pmatrix} \tag{4.17}$$

At the DFE point, we obtain:

$$S_h = \frac{b_h}{d_h}, \quad S_m = \frac{b_m}{d_m}, \quad N_h = \frac{b_h}{d_h} \quad (\text{since } I_h = R_h = 0).$$

Simplification of equation (4.18), we obtain

$$F = \begin{pmatrix} 0 & \beta \alpha_h \\ \frac{\beta \alpha_m b_m}{d_m} & 0 \end{pmatrix} \quad (4.18)$$

For V , we take partial derivatives of the transition terms (removal from infected compartments):

$$v = \begin{cases} v_1 = (d_h + \delta)I_h - \gamma_2 M_h, \\ v_2 = d_m I_m. \end{cases} \quad (4.19)$$

Hence, evaluation of equation () at DFE point yields

$$V = \begin{pmatrix} d_h + \delta & 0 \\ 0 & d_m \end{pmatrix}.$$

The inverse of V is:

$$V^{-1} = \begin{pmatrix} \frac{1}{d_h + \delta} & 0 \\ 0 & \frac{1}{d_m} \end{pmatrix}.$$

The next generation matrix is:

$$K = FV^{-1} = \begin{pmatrix} 0 & \frac{\beta \alpha_h}{d_m} \\ \frac{\beta \alpha_m b_m}{d_h + \delta} & 0 \end{pmatrix}.$$

At DFE, $S_m = \frac{b_m}{d_m}$, hence:

$$K = \begin{pmatrix} 0 & \alpha \\ b & 0 \end{pmatrix},$$

where $\alpha = \frac{\beta \alpha_h}{d_m}$, $b = \frac{\beta \alpha_m b_m / d_m}{d_h + \delta}$

The eigenvalues of K are:

$$\lambda = \pm \sqrt{\alpha b}.$$

Therefore, the basic reproduction number is:

$$R_0 = \sqrt{\alpha b} = \sqrt{\frac{\beta^2 \alpha_h \alpha_m b_m}{d_m^2 (d_h + \delta)}}. \quad (4.20)$$

From equation (4.18), we noticed that if $R_0 < 1$, the dengue fever dies out of the society, but if $R_0 > 1$ the disease invade the population.

4.4.3 Local stability of DFE

in this subsection, we present the local asymptotically stability of DFE point of our model would be considered. We utilize the desired Jacobian matrix evaluated at DFE point of our model along with the Routh-Hurwitz criteria.

Theorem 4.4.1. *If $R_0 < 1$, then E^0 is locally asymptotically stable in Ω , and unstable otherwise.*

Proof. We first compute the Jacobian matrix of model (4.1) as follows:

$$J = \begin{pmatrix} -(\gamma_1 + \gamma_2) & 0 & 0 & 0 & 0 & 0 \\ \gamma_1 & \frac{-\beta\alpha_h I_H}{N_h} - d_h & 0 & \omega & -\beta(\alpha_m I_h) & -\beta\alpha_h \\ \gamma_2 & \frac{\beta\alpha_h I_H}{N_h} & \frac{\beta\alpha_h I_H}{N_h} - p & 0 & 0 & \beta\alpha_h \\ 0 & 0 & \delta & -(d_h + \omega) & 0 & 0 \\ 0 & 0 & -\beta\alpha_m S_m & 0 & -\beta\alpha_m I_h - d_m & 0 \\ 0 & 0 & -\beta\alpha_m I_h & 0 & \beta\alpha_m S_m & -d_m \end{pmatrix}, \quad (4.21)$$

where $p = (d_h + \delta)$.

We now calculate the desired Jacobian matrix J at E^0 , which is expressed as:

$$J(E^0) = \begin{pmatrix} -(\gamma_1 + \gamma_2) & 0 & 0 & 0 & 0 & 0 \\ \gamma_1 & -d_h & 0 & \omega & 0 & -\beta\alpha_h \\ \gamma_2 & 0 & -p & 0 & 0 & \beta\alpha_h \\ 0 & 0 & \delta & -(d_h + \omega) & 0 & 0 \\ 0 & 0 & -\frac{\beta\alpha_m b_m}{d_m} & 0 & -d_m & 0 \\ 0 & 0 & \frac{\beta\alpha_m b_m}{d_m} & 0 & 0 & -d_m \end{pmatrix}, \quad (4.22)$$

From equation (4.22), we can compute the eigenvalues of the characteristic equation from the Jacobian $J(E^0)$ (i.e., $\det(J(E^0) - \lambda I) = 0$) which becomes

$$\lambda_1 = -(\gamma_1 + \gamma_2), \quad \lambda_2 = -d_h, \quad \lambda_3 = -d_m.$$

while the sign of the remaining eigenvalues can be obtained from the following reduced matrix:

$$\begin{pmatrix} -p & 0 & \beta\alpha_h \\ \delta & -(d_h + \omega) & 0 \\ \frac{\beta\alpha_m b_m}{d_m} & 0 & -d_m \end{pmatrix} \quad (4.23)$$

From equation (4.23), we obtain the third order polynomial in λ of the form:

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \quad (4.24)$$

where

$$a_1 = 2d_h + \delta + \omega + d_m.$$

$$a_2 = d_h^2\delta + \delta d_h + \omega d_h + \delta\omega + 2d_h d_m + \delta d_m + \omega d_m + \frac{\beta^2 \alpha_h \alpha_m b_m}{d_m}.$$

$$a_3 = d_h^2 d_m + \delta d_h d_m + \omega d_h d_m + \delta \omega d_m + \frac{\beta^2 \alpha_h \alpha_m b_m}{d_m} (d_h + \omega).$$

From equation (4.37), we see that the sign of coefficient a_i ($i = 1, 2, 3$) are positive, thererore, all eigenvalues have negative real parts by using the Routh-Hurwitz criteria. Consequently, E_0 is locally asymptotically stable in the biologically feasible region. Applying the Routh-Hurwitz criteria, and after some simplifications, it can be shown that the eigen values of reduced matrix have negative real parts if $R_0 < 1$. If $R_0 > 1$, then $a_3 > 0$ and the matrix J1 has at least one eigen value with positive real part. Therefore the disease free equilibrium R_0 of the system is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. Since all eigenvalues have negative real parts, the DFE is locally asymptotically stable when $R_0 < 1$. $\square$

4.4.4 Global Stability of DFE

We provide a global stability analysis for the model. We determine the value of the basic reproduction number R_0 and prove that the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$ condition under which a positive endemic equilibrium exists. Global stability involves By constructing the lyapunov function and By using LaSalle invariant principle.

Theorem 4.4.2. *The disease free equilibrium of the system is globally asymptotically stable in $R_0 < 1$ and otherwise unstable.*

Proof. Global stability of DFE using Lyapunov function Let define lyapunov function involving the infected compartment. Total Infectious Population Let the total number of infected individuals be defined as:

$$L = I_h + I_m. \quad (4.25)$$

where I_h : number of infected humans I_m : number of infected mosquitoes

Differentiating L with respect to time

$$\frac{dL}{dt} = \frac{dI_h}{dt} + \frac{dI_m}{dt}. \quad (4.26)$$

Substituting Dynamics (assuming $M_h = 0$). From the model equations:

$$\frac{dI_h}{dt} = \frac{\beta \alpha_h S_h I_h}{N_h} - (d_h + \delta) I_h.$$

$$\frac{dI_m}{dt} = \beta \alpha S_m I_h - d_m I_m.$$

$$\frac{dL}{dt} = \frac{dI_h}{dt} + \frac{dI_m}{dt}.$$

$$\frac{dL}{dt} = \left(\frac{\beta \alpha_h S_h I_h}{N_h} - (d_h + \delta) I_h + \gamma_2 M_h \right) + (\beta \alpha S_m I_h - d_m I_m).$$

Group them

$$\frac{dL}{dt} = \frac{\beta \alpha_h S_h I_h}{N_h} + \beta \alpha S_m I_h - d_m I_m - (d_h + \delta) I_h.$$

Near to equilibrium.

$$S_h \leq \frac{b_m}{d_h} \quad \text{and} \quad S_m \leq \frac{b_m}{d_m}.$$

Let bound

$$\frac{b_m}{d_h} \leq 1.$$

$$\begin{aligned} \frac{dL}{dt} &\leq \frac{\beta \alpha_h S_h I_h}{N_h} + \beta \alpha S_m I_h - d_m I_m - (d_h + \delta) I_h. \\ &= - \left[(d_h + \delta) + \beta \alpha \frac{b_m}{d_m} \right] I_h - [d_m - \beta \alpha_h] I_m \end{aligned}$$

Thus, if $R_0 < 1$, the above expression is negative definite. Hence,

$$\frac{dL}{dt} < 0 \quad \text{for all } I_h > 0, I_m > 0.$$

By applying LaSalle Invariant Principle, if

$$\frac{dL}{dt} \leq 0,$$

then

$$\frac{dL}{dt} = 0 \quad \text{only when } I_h = I_m = 0.$$

Since the system evolves in a positively invariant region, the solutions approach the largest invariant set where

$$I_h = I_m = 0.$$

which corresponds to the disease-free equilibrium (DFE).

Therefore, if $R_0 < 1$. the disease-free equilibrium is globally asymptotically stable. $\square$

4.5 Existence of Endemic Equilibrium

In epidemiological modeling, the endemic equilibrium (EE) refers to a steady-state condition where the disease persists within the population at a constant, non-zero level. This implies that infections remain present indefinitely rather than dying out.

The endemic equilibrium occurs when both the human and mosquito populations have a positive number of infected individuals, i.e.,

$$I_h > 0 \quad \text{and} \quad I_m > 0.$$

At this point, the system of differential equations simplifies because $M_h = 0$, meaning all terms involving M_h drop out.

From the fourth equation of system (4.1), we get:

$$R_h = \frac{\delta I_h}{d_h + \omega}.$$

From the sixth equation, we obtain:

$$I_m = \frac{\beta \alpha_h S_m I_h}{d_m}.$$

Also, we have

$$S_m = \frac{b_m}{\beta \alpha_h I_h + d_m}.$$

Similarly, from the third equation, and using $M_h = 0$, we have:

$$\frac{\beta \alpha_h S_h I_m}{N_h} - (d_h + \delta) I_h + \gamma_2 M_h = 0.$$

Substituting $M_h = 0$, the equation reduces to:

$$\frac{\beta \alpha_h S_h I_m}{N_h} - (d_h + \delta) I_h = 0. \tag{4.27}$$

Rearranging equation (4.27) gives:

$$S_h = \frac{(d_h + \delta) N_h}{\beta \alpha_h I_m}. \tag{4.28}$$

Now substitute equation (4.28) into the sixth equation of model (4.27)

$$I_m = \frac{\beta \alpha_h S_m I_h}{d_m}.$$

Thus,

$$S_h = \frac{(d_h + \delta)N_h d_m}{\beta^2 \alpha \alpha_h I_h S_m}.$$

Finally, from the second equation of system (4.1) with $M_h = 0$, and substituting $R_h = \frac{\delta I_h}{d_h + \omega}$, we have:

$$0 = b_h - \frac{\beta \alpha_h S_h I_h}{N_h} + \omega R_h - d_h S_h. \quad (4.29)$$

This set of expressions forms the basis for characterizing the endemic equilibrium state. To show existence of EE we must show that $F(I_h) = 0$ has positive root that is $I_h = 0$. We check if the sign of $(F(I_h))$ and I_h increases by intermediate value theorem if $F(0) > 0$, $F(I_h) > 0$ for large I_h there exist $I_h > 0$ such that $F(I_h) = 0$. The basic reproduction number is given by $R_0 = \frac{\beta^2 \alpha_h S_h S_m}{(d_h + \delta)(d_m N_h)}$. If $R_0 > 1$, a positive endemic equilibrium typically exists. Therefore, the existence of an endemic equilibrium is generally guaranteed whenever $R_0 > 1$.

4.5.1 Local stability of EE

The local Stability of our model can be determined as follows

$$N_h = S_h + I_h + R_h.$$

All parameters are assumed to be positive constants.

The endemic equilibrium satisfies:

$$I_h^* > 0, \quad I_m^* > 0.$$

Disease Dynamics to analyze the local stability at the endemic equilibrium, we focus on the infected compartments:

- I_h : infected humans
- I_m : infected mosquitoes

We reduce the full model to the following disease subsystem:

$$\begin{cases} \frac{dI_h}{dt} = \frac{\beta \alpha_h S_h^* I_m}{N_h^*} - (d_h + \delta)I_h, \\ \frac{dI_m}{dt} = \beta \alpha S_m^* I_h - d_m I_m. \end{cases}$$

This simplification is valid since at equilibrium, the values S_h^* and S_m^* are constants (i.e., they do not vary with time).

$$x_1 = I_h, \quad x_2 = I_m.$$

Then the system can be written in vector-matrix form as:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(d_h + \delta) & \frac{\beta \alpha_h S_h^*}{N_h^*} \\ \beta \alpha S_m^* & -d_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Call the Jacobian matrix:

$$J = \begin{bmatrix} -(d_h + \delta) & \frac{\beta \alpha_h S_h^*}{N_h^*} \\ \beta \alpha S_m^* & -d_m \end{bmatrix}$$

the Characteristic Equation to determine the local stability of the system, we compute the characteristic equation of the Jacobian J by solving:

$$\det(J - \lambda I) = 0.$$

$$\begin{vmatrix} -(d_h + \delta) - \lambda & \frac{\beta \alpha_h S_h^*}{N_h^*} \\ \beta \alpha S_m^* & -d_m - \lambda \end{vmatrix} = 0$$

Compute the determinant:

$$[(d_h + \delta + \lambda)(d_m + \lambda)] - \frac{\beta^2 \alpha_m \alpha_h S_h^* S_m^*}{N_h^*} = 0.$$

$$\Rightarrow \lambda^2 + (d_h + \delta + d_m)\lambda + (d_h + \delta)d_m - \frac{\beta^2 \alpha_m \alpha_h S_h^* S_m^*}{N_h^*} = 0.$$

This is the characteristic equation for the disease subsystem near the endemic equilibrium.

$$x_1 = I_h, \quad x_2 = I_m$$

Then the system can be written in vector-matrix form as:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(d_h + \delta) & \frac{\beta \alpha_h S_h^*}{N_h^*} \\ \beta \alpha S_m^* & -d_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Call the Jacobian matrix:

$$J = \begin{bmatrix} -(d_h + \delta) & \frac{\beta \alpha_h S_h^*}{N_h^*} \\ \beta \alpha_m S_m^* & -d_m \end{bmatrix}$$

Application of Routh-Hurwitz Criteria

Let us define the coefficients of the characteristic equation:

$$\lambda^2 + A\lambda + B = 0.$$

where:

$$A = d_h + \delta + d_m > 0.$$

$$B = (d_h + \delta)d_m - \frac{\beta^2 \alpha_m \alpha_h S_h^* S_m^*}{N_h^*}.$$

According to the Routh-Hurwitz stability criteria, the roots of the characteristic polynomial will have negative real parts (i.e., the system is locally asymptotically stable) if:

$$A > 0 \quad \text{and} \quad B > 0.$$

We already know that $A > 0$ since all parameters are positive.

Thus, for local stability, we require:

$$(d_h + \delta)d_m > \frac{\beta^2 \alpha_m \alpha_h S_h^* S_m^*}{N_h^*}.$$

This inequality ensures that both coefficients of the characteristic equation are positive, implying that all eigenvalues have negative real parts, and hence the endemic equilibrium is locally asymptotically stable.

Using the Endemic Expression for R_0

Previously, the basic reproduction number was given by:

$$R_0 = \sqrt{\frac{\beta^2 \alpha_m \alpha_h b_m}{d_m^2 (d_h + \delta)}}.$$

Squaring both sides:

$$R_0^2 = \frac{\beta^2 \alpha_m \alpha_h b_m}{d_m^2 (d_h + \delta)}.$$

At the endemic equilibrium, we assume $R_0 > 1$, which implies:

$$\frac{\beta^2 \alpha_m \alpha_h b_m}{d_m^2 (d_h + \delta)} > 1.$$

Multiplying both sides by $\frac{S_h^* S_m^*}{b_m N_h^*}$, we obtain:

$$\frac{\beta^2 \alpha_m \alpha_h S_h^* S_m^*}{N_h^*} < (d_h + \delta) d_m.$$

This inequality corresponds to:

$$B = (d_h + \delta) d_m - \frac{\beta^2 \alpha_m \alpha_h S_h^* S_m^*}{N_h^*} > 0.$$

Hence, the second Routh–Hurwitz condition $B > 0$ is satisfied.

Since both conditions

$$A = d_h + \delta + d_m > 0 \quad \text{and} \quad B > 0.$$

are satisfied, we conclude that the endemic equilibrium is **locally asymptotically stable**.

4.5.2 Global stability of EE

In this section, we examine the existence of endemic equilibrium (EE) of model (4.1).

Theorem 4.5.1. *If $R_0 > 1$, there exists a unique EE of system (4.1).*

Proof. We first find E^* by solving the algebraic system where derivative are zero. Lyapunov function construction A common method for vector host model is constructed by lyapunov function that measure the deviation from EE Define variable normalize by either EE value Let

$$x_1 = \frac{S_h}{S_h^*}, x_2 = \frac{I_h}{I_h^*}, x_3 = \frac{R_h}{R_h^*}, x_4 = \frac{S_m}{S_m^*}, x_5 = \frac{I_m}{I_m^*}.$$

Consider lyapunov candidate

$$V = \sum_{j=1}^5 (x_j - 1) - \ln x_j.$$

$$f(z) = z - 1 - \ln z \geq 0.$$

with equation only at $z = 1$. Calculate $\frac{dv}{dt}$ take the derivative along solution where the function

Here's your expression, nicely formatted in:

$$\frac{dV}{dt} = \sum_{i=1}^5 \left(1 - \frac{1}{x_i} \right) \frac{dx_i}{dt}.$$

Consider the Lyapunov function:

$$V = \sum_{i=1}^5 (x_{i-1} - \ln x_i).$$

Differentiating V with respect to time t we get:

$$\frac{dV}{dt} = \sum_{i=1}^5 \left(1 - \frac{1}{x_i}\right) \frac{dx_i}{dt}.$$

Each term $\frac{dx_i}{dt}$ can be expressed using the original system divided by its corresponding equilibrium point, through careful algebraic manipulation and using the equilibrium condition. By LaSalle's Invariance Principle, we analyze the behavior of V over time t : $V \geq 0$ with equality $V = 0$ only at the endemic equilibrium. $\frac{dV}{dt} \leq 0$, with equality only at the endemic equilibrium. The solution remains in the positively invariant region Ω . Hence, by LaSalle's Invariance Principle, all trajectories starting in Ω converge globally to the endemic equilibrium. $\square$

4.6 Sensitivity analysis

Sensitivity analysis determines how sensitive your model outcomes like basic reproduction number (R_0) equilibrium values are to changes in parameters. It's crucial in identifying which parameters most affect disease spread, guiding control strategies (e.g. personal prevention, insecticide, treatment). In epidemiological models, we commonly focus on the basic reproduction number but you can also analyze sensitivity for infected populations or other quantities of interest.

We utilize the following steps in Sensitivity Analysis. We apply these steps to show sensitivity: step1. Define basic reproduction number of your model model step2. Calculate normalized forward sensitivity indices step3. Interpret the results Step 1: Define Basic Reproduction Number (R_0)

Definition 4.6.1 (see ([Khatua and Kar, 2020](#); [Medeiros et al., 2011](#))). *For a variable R_0 , the normalized forward sensitivity index with respect to a parameter p is given by*

$$\Gamma_p^{R_0} = \left(\frac{\partial R_0}{\partial p} \right) \left(\frac{p}{R_0} \right) \quad (4.30)$$

From the model under discussion, and its basic reproduction number R_0 . we differentiate it with respect to its associated parameters. We have the basic reproduction number R_0 , and let β be a parameter. Then we can find the sensitivity index:

$$\Gamma_{\beta}^{R_0} = \left(\frac{\partial R_0}{\partial \beta} \right) \left(\frac{\beta}{R_0} \right).$$

where the expression for R_0 from equation (4.20) is given by:

$$R_0 = \sqrt{\frac{\beta^2 \alpha_h \alpha_m b_m}{d_m^2 (d_h + \delta)}}.$$

Now, compute the sensitivity index $\Gamma_{\beta}^{R_0}$ by differentiating the expression that yields:

$$\frac{\partial R_0}{\partial \beta} = \frac{1}{2} \sqrt{\frac{d_m^2 (d_h + \delta)}{\beta^2 \alpha_h \alpha_m b_m}} \cdot \frac{2\beta \alpha_h \alpha_m b_m}{d_m^2 (d_h + \delta)}.$$

Then plug the above equation into $\Gamma_{\beta}^{R_0}$ that yields:

$$\Gamma_{\beta}^{R_0} = \left(\frac{1}{2} \sqrt{\frac{d_m^2 (d_h + \delta)}{\beta^2 \alpha_h \alpha_m b_m}} \right) \times \frac{2\beta \alpha_h \alpha_m b_m}{d_m^2 (d_h + \delta)} \times \left(\beta \sqrt{\frac{d_m^2 (d_h + \delta)}{\beta^2 \alpha_h \alpha_m b_m}} \right).$$

Now simplifying equation (4.6), we get

$$\Gamma_{\beta}^{R_0} = \frac{\beta}{2} \times \frac{d_m^2 (d_h + \delta)}{\beta^2 \alpha_h \alpha_m b_m} \times \frac{2\beta \alpha_h \alpha_m b_m}{d_m^2 (d_h + \delta)}.$$

Simplifying it further, we obtain

$$\begin{aligned} \Gamma_{\beta}^{R_0} &= \frac{1}{2} \cdot \left(\frac{d_m^2 (d_h + \delta)}{\beta^2 \alpha_h \alpha_m b_m} \right) \cdot \frac{2\beta^2 \alpha_h \alpha_m b_m}{d_m^2 (d_h + \delta)} \\ &= \frac{1}{2} \cdot \frac{2\beta^2 \alpha_h \alpha_m b_m}{\beta^2 \alpha_h \alpha_m b_m} = 1 \end{aligned}$$

Again using formula in equation (4.31), we can find the sensitivity index for b_m as follows:

$$\begin{aligned} \Gamma_{b_m}^{R_0} &= \left(\frac{\partial R_0}{\partial b_m} \right) \left(\frac{b_m}{R_0} \right) \\ &= \left(\frac{1}{2} \sqrt{\frac{d_m^2 (d_h + \delta)}{\beta^2 \alpha_h \alpha_m b_m}} \right) \times \frac{\beta^2 \alpha_h \alpha_m}{d_m^2 (d_h + \delta)} \times \left(b_m \sqrt{\frac{d_m^2 (d_h + \delta)}{\beta^2 \alpha_h \alpha_m b_m}} \right) \\ &= \frac{1}{2} \left(\frac{d_m^2 (d_h + \delta)}{\beta^2 \alpha_h \alpha_m b_m} \right) \times \frac{\beta^2 \alpha_h \alpha_m b_m}{d_m^2 (d_h + \delta)} \\ &= \frac{1}{2} \cdot \frac{\beta^2 \alpha_h \alpha_m b_m}{\beta^2 \alpha_h \alpha_m b_m} = \frac{1}{2}. \end{aligned}$$

Thus, the sensitivity indices of the remaining parameters associated with the basic reproduction number is summarized in Table 4.1.

Table 4.1: Sensitivity indices of the basic reproduction number associated with basic model parameter.

Parameter	Value	Sensitivity index	Result
β	0.5	$\Gamma_{\beta}^{R_0}$	1
α_m	0.06	$\Gamma_{\alpha_m}^{R_0}$	0.64
α_h	0.05	$\Gamma_{\alpha_h}^{R_0}$	0.044
δ	0.18	$\Gamma_{\delta}^{R_0}$	-0.37
d_m	0.01	$\Gamma_{d_m}^{R_0}$	-0.5
b_m	0.95	$\Gamma_{b_m}^{R_0}$	0.5
d_h	0.02	$\Gamma_{d_h}^{R_0}$	-0.35

The result from Table 4.1 shows that small increase in the value of parameter β will highly increase the basic reproduction number, which is not advisable. Thus, it is required to decrease the value of parameter β since it highly contribute to the incidence of dengue virus disease. The result shows that the dominant parameter β most spread the disease rather than the other parameter.

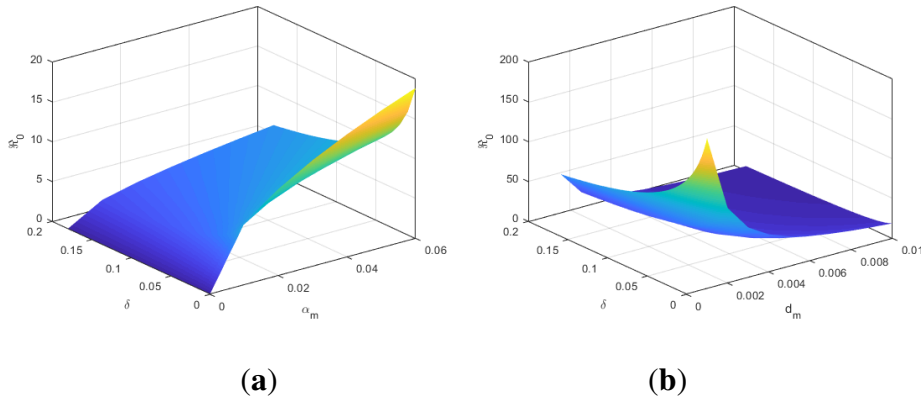


Figure 4.2: The effect of parameters δ and d_m on the basic reproduction number.

Figure 4.2 demonstrates the effect of parameters δ , d_m and α_m on the basic reproduction number. It is evident from the figure that small increase in the value of parameter α_m and a small decrease in the value of parameter δ will significantly increase the basic reproduction number (see Figure 4.2)..

4.7 Optimal Control Analysis

In this section, our objective is to minimize both the disease burden and the associated costs of personal prevention, treatment and insecticide chemical spray control measures.

To achieve this, we define the following objective functional:

$$\mathcal{J}(u) = \int_0^{t_f} (a_1 I_h(t) + a_2 I_m(t) + \frac{1}{2} b_1 u_1^2 + \frac{1}{2} b_2 u_2^2 + \frac{1}{2} b_3 u_3^2) dt. \quad (4.31)$$

subject to dynamical control system

$$\begin{cases} \frac{dM_h}{dt} = -(\gamma_1 + \gamma_2 + d_h)M_h, \\ \frac{dS_h}{dt} = b_h + \gamma_1 M_h - \frac{(1-u_1)\beta\alpha_h S_h I_m}{N_h} - d_h S_h + \omega R_h, \\ \frac{dI_h}{dt} = \frac{(1-u_1)\beta\alpha_h S_h I_m}{N_h} - (d_h + \delta + u_2)I_h + \gamma_2 M_h, \\ \frac{dR_h}{dt} = (\delta + u_2)I_h - (d_h + \omega)R_h, \\ \frac{dS_m}{dt} = b_m - (1-u_1)\beta\alpha S_m I_h - (d_m + u_3)S_m, \\ \frac{dI_m}{dt} = (1-u_1)\beta\alpha S_m I_h - (d_m + u_3)I_m. \end{cases} \quad (4.32)$$

with given initial conditions

$$M_h(0) \geq 0, S_h(0) > 0, I_h(0) \geq 0, R_h(0) \geq 0, S_m(0) > 0, I_m(0) \geq 0. \quad (4.33)$$

Here, a_1 and a_2 are positive constants representing the costs associated with infected human and infected mosquito, respectively. The parameters b_1, b_2 , and b_3 are weight constants reflecting the relative cost of implementing the control measures u_1, u_2 , and u_3 . The variable t_f denotes the final time of intervention.

The terms $a_1 I_h(t)$ and $a_2 I_m(t)$ quantify the economic impact of dengue infections. To model the cost of control, we adopt a quadratic form as an approximation to the nonlinear functional, assuming that cost grows nonlinearly with control intensity.

The goal is to determine an optimal control strategy $u^* = (u_1^*, u_2^*, u_3^*)$ such that the system trajectories satisfy model (4.5) for all $t \in [0, t_f]$ and minimize the objective functional (4.2). Mathematically, this can be expressed as:

$$\mathcal{J}(u^*(t)) = \min(\mathcal{J}(u) : u \in U). \quad (4.34)$$

where the admissible control set is defined as:

$$U = \{u = (u_1, u_2, u_3) : \text{each } u_i(t) \text{ is measurable and } 0 \leq u_i(t) \leq 1, i = 1, 2, 3, \forall t \in [0, t_f]\}. \quad (4.35)$$

is an admissible control set.

4.7.1 Existence of Optimal Control

This section presents the existence of optimal controls that helps to minimize the cost functional in the finite time. For this, the boundedness of state solutions to control-induced system (4.32) is necessary to determine the existence and uniqueness of the optimal control.

Theorem 4.7.1. *Given the objective functional*

$$\mathcal{J}(u) = \int_0^{t_f} \left(a_1 I_h(t) + a_2 I_m(t) + \frac{1}{2} b_1 u_1^2 + \frac{1}{2} b_2 u_2^2 + \frac{1}{2} b_3 u_3^2 \right) dt. \quad (4.36)$$

subjected to the governing systems (4.32) such that

$$U = \left\{ u = (u_1, u_2, u_3) : \text{are measurable and } 0 \leq u_i(t) \leq 1, i = 1, 2, 3, \forall t \in [0, t_f] \right\}. \quad (4.37)$$

Then there exists optimal control triple $u^ = (u_1^*, u_2^*, u_3^*)$ and compatible state solutions $(M_h^*, S_h^*, I_h^*, R_h^*, S_m^*, I_m^*)$ such that*

$$\mathcal{J}(u^*) = \min_{u \in U} \mathcal{J}(u). \quad (4.38)$$

provided the following conditions hold:

- (i) *The solution set of system (4.1) with the associated control functions in (4.37) is non-empty.*
- (ii) *The admissible control set U in (4.37) is convex and closed.*
- (iii) *The right-hand side of the control-induced system (4.32) is continuous, bounded by a combination of the state and bounded controls, and can be expressed as a linear function of u with coefficients depending on time and the state.*
- (iv) *The integrand in equation (4.37) is convex with respect to the control variables.*
- (v) *There exist positive constants c_1, c_2 and a constant $\chi > 1$ such that*

$$F(t, M_h, S_h, I_h, R_h, S_m, I_m, u) \geq b_2 |u|^\chi - b_1. \quad (4.39)$$

Proof. For the existence of the given optimal control u^* , we need to verify the above conditions proved by (Fleming and Rishel, 2012). The state variables in the proposed model are continuously differentiable. Besides, it is proved in Theorem 4.3.2 that the solutions of the

state system are continuous and bounded. Furthermore, the right-hand side function of the model equation (4.32) satisfies the Lipschitz condition with respect to the state variables. Therefore, using Picard-Lindelof's existence theorem (Coddington and Levinson, 1955), the solution set of the system (4.32) is non-empty. Thus condition (i) is achieved. To prove (ii), let us take any controls $u_1, u_2 \in \Omega_{ad}$. This implies $|u_1| \leq 1$ and $|u_2| \leq 1$. Then for any $\theta \in [0, 1]$,

$$|\theta u_1 + (1 - \theta)u_2| \leq \theta|u_1| + (1 - \theta)|u_2| \leq 1.$$

This implies that Ω_{ad} is convex and closed. Let $x = (S, E, Q, A, I, J, R)$. and $u = (u_1, u_2, u_3)$. The control system (1) can be rewritten as

$$F(t, x, u) = a(t, x) + b(t, x)u. \quad (4.40)$$

where the expressions for $a(t, x(t))$ and $b(t, x(t))u(t)$ are given as below.

$$a(t, x) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\beta \alpha_h S_h I_m}{N_h} & 0 & 0 \\ -\frac{\beta \alpha_h S_h I_m}{N_h} & -I_h & 0 \\ 0 & I_h & 0 \\ \beta \alpha_m S_m I_h & 0 & -S_m \\ -\beta \alpha_m S_m I_h & 0 & S_m \end{pmatrix} \quad \text{and} \quad b(t, x)u = \begin{pmatrix} -(\gamma_1 + \gamma_2)M_h \\ b_h - \left(\frac{\beta \alpha_h I_m}{N_h} + d_h\right)S_h + \omega R_h + \gamma_1 M_h \\ \frac{\beta \alpha_h S_m I_m}{N_h} + (d_h + \delta)I_h + \gamma_2 M_h \\ \delta I_h - (d_h + \omega)R_h \\ b_m - \beta \alpha_m S_m I_h - d_m S_m \\ \beta \alpha_m S_m I_h - d_m I_m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

The control system (1) can be rewritten as $F(t, x, u) = a(t, x) + b(t, x)u$, or

$$|\theta u_1 + (1 - \theta)u_2| \leq \theta|u_1| + (1 - \theta)|u_2| \leq 1,$$

this implies that the admissible control set $\mathcal{A}_d$ is convex and closed.

Let the state vector be:

$$x = (S, E, Q, A, I, J, R),$$

and the control vector:

$$u = (u_1, u_2, u_3).$$

Then, the control system 4.32 can be rewritten in the compact form:

$$F(t, x, u) = a(t, x) + b(t, x)u. \quad (4.41)$$

Equation (4.41) implies the linear dependence of the right-hand side functions of the control system (4.32) on the controls u_1, u_2 , and u_3 . Further, by definition, each right-hand side of the control induced system (4.32) is continuous, and bounded by the sum of bounded controls and the state. Thus condition (iii) also holds.

Being a quadratic function of the control variables $u = (u_1, u_2, u_3)$, the integrand function in equation (4.31) is convex on Ω_{ad} . Thus condition (iv) is proved. Furthermore, for proving the condition in (v), it can be seen that, $b_3 u_3^2 \leq b_3$ as $b_3 \in [0, 1]$, that is,

$$\frac{b_3 u_3^2}{2} \leq \frac{b_3}{2}.$$

Thus

$$\begin{aligned} F(t, x, u) &> \frac{b_1}{2} u_1^2 + \frac{b_2}{2} u_2^2 + \frac{b_3}{2} u_3^2. \\ &\geq \frac{b_1}{2} u_1^2 + \frac{b_2}{2} u_2^2 + \frac{b_3}{2} u_3^2 - \frac{b_3}{2}. \\ &\geq \min\left(\frac{b_1}{2}, \frac{b_2}{2}, \frac{b_3}{2}\right) (u_1^2 + u_2^2 + u_3^2) - \frac{b_3}{2}. \\ &= \min\left(\frac{b_1}{2}, \frac{b_2}{2}, \frac{b_3}{2}\right) |u_1, u_2, u_3|^2 - \frac{b_3}{2}. \end{aligned}$$

Choosing $b_2 = \min\left(\frac{b_1}{2}, \frac{b_2}{2}, \frac{b_3}{2}\right)$ and $b_1 = \frac{b_3}{2}$, we get

$$F(t, x, u) \geq b_2 |u_1, u_2, u_3|^\chi - b_1$$

with $\chi > 1$. This completes the proof. $\square$

4.7.2 Characterization of optimality

To tackle the optimization problems (4.31), (4.32), and (4.35), we apply Pontryagin's Minimum Principle which transforms the original problem into minimizing the Hamiltonian $\mathcal{H}$, defined as:

$$\mathcal{H} = a_1 I_h(t) + a_2 I_m(t) + \frac{1}{2} b_1 u_1^2 + \frac{1}{2} b_2 u_2^2 + \frac{1}{2} b_3 u_3^2 + \sum_{i=1}^6 \lambda_i(t) g_i(t, x, u). \quad (4.42)$$

where $g_i = \frac{dx}{dt}$ denotes the right-hand side functions of the state system (4.34). The term $\lambda : [0, t_f] \rightarrow \mathbb{R}^6$, given by $\xi = (\xi_1(t), \xi_2(t), \dots, \xi_7(t))$ represents a nontrivial, absolutely continuous vector function known as the adjoint vector.

Theorem 4.7.2. Let $u^* = (u_1^*, u_2^*, u_3^*)$ be optimal solution to (4.32), (4.33) – (4.35) and $x^* = (M_h^*, S_h^*, I_h^*, R_h^*, S_m^*, I_m^*)$ be the associated optimal state. Then there exist adjoint functions $\lambda_1^*, \dots, \lambda_6^*$ that satisfy the adjoint system

$$\begin{cases} \frac{d\lambda_1}{dt} = (\lambda_1 - \lambda_2)\gamma_1 + (\lambda_1 - \lambda_3)\gamma_2 + (\lambda_3 - \lambda_2) \left(\frac{(1-u_1)\beta\alpha_h S_h I_m}{N_h^2} \right), \\ \frac{d\lambda_2}{dt} = \frac{(\lambda_3 - \lambda_2)(1-u_1)\beta\alpha_h I_m}{N_h} + \lambda_2 d_h, \\ \frac{d\lambda_3}{dt} = -a_1 + (\lambda_3 - \lambda_2) \frac{(1-u_1)\beta\alpha_h S_h I_m}{N_h^2} + (\lambda_6 - \lambda_5)(1-u_1)\beta\alpha_m S_m + \lambda_4(\delta + u_2), \\ \frac{d\lambda_4}{dt} = (\lambda_3 - \lambda_2) \frac{(1-u_1)\beta\alpha_h S_h I_m}{N_h^2} + (\lambda_4 - \lambda_2)\omega + \lambda_4 d_h, \\ \frac{d\lambda_5}{dt} = (\lambda_3 - \lambda_2) \frac{(1-u_1)\beta\alpha_h S_h I_m}{N_h^2} + (\lambda_5 - \lambda_6)(1-u_1)\beta\alpha_m I_h + \lambda_5(d_m - U_3), \\ \frac{d\lambda_6}{dt} = a_2 + \frac{(\lambda_2 - \lambda_3)\beta\alpha_h S_h}{N_h} + (\lambda_3 - \lambda_2) \frac{1-u_1\beta\alpha_h S_h}{N_h}. \end{cases} \quad (4.43)$$

subject to transversality condition

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = \lambda_5(t_f) = \lambda_6(t_f) = 0.$$

Furthermore, the optimal control $u^* = (u_1^*, u_2^*, u_3^*)$ is given by

$$\begin{aligned} u_1^* &= \min \left\{ \max \left\{ 0, \lambda_3 - \lambda_2 \frac{\beta\alpha_h S_h I_m}{b_1 N_h} + (\lambda_6 - \lambda_5) \frac{\beta\alpha_m S_m I_h}{b_1} \right\}, 1 \right\}. \\ u_2^* &= \min \left\{ \max \left\{ 0, \frac{\lambda_3(d_h I_h + \delta I_h) - \lambda I_h}{b_2} \right\}, 1 \right\}. \\ u_3^* &= \min \left\{ \max \left\{ 0, \frac{\lambda_5 S_m + \lambda_6 I_m}{b_3} \right\}, 1 \right\}. \end{aligned} \quad (4.44)$$

Proof. The result can be easily established by applying PMP such that

$$\frac{d\lambda_1}{dt} = -\frac{\partial \mathcal{H}}{\partial M_h}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial \mathcal{H}}{\partial S_h}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial \mathcal{H}}{\partial I_h}, \quad \frac{d\lambda_4}{dt} = -\frac{\partial \mathcal{H}}{\partial R_h}, \quad \frac{d\lambda_5}{dt} = -\frac{\partial \mathcal{H}}{\partial S_m}, \quad \frac{d\lambda_6}{dt} = -\frac{\partial \mathcal{H}}{\partial I_m}.$$

Which means that

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial \mathcal{H}}{\partial M_h}. \\ &= - \left[\lambda_1(-(\gamma_1 + \gamma_2)) + \lambda_2 \left(\gamma_1 + \frac{(1-u_1)\beta\alpha_h S_h I_m}{N_h^2} \right) + \lambda_2 \left(\gamma_2 - \frac{(1-u_1)\beta\alpha_h S_h I_m}{N_h^2} \right) \right]. \\ &= (\lambda_1 - \lambda_2)\gamma_1 + (\lambda_1 - \lambda_3)\gamma_2 + (\lambda_3 - \lambda_2) \left(\frac{(1-u_1)\beta\alpha_h S_h I_m}{N_h^2} \right). \end{aligned}$$

$$\begin{aligned} \frac{d\lambda_2}{dt} &= \frac{\partial \mathcal{H}}{\partial S_h}. \\ &= - \left[- \left(\frac{\lambda_2(1-U_1)\beta\alpha_h I_m}{N_h} - d_h \right) + \frac{\lambda_3(1-U_1)\beta\alpha_h I_m}{N_h} \right]. \\ &= (\lambda_2 - \lambda_3) \frac{(1-u_1)\beta\alpha_h I_m}{N_h} + \lambda_2 d_h. \end{aligned}$$

We can obtain the other in the same manner

with $\lambda_i(t_f) = 0$, for $i = 1, 2, 3, \dots, 6$. Moreover, the convexity of the objective functional in (3) w.r.t u guarantees the existence of the optimal controls. Thus, from the optimality conditions which is given by $\frac{\partial \mathcal{H}}{\partial u_i} = 0$, $i = 1, 2, 3$, we can explicitly obtain the control u^* in terms of state and adjoint variables. Mathematically,

$$\frac{\partial \mathcal{H}}{\partial u_i} = 0, i = 1, 2, 3$$

Which implies to

$$\begin{aligned} u_1^* &= \lambda_3 - \lambda_2 \frac{\beta \alpha_h S_h I_m}{b_1 N_h} + (\lambda_6 - \lambda_5) \frac{\beta \alpha_m S_m I_h}{b_1}. \\ u_2^* &= \frac{\lambda_3 (d_h I_h + \delta I_h) - \lambda_4 I_h}{b_2}. \\ u_3^* &= \frac{\lambda_5 S_m + \lambda_6 I_m}{b_3}. \end{aligned}$$

From the boundedness of $u_i^*(t)$ on $(0,1)$ and the minimality condition, yields

$$u_1^*(t) = \begin{cases} 0, & \text{if } \frac{\partial \mathcal{H}}{\partial u_1} > 0, \\ \lambda_3 - \lambda_2 \frac{\beta \alpha_h S_h I_m}{b_1 N_h} + (\lambda_6 - \lambda_5) \frac{\beta \alpha_m S_m I_h}{b_1}, & \text{if } \frac{\partial \mathcal{H}}{\partial u_1} = 0, \\ 1, & \text{if } \frac{\partial \mathcal{H}}{\partial u_1} < 0, \end{cases} \quad u_2^* = \begin{cases} 0, & \text{if } \frac{\partial \mathcal{H}}{\partial u_2} > 0, \\ \frac{\lambda_3 (d_h I_h + \delta I_h) - \lambda_4 I_h}{b_2}, & \text{if } \frac{\partial \mathcal{H}}{\partial u_2} = 0, \\ 1, & \text{if } \frac{\partial \mathcal{H}}{\partial u_2} < 0, \end{cases} \quad u_3^* =$$

Thus the optimal control functions $u_i^*(t)$ can be expressed in compact form as given in (11). ■.

To verify that the optimal control is indeed a minimizer, one can compute the Hessian matrix of the Hamiltonian $\mathcal{H}$ function and check its positive definiteness. That is, the Hessian matrix with respect to the control u becomes

$$\frac{\partial^2 \mathcal{H}}{\partial u^2} = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}.$$

Clearly, this matrix is a positive definite due to b_1, b_2 and b_3 are positive constants. Hence, $\mathcal{H}$ is convex and the optimal control $u = (u_1, u_2, u_3)$ is a minimizer.

The system (4.35) is an initial value problems while the adjoint system (10) subject to the transversality conditions $\lambda_i(t_f) = 0, i = 1, \dots, 6$ is a terminal value problem. Therefore,

boundary value problems given by Theorem 4.2 need to be solved numerically using forward iterations in the state system, and adjoint system equations (10) require backward integration. □

4.8 Numerical simulation and discussion

This section is devoted to addressing the numerical simulations and discussion of the results. We firstly perform the numerical simulations. Numerical simulations of model (4.2) were conducted to corroborate some analytical findings. We make use of the parameters that are listed in Table 4.2. We assume the initial values for each subpopulation as follows:

$$M_h(0) = 50, S_h(0) = 100, I_h(0) = 90, R_h(0) = 50, S_m(0) = 100, I_m(0) = 90.$$

Moreover, we utilized the MATLAB software is used for all simulations. To obtain the optimal solution to the dynamic system with control in equation (4.3), we apply the forward-backward sweep method. This iterative method consists in solving the system (4.6) with the guess for the controls over the time interval $[0, t_f]$ using a forward fourth-order Runge Kutta scheme and the transversality conditions. Then, the adjoint system is solved by applying a backward fourth-order Runge Kutta scheme to the current iteratin solution of the system (4.6). The controls are updated by means of a convex combination of the previous controls and the values in the characterization process (4.6). To repeat the solutions of state variables in the system (4.6), and adjoint systems, the updated controls are used. This situation is continued until the values of unknowns at the previous iteration and at the present iteration are close enough.

In Figure 4.3, we presented the graphical representation of the dynamical behavior of the human mobility, susceptible human, infected human, recovered human, susceptible mosquito and infected mosquito. The trajectories indicate that the solution of the model converge to the endemic

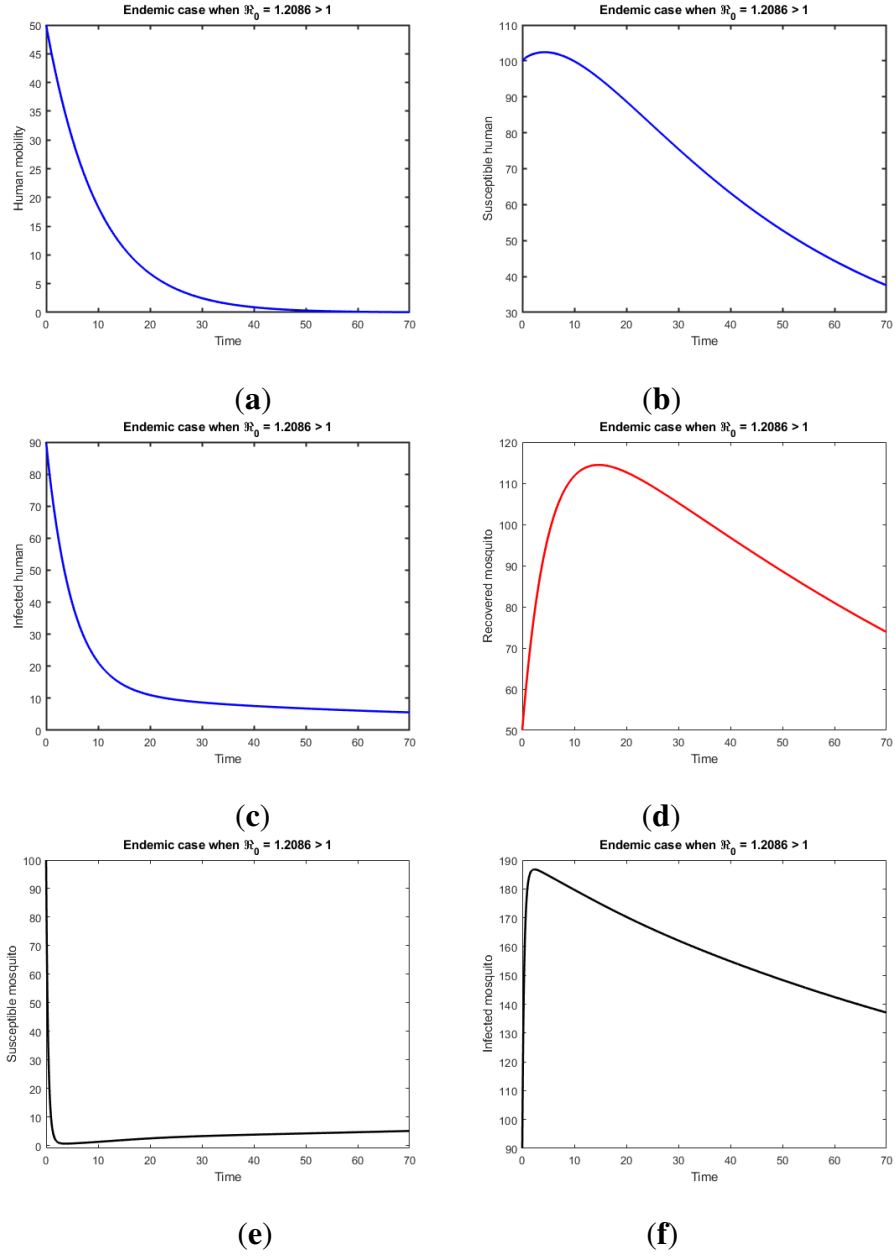


Figure 4.3: Dynamic behavior of (a) human mobility, (b) susceptible human, (c) infected human, (d) recovered human, (e) susceptible mosquito and (f) infected mosquito for the endemic case for when $R_0 = 1.2086 > 1$ using parameter values in Table 4.2.

Now, we study the impact of intervention strategies considering the following scenarios for the optimal control problem.

- **Scenario S-I:** Use of personal preventive measure only ($U_1 \neq 0$)
- **Scenario S-II:** Use of insecticide control measure only ($u_3 \neq 0$)
- **Scenario S-III:** Use of personal preventive and treatment control measures only

- **Scenario S-IV:** Use of personal preventive and insecticide control measures
- **Scenario S-V:** Use of personal preventive, treatment and insecticide spray only

4.8.1 Use of personal preventive control measure only

In this subsection, we study the impact of personal preventive control measure only on the transmission dynamics of dengue fever in the absence of treatment and insecticide control measures. It is seen from Figure 4.4 that the number of infected human and infected mosquito decrease significantly in less than 30 weeks.

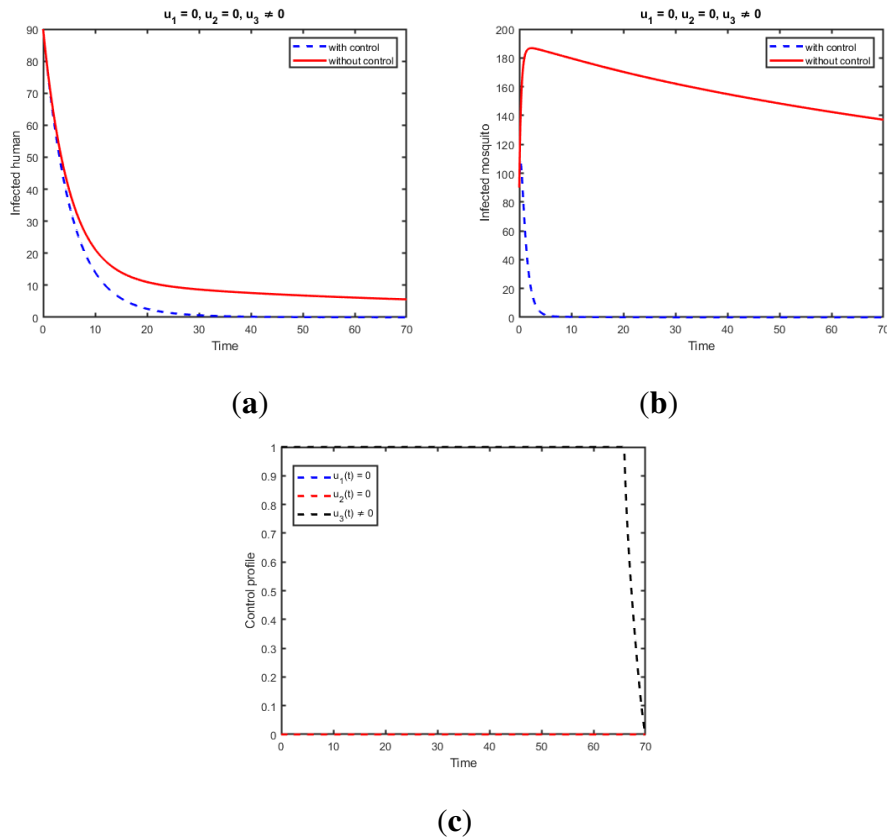


Figure 4.4: The impact of personal prevention control measure on the dynamic behavior of (a) infected human, (b) infected mosquito and its control profile.

4.8.2 Use of insecticide control measure only

This subsection presents the effect of insecticide spray control measure in fighting the spread of dengue virus infection in the tropical or subtropical regions. It is evident that Figure 4.5 shows significant decrease in the number of infected human and infected mosquito after the application of this control measure.

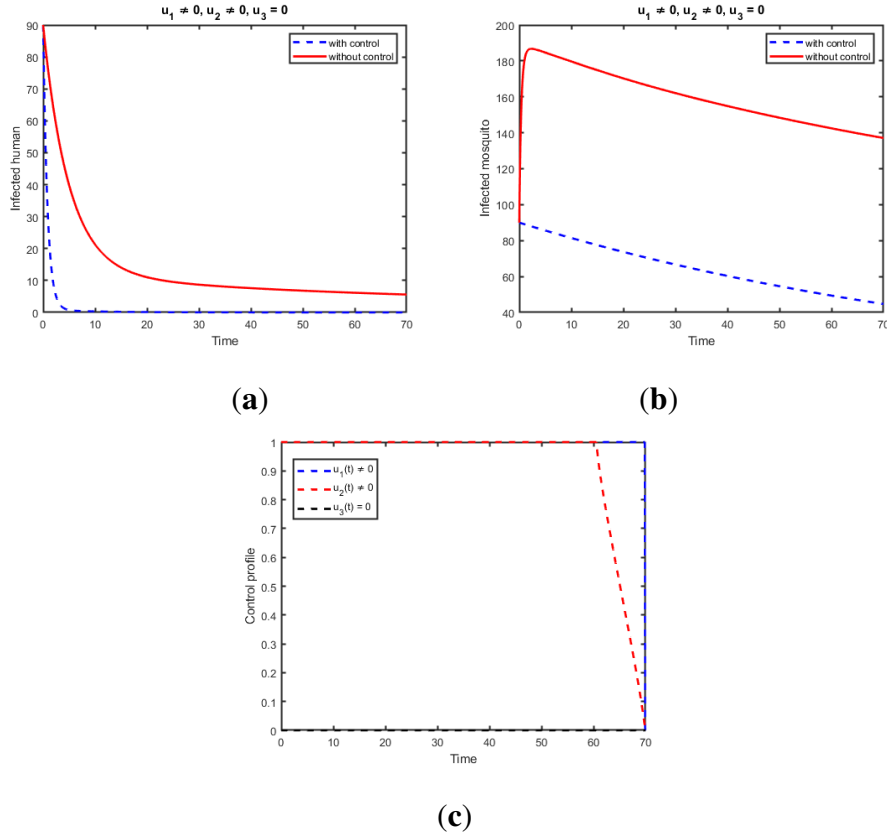


Figure 4.5: The impact of insecticide chemical spray control measure on the dynamic behavior of (a) infected human and (b) infected mosquito with its (c) control profile.

4.8.3 Use of personal preventive and treatment control measures only

In this subsection, we studied the effect of using the combination of personal preventive and treatment control measures in combating the spread of dengue fever in the society living under the outbreak of dengue fever. From Figure 4.5, we see that the number of infected human and infected mosquito considerable decrease under the application of this combination.

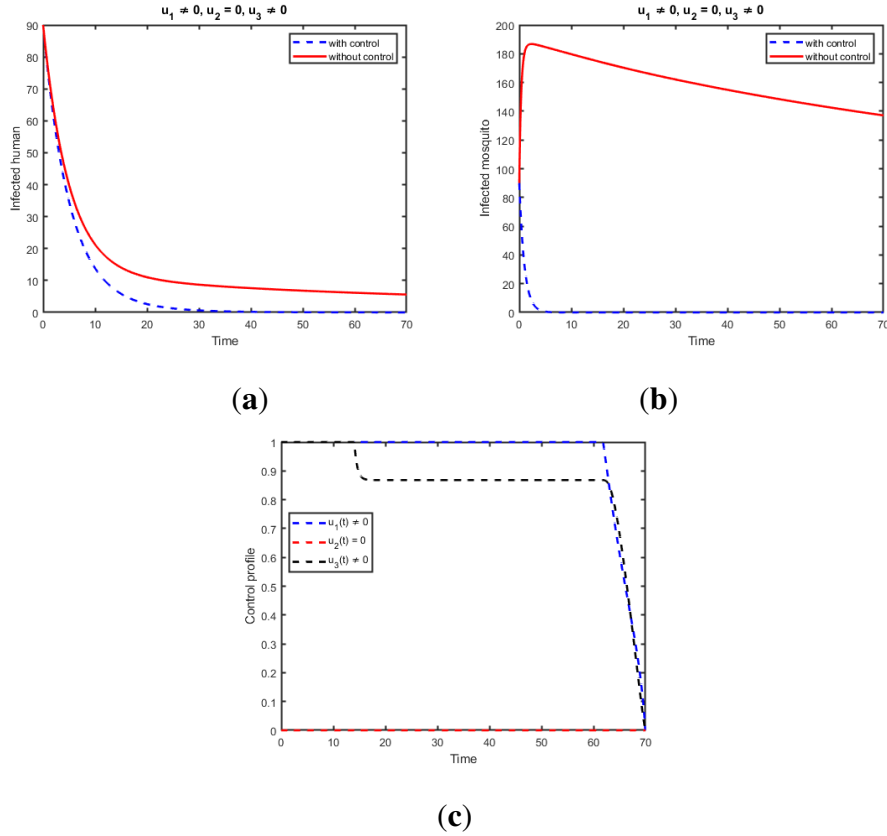


Figure 4.6: The Impact of personal preventive and treatment control measures on the dynamic behavior of (a) infected humans, (b) infected mosquitoes, and (c) the corresponding control profile.

This study analyzes the impact of combining personal preventive and treatment control measures on the dynamic behavior of infected humans and infected mosquitoes, as well as on the corresponding control profile.

4.8.4 Use of personal preventive, treatment and insecticide spray only

In this subsection, we present the impact of using personal preventive, treatment and insecticide spray control measures to reduce the transmission of dengue fever in the society. This study highlights the combined effect of applying three control measures to eliminate dengue virus infection. Figure 4.7 demonstrates the simulation results of the effect of applying three control measures all simultaneous to reduce the spread of dengue fever. It is clearly seen that the number of infected humans and infected mosquitoes fall tremendously in less than the expected time.

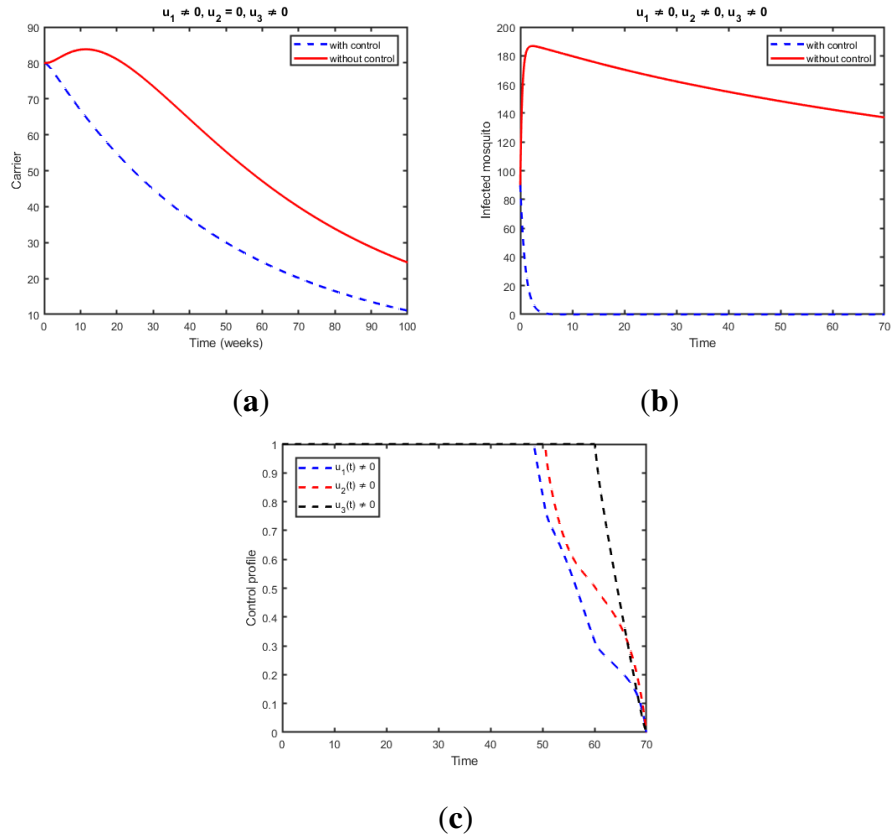


Figure 4.7: The Impact of personal preventive, treatment and insecticide control measures on the dynamic behavior of (a) infected humans, (b) infected mosquitoes, and (c) the corresponding control profile.

4.9 Local and Global Stability of Numerical Simulation

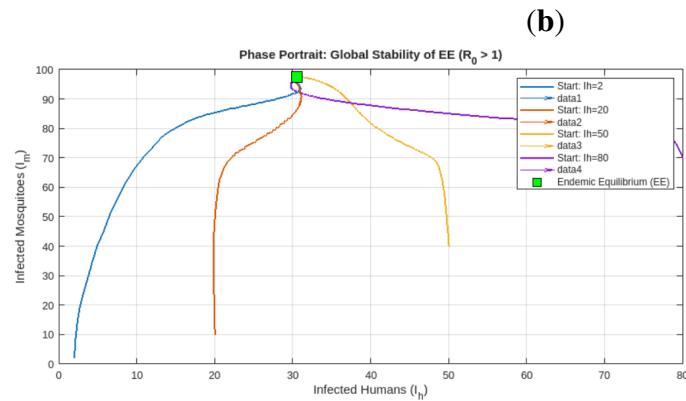
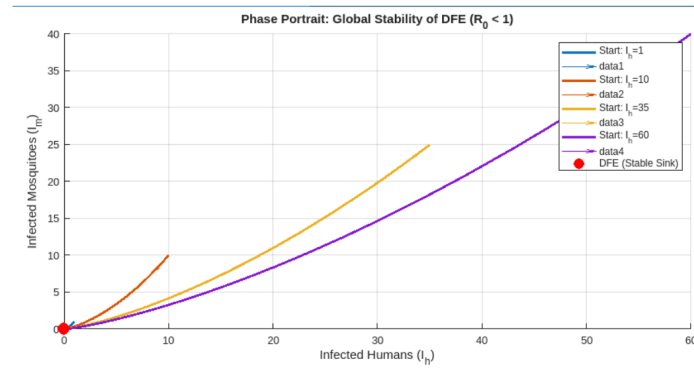
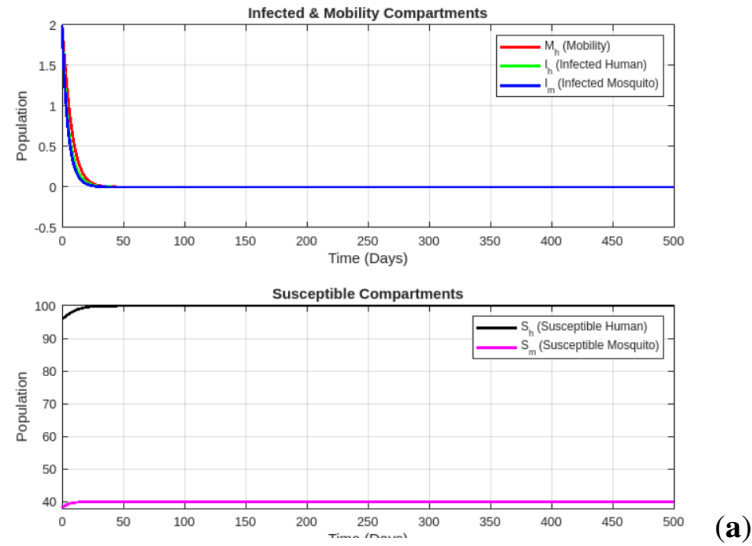


Figure 4.8: The local stability of the disease at DFE (a) infected humans, (b)The global stability at DFE infected mosquitoes, and (c) The Global stability of the infected host vrs infected mosquito.

4.10 Cost-Effectiveness Analysis

It is important to identify the most cost-effective strategies for prevention, environmental hygiene, and insecticide control, as well as their combined control policies, to effectively reduce the spread of dengue at the lowest possible cost. This can be done by comparing the differences in the costs and outcomes of each intervention using the Incremental Cost-Effectiveness Ratio (ICER). The ICER measures the additional cost required for each extra unit of benefit achieved by a new intervention compared with a less effective one.

When comparing two or more intervention strategies, each is evaluated against the next less effective option. The numerator of the ICER represents the total difference in intervention costs, including costs related to averted dengue cases and productivity losses (if applicable), between the two compared strategies. The **denominator** represents the difference in the total number of active dengue cases averted. Thus, the ICER is given by:

$$\text{ICER} = \frac{\text{Difference in total cost between control strategies}}{\text{Difference in total number of active cases averted by control strategies}}$$

We also calculated the Average Cost-Effectiveness Ratio (ACER), which evaluates the cost-effectiveness of each individual intervention. The ACER is the ratio between the total cost incurred and the total number of active dengue cases averted by that intervention:

$$\text{ACER} = \frac{\text{Total cost}}{\text{Total active cases averted}}$$

The total costs for each control strategy: personal preventive, treatment, insecticide and their possible combinations - are obtained from the objective functional (4.3). The number of cases averted is found by comparing the number of infectious individuals with and without the control strategy. Let S-I, S-II, S-III, S-IV, and S-V, respectively, represent prevention control measure onl $u_1(t)$, insecticide chemical spray only control $u_3(t)$, combination of personal preventive and treatment control measures, combination of personal preventive and insecticide control measures and combination of personal preventive, treatment and insecticide chemical spray control measures.

From the numerical simulations, we obtained the total infections averted and the total cost associated with implementing the intervention strategies, which we summarize in the Table 4.2.

Table 4.2: Infection averted and the total cost

Strategy	Infection averted	total cost	ICER
S-I	6891.4	1792	
S-II	6786.8	2367	
S-III	12672.3	1648	
S-IV	6891.5	2068	
S-V	12672.5	6273	

Now, we arrange the results in Table 4.2 in ascending order of infectious averted, which is summarized in Table 4.3.

Table 4.3: Infection averted and the total cost

Strategy	Infection averted	total cost	ICER
S-II	6786.8	2367	0.3488
S-I	6890.4	1792	-5.5508
S-IV	6891.5	2068	
S-III	12671.3	1648	
S-V	12672.5	6273	

Using the results in Table 4.3, we compute the ICER of subsequent intervention strategies enlisted in the table.

$$ICER(S-II) = \frac{2367}{6786.8} = 0.3488.$$

$$ICER(S-I) = \frac{1792 - 2367}{6890.4 - 6786.8} = -5.5508.$$

The result from the ICER of S-II and S-i shows that $ICER(S-II) > ICER(S-I)$. This indicates that S-II consumes the limited resources, therefore, we exclude S-II from the list of alternative strategies and arrange the remaining intervention strategies in ascending order as summarized in Table 4.4.

Table 4.4: Infection averted and the total cost

Strategy	Infection averted	total cost	ICER
S-I	6890.4	1792	0.2601
S-IV	6891.5	2068	
S-III	12671.3	1648	
S-V	12672.5	6273	

Using the results in Table 4.4, we compute the ICER of subsequent intervention strategies enlisted in the table.

$$ICER(S-I) = \frac{1792}{6890.4} = 0.2601.$$

$$ICER(S-IV) = \frac{2068 - 1792}{6891.5 - 6890.4} = 250.9.$$

The result from the ICER of S-I and S-IV shows that $ICER(S-IV) > ICER(S-I)$. This indicates that S-IV consumes the limited resources, therefore, we exclude S-IV from the list of alternative strategies and arrange the remaining intervention strategies in ascending order as summarized in Table 4.5.

Table 4.5: Infection averted and the total cost

Strategy	Infection averted	total cost	ICER
S-I	6890.4	1792	0.2601
S-III	12671.3	1648	-0.0249
S-V	12672.5	6273	

Using the results in Table 4.5, we compute the ICER of subsequent intervention strategies enlisted in the table.

$$ICER(S-I) = \frac{1792}{6890.4} = 0.2601.$$

$$ICER(S-III) = \frac{1648 - 1792}{12671.3 - 6890.4} = -0.0249.$$

The result from the ICER of S-I and S-III shows that $ICER(S-I) > ICER(S-III)$. This indicates that S-I consumes the limited resources, therefore, we exclude S-IV from the list

of alternative strategies and arrange the remaining intervention strategies in ascending order as summarized in Table 4.6.

Table 4.6: Infection averted and the total cost

Strategy	Infection averted	total cost	ICER
S-III	12671.3	1648	0.1301.
S-V	12672.5	6273	.

Using the results in Table 4.6, we compute the ICER of subsequent intervention strategies enlisted in the table.

$$ICER(S - III) = \frac{1648}{12671.3} = 0.1301.$$

$$ICER(S - V) = \frac{6273 - 1648}{12672.5 - 12671.3} = 3854.1615.$$

The result from the ICER of S-III and S-V shows that $ICER(S - V) > ICER(S - III)$. This indicates that S-V consumes the limited resources, therefore, we exclude S-V from the list of alternative strategies. Therefore, the result from the const-effectiveness analysis we conclude that personal preventive and treatment with medication is the least cost and most effective intervention strategy to combat dengue fever in the society.

Chapter 5

Summary, Conclusions and Recommendations

This chapter presents the summary, conclusions and recommendation as well as future work. In the next sections, we will present as follow.

5.1 Summary

In this section, we provide detail summary of the work in the first chapter upto the last chapter of the thesis. In chapter 1, we present introduction to dengue fever focusing on its mode of transmission, causes of dengue fever, how to control and prevent it. Additionally, we identified the gap to be filled in this section along with the research objectives. Moreover, we present the significance of the study and some important mathematical and epidemiological preliminaries which helpful throughout the thesis.

In chapter 2, we present the review of related literature on the transmission dynamics of dengue fever. We reviewed a large scale of literature in the field to identify what is done and what is not yet done. For this matter, we reviewed the most recent publications mainly publications on the classical epidemiological models of dengue fever.

In chapter 3, we provide detail about the research methodology. This chapter mainly captures the study area and period, study design, study procedures and source of information. We also present the framework for dengue fever transmission dynamics.

In chapter 4, we provide the main results of the thesis on the optimal control and cost-effectiveness analysis of dengue fever epidemic model. In this chapter, we established the existence, uniqueness, positivity and boundedness of the solution of the model. We analyzed the stability of both equilibria. Additionally, we compute sensitivity analysis in order to impose control measures on the most sensitive parameters. Furthermore, we extended the model to optimal control problem and conducted the cost-effectiveness of intervention

strategies.

5.2 Conclusions

In this thesis, we investigated a classical deterministic model for dengue fever transmission incorporating six compartments: human mobility, susceptible humans, infected humans, recovered humans, susceptible mosquitoes, and infected mosquitoes. We established the fundamental analytical properties of the model, including the existence, uniqueness, positivity, and boundedness of its solutions, ensuring that the model is mathematically and epidemiologically well-posed.

The basic reproduction number (R_0) was derived to characterize the transmission potential of the disease. Furthermore, both the disease-free equilibrium (DFE) and the endemic equilibrium (EE) points of the system were determined, and their local stability properties were rigorously analyzed. Sensitivity analysis was carried out to identify key parameters that most strongly influence (R_0). Based on these insights, appropriate control measures were imposed on the most sensitive parameters to reduce disease spread.

The model was then extended to an optimal control framework by incorporating three time-dependent control strategies. A comprehensive set of numerical simulations was conducted, particularly for the case where ($R_0 > 1$), representing scenarios where dengue can persist in the population. Seven intervention strategies were tested, and their cost-effectiveness was evaluated. The analysis revealed that the combined application of personal preventive and treatment with medication control measures is the most effective and economically efficient intervention strategy.

Although the present study provides significant insights into dengue transmission dynamics and control, it is not exhaustive. Future researchers may extend, modify, or generalize the results by employing stochastic modeling approaches to capture randomness in disease spread or by applying fractional calculus techniques to account for memory effects and other complexities not captured by classical models. Such extensions would deepen the understanding of dengue epidemiology and contribute to the development of more robust control strategies.

5.3 Recommendations

In this section, we provide the recommendations from our research output for the concerned bodies. It is based on the findings of this study, the following recommendations are proposed: For Society we recommend that communities should actively participate in dengue prevention by eliminating mosquito breeding sites, using personal protective measures, and engaging in awareness campaigns to reduce exposure to mosquito bites.

For Public Health Officers suggest that strengthen routine surveillance, improve early detection systems, and implement integrated vector management strategies targeting the most sensitive parameters identified in the model.

For Policy Makers and Decision Makers, we recommend them that they have prioritize the allocation of resources to combined intervention strategies, as the cost-effectiveness analysis shows that simultaneous application of multiple control measures yields the highest impact.

For Government Officials, we suggest that they should support long-term policies that enhance environmental sanitation, provide adequate health infrastructure, and ensure sustainable funding for vector control programs and community education.

For researchers and health institutions, we recommend that they should encourage further investigation using stochastic and fractional modeling approaches to better capture real-world complexities and improve future dengue prevention and control strategies.

Bibliography

- Abidemi, A., Fatoyinbo, H. O., and Asamoah, J. K. K. (2020). Analysis of dengue fever transmission dynamics with multiple controls: a mathematical approach. In *2020 International Conference on Decision Aid Sciences and Application (DASA)*, pages 971–978. IEEE.
- Adegbite, G., Edeki, S., Isewon, I., Emmanuel, J., Dokunmu, T., Rotimi, S., Oyelade, J., and Adebisi, E. (2023). Mathematical modeling of malaria transmission dynamics in humans with mobility and control states. *Infectious Disease Modelling*, 8(4):1015–1031.
- Ahman, Q. O., Aja, R. O., Omale, D., and Okpara, P. A. (2025). Mathematical modeling of dengue virus transmission: exploring vector, vertical, and sexual pathways with sensitivity and bifurcation analysis. *BMC Infectious Diseases*, 25(1):1–28.
- Aldila, D., Aulia Puspadani, C., and Rusin, R. (2023). Mathematical analysis of the impact of community ignorance on the population dynamics of dengue. *Frontiers in Applied Mathematics and Statistics*, 9:1094971.
- Barmak, D. H., Dorso, C. O., and Otero, M. (2016). Modelling dengue epidemic spreading with human mobility. *Physica A: Statistical Mechanics and its Applications*, 447:129–140.
- Boyce, W. E., DiPrima, R. C., and Meade, D. B. (2017). *Elementary differential equations and boundary value problems*. John Wiley & Sons.
- Cai, L., Guo, S., Li, X., and Ghosh, M. (2009). Global dynamics of a dengue epidemic mathematical model. *Chaos, Solitons & Fractals*, 42(4):2297–2304.
- CDC, A. W. (2020). Centers for disease control and prevention.
- Coddington, E. A. and Levinson, N. (1955). *Theory of ordinary differential equations*. McGraw-Hill New York.

- Din, A., Khan, T., Li, Y., Tahir, H., Khan, A., and Khan, W. A. (2021). Mathematical analysis of dengue stochastic epidemic model. *Results in Physics*, 20:103719.
- Draidi Areed, W., Thanh Thao Nguyen, T., Do, K. Q., Nguyen, T., Bui, V., Nelson, E., Warren, J. L., Doan, Q.-V., Sinh, N. V., Osborne, N., et al. (2024). A district-level ensemble model to enhance dengue prediction and control for the mekong delta region of vietnam. *arXiv e-prints*, pages arXiv–2412.
- Dwivedi, A., Keval, R., and Khajanchi, S. (2022). Modeling optimal vaccination strategy for dengue epidemic model: a case study of india. *Physica Scripta*, 97(8):085214.
- Eegunjobi, A. S., Anyanwu, M. C., and Neossi-Nguetchue, S. (2023). Modelling the super-infection of two strains of dengue virus. *Journal of the Egyptian mathematical Society*, 31(1):1.
- Esteva, L. and Vargas, C. (1998). Analysis of a dengue disease transmission model. *Mathematical biosciences*, 150(2):131–151.
- Fleming, W. H. and Rishel, R. W. (2012). *Deterministic and stochastic optimal control*, volume 1. Springer Science & Business Media.
- Gyasi, P., Bright Yakass, M., and Quaye, O. (2023). Analysis of dengue fever disease in west africa. *Experimental Biology and Medicine*, 248(20):1850–1863.
- Hamdan, N. and Kilicman, A. (2021). The development of a deterministic dengue epidemic model with the influence of temperature: a case study in malaysia. *Applied Mathematical Modelling*, 90:547–567.
- Harjule, P. et al. (2024). Mathematical modelling and analysis of dengue transmission dynamics. *Procedia Computer Science*, 235:539–548.
- Islam, M. S., Shahrear, P., Saha, G., Ataulhha, M., and Rahman, M. S. (2024). Mathematical analysis and prediction of future outbreak of dengue on time-varying contact rate using machine learning approach. *Computers in biology and medicine*, 178:108707.
- Jan, R. and Xiao, Y. (2019). Effect of partial immunity on transmission dynamics of dengue disease with optimal control. *Mathematical Methods in the Applied Sciences*, 42(6):1967–1983.

- Khatua, A. and Kar, T. K. (2020). Dynamical behavior and control strategy of a dengue epidemic model. *The European Physical Journal Plus*, 135(8):643.
- Kumar, R. P., Mahapatra, G., Santra, P., and Nieto, J. J. (2024). Optimal control for dengue transmission based on a model with reinfection and treatment. *Mathematical Population Studies*, 31(3):165–203.
- Martcheva, M. (2015). *An introduction to mathematical epidemiology*, volume 61. Springer.
- Medeiros, L. C. d. C., Castilho, C. A. R., Braga, C., de Souza, W. V., Regis, L., and Monteiro, A. M. V. (2011). Modeling the dynamic transmission of dengue fever: investigating disease persistence. *PLOS neglected tropical diseases*, 5(1):e942.
- O'Neill, S. L., Ryan, P. A., Turley, A. P., Wilson, G., Retzki, K., Iturbe-Ormaetxe, I., Dong, Y., Kenny, N., Paton, C. J., Ritchie, S. A., et al. (2019). Scaled deployment of wolbachia to protect the community from dengue and other aedes transmitted arboviruses. *Gates open research*, 2:36.
- Pontryagin, L. S. (2018). *Mathematical theory of optimal processes*. Routledge.
- Sheet, F. (2024). Dengue and severe dengue. *The Worlds Health Organization: Geneva, Switzerland*.
- Sk, T., Bal, K., Biswas, S., and Sardar, T. (2024). Global stability and optimal control in a single-strain dengue model with fractional-order transmission and recovery process. *arXiv preprint arXiv:2402.11974*.
- Sow, A., Diallo, C., and Cherifi, H. (2024). Interplay between vaccines and treatment for dengue control: An epidemic model. *Plos one*, 19(1):e0295025.
- Xue, L., Ren, X., Magpantay, F., Sun, W., and Zhu, H. (2021). Optimal control of mitigation strategies for dengue virus transmission. *Bulletin of Mathematical Biology*, 83(2):8.
- Yoda, Y., Ouedraogo, H., Ouedraogo, D., and Guiro, A. (2024). Mathematical analysis and optimal control of dengue fever epidemic model. *Advances in Continuous and Discrete Models*, 2024(1):11.