WOLLEGA UNIVERSITY

SCHOOL OF GRADUATE STUDIES

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE

DEPARTMENT OF MATHEMATICS

THESIS ON

FIXED POINT THEOREM FOR SET-VALUED CONTRACTION MAPPING IN b-METRIC SPACE.

BY:

MENGISTU TADESE

ADVISOR: DULA TOLERA (PhD)

OCTOBER,2019

NEKEMTE, ETHIOPIA

WOLLEGA UNIVERSITY

SCHOOL OF GRADUATE STUDIES

FIXED POINT THEOREM FOR SET-VALUED CONTRACTION MAPPING IN b-METRIC SPACE.

THESIS SUBMITTED TO SCHOOL OF GRADUATE STUDIES, WOLLEGA UNIVERSITY, COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES, DEPARTMENT OF MATHEMATICS FOR THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE MASTERS OF DEGREE (MSC) IN MATHEMATICS

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OCTOBER 2019

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STATEMENT OF THE AUTHOR

I declare and affirm that this Thesis entitled**" Fixed point theorem for set valued contraction mapping in b-metric space"** is my work and that all sources of materials used for this thesis have been appropriately acknowledged. This thesis is submitted in partial fulfillment of the requirements for M.Sc. degree in Mathematics at Wollega University. I declare that this thesis not submitted to any other institution anywhere for the award of any degree or diploma. The thesis deposited at the University Library to make available to borrowers under rules of the Wollega University Library.

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Name: Mengistu Tadese Signature: __________ Date of Submission____________

Place: Wollega University, Nekemte

DECLARATION

This is to certify that this thesis entitled **“Fixed point theorem for set valued contraction mapping in b-metric space** "accepted in partial fulfillment of the requirements for the award of the Degree **of Master of science** in **Mathematics** by the school of Graduate Studies, Wollega University through the college of **Natural and Computational Science**, done by MengistuTadese is a genuine work carried out by him under my guidance. The matter embodied in this Thesis work has not been submitted earlier for the award of any degree or diploma.

The assistance and help received during the course of this investigation have been duly acknowledged. Therefore, I recommend that it can be accepted as fulfilling the Thesis requirements.

______________________	______________________	______________
Major Advisor	**Signature**	**Date**

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FINAL THESIS APPROVAL FORM

As members of the Board of Examining of the Final M. A. /M. Sc. Thesis open defense, we certify that we have read and evaluated the thesis prepared by **MengistuTadese** under the title **“Fixed point theorem for set valued contraction mapping in b-metric space”** and recommend that the project be accepted as fulfilling the thesis requirement for the **Degree of Master of science** in Mathematics (Real Analysis).

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Certification of the final Thesis

I hereby certify that all the correction and recommendation suggested by the board of examiners are incorporated into the final Thesis entitled **“Fixed point theorem for set valued contraction mapping in b-metric space”** by **MengistuTadese**.

Dean of SGS	**Signature**	**Date**

Acknowledgment

First, I would like to express my deepest gratitude to God for the reason that he gives me long life, strength, to reach this precise time, and blessing me in all aspects of my life. Secondly, I would like express my heartfelt gratitude and thanks to my advisor **Dr. DulaTolera**, for helpful discussion, excellent guidance, caring, patience, constructive comments , providing the necessary materials, suggestions immediate responses and encouragements from the beginning up to the end of the study.

Finally, I would like to extend gratefulness thanks to my wife GelaneTolesa for her understanding and patience while I spend much time on my study, as well as for her unreserved support and encouragement all the way through and my family who support me to and giving advice to study my lesson properly.

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Abstract

In this thesis, we state fixed point theorems for set-valued mapping satisfies some contraction condition and prove the existence of fixed points for such mapping in b-metric space. We extend the results of Seong-Hoon[9] from metric space to b-metric spaces. We deduce some corollaries from our mainresult.

Keywords: b- Metric space, set-valued mapping, contraction.

Abbreviation and Notation

Abbreviation

Inf : Infrimum

Lim : Limit

Max : Maximum

Min : Minimum

Sup : Supreme

Notation

ϵ : $Element$

ε : $Epsilo$

$\mathbb{N}$: $Natural\ number$

n (ε) : $NDepends\ no\ epsilon$

$\mathbb{R}$: $The\ set\ of\ real\ number$

$\subset$: $Subsetof$

$\leq$: $Less\ than\ or\ equalto$

$<$: $Strictlly\ less\ than$

∞ : $Infinity$

$\neq$: Not equal to

CHAPTER ONE

Introduction

1.1 Background

In many branches of Science, Economics, Computer Science, Engineering and the development of non linear Analysis the fixed point theory is one of the important tools. We say that f(x) is a set-valued map (or point to set map or multi-valued function or simply multifunction) from a set X to another set Y, denoted by f(x): X$\rightarrow Y$, if for every xϵX, f(x) is subset of Y. Let T be a self map of a non-empty set X a point x ϵX such that Tx=x is called a fixed point on the map T. The theorems concerning the properties and existence of fixed point are known as fixed point theorem. Fixed point theorem is one of the most dynamic research subjects in non leaner Science. The feasibility of application of it to the various disciplines increases its beauty more and more. In 1922, the famous Banach contraction principle which state that if (X, d) is complete metric space and T: X $\rightarrow$X such that D $(Tx, Ty) \leq k\ d(x, y)$ for all x, yϵX where k$\epsilon[0,1)$, then T has unique fixed point was introduced by Banach [14].So Banach fixed point theorem in 1922 [14] says that every contractive mapping on a complete metric space has a unique fixed point. The Banach contraction principle is one of the pivotal results in the metric fixed theory. Metric fixed point has been fruit full and promising area of research for many Mathematics during the last several decades. The study of fixed point in difference spaces is an interesting area and it has became a use full tool because the study finds an application in different branch of mathematics as well as in other fields like Physical Science, Life Sciences and Economics. The field of fixed point theory is expanding is domain; there by leading to the emergency of plethora of techniques and ideas. As it was well known, the Banach fixed point theorem is very useful, simple and classical tool in modern Analysis. The concept of $b-metric$space was introduced by Bakhtain [4] in 1989, which used it to prove a generalization of the Banach contraction principle in space endowed with such kind of metrics. Since then, this notion has been used by many authors to obtain various fixed point theorems. This direction was the source of several fixed point and coincidence point theorems in various ambient spaces. The concept of compatibility was used by many authors to prove existence theorems in fixed point theory. On other hand, Czerwik [12] extends the concept of b-metric spaces which is the generalization of metric spaces and generalized the Banach contraction principle in the context of complete b- metric space.

On the other side hand, in 1993, Czerwik, [12] develop the notion of $b-metric$ space and established some fixed point theorems in $b-metric$ spaces. Using these concepts, several researchers presented generalization of the renowned Banach fixed point theorem in a $b-metric$space.

In 1998 S.Czerwik, [11], introduce a nonlinear set-valued contraction mapping in b-metric space. The concepts of b-metric space were the generalization of metric spaces and generalized the Banach contractions principles in the contexts of complete b- metric space.

Subsequently,several results appeared in this direction like,[3,6,14,15],In 2009,Boriceanu,M.Molnar,[1],study a fixed point theory for multi valued generalized contraction on set-valued with b-metric space.

In other hand Seong-Hoon ,[13], introduces new set-valued contraction mapping. Based on these concept we extends a fixed point theorem for set- valued contraction mapping in b-metric space and study the existence of fixed point for such mapping in b-metric space. We deduce some corollaries and provide some examples which support the main result.

1.2 Statement of the problem

In this thesis, we are going to establish the fixed point theorems for set-valued contraction mapping in b-metric spaces. Owing to this, the present study attempt to answer the following questions

- How we define fixed point?
- What are the properties of fixed point theorems in set-valued contractive mapping?
- How we prove the fixed point theorem for set-valued contraction mapping in b-metric space?

1.3 Objective of the study.

1.3.1 The general objective of this study is:

- To study fixed point theorem for set-valued contraction mapping in b-metric space and prove the theorem in our results

1.3.2 The specific objective of this study is:

- To understand the techniques of finding fixed point

- To know the properties of fixed point theorems for set -valued contractive mapping under b-metric space.
- To study how to generalize and extend fixed point for set valued contractive mapping.

1.4 Delimitation of the study

This study delimited to prove the fixed point theorems on set-valued mapping in b-metric space which generalize and extends fixed points of many research papers in literature.

1.5. Significance of the study

Fixed point theorem becomes, in the last decades not only a field with a huge development ,but also strong tool for solving various problems arising in different fields of pure and applied mathematics. The concept of fixed point plays a key role in Analysis. Thus the study of fixed point theory has been researched extensively.

CHAPTER TWO

REVIEW OF RELATED LITERATURE

2.1 INTRODUCTION AND PRELIMINARIE

In this section we mention some definition, notations theorems and Lemmas which supports our main results in the literature that will be needed to prove our main results.

Let (X, d) be a metric space. Then the followings are concepts from set valued:

$$\text{N(X)}= \{A: A \text{ is a nonempty subset of } X\}$$

$$\text{B(X)}= \{A: A \text{ is a nonempty bounded subset of } X\}$$

$$\text{C(X)}= \{A: A \text{ is a nonempty compact subset of } X\}$$

$$\text{CB(X)}= \{A: A \text{ is a nonempty closed bounded subset of } X\}$$

For $A, B \in CB(X)$, define the function $H: CB(X) \times CB(X) \to \mathbb{R}^+$ by

H(A,B) =max$\left\{\begin{matrix} supD(a,B) \\ a\epsilon A \end{matrix}, \begin{matrix} supD(b,A) \\ b\epsilon B \end{matrix}\right\}$, for A, B$\epsilon$CB(X).Where

$\delta(\text{A, B}) = \sup\{D(a, B), a \in A\}$, and

$\delta(\text{B, A}) = \sup\{D(b, A), b \in B\}$With;$D(a, C) = \inf\{d(a, x), x \in C\}$

$\delta(x, A) = sup\{d(x, y): y\epsilon A\}$Where A ϵ $CB(X)$.

D (a, B) = inf $\{d(a, b): b\epsilon B\}$, D is the distance from the point a to the subset B.

The author of [15] give a generalization of Banach contraction principle [9] to the case of set-valued mapping, then many authors generalize and extended the result of[10] in many direction, espetailly authors of [8] obtained a generalization of the result of [10] they prove the following result.

Theorem 2.1 [8] Let (X, d) be a complete metric space. Suppose that a set-valued mapping T: X→ CB(X) satisfies the following condition:

$$\text{H } (Tx, Ty) \leq kM(x, y) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 2.1$$

For all x, y ∈X, where k ϵ [0, 1) and

M(x, y) =max$\left\{\text{d(x,y)}, \text{D(x,}Tx), D(y, Ty), \frac{1}{2}[D(x, Ty) + D(y, Tx)]\right\}$. If x→ D (x,$Tx$) is lower semi-continuous, then T has a fixed point in X. that is there exist $x_* \epsilon X$ such that $x_* \epsilon Tx$.

Let (X, d) be metric space and $x_O \epsilon X$ and X→ CB(X) be a set valued mapping. Then a sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_n \epsilon\ Tx_{n-1}$, n=1, 2, 3…is called a (Picard) orbit of T at initial point x_o.

Theorem 2.2 [5] Let (X, d) be a complete metric space. If a set-valued mapping

T: X→ CB(X) satisfies the following condition: there exist two constant $\theta \epsilon$ (0,1) and L≥ 0 such that

$$H\,(Tx, Ty) \leq \theta d(x, y) + L\, D\,(x, Ty) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2$$

For all $x, y \in X$ then T has a fixed point in X. More Ever for any $x_o \epsilon X$, there exists an orbit $\{x_n\}$ Of T at the initial point x_o that converges to the fixed point x_* of T for which the following estimate holds:

There exist a constant k ∈ (0, 1) such that

(i) For n=0, 1, 2...

$$d\,(x_n, x_*) \leq \frac{k^n}{1-k} d(x_{0,} x_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.3$$

(ii) For n=1, 2, 3…

$$d(x_n, x_*) \leq \frac{k}{1-k} d(x_{n-1,} x_n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.4$$

The authors [2] obtained the following result**:**

Theorem 2.3 [2] Let (X, d) be a complete metric space. If a set-valued mapping

T: X→ CB(X) satisfies the following condition:

$$H\,(Tx, Ty) \leq \frac{D(x,Ty)+D(y,Tx)}{\sigma(x,Tx)+D(y,Ty)+1} d(x, y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.5$$

For all $x, y \in X$ then T has a fixed point in X. Now using the result of [2, 5, 8, 15] we introduce the new set valued contraction mapping in b- metric space and establish a new fixed point theorem for such mapping.

Theorem 2.4 [13] Let (X, d) be a metric space and T: X→ CB(X) be a set-valued contraction mapping .suppose that X is T-orbitally complete if T satisfies the following condition.

$$H\,(Tx, Ty) \leq \frac{D\,(x,Ty)+D(y,Tx)}{\sigma(x,Tx)+D(y,Ty)+1]} M(x, y) + L\, D(y,Tx) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.6$$

For all x, y∈ X, L≥ O,.then T has a fixed point in X. More ever for any,$x_o \epsilon X$, there exist an orbit $\{x_n\}$ of T at the initial point x_o that converges to a fixed pointx_* of T, for which (2.3) and(2.4) hold

Let (X, d) be a complete metric space and T: X→ CB(X) be a set-valued mapping and let $x_o \epsilon X$.Then X is called T-orbitally complete if any Cauchy subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ converges to same fixed point in X. where $\{x_n\}$ is a Picard orbit of T at initial point $x_o \epsilon X$.

Note that if T: X→ $X$is a single valued mapping, then X is T-orbitally complete if any Cauchy subsequence $\{x_{n(k)}\}$ of the sequence$\{x_n\}$ defined by

x_{n+1}=T_{xn}, n=0, 1, 2, 3…, $x_o \epsilon X$ converges to some point in X.

Also, note that completeness implies T-orbitally completeness, for set (or single)-valued mapping T.

Definition2.5 [9] Let (X, d) be a metric space. A mapping $T: X \to CB(X)$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$:

$d(Tx, Ty) \leq \alpha d(x, y)$Where$(\alpha < 1)$

Before starting our results, we present some definition and propositions in a $b - metric$space.

Definition 2.6 [12] Let X be a nonempty set and s≥ 1 a given real number.

A function $d: XxX \to R^+$is called a $b - metric$ if the following condition are satisfied:

(1) $d(x, y) = 0 \quad iff x = y$,

(2) $d(x, y) = d(y, x)$ for all$x, y \in X$,

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$

A pair $(X, d)$is called a $b - metric$ space with (constant s).

It should be noted that the class of b-metric space is effectively larger than that of metric spaces. Indeed, a $b - metric$ is metric if $s = 1$.

Example 2.7.1 Let (X , d) be a metric space and m$(x, y) = (d(x, y))^p$, where p>1 is a real number is a b-metric space with $s = 2^{p-1}$.

Example 2.7.2.Let X= {1, 2, 3} and the mapping T: X ×X→ $\mathbb{R}^+$ defined by

d (1,1) = d(2,2)= d(3,3)=0

d(1,2)= d(2,1) =d(2,3)= d(3,2)=1 and

d(1,3)= d(3,1)=m

Where m is given real number such that m≥ 2, it is easy to see that d(x,y)≤ $\frac{m}{2}$ [d(x ,z)+d(z ,y)]

for all x, y, z ϵX,therefore ,(X ,d) is b-metric space with coefficient s=$\frac{m}{2}$ we obtain that the ordinary triangular inequalities does not hold if m>2 and then (X, d) is not metric space.

Definition 2.8 [1] Let(X, d) be a $b - metric$ space and $\{x_n\}$ be sequence of X such that

(1). $\{x_n\}$ is convergent if there exist an x in X such that for any ε>0, there exist an n(ε)$\epsilon\mathbb{N}$,such that n≥ $n(\varepsilon), d(x_n, x) < \varepsilon$.

(2) $\{x_n\}$ is Cauchy sequence if for any ε>0 there exist an n(ε)$\epsilon\mathbb{N}$,such that m ,n≥ $n(\varepsilon)$, $d(x_m, x_n) < \varepsilon$

(3) (X,d) is called complete if and only if every Cauchy sequence in X is convergent.

Definition 2.9[1] Let (X, d) be a $b - metric$ space. Then a sequence $\{x_n\}$ in X is called:

(a) Convergent if and only if there exists $x \in X$ such that $d(x, y) \to 0$, as $n \to +\infty$ .In this case, we write $\lim_{n\to\infty} x_n = x$

(b) Cauchy if and only if $d(x_n , x_m) \to 0$ as,$n, m \to +\infty$

Proposition2.10. in a $b - metric$ space (X, d) the following assertions hold:

(i) A convergent sequence has a unique limit,

(ii) Each convergent sequence is Cauchy,

(iii) In general, a b-metric is not continuous.

Definition 2.11[1]. The $b - metric$ space (X, d) is complete if every Cauchy sequence in X converges.

In general a $b - metric$ function $d \; for \; s > 1$ is not jointly continuous in all of its two variables.

A $b - metric$ is not continuous in general, so we need the following simple lemma about the b-convergent sequence.

Also, we present the following simple lemma needed in the proof of our main result.

Lemma 2.12 Let (X, d) be a b-metric space .suppose that A, BϵCB(X) and c> 0. if H (A, B) < c and a$\epsilon A, then \; there \; exist \; b\epsilon B such \; that \; d(a, b) < c.$

Lemma 2.13[12] Let(X, d) be a b-metric space with $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y , respectively. Then, we have

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y)$$

In particular, if x=y, we have then $\lim_{n\to\infty} d(x_n, y_n)$ =0

Moreover for each $z \in X$ we have

$$\frac{1}{s} d(x,z) \le \liminf_{n\to\infty} d(x_n,z) \le \limsup_{n\to\infty} d(x_n,z) \le s\, d(x,z).$$

Lemma 2.14 let (X,d) be a $b-metric$ space. If there exist two sequence$\{x_n\}$and$\{y_n\}$ such that $\lim_{n\to\infty} d(x_n,y_n) = 0, whenever$ $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$

For some $t \in X$ then, that $\lim_{n\to\infty} y_n$=t

Proof: By a triangle inequality in a $b-metric$ space, we have

$$d(y_n,t) \le s(d(y_n,x_n) + d(x_{n,}t))$$

Now, by taking the upper limit when $n \to \infty$ in the above inequality we get

$$\limsup_{n\to\infty} d(y_n,t) \le s(\limsup_{n\to\infty} d(y_n,x_n) + \limsup_{n\to\infty} d(x_n,t)) = 0$$

Hence, $\lim_{n\to\infty} y_n = t$

Lemma 2.15.Let (X,d,s) be a b-metric space. For any $A,B \in CB(X)$ and any $x,y \in X$, one has the following:

(i) $d(x,B) \le d(x,b), for\ any\ b \in B$;

(ii) $\delta(A,B) \le H(A,B)$;

(iii) $d(x,B) \le H(A,B), for\ any\ x \in A$;

(iv) $H(A,A) = 0$;

(v) $H(A,B) = H(B,A)$;

(vi) $H(A,C) \le s\big(H(A,B) + H(B,C)\big)$;

(vii) $D(x,A) \le s\big(d(x,y) + D(y,A)\big)$.

CHAPTER THREE

PROJECT DESIGN AND METHODOLOGY

3.1 Study Area and Period

The study was conducted at Wollega University Nekemte campus under the department of Mathematics from November 2018 G.C. to October, 2019 G.C. Consequently the study focuses on common fixed point theorems on multi-valued mapping.

3.2. Study Design

This study would employ documentary review design on fixed point theorems for set-valued mapping in b-metric space.

3.3 Sources of Information

The relevant sources of information were obtained mainly from internet based sources and some library books and published articles. Authenticity of the internet sources were, thoroughly checked. Moreover, mostly pdf format books and articles from trustworthy journals were included.

3.4 Mathematical Procedure

Important materials and data for the study would be collected by means of documentary review. Hence in order to achieve the stated objectives, the study will follow the following procedures:-

- Defining the Problem.
- Analyzing the information obtained about it.
- Reviewing literature and extended the set-valued mapping from metric to b-metric.
- Proving the fixed point theorems for set -valued contractive mapping under b-metric space.

3.5 Limitation of the thesis

There were various limitations to conduct the study. Lack of the necessary reference materials was a problem. Even if an attempt was made to bridge the gap by referring to internet sources, in ability to access the original works mentioned these sources were major problems. Lack of internet for some time during the study period has also had a negative impact on the study.

CHAPTER FOUR
RESULTS AND DISCUSSION

4.1 Main results

Theorem 4.1 Let (X, d) be a complete b-metric space with coefficient s$\geq$ 1 and T: X$\rightarrow$ CB(X) be a set-valued contraction mapping .Suppose that X is T-orbitally complete if T satisfies the following condition.

$H(Tx, Ty) \leq \frac{D\,(x,Ty)+D(y,Tx)}{s^{2}[\sigma(x,Tx)+D(y,Ty)+1]}$ M(x, y) +Lmin$\{d(x,y), D(y,Tx)\}$ ……………………………4.1

For all x, y$\in$ X, L$\geq O$ and M (x, y) =max$\Big\{$d(x,y), D(x,Tx), $D(y,Ty)$, $\frac{1}{2s}[D(x,Ty)+d(y,Tx)]\Big\}$.then T has a fixed point in X. Moreover for any,$x_o \epsilon X$, there exist an orbit $\{x_n\}$ of T at the initial point x_o that converges to a fixed pointx_* of T, for which (2.3) and(2.4) hold.

Proof: let $x_o \epsilon X$ and let $x_1 \epsilon$ Tx_0

If $x_{o=}x_1$ then,$x_o \epsilon$ Tx_0, implies x_o is the fixed point of T then the proof is complete.

Assume that $x_{o\neq}x_1$

From 4.1 D $(x_1, \mathrm{T}x_1) \leq$ H(Tx_0, Tx_1)

$$\leq \frac{D(x_{o,}\mathrm{T}x_1)+D(x_1,\mathrm{T}x_0)}{s^2[\sigma(x_{o,}\mathrm{T}x_0)+D(x_1,\mathrm{T}x_1)+1]}\,\mathrm{M}(x_o,x_1) \quad +\mathrm{L}\,\min\{d(x_o,x_1), D(x_1,\mathrm{T}x_0)\}$$

$$\leq \frac{D(x_{o,}\mathrm{T}x_1)+D(x_1,\mathrm{T}x_0)}{s^2[\sigma(x_{o,}\mathrm{T}x_0)+D(x_1,\mathrm{T}x_1)+1]}\,\mathrm{M}(x_o,x_1) \quad +\mathrm{L}\,\min\{d(x_o,x_1), D(x_1,\mathrm{T}x_0)\}$$

$$= \frac{D(x_{o,}\mathrm{T}x_1)+d(x_1,x_1)}{s^2[\sigma(x_{o,}\,\mathrm{T}x_0)+D(x_1,\mathrm{T}x_1)+1]}\,\mathrm{M}(x_o,x_1) \quad +\mathrm{L}\,\min\{d(x_o,x_1), d(x_1,x_1)\}$$

$$\leq \frac{s[D(x_{o,}x_1)+D(x_1,T_{x1})]}{s^2\sigma(x_{o,}\,\mathrm{T}x_0)+D(x_1,\mathrm{T}x_1)+1}\,\mathrm{M}(x_o,x_1)$$

$$\leq \frac{[D(x_{o,}x_1)+D(x_1,T_{x1})]}{s[\sigma(x_{o,}\,\mathrm{T}x_0)+D(x_1,\mathrm{T}x_1)+1]}\,\mathrm{M}(x_o,x_1)$$

$$\leq \frac{D(x_{o,}x_1)+D(x_1,T_{x1})}{\sigma(x_{o,}\,\mathrm{T}x_0)+D(x_1,\mathrm{T}x_1)+1}\,\mathrm{M}(x_o,x_1)$$

$=\beta_o$ M(x_o, x_1) ,where $\beta_o = \frac{d(x_{o,}x_1)+D\,(x_1,\mathrm{T}x_1)}{\sigma(x_{o,}\,\mathrm{T}x_0)+D\,(x_1,\mathrm{T}x_1)+1}$………………………………… 4.2

We deduce that

M $(x_{o,}x_1)$ =max$\Big\{$d(x_o, x_1), D $\big(, x_{o,}\,\mathrm{T}x_0\big)$, $D\,(x_1, \mathrm{T}x_1)$, $\frac{1}{2s}[D\,\big(x_{o,}\mathrm{T}x_1\big) + D\,(x_1, \mathrm{T}x_0)]\Big\}$.

$$\leq \max\Big\{\mathrm{d}(x_o,x_1), d(x_o,x_1), D\,(x_1,\mathrm{T}x_1), \frac{1}{2s}[D\big(x_{o,}\mathrm{T}x_1\big)+d(x_1,x_1)]\Big\}$$

$$= \max\Big\{\mathrm{d}(x_o,x_1), D\,(x_1,\mathrm{T}x_1), \frac{1}{2s}[D\,(x_o,\mathrm{T}x_1)]\Big\}$$

$$\leq \max\left\{d(x_o, x_1), D\ (x_1, Tx_1), \frac{1}{2s}[s[d(x_o, x_1) + D\ (x_1, Tx_1)\right\}$$

$$= \max\left\{d(x_o, x_1), D\ (x_1, Tx_1), \frac{1}{2}[d(x_o, x_1) + D\ (x_1, Tx_1)]\right\}$$

$= \max\{d(x_o, x_1), D\ (x_1, Tx_1)\}$……………………………………………4.3

From (4.2) and (4.3) we obtain

$D\ (x_1\, , Tx_1) \leq H(\ Tx_0, Tx_1) \leq \beta_o \max\{d(x_o, x_1), D\ (x_1, Tx_1)\}$ … … … … … … … … … … … … … (1 ∗)

from equation (1 ∗) we have

suppose $\max\{d(x_o, x_1), D\ (x_1, Tx_1)\} = D(x_1, Tx_1)$

$D\ (x_1\, , Tx_1) \leq H(\ Tx_0, Tx_1) \leq \beta_o D(x_1, Tx_1) < D\ (x_1, Tx_1)$ because $\beta_o < 1$

which is contradiction

hence $d(x_o, x_1) > D(x_1, Tx_1)$

i.e.

$D\ (x_1\, , Tx_1) \leq H(\ Tx_0, Tx_1) \leq \beta_o d(x_o, x_1)$

Let $r \epsilon (\beta_o, 1)$ be fixed

Then $H\ (\ Tx_0, Tx_1) < r d(x_o, x_1)$

By Lemma 2.12 there exists

$x_2 \epsilon Tx_1$ Such that

$d(x_2, x_1) < r d(x_o, x_1)$…………………………………………………………………… 4.4

If$x_1 = x_2$, then T has a fixed point and the proof is complete.

Assume that $x_{2 \neq} x_1$

Le t $\alpha_o = \frac{d(x_o, x_1) + d(x_1, x_2)}{d(x_o, x_1) + d(x_1, x_2) + 1}$

Then $\beta_o < \alpha_o$, let k be a fixed such that

$\text{Max}\{\alpha_o, r\} < k < \frac{1}{s}$

Then from(4.4)

$d(x_2, x_1) < k d(x_o, x_1)$……………………………………………………………………..4.5

Again from 4.1 we have

$D\ (x_2\, , Tx_2) \leq H(Tx_1, Tx_2)$

$\leq \frac{D\ (x_1, Tx_2) + D\ (x_2, Tx_1)}{s^2[\sigma(x_1, Tx_1) + D\ (x_2, Tx_2) + 1]} M\ (x_2, x_1)\ \ + L \min\{d(x_2, x_1), D\ (x_2, Tx_1)\}$

$\leq \frac{D\,(x_{1,}\mathrm{T}x_2)+D\,(x_2,x_2)}{s^2[\sigma(x_{1,}\mathrm{T}x_1)+D\,(x_2,\mathrm{T}x_2)+1]}$ M (x_2, x_1) +L min$\{d(x_2,x_1), d\,(x_2,x_2)\}$

$= \frac{D\,(x_{1,}\mathrm{T}x_2)}{s^2[\sigma(x_{1,}\mathrm{T}x_1)+D\,(x_2,\mathrm{T}x_2)+1]}$ M (x_2, x_1)

$\leq \frac{s[d(x_{1,}x_2)+D\,(x_2,\mathrm{T}x_2)]}{s^2[\sigma(x_{1,}\mathrm{T}x_1)+D\,(x_2,\mathrm{T}x_2)+1]}$M (x_2, x_1)

$\leq \frac{d(x_{1,}x_2)+D\,(x_2,\mathrm{T}x_2)}{s[\sigma(x_{1,}\mathrm{T}x_1)+D\,(x_2,\mathrm{T}x_2)+1]}$M (x_2, x_1)

$\leq \frac{d(x_{1,}x_2)+D\,(x_2,\mathrm{T}x_2)}{\sigma(x_{1,}\mathrm{T}x_1)+D\,(x_2,\mathrm{T}x_2)+1}$M (x_2, x_1)

$=\beta_1$ M (x_1, x_2), whereβ_1=$\frac{d(x_{1,}x_2)+D\,(x_2,\mathrm{T}x_2)}{\sigma(x_{1,}\mathrm{T}x_1)+D\,(x_2,\mathrm{T}x_2)+1}$ 4.6

We deduce that

M $(x_{1,}x_2)$ =max$\left\{\mathrm{d}(x_1,x_2), \mathrm{D}\,(x_{1,}Tx_1), D\,(x_2,\mathrm{T}x_2), \frac{1}{2s}[D\,(x_{1,}\mathrm{T}x_2) + D\,(x_2,\mathrm{T}x_1)]\right\}$.

$\leq$ max$\left\{\mathrm{d}(x_1,x_2), d(x_1,x_2), D\,(x_2,\mathrm{T}x_2), \frac{1}{2s}[D\,(x_{1,}\mathrm{T}x_2) + 0]\right\}$

=max$\left\{\mathrm{d}(x_1,x_2), D\,(x_2,\mathrm{T}x_2), \frac{1}{2s}[D\,(x_{1,}\mathrm{T}x_2)]\right\}$

$\leq$ Max$\left\{\mathrm{d}(x_1,x_2), D\,(x_2,\mathrm{T}x_2), \frac{1}{2}[\mathrm{d}(x_1,x_2), D\,(x_2,\mathrm{T}x_2)]\right\}$

= max$\{\mathrm{d}(x_1,x_2), D\,(x_2,\mathrm{T}x_2)\}$..4.7

From (4.6) and (4.7)

D $(x_2\,,\mathrm{T}x_2) \leq$ H(Tx_1, Tx_2)$\leq \beta_1$ max$\{\mathrm{d}(x_1,x_2), D\,(x_2,\mathrm{T}x_2)\}$ … … … … … … … … … . $(2*)$

From equation $(2*)$ *we have*

Suppose max$\{\mathrm{d}(x_1,x_2), D\,(x_2,\mathrm{T}x_2)\}$=$D\,(x_2,\mathrm{T}x_2)$

D $(x_2\,,\mathrm{T}x_2) \leq$ H(Tx_1, Tx_2)$\leq \beta_1$D $(x_2\,,\mathrm{T}x_2) < D\,(x_2\,,\mathrm{T}x_2)$ because $\beta_1 <1$

This is contradiction hence $\mathrm{d}(x_1,x_2) > d(x_2,\mathrm{T}x_2)$

H (Tx_1, Tx_2) $\leq \beta_1$ $\mathrm{d}(x_1,x_2)$ 4.8

We deduce that $\beta_1 \leq \alpha_o$in fact we obtain

$\beta_1 - \alpha_o$=$\frac{D\,(x_2,\mathrm{T}x_2)-\mathrm{d}(x_o,x_1)}{[\sigma(x_{1,}\mathrm{T}x_1)+D\,(x_2,\mathrm{T}x_2)+1][d(x_{o,}x_1)+d(x_1,x_2)+1]} < 1$

Because D $(x_2\,,\mathrm{T}x_2) \leq \mathrm{d}(x_1,x_2)$<$d(x_{o,}x_1)$

Thus H (Tx_1, Tx_2) $\leq \alpha_o$ $\mathrm{d}(x_1,x_2)$< k $\mathrm{d}(x_1,x_2)$

By Lemma 2.12 with $x_2\epsilon$ Tx_1 we choose $x_3\epsilon$ Tx_2 such that

$\mathrm{d}(x_2,x_3)$<k $\mathrm{d}(x_1,x_2)$

If $x_2 = x_3$ then $x_2 \epsilon$ Tx_2 then the proof so complete.

Assume that $x_2 \neq x_3$ from 4.5

We have d$(x_2, x_3) < k^2$ d(x_0, x_1)……………………………………………………………...4.9

Let $\alpha_1 = \frac{d(x_1,x_2)+d(x_2,x_3)}{d(x_1,x_2)+d(x_2,x_3)+1}$,

Then it is easy that $\alpha_1 \leq \alpha_o$

Let $\beta_2 = \frac{d(x_2,x_3)+D(x_3,\mathrm{T}x_3)}{\delta(x_2,\mathrm{T}x_2)+D(x_2,\mathrm{T}x_3)+1}$

Then we deduce that $\beta_2 < \alpha_1$

From 4.1 we have

$$D(x_3, \mathrm{T}x_3) \leq H(\mathrm{T}x_2, \mathrm{T}x_3)$$

$$\leq \frac{D(x_2,\mathrm{T}x_3)+D(x_3,\mathrm{T}x_2)}{s^2[\sigma(x_2,\mathrm{T}x_2)+D(x_3,\mathrm{T}x_3)+1]} M(x_3, x_2) + L \min\{d(x_3, x_2), D(x_3, \mathrm{T}x_2)\}$$

$$= \frac{D(x_2,\mathrm{T}x_3)}{s^2[\sigma(x_2,\mathrm{T}x_2)+D(x_3,\mathrm{T}x_3)+1]} M(x_2, x_3) + L \min\{d(x_2, x_3), d(x_3, x_3)\}$$

$$\leq \frac{D(x_2,\mathrm{T}x_3)}{\sigma(x_2,\mathrm{T}x_3)+D(x_3,\mathrm{T}x_3)+1} M(x_2, x_3)$$

$$=\beta_2 M(x_2, x_3), \text{ where } \beta_2 = \frac{D(x_2,\mathrm{T}x_3)}{\sigma(x_2,\mathrm{T}x_2)+D(x_3,\mathrm{T}x_3)+1} \ldots\ldots 4.10$$

We deduce tha

$$M(x_2, x_3) = \max\left\{d(x_2, x_3), D(x_2, \mathrm{T}x_2), D(x_3, \mathrm{T}x_3), \frac{1}{2s}[D(x_2, \mathrm{T}x_3) + D(x_3, \mathrm{T}x_2)]\right\}.$$

$$\leq \max\left\{d(x_2, x_3), d(x_2, x_3), D(x_3, \mathrm{T}x_3), \frac{1}{2s}[D(x_2, \mathrm{T}x_3) + d(x_3, x_3)]\right\}$$

$$=\max\left\{d(x_2, x_3), D(x_3, \mathrm{T}x_3), \frac{1}{2s}[D(x_2, \mathrm{T}x_3)]\right\}$$

$$\leq \max\left\{d(x_2, x_3), D(x_3, T_{x3}), \frac{1}{2}[d(x_2, x_3) + D(x_3, \mathrm{T}x_3)]\right\}$$

$$= \max\{d(x_2, x_3), D(x_3, \mathrm{T}x_3)\} \ldots\ldots 4.11$$

From (4.10) and (4.11)

$$D(x_3, \mathrm{T}x_3) \leq H(\mathrm{T}x_2, \mathrm{T}x_3) \leq \beta_2 \max\{d(x_2, x_3), D(x_3, \mathrm{T}x_3)\} \ldots\ldots (3*)$$

From equation $(3*)$,Suppose that

$$\mathrm{Max}\{d(x_2, x_3), D(x_3, \mathrm{T}x_3)\} = D(x_3, \mathrm{T}x_3)$$

$D(x_3, \mathrm{T}x_3) \leq H(\mathrm{T}x_2, \mathrm{T}x_3) \leq \beta_2 D(x_3, \mathrm{T}x_3) < D(x_3, \mathrm{T}x_3)$because $\beta_2 < 1$

This is contradiction hence $\mathrm{d}(x_2, x_3) > D\ (x_3, \mathrm{T}x_3)$

Implies $\mathrm{H}\ (\mathrm{T}x_2, \mathrm{T}x_3) \leq \beta_2 \mathrm{d}(x_2, x_3)$ ……………………………………………………..4.12

Thus we have

$\mathrm{H}\ (\mathrm{T}x_2, \mathrm{T}x_3) \leq \beta_2 \mathrm{d}(x_2, x_3) \leq \alpha_1 \mathrm{d}(x_2, x_3) \leq \alpha_o \mathrm{d}(x_2, x_3) < k\mathrm{d}(x_2, x_3)$

By applying Lemma (2.12),$x_3 \epsilon\ \mathrm{T}x_2$if, then we can choose $x_4 \epsilon\ \mathrm{T}x_3$such that

$\mathrm{d}(x_3, x_4) < k\mathrm{d}(x_2, x_3)$

Assume that $x_3 \neq x_4$

From 4.9 we have

$\mathrm{d}(x_3, x_4) < k^3\ \mathrm{d}(x_0, x_1)$

Continuing this process, we obtain a sequence $\{x_n\} \subset$X such that for all n=1, 2, 3…

$x_n \epsilon\ \mathrm{T}x_{n-1}$,$x_{n-1} \neq x_n$and

$$\mathrm{d}(x_n, x_{n+1}) \leq \mathrm{kd}(x_{n-1}, x_n) \leq k^2\ \mathrm{d}(x_{n-2}, x_{n-3}) \leq k^3 \mathrm{d}(x_{n-3}, x_{n-2}) \leq \ldots \quad 4.13$$

and hence

$$\mathrm{d}(x_n, x_{n+1}) \leq k^n\ \mathrm{d}(x_o, x_1) \quad \ldots\ldots 4.14$$

Note that$\{x_n\}$ is an orbit of T at the initial pointx_o.

for m>n we obtain

$$\begin{aligned}
\mathrm{d}(x_n, x_m) &\leq \mathrm{s}[\mathrm{d}(x_n, x_{n+1}) + \mathrm{d}(x_{n+1}, x_m)] \\
&\leq \mathrm{s}[\mathrm{d}(x_n, x_{n+1}) + \mathrm{s}[\mathrm{d}(x_{n+1}, x_{n+2}) + \mathrm{d}(x_{n+2}, x_m)]] \\
&\leq (\mathrm{s}k^n + \mathrm{s}^2 k^{n+1} + \mathrm{s}^3 k^{n+2} + \mathrm{s}^4 k^{n+3} + \cdots + \mathrm{s}^{\mathrm{m}} k^{m-1})\ \mathrm{d}(x_o, x_1) \\
&\leq k^n s(1 + \mathrm{sk} + \mathrm{s}^2 k^2 + \mathrm{s}^3 k^3 + \mathrm{s}^4 k^4 + \cdots)\ \mathrm{d}(x_o, x_1) \\
&\leq \frac{k^n s}{1-sk}\ \mathrm{d}(x_o, x_1)
\end{aligned}$$

As m, n→ ∞, then $\lim_{m,n\to\infty} \mathrm{d}(x_n, x_m)=0$

Thus $\{x_n\}$ is a Cauchy sequence in X.

By T orbital of completeness of X, there exist $x_* \epsilon X$ such that

$\lim_{n\to\infty} x_n = x_*$

From 4.1 we have

$\mathrm{D}\ (x_{n+1}, \mathrm{T}x_*) \leq \mathrm{H}\ (\mathrm{T}x_n, \mathrm{T}x_*)$

$\leq \frac{D\ (x_n, \mathrm{T}x_*) + D\ (x_*, \mathrm{T}x_n)}{s^2 \sigma(x_n, \mathrm{T}x_n) + D\ (x_*, \mathrm{T}x_*) + 1} \mathrm{M}\ (x_n, x_*) \ \ + \mathrm{L}\ \mathrm{Min}\{d(x_n, x_*), D\ (x_*, \mathrm{T}x_n)\}$

$\leq \frac{D\ (x_n, T_{x_*}) + D\ (x_*, \mathrm{T}x_n)}{s^2 [\sigma(x_n, \mathrm{T}x_n) + D\ (x_*, \mathrm{T}x_*) + 1]} \mathrm{M}\ (x_n, x_*) \ \ + \mathrm{L}\ \mathrm{Min}\{d(x_n, x_*), D\ (x_*, \mathrm{T}x_n)\}$

$\leq \frac{D\ (x_{n,}\mathrm{T}x_*)+d(x_*,x_{n+1})}{s^2[D\ (x_*,T_{x_*})+1]}$ M (x_n, x_*) +L Min$\{d(x_n, x_*), d(x_*, x_{n+1})\}$

$\leq \frac{D\ (x_{n,}\mathrm{T}x_*)+d(x_*,x_{n+1})}{s^2[D\ (x_*,\mathrm{T}x_*)+1]}$ M (x_n, x_*) +L Min$\{d(x_n, x_*), d(x_*, x_{n+1})\}$

$\leq \frac{s[d(x_{n,}x_*)+D\ (x_{*,}\mathrm{T}x_*)]+d(x_*,x_{n+1})}{s^2[D\ (x_*,\mathrm{T}x_*)+1]}$ M (x_n, x_*) +L Min$\{d(x_n, x_*), d(x_*, x_{n+1})\}$

Then we have

$D\ (x_*, \mathrm{T}x_*) \leq s[d(x_*x_{n+1})+D\ (x_{n+1}, \mathrm{T}x_*)]$

$\frac{1}{s} D\ (x_*, \mathrm{T}x_*) \leq [d(x_*x_{n+1})+D\ (x_{n+1}, \mathrm{T}x_*)]$

$\leq d(x_*, x_{n+1}) + \frac{s[d(x_{n,}x_*)+D\ (x_{*,}\mathrm{T}x_*)]+d(x_*,x_{n+1})}{s^2[D\ (x_*,\mathrm{T}x_*)+1]}$ M (x_n, x_*) +L Min$\{d(x_n, x_*), d(x_*, x_{n+1})\}$ 4.16

We deduce that

M (x_n, x_*) =Max$\left\{\mathrm{d}(x_n, x_*), \mathrm{D}\left(x_{n,}\mathrm{T}x_n\right), D\left(x_*, T_{x_*}\right), \frac{1}{2s}[D\ (x_*, \mathrm{T}x_n) + D\ (x_n, \mathrm{T}x_*)]\right\}$.

$= \mathrm{Max}\left\{\mathrm{d}(x_n, x_*), \mathrm{d}\left(x_{n,}x_{n+1,}\right), D\ (x_*, \mathrm{T}x_*), \frac{1}{2s}[s[d(x_n, x_*) + D\ (x_*, \mathrm{T}x_*)] + D\ (x_*, \mathrm{T}x_n)]\right\}$.

$= \mathrm{Max}\left\{\mathrm{d}(x_n, x_*), \mathrm{d}\left(x_{n,}x_{n+1,}\right), D\ (x_*, \mathrm{T}x_*), \frac{1}{2s}[s[d(x_{xn}, x_*) + D\ (x_*, \mathrm{T}x_*)] + d\left(x_*x_{n+1,}\right)]\right\}$

Taking the limit as n→ ∞ we have

$$\lim_{n\to\infty} \mathrm{M}\ (x_n, x_*) = \mathrm{Max}\left\{0,0, D\ (x_*, \mathrm{T}x_*), \frac{D\ (x_*, \mathrm{T}x_*)}{2}\right\}$$

$$\lim_{n\to\infty} \mathrm{M}\ (x_n, x_*) = \mathrm{Max}\left\{D\ (x_*, \mathrm{T}x_*), \frac{D\ (x_*, \mathrm{T}x_*)}{2}\right\}$$

$$= \mathrm{Max}\left\{D(x_*, \mathrm{T}x_*), \frac{D(x_*,\mathrm{T}x_*)}{2}\right\} = D(x_*, \mathrm{T}x_*)$$

$$\lim_{n\to\infty} \mathrm{M}(x_n, x_*) = D\ (x_*, \mathrm{T}x_*)$$

Therefore

$\lim_{n\to\infty} \mathrm{M}(x_n, x_*) = D\ (x_*, \mathrm{T}x_*)$.. 4.17

Then we obtain

$\frac{1}{s} D(x_*, \mathrm{T}x_*) \leq d(x_*, x_{n+1}) + \frac{s[d(x_{n,}x_*)+D\ (x_{*,}\mathrm{T}x_*)]+d(x_*,x_{n+1})}{s^2[D\ (x_*,\mathrm{T}x_*)+1]}$ D (x_*, Tx_*)

+L Min$\{d(x_n, x_*), d(x_*, x_{n+1})\}$

$\frac{1}{s} D(x_*, \mathrm{T}x_*) \leq 0 + \frac{s(D(x_*,\mathrm{T}x_*))}{s^2[D(x_*,\mathrm{T}x_*)+1]}$D (x_*, Tx_*)

$\frac{1}{s} D(x_*, \mathrm{T}x_*) \leq \frac{(D(x_*,\mathrm{T}x_*))}{s[D(x_*,\mathrm{T}x_*)+1]}$D (x_*, Tx_*) =$\frac{(D(x_*,\mathrm{T}x_*))}{s[D(x_*,\mathrm{T}x_*)+1]}$ D (x_*, Tx_*)

$D(x_*, \mathrm{T}x_*) \leq \frac{(D(x_*, T_{x_*})}{D(x_*, T_{x_*})+1} \mathrm{D}\,(x_*, Tx_*) < D(x_{*,} \mathrm{T}x_*)$

Which is a contradiction if $D(x_{*,} \mathrm{T}x_*) \neq 0$

Therefore $D\big(x_{*,} \mathrm{T}x_*\big) = 0$

This implies, $x_* \,\epsilon \mathrm{T}x_*$, and so T has a fixed point

We now show that (2.3) and (2.4) are satisfied.

From (4.15) we have

$$d(x_n, x_*) \; \leq s[\mathrm{d}(x_n, x_m) + d(x_m, x_*)]$$

$\leq \frac{sk^n}{1-ks}$ $\mathrm{d}(x_o, x_1)$+ $d(x_m, x_*)$

Where m > n letting m→ ∞ in the above inequality, we have (2.3)

From (4.13) we have

$$d\big(x_n, x_{n+p}\big) \leq \mathrm{s}[\mathrm{d}(x_n, x_{n+1}) + \mathrm{d}\big(x_{n+1}, x_{n+p}\big)]$$

$$\leq \mathrm{s}[\mathrm{d}(x_n, x_{n+1}) + \mathrm{s}[\mathrm{d}(x_{n+1}, x_{n+2}) + \mathrm{d}\big(x_{n+2}, x_{n+p}\big)]$$

$$\leq \; (\mathrm{s}k + s^2k^2 + \; s^3k^3 + s^4k^4 + \cdots + s^pk^p \;)\; \mathrm{d}(x_{n-1}, x_n)$$

$$\leq \frac{ks(1 - s^pk^p)}{1 - ks} \; \mathrm{d}(x_{n-1}, x_n)$$

Letting p→ ∞ in the above inequality, we have (2.4)

4.2 Corollaries

COROLLARY 4.2.1: Let (X, d) be a complete b-metric space with coefficient s≥ 1 and T: X→ CB(X) be a set-valued mapping .suppose that X is T-orbitally complete if T satisfies the following condition. H $(T_x, T_y) \leq \frac{d(x,T_y)+d(y,T_x)}{s^2[\sigma(x,\mathrm{T}x_*)+d(y,\mathrm{T}y)+1]}$ N(x, y) +L Min$\{d(x, y), d(y, T_x)\}$

For all $x, y \in X$, L≥ O *and*

N(x, y) =Max$\Big\{\mathrm{d}(\mathrm{x}, y), \frac{1}{2\mathrm{s}}[\mathrm{d}(\mathrm{x}, T_x) + d\big(y, T_y\big)], \frac{1}{2s}[d\big(x, T_y\big) + d(y, T_x)]\Big\}$.then T has a fixed point in X.

COROLLARY 4.2.2: Let (X, d) be a complete b-metric space with coefficient s≥ 1 and T: X→ CB(X) be a set-valued mapping .suppose that X is T-orbitally complete if T satisfies the following condition. H $(Tx, Ty) \leq \frac{D(x,Ty)+D(y,Tx)}{s^2[\sigma(x,Tx)+d(y,Ty)+1]}$ d(x, y) +L Min$\{d(x, y), D(y, Tx)\}$

For all $x, y \in X$, L≥ O then T has a fixed point in X.

COROLLARY 4.2.3: Let (X, d) be a complete b-metric space with coefficient s≥ 1 and

T: X→ X be a mapping .suppose that X is T-orbitally complete.

If T satisfies the following condition:

$$H\ (Tx, Ty) \leq \frac{D(x,Ty)+D(y,Tx)}{s^2[\sigma(x,Tx)+d(y,Ty)+1]} \text{Max}\left\{\text{d(x, y), D(x,} Tx), D(y, Ty), \frac{1}{2s}[D(x, Tx) + D(y, Ty)]\right\} + \text{L}$$

$\min\{d(x, y), D(y, Tx)\}$

For all $x, y \in X$, L$\geq O$ then

(a) T has a fixed point in X

(b) d $(x_*, y_*) \geq \frac{1-L}{2}$,when ever x_*andy_* are two distinct fixed point of T;

(c) For each $x_o \epsilon$ X, the Picard iteration $\{x_n\}$ given by x_{n+1}=T_{xn}, n=0, 1, 2… converge to the fixed point x_* of T and the following estimate holds:

There exist a constant k ϵ (0, 1) such that

(i) for n=0,1,2,..,

$$\text{d}(x_n, x_*) \leq \frac{sk^n}{1-sk} \text{d}(x_0 x_1)$$

(ii) for n=1,2,3,…,

$$\text{d}\ (x_n, x_*) \leq \frac{sk}{1-sk} \text{d}\ (x_{n-1} x_n)$$

Proof: By taking single valued mapping in Theorem 4.1, we obtain (a) and (c)

To proof (b),

Let $x_*$and y_* are two distinct fixed point of T

We have

d (x_*, y_*) =D (Tx,Ty)

$$\leq \frac{D(x_*,Ty_*)+D(y_*,Tx_*)}{s^2 D(x_*,Tx_*)+dD(y_*,Ty_*)+1} \text{Max}\left\{\text{d}\ (x_*, y_*), D(x_*, Tx_*), D(y_*, Ty_*), \frac{1}{2s}[D(x_*, Ty_*) + D(y_*, Tx_*)]\right\}$$

+L Min{d $(x_*, y_*), D(y_*, Tx_*)\}$

$$\leq \frac{D(x_*,Ty_*)+D(y_* Tx_*)}{D(x_*,Tx_*)+D(y_*,Ty_*)+1} \text{Max}\left\{\text{d}\ (x_*, y_*), D(x_*, Tx_*), D(y_*, Ty_*), \frac{1}{2s}[D(x_*, Ty_*) + D(y_*, Tx_*)]\right\} +$$

L Min{d $(x_*, y_*), D(y_*, Tx_*)\}$

$$\leq \frac{\text{d}\ (x_*,y_*)+\text{d}\ (x_*,y_*)}{D(x_*,Tx_*)+D(y_*,Ty_*)+1} \text{Max}\left\{\text{d}\ (x_*, y_*), D(x_*, Tx_*), D(y_*, Ty_*), \frac{1}{2s}[D(x_*, \text{T}y_*) + d(y_*, y_*)]\right\} +$$

L Min{d $(x_*, y_*), D(y_*, Tx_*)\}$

$\leq \frac{d(x_*,y_*)+d(x_*,y_*)}{d(x_*,x_*)+d(y_*,y_*)+1}\text{Max}\left\{d(x_*,y_*), d(x_*,x_*), d(y_*,y_*), \frac{1}{2}[d(x_*,y_*)+d(y_*,y_*)]\right\}$ +

L $\text{Min}\{d(x_*,y_*), d(y_*,x_*)\}$

$= d(x_*,y_*) + d(x_*,y_*)\ \text{Max}\left\{d(x_*,y_*),0,0,\frac{1}{2}[d(x_*,y_*)+0]\right\}$ $+L\ \text{Min}\{d(x_*,y_*), d(y_*,x_*)\}$

$=2\ d(x_*,y_*)\ d(x_*,y_*) + Ld(x_*,y_*)$

$d(x_*,y_*) \leq d(x_*,y_*)(2\ d(x_*,y_*) + L)$

Hence $d(x_*,y_*) \geq \frac{1-L}{2}$ ■

CHAPTER 5

5.1 SUMMARY AND CONCLUSSIONS

In this Thesis, we study the contraction mapping, set valued contraction mapping, set valued contraction mapping in b-metric space and prove the existence of fixed point theorem for set valued contraction mapping in b-metric space. The result on this thesis generalized and extends several fixed point results in the literature.

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