

# **POSITIVE LINEAR FUNCTIONALS AND BOREL MEASURES**

**M. Sc. Graduate Seminar**

**By**

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**January 2011**

**Haramaya University**

# **POSITIVE LINEAR FUNCTIONALS AND BOREL MEASURES**

**A Graduate Seminar Submitted to the College of Natural and  
Computational Sciences, Department of Mathematics**

**School of Graduate Studies**

**HARAMAYA UNIVERSITY**

**In Partial Fulfillment of the Requirements for the Degree of MASTER OF  
SCIENCE IN MATHEMATICS (ANALYSIS)**

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**January 2011**

**Haramaya University**

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## **PREFACE**

The main purpose of this seminar is to discuss Borel measures and positive linear functionals. In addition to this it deals with the relationship between these two concepts.

This report basically consists of two chapters. In the first chapter, the introduction and preliminary concepts are included. The second chapter deals with the main concepts about positive linear functional and Borel measures.

## **ACKNOWLEDGEMENTS**

I would like to express my thanks and appreciation to my advisor Dr. Taddesse Zegeye for his unlimited and kind support and commitment starting from the beginning until the completion of the seminar.

I would like to thank Haramaya University Department of Mathematics for providing materials that are important in preparing the seminar.

My special thanks goes to my friends for their constructive ideas and suggestions in preparing the seminar.

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# CHAPTER 1

## INTRODUCTION AND PRELIMINARY CONCEPTS

### 1.1 INTRODUCTION

The importance of linear spaces lies mainly in the linear operators they carry, for vast parts of algebra and analysis, when placed in their proper context, reduce to the study of linear transformations of one linear space into another. The theory of matrices, for instance, is one small corner of this subject as are the theories of certain types of differential and integral equations.

A linear functional is a linear operator from a linear space to the set of real numbers. Indeed a linear functional is a homomorphism from the first space into the set of real numbers, as it is a mapping which preserves operations.

Measure and integration can be interpreted in two ways: by starting either from a measure and introducing integration as a derived concept or from a continuous linear real valued operator, an integral on a space of functions and considering measure as a derived concept. There are several fundamental and essential theorems that relate these two concepts. The Reisz Representation Theorem for positive linear functionals is among these theorems. In this seminar, this theorem will be discussed.

Though the theory can work in much generality, we shall only consider topological spaces which are locally compact and Hausdorff.

Given a locally compact Hausdorff space  $X$ . By  $C_c(X)$  we shall mean the space of continuous real-valued functions with compact support. A real-valued linear functional  $F$  on  $C_c(X)$  said to be positive if  $F(f) \geq 0$  whenever  $f \geq 0$ .

To this end, let  $C_c(X)$  be the space of continuous real-valued functions with compact support and  $M$  be the  $\sigma$  – algebra of Borel sets in  $X$ . If  $\mu$  is a positive measure on  $M$  which is finite on compact sets, then  $\mu$  gives rise to a positive linear functional  $F$  on  $C_c(X)$  and is given by

$$F(f) = \int f d\mu.$$

The study that positive linear functionals can be represented as an integral with respect to a suitable Borel measure was firstly established by Markov for the case when  $F$  is a bounded positive linear functional.

It was established in 1909 by F. Riesz for the case when  $X = [a, b]$ , by Radon in 1913 when  $X$  is a compact subset contained in  $\mathbb{R}^n$ , and by Banach in 1937 when  $X$  is a compact metric space. [3]

E. Borel in 1898 was the first to establish a measure theory on the subsets of the real number today known as Borel sets.

By a Borel measure, we shall mean the measure which is defined on the  $\sigma$  – algebra of Borel sets. Some Borel measures can be obtained out of some nonnegative extended real-valued functions.



## 1.2 PRELIMINARY CONCEPTS

**Definition 1.2.1** A set  $X$  of elements is called a vector space or a linear space or a linear vector space over the set of real numbers if we have a function  $+$  on  $X \times X$  and a function  $\cdot$  on  $\mathbb{R} \times X$  to  $X$  that satisfy the following conditions:

- i)  $x+y=y+x$
- ii)  $(x+y)+z=x+(y+z)$
- iii) There is a vector  $\theta$  in  $X$  such that  $x+\theta=x$ , for all  $x \in X$ .
- iv)  $(\alpha+\beta)x=\alpha x+\beta x$ , for  $\alpha, \beta \in \mathbb{R}$  and  $x, y \in X$ .
- v)  $(\alpha\beta)x=\alpha(\beta x)$ ,  $\alpha, \beta \in \mathbb{R}$ ,  $x \in X$ .
- vi)  $(\alpha\beta)x=\alpha(\beta x)$ ,  $\alpha, \beta \in \mathbb{R}$ ,  $x \in X$ .
- vii)  $\theta.x=\theta$  and  $1.x=x$ .

We call ' $+$ ' addition, and ' $\cdot$ ' multiplication by scalars.

**Remark 1.2.2** The element  $\theta$  defined in (iii) is unique.

**Proof:** Let  $\beta$  also has the same property with  $\theta$ .

$$\text{Then, } \theta = \theta + \beta = \beta + \theta = \beta.$$

**Definition 1.2.3** A nonnegative real-valued function  $\| \cdot \|$  defined on a vector space is called a norm if:

- i)  $\|x\| = 0 \Leftrightarrow x = \theta$ .
- ii)  $\|x+y\| \leq \|x\| + \|y\|$
- iii)  $\|\alpha x\| = |\alpha| \|x\|$

**Definition 1.2.4** A vector space together with the function  $\| \cdot \|$  defined on it is called a normed vector space.

**Example 1.2.5** Let  $E$  be the space of continuous functions on  $[0, 1]$ . For  $f \in E$ , define

$$\|f\|_1 = \int_0^1 |f(x)| dx.$$

Then,  $\| \cdot \|_1$  is a norm on  $E$  called the  $L^1$  - norm.

**Definition 1.2.6** A set  $X$  is called a metric space if we define a real-valued function  $d$  on  $X \times X$  such that for all  $x, y$ , and  $z$  in  $X$ :

- i)  $d(x, y) \geq 0$ ;
- ii)  $d(x, y) = 0$  if and only if  $x = y$ ;
- iii)  $d(x, y) = d(y, x)$ ;
- iv)  $d(x, z) \leq d(x, y) + d(y, z)$ .

**Definition 1.2.7** The function  $d$  satisfying the above four properties is called a metric.

**Example 1.2.8** The set of real numbers  $\mathbb{R}$  together with the function

$$d(x, y) = |x - y| \text{ is a metric space.}$$

**Example 1.2.9** The  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  whose points are  $n$ -tuples

$x = \langle x_1, x_2, \dots, x_n \rangle$  of real numbers together with the metric

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

is a metric space.

**Definition 1.2.10** A topology on non-empty set  $X$  is a collection of subsets of  $X$  with the following properties:

- i)  $\emptyset, X \in \mathcal{T}$
- ii) If  $\{U_i\}$  is a family of elements of  $\mathcal{T}$ , then  $\bigcup_{i=1}^{\infty} U_i \in \mathcal{T}$ , that is  $\mathcal{T}$  is closed under arbitrary union.
- iii) If  $\{U_1, U_2, U_3, \dots, U_n\}$  is a family of finite set of elements of  $\mathcal{T}$ , then  $\bigcap_{i=1}^n U_i \in \mathcal{T}$ . That is  $\mathcal{T}$  is closed under finite intersection.

**Definition 1.2.11** A set  $X$  together with the topology  $\mathcal{T}$  is called a topological space and is denoted by  $(X, \mathcal{T})$ .

**Example 1.2.12**  $\mathcal{T}_1 = \{\emptyset, X\}$  is a topology on  $X$  called the indiscrete topology.

**Example 1.2.13**  $\mathcal{T}_2 = P(X) = 2^X$ , the set consisting of all subsets of  $X$ , is a topology on  $X$  called the discrete topology.

**Definition 1.2.14** Let  $X$  be a topological space with topology  $\tau$ , we say a subset  $U$  of  $X$  is open in  $X$  if  $U \in \tau$ .

**Definition 1.2.15** A set is said to belong to  $G$  if it is a countable intersection of open sets.

**Definition 1.2.16** We say that a collection  $U$  of open sets in a topological space  $X$  is an open covering of  $X$  if  $X$  is contained in the union of the sets in  $U$ .

**Definition 1.2.17** A topological space  $X$  is said to be compact if every open covering of  $X$  has a finite sub-collection which also covers  $X$ , that is, if we can find a finite sub-collection  $\{O_1, O_2, \dots, O_n\} \subseteq U$  such that  $X = \bigcup_{i=1}^n O_i$

**Example 1.2.18** The space  $[0,1]$  is compact.

**Definition 1.2.19** In a topological space a subset is called  $\sigma$ -compact if it is the union of a countable number of compact sets.

**Definition 1.2.20** Given two topological spaces  $X$  and  $Y$ . A function  $f : X \rightarrow Y$  is said to be continuous if for every open subset  $O$  of  $Y$ ,  $f^{-1}(O)$  is open in  $X$ .

**Definition 1.2.21** Let  $E$  be a subset of a space  $X$ . A point  $x$  of  $X$  is called a limit point of  $E$  if the intersection of any open set containing  $x$  and  $E$  is non-empty.

**Definition 1.2.22** The set of all limit points of a set  $E$  is called the closure of  $E$  and is denoted by  $\bar{E}$ .

**Definition 1.2.23** A set is said to be closed if it is equal to its closure.

**Definition 1.2.24** If  $A$  is a subset of a space  $X$ , we define the complement  $A^c$  of  $A$  (relative to  $X$ ) to be the set of elements not in  $A$ , that is,  $A^c = \{x : x \notin A\}$

**Definition 1.2.25** Let  $X$  be a metric space. A subset  $O$  of  $X$  is called an open set if for every  $x \in O$ ,  $\exists \delta > 0$  such that every  $y$  with  $\rho(x, y) < \delta$  belongs to  $O$ .

**Definition 1.2.26** If  $f$  is a real-valued function on a topological space, the support of the function  $f$ , denoted by  $Supp f$ , is the closure of the set  $\{x : f(x) \neq 0\}$ , that is,

$$Supp f = \overline{\{x : f(x) \neq 0\}}$$

**Definition 1.2.27** A topological space  $X$  is called locally compact if for each  $x$  in  $X$ , there is an open set  $U$  containing  $x$  such that  $\bar{U}$  is compact. Thus,  $X$  is called locally compact if and only if the collection of open sets with compact closures forms a base for the topology of  $X$ .

**Example 1.2.28** The set of real numbers  $\mathbb{R}$  is locally compact but not compact.

**Definition 1.2.29** A topological space  $X$  is said to be Hausdorff space if for each pair  $x_1$  and  $x_2$  of distinct points of  $X$ , there exist neighborhoods  $U_1$  and  $U_2$  of  $x_1$  and  $x_2$  respectively such that

$$U_1 \cap U_2 = \emptyset$$

We say that a collection  $\{f_i\}$  of real-valued functions on  $X$  is subordinate to a covering  $\{O_i\}$  of a topological space  $X$  if the support of each  $f_i$  is contained in some  $O_i$ .

**Proposition 1.2.30**[3] Let  $\{O_i\}$  be an open covering of a compact subset  $K$  of a locally compact Hausdorff space  $X$ . Then, there is a finite collection  $\{f_1, f_2, \dots, f_n\}$  of continuous nonnegative real-valued functions subordinate to the collection  $\{O_i\}$  and such that

$$f_1 + f_2 + \dots + f_n = 1 \text{ on } K.$$

**Proof:** Let  $O$  be an open set with  $K \subseteq O$  and  $\bar{O}$  compact.

For each  $x_0 \in K$ , there is a continuous real-valued function  $f_{x_0}$  such that

$$f_{x_0}(x_0) = 1, 0 \leq f_{x_0} \leq 1 \text{ and } \text{supp } f_{x_0} \subset O \cap O_\lambda \text{ for some } \lambda.$$

For each  $x_0 \in \bar{O} - K$ , let  $g_{x_0}$  be a continuous real-valued function with

$$g_{x_0}(x_0) = 1, 0 \leq g_{x_0} \leq 1 \text{ and } \text{supp } g_{x_0} \subset \bar{O} - K.$$

By the compactness of  $\bar{O}$  we may choose a finite number  $f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_m$ , of these functions such that the sets where they are positive cover  $\bar{O}$ .

Set  $f = \sum_{i=1}^n f_i$  and  $g = \sum_{i=1}^m g_i$ .

Then,  $f > 0$  on  $K$ ,  $\text{supp } f \subset O$ ,  $f + g > 0$  on  $\bar{O}$ , and,  $g = 0$  on  $K$ .

Thus,  $\frac{f}{f+g}$  is continuous and equal to 1 on  $K$ . Taking  $f_i = \frac{fi}{f+g}$ ,

$\{f_1, f_2, \dots, f_n\}$  are the required functions. ■

**Definition 1.2.31** Let  $X$  be a non-empty set. A collection  $S$  of subsets of  $X$  is called a semi-ring if it satisfies the following conditions:

- i)  $\phi \in S$ .
- ii)  $A, B \in S \Rightarrow A \cap B \in S$ , that is,  $S$  is closed under finite intersection.
- iii) The set difference of any two sets in  $S$  can be written as a finite union of pair wise disjoint members of  $S$ , that is,  $A-B = \bigcup_{i=1}^n U_i$  whenever  $U_i \cap U_j = \phi$  for  $i \neq j$ .

**Definition 1.2.32** A non-empty collection  $S$  of subsets of a set  $X$  which is closed under finite intersection and complementation is called an algebra of sets (or simply an algebra). That is  $S$  is an algebra whenever it satisfies the following properties:

- i) If  $A, B \in S$ , then  $A \cap B \in S$ .
- ii) If  $A \in S$ , then  $A^c \in S$ .

**Example 1.2.33** For any non-empty set  $X$ ,  $S = \{\phi, X\}$  is an algebra. This is the smallest possible algebra.

**Example 1.2.34** For any non-empty set  $X$ , its power set,  $P(X)$  (i.e. the collection of all subsets of  $X$ ) forms an algebra. This is the largest possible algebra.

**Definition 1.2.35** An algebra  $S$  of subsets of a set  $X$  is called a  $\sigma$ -algebra if every union of a countable collection of members of  $S$  is again in  $S$ . That is,  $\bigcup_{n=1}^{\infty} A_n \in S$ .

From this one can see that  $\bigcap_{n=1}^{\infty} A_n = (\bigcup_{n=1}^{\infty} A_n^c)^c \in S$ , that is,  $S$  is closed under countable intersection. By a measurable space, we mean a couple  $(X, S)$  consisting of a set  $X$  and a  $\sigma$ -algebra  $S$  of subsets of  $X$ . A subset  $A$  of  $X$  is called measurable (or measurable with respect to  $S$ ) if  $A \in S$ .

**Example 1.2.36** Let  $X = \{a, b, c\}$  and let  $\beta = \{X, \emptyset, \{a\}, \{b, c\}\}$ . Then,  $(X, \beta)$  is a measurable space.

**Definition 1.2.37** Let  $X$  be a measurable space and  $Y$  be a topological space. A function  $f: X \rightarrow Y$  is said to be measurable if  $f^{-1}(O)$  is a measurable set for every open set  $O$  of  $Y$ . The constant function is seen to be measurable. To see this, let  $f(x) = c$  for all  $x \in X$  and  $O$  is open set of  $\mathbb{R}$ , then

$$f^{-1}(O) = \begin{cases} \emptyset & \text{if } c \notin O \\ X & \text{if } c \in O \end{cases}$$

Since  $X$  and  $\emptyset$  are measurable sets, we have  $f$  is measurable function.

**Definition 1.2.38** Let  $(X, \beta)$  be a measurable space, a measure  $\mu$  on  $(X, \beta)$  is a nonnegative set function defined for all elements of  $\beta$  and satisfying the following conditions:

- i)  $\mu(\emptyset) = 0$
- ii)  $\mu(\bigcup_{n=1}^{\infty} E_i) = \sum_{n=1}^{\infty} \mu(E_i)$  for any sequence  $E_i$  of pair wise disjoint measurable sets. By a measure space we mean a measurable space  $(X, \beta)$ , together with a measure  $\mu$  defined on  $\beta$  and is denoted by  $(X, \beta, \mu)$ .

The second property of  $\mu$  is often referred to as countable additivity of  $\mu$ . These properties show that  $\mu$  is also finitely additive.

**Example 1.2.39** Let  $X$  be a set,  $x_0 \in X$ .

If  $A$  is a subset of  $X$  containing  $x_0$ , we define  $\mu(A) = 1$ . If  $A$  does not contain  $x_0$ , we define  $\mu(A) = 0$ . Then,  $\mu$  is a measure called Dirac measure at  $x_0$ .

**Verification:**

- i) Since  $\emptyset$  does not contain any element of  $X$ , in particular it does not contain  $x_0$ . Hence  $\mu(\emptyset) = 0$ .
- ii) Let  $\{E_i\}$  be a sequence of pair wise disjoint measurable sets in  $X$ . Since they are disjoint,  $x_0$  is an element of at most one  $E_i$ . Here, we have two cases:
  - a) If  $x_0$  does not belong to  $E_i$  for all  $i$ , we have  $\mu(E_i) = 0$  for each  $i$ . Thus,

$$\mu(\bigcup_{i=1}^{\infty} E_i) = 0 \neq \sum_{i=1}^{\infty} \mu(E_i)$$

b) If  $x_0$  is an element of  $E_i$  for only one  $i$ , we have  $x_0 \in \bigcup_{i=1}^{\infty} E_i$ .

Thus,

$$\begin{aligned}\mu(\bigcup_{i=1}^{\infty} E_i) &= 1 = \mu(E_1) + \mu(E_2) + \mu(E_3) + \dots + \mu(E_i) + \mu(E_{i+1}) + \dots \\ &= 0 + 0 + 0 + \dots + 1 + 0 + 0 + \dots = \sum_{i=1}^{\infty} \mu(E_i)\end{aligned}$$

Hence,  $\mu$  is a measure.

**Example 1.2.40** Given any set  $X$ , if a subset  $E$  of  $X$  is finite, we define  $\mu E$  to be the number of elements in it, and if it is infinite,  $\mu E = \infty$ . This defines a measure called the counting measure.

**Example 1.2.41** Let  $S$  be a semi-ring in  $\mathbb{R}^n$  consisting of the empty set and all sets of the form

$A = \bigcup_{i=1}^n [a_i, b_i)$  where  $- \infty < a_i < b_i < \infty$  for each  $i$ . The set function

$\lambda : S \rightarrow [0, \infty)$  defined by  $\lambda(\emptyset) = 0$  and

$$\lambda\left(\bigcup_{i=1}^n [a_i, b_i)\right) = \sum_{i=1}^n (b_i - a_i)$$

is a measure called the Lebesgue measure on  $S$ .

**Theorem 1.2.42** [1] A subset  $E$  of  $\mathbb{R}^n$  is Lebesgue measurable if and only if for each  $\epsilon > 0$  there exists an open set  $O$  such that  $E \subseteq O$  and  $\lambda(O \setminus E) < \epsilon$ .

**Definition 1.2.43** A set function  $\mu : P(X) \rightarrow [0, \infty]$  defined on the power set,  $P(X)$ , of some set  $X$  is called an outer measure if it satisfies the conditions:

- i)  $\mu(\emptyset) = 0$
- ii)  $\mu(A) \leq \mu(B)$  if  $A \subset B$ , that is  $\mu$  is monotone.
- iii)  $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$  holds for every sequence  $\{A_n\}$  of subsets of  $X$ , that is,  $\mu$  is  $\sigma$ -subadditive.

**Definition 1.2.44** A subset  $E$  of  $X$  is called measurable with respect to  $\mu$  whenever

$$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c) \text{ for any subset } A \text{ of } X.$$

**Definition 1.2.45** An outer-measure  $\mu^*$  on a locally compact Hausdorff space  $X$  is said to be (topologically) regular if the following conditions are satisfied:

- i) For each  $E \subseteq X$ ,  $\mu^*(E) = \inf \{ \mu^*(O) : O \text{ is open, } E \subseteq O \}$ .
- ii)  $\mu^*(O_1 \cup O_2) = \mu^*(O_1) + \mu^*(O_2)$  if  $O_1$  and  $O_2$  are disjoint open sets.
- iii)  $\mu^*(O) = \sup \{ \mu^*(K) : K \subseteq O, K \text{ is compact} \}$ , for  $O$  open.

**Lemma 1.2.46** [3] If  $\mu^*$  is an outer measure on  $X$ , then each of the following is equivalent to (iii) of the definition of regularity of  $\mu^*$ .

- a)  $\mu^*(O) = \sup \{ \mu^*(K) : K \subseteq O, K \text{ is compact} \}$  for  $O$  open.
- b)  $\mu^*(O) = \sup \{ \mu^*(U) : \bar{U} \subseteq O, \bar{U} \text{ is compact, } U \text{ open} \}$  for  $O$  open.

**Definition 1.2.47** If  $X$  is any topological space, a Borel set in  $X$  is a set that can be formed from open sets through the operations of countable union, countable intersection, or complements.

**Example 1.2.48** Let  $X$  be any topological space. The collection  $\{ \emptyset, X \}$  is a topology on  $X$ . The open sets of  $X$  with respect to this topology are  $\emptyset$  and  $X$  itself. Now,  $\{ \emptyset, X \}$  is the smallest  $\sigma$ -algebra containing the open sets and hence it is the  $\sigma$ -algebra of Borel sets.

**Lemma 1.2.49** [1] Let  $\mu^*$  be a topologically regular outer measure. An arbitrary set  $E \subset X$  is  $\mu^*$ -measurable if and only if

$$\mu^*(O) = \mu^*(O \cap E) + \mu^*(O \cap E^c) \text{ for each open set } O \text{ of } X \text{ with } \mu^*(O) < \infty.$$

**Proof:** ( $\Rightarrow$ ) Suppose  $E$  is  $\mu^*$ -measurable, that is,

$$\mu^*(O) = \mu^*(O \cap E) + \mu^*(O \cap E^c).$$

$$\Rightarrow \mu^*(O) = \mu^*(O \cap E) + \mu^*(O \cap E^c)$$

( $\Leftarrow$ ) Suppose  $\mu^*(O) = \mu^*(O \cap E) + \mu^*(O \cap E^c)$  for each open set  $O$  in  $X$ , .....(i)

Now,  $\mu^*(O) = \mu^*(O \cap (E \cup E^c)) = \mu^*((O \cap E) \cup (O \cap E^c))$

$$= \mu^*(O \cap E) + \mu^*(O \cap E^c), \text{ by the } \mu^* \text{-subadditivity of } \mu^* \text{ .....(ii)}$$



Now, from (i), and (ii) we have

$$\mu^*(O) = \mu^*(O \cap E) + \mu^*(O \cap E^c).$$

Therefore,  $E$  is measurable with respect to  $\mu^*$ . ■

**Definition 1.2.50** A set  $E$  in a locally compact Hausdorff topological space  $X$  is called (topologically) bounded if  $E$  is contained in some compact set. That is,  $\bar{E}$  is compact.

**Definition 1.2.51** A set  $E$  in a topological space  $X$  is said to be  $\sigma$ -bounded if it is the union of a countable collection of bounded sets.

**Definition 1.2.52** A mapping  $F$  of a vector space  $X$  in to a vector space  $Y$  is called a linear operator if  $F(a_1x_1+a_2x_2) = a_1F(x_1)+a_2F(x_2)$ , for all  $x_1, x_2 \in X$  and all  $a_1, a_2 \in \mathbb{R}$ .

**Definition 1.2.53** If  $X$  and  $Y$  are normed vector spaces and  $F$  is a linear operator from  $X$  to  $Y$ , we say that  $F$  is bounded if there exists a constant  $M$  such that for all  $x$  we have  $\|Fx\| \leq M \|x\|$ . We call the least such  $M$  the *norm* of  $F$  and denote it by  $\|F\|$ . Thus,

$$\|F\| = \sup_{x \neq 0} \frac{\|Fx\|}{\|x\|}$$

**Example 1.2.54** Consider the linear space  $\mathbb{R}^2$ . The following maps are seen to be linear operators of  $\mathbb{R}^2$  on to itself.

i)  $T_1((x_1, x_2)) = (\eta x_1, \eta x_2)$  where  $\eta$  is a real number. Now,

$$\begin{aligned} T_1(\alpha(x_1, x_2) + \beta(x_3, x_4)) &= T_1((\alpha x_1 + \beta x_3), (\alpha x_2 + \beta x_4)) \\ &= (\eta(\alpha x_1 + \beta x_3), \eta(\alpha x_2 + \beta x_4)) \\ &= (\eta\alpha x_1 + \eta\beta x_3, \eta\alpha x_2 + \eta\beta x_4) \\ &= (\eta\alpha x_1, \eta\alpha x_2) + (\eta\beta x_3, \eta\beta x_4) \\ &= \alpha(\eta x_1, \eta x_2) + \beta(\eta x_3, \eta x_4) \\ &= \alpha T_1(x_1, x_2) + \beta T_1(x_3, x_4). \end{aligned}$$

Therefore,  $T_1$  is a linear operator.

- ii)  $T_2((x_1, x_2)) = (x_2, x_1)$  is a linear operator which reflects  $\mathbb{R}^2$  about the diagonal line  $x_1 = x_2$ .
- iii)  $T_3((x_1, x_2)) = (x_1, 0)$  is a linear operator which projects  $\mathbb{R}^2$  on to the  $x_1$ -axis.
- iv)  $T_4((x_1, x_2)) = (0, x_2)$  is a linear operator which projects  $\mathbb{R}^2$  on to the  $x_2$ -axis.

**Example 1.2.55** Consider the linear space  $P$  of all polynomials with real coefficients defined on  $[0, 1]$ .

The mapping  $D$  defined by  $D(p) = \frac{dp}{dx}$  is a linear operator of  $P$  in to itself.

**Definition 1.2.56** A measurable function  $f$  defined on  $[0, 1]$  is said to belong to the space

$L^p = L^p[0, 1]$  if the integral  $\int_0^1 |f|^p$  is finite. That is,  $\int_0^1 |f|^p < \infty$ , that is  $|f|^p$  is integrable on the interval  $[0, 1]$ .

**Definition 1.2.57** Let  $X$  be locally compact Hausdorff space. By  $C_c(X)$ , we denote the space of continuous real-valued functions with compact support (that is, real valued functions which vanish outside a compact set.)

**Definition 1.2.58** An  $n \times n$  symmetric matrix  $M$  is said to be positive definite if  $z^T M z > 0$  for all non-zero vectors  $z$  with real entries ( $z \in \mathbb{R}^n$ ), where  $z^T$  is the transpose of  $z$ .

**Definition 1.2.59** An  $n \times n$  Hermitian matrix is positive definite if  $z^* M z$  is positive for all non-zero complex vectors  $z$ , where  $z^*$  denotes the conjugate transpose of  $z$ . A Hermitian matrix or self adjoint matrix is a square matrix with complex entries that is equal to its own conjugate transpose.

**Example 1.2.60** Let  $A = \begin{bmatrix} 3 & 2+i \\ 2-i & 1 \end{bmatrix}$ . Then, the transpose  $A^T$  is given by

$$A^T = \begin{bmatrix} 3 & 2-i \\ 2+i & 1 \end{bmatrix} \text{ and the conjugate of the transpose } \overline{A^T} \text{ is given by}$$

$$\overline{A^T} = \begin{bmatrix} 3 & 2+i \\ 2-i & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2+i \\ 2-i & 1 \end{bmatrix} = A$$

The quantity  $z^* M z$  is always real because  $M$  is Hermitian.

**Example 1.2.61** Let  $M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Let  $z$  be a vector with entries  $z = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$ .

Then  $\begin{bmatrix} z_0 & z_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = z_0^2 + z_1^2$ , where the entries  $z_0, z_1$  are real and at least one of them is non-zero. Hence,  $M_0$  is positive definite matrix.

**Definition 1.2.62** A topological space  $X$  is said to be normal if it is Hausdorff, and if given two disjoint closed sets  $A, B$  in  $X$ , then there exist disjoint open sets  $U, V$  such that  $A \subset U$  and  $B \subset V$ .

**Theorem 1.2.63** [4] (**Urysohn's Lemma version I**)

Let  $X$  be a normal space and  $A, B$  be disjoint closed sets in  $X$ . Then, there exists a continuous function  $f$  on  $X$  with values in the interval  $[0, 1]$  such that  $f(A) = 0$  and  $f(B) = 1$ .

**Corollary 1.2.64** [4] Let  $X$  be a locally compact Hausdorff space, and  $K$  a compact subset. There exists a continuous function  $g$  on  $X$  which is 1 on  $K$  and which is equal to 0 outside a compact set.

**Proof:** Since  $X$  is locally compact, each  $x \in K$  has an open neighborhood  $V_x$  with compact closure. Since  $K$  is compact, a finite number of such neighborhoods  $V_{x_1}, V_{x_2}, \dots, V_{x_n}$  covers  $K$ . Let

$$V = V_{x_1} \cup V_{x_2} \cup \dots \cup V_{x_n}.$$

Then the closure of  $V$  is compact. There exists a continuous function  $g = 0$  on  $\bar{V}$  (compact, Hausdorff and hence normal) which is 1 on  $K$  and 0 outside  $V$ , that is 0 on  $\bar{V} \cap V^c$ . Now, we define  $g$  to be 0 on the complement of  $\bar{V}$  in  $X$ . Then,  $g$  is continuous at every point in the complement of  $\bar{V}$ , and as function on  $X$  is also continuous on  $\bar{V}$ . ■

**Proposition 1.2.65** [4] Let  $\mathcal{A}$  be the collection of all subsets  $A$  of  $X$  such that  $\mu(A) < \infty$ , and

$$\mu(A) = \sup\{\mu(K) \text{ for } K \subseteq A, K \text{ compact}\}.$$

Then,  $\mathcal{A}$  is an algebra containing all compact sets and all open sets of finite measure.

Furthermore,  $\mu$  is a positive measure on  $\Omega$ . In fact, if  $\{A_n\}$  is a disjoint sequence of elements of  $\mathcal{A}$ , and  $A = \bigcup A_n$ , then  $\mu(A) = \sum \mu(A_n)$ . If in addition  $\mu(A) < \infty$ , then  $A \in \mathcal{A}$ .

**Proposition 1.2.66** [4] Let  $\{A_i\}$  be a countable family of sets whose union is equal to  $X$ . For each  $i$ , let  $N_i$  be a  $\sigma$ -algebra of subsets of  $A_i$ . Let  $N$  be the collection of subsets  $Y$  of  $X$  such that

$Y \cap A_i \in N_i$  for all  $i$ . Then,  $N$  is a  $\sigma$ -algebra in  $X$ .

**Definition 1.2.67** Let  $X$  be a measurable space and  $E \subset X$ . The characteristic function  $\chi_E$  of  $E$  is the function defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \text{ is in } E \\ 0 & \text{otherwise} \end{cases}.$$

## CHAPTER 2

### POSITIVE LINEAR FUNCTIONALS AND BOREL MEASURES

#### 2.1 BOREL MEASURES

**Definition 2.1.1** Let  $X$  be a locally compact Hausdorff space and  $\mathcal{M}$  be the  $\sigma$ -algebra of all Borel sets on  $X$ . A measure  $\mu$  defined on  $\mathcal{M}$  is called a Borel measure on  $X$ .

We shall assume that  $\mu$  is always finite on compact sets.

**Definition 2.1.2** A measure  $\mu$  on the  $\sigma$ -algebra of Borel sets is called a regular Borel measure if it satisfies the following conditions:

- i)  $\mu(K) < \infty$  for every compact subset  $K$  of  $X$ .
- ii) If  $B$  is a Borel subset of  $X$ , then
$$\mu(B) = \inf \{ \mu(O) : O \text{ open and } B \subseteq O \}.$$
- iii) If  $O$  is an open subset of  $X$  or  $O$  is any element of  $\mathcal{M}$ , then
$$\mu(O) = \sup \{ \mu(K) : K \text{ compact and } K \subseteq O \}.$$

In condition (ii), we say that  $B$  is outer regular for  $\mu$ , or  $\mu$  is outer regular for  $B$ , and in (iii),  $\mu$  is inner regular for  $O$ .

$\mu$  is said to be quasi-regular if it is outer regular and each open Borel set  $O$  is inner regular for  $\mu$ .

The following theorem realizes that Lebesgue measure is a Borel regular measure.

**Theorem 2.1.3** The Lebesgue measure on  $\mathbb{R}^n$  is a Borel regular measure.

**Proof:** In order to show this, we need to show that the three conditions in the definition of Borel regular measure are satisfied.

- (1) Let  $K$  be a compact subset of  $\mathbb{R}^n$ . Then,  $K$  is bounded and so there exists some

$$A = \bigcup_{i=1}^n [a_i, b_i) \text{ with } K \subseteq A.$$

Therefore,  $\lambda(K) = \lambda(A) < \infty$ .

- (2). Let  $B$  be a Borel set and let  $\epsilon > 0$ . By Theorem 1.2.40 there exists an open set  $V$  such that

$B \subseteq V$  and  $\lambda(V \setminus B) < \epsilon$  where  $\lambda$  is the Lebesgue measure. There fore,

$$\lambda(B) = \inf \{ \lambda(O) : O \text{ is open and } B \subseteq O \}$$

$$\lambda(V) = \lambda(V \setminus B) + \lambda(B)$$

$$\lambda(B) +$$

holds for all  $\epsilon > 0$  from which the desired equality follows.

(3). Let  $O$  be an open subset of  $\mathbb{R}^n$ . Pick a sequence  $\{K_i\}$  of compact sets with

$O = \bigcup_{i=1}^{\infty} K_i$ ; for instance, let  $\{K_1, K_2, \dots\}$  be an enumeration of all closed balls with rational centers and rational radii that are included in  $O$ . Now for each  $i$ , let  $C_i = \bigcup_{m=1}^i K_m$ , and note that each  $C_i$  is compact and  $C_i \uparrow O$ .

$$\Rightarrow \lambda(C_i) \uparrow \lambda(O), \text{ and so}$$

$$\lambda(O) = \sup \{ \lambda(K) : K \text{ is compact and } K \subseteq O \} \text{ holds.}$$

There fore, from (1), (2), and (3) we have the Lebesgue measure on  $\mathbb{R}^n$  is Borel regular measure. ■

**Theorem 2.1.4** Let  $\mu^*$  be a topologically regular outer measure on  $X$ . Then, each Borel set in  $X$  is  $\mu^*$ -measurable, that is,  $\mu^*$  is Borel measure.

**Proof:** Since the  $\mu^*$ -measurable sets form a  $\sigma$ -algebra, it is sufficient to show that each closed set  $F$  is  $\mu^*$ -measurable. To see this, let  $O$  be any open set with  $\mu^*(O) < \infty$  and  $\epsilon$  an arbitrary positive number. Then,  $O \cap F^c$  is an open set of finite outer measure. Then, by property (b) of lemma 1.2.46, there exists an open set  $U$  with  $\bar{U} \subset O \cap F^c$  and  $\mu^*(U) > \mu^*(O \cap F^c) - \epsilon$ .

Let  $V = O - \bar{U}$ , then  $V \cap U = \emptyset$  and  $O \cap F \subset V$ . Hence,

$$\begin{aligned} \mu^*(O \cap F) + \mu^*(O \cap F^c) &< \mu^*V + \mu^*U + \epsilon \\ &= \mu^*(U \cup V) + \epsilon \end{aligned}$$

$$< \mu^*(O) +$$

Since  $\epsilon$  is arbitrary, we have

$$\mu^*(O \cap F) + \mu^*(O \cap F^c) = \mu^*(O) \text{ and so } F \text{ is } \mu^* \text{-measurable. } \blacksquare$$

In the next proposition, we will define a topologically regular outer measure in terms of a nonnegative extended real-valued function and this outer measure in turn becomes a regular Borel measure.

**Proposition 2.1.5** Let  $\bar{\mu}$  be a nonnegative extended real-valued function defined on the class of open sets of  $X$  and satisfying;

- i)  $\bar{\mu}(O) < \infty$  if  $\bar{O}$  is compact,
- ii)  $\bar{\mu}(O_1) \leq \bar{\mu}(O_2)$  if  $O_1 \subseteq O_2$ ,
- iii)  $\bar{\mu}(O_1 \cup O_2) = \bar{\mu}O_1 + \bar{\mu}O_2$  for disjoint  $O_1$  and  $O_2$ .
- iv)  $\bar{\mu}(\bigcup O_i) \leq \sum \bar{\mu}O_i$ ,
- v)  $\bar{\mu}(O) = \sup \{ \bar{\mu}U : \bar{U} \subseteq O, \bar{U} \text{ is compact} \},$

Then, the set function  $\mu^*$  defined by

$$\mu^*(E) = \inf \{ \bar{\mu}O : E \subseteq O \} \text{ is topologically regular outer measure and hence Borel measure.}$$

**Proof:** The monotonicity and countable subadditivity of  $\mu^*$  follow directly from (ii) and (iv) and the definition of  $\mu^*$ . Also  $\mu^*O = \bar{\mu}O$  for  $O$  open and so condition (b) of lemma 1.2.46 follows from (v). Condition (ii) of the definition of regularity follows from hypothesis (iii) of the proposition and condition (i) from the definition of  $\mu^*$ . Since  $\bar{\mu}O < \infty$  for  $\bar{O}$  compact, we have  $\mu^*E < \infty$  for each bounded set  $E$ .

$$\Rightarrow \mu^* \text{ is topologically regular outer-measure.}$$

$$\Rightarrow \text{By theorem 2.1.4, } \mu^* \text{ is Borel measure. } \blacksquare$$

**Theorem 2.1.6** If  $\mu$  is a regular Borel measure on a topological space, then

$$\mu(B) = \sup\{\mu(K): K \text{ is compact } K \subset B\}$$

holds for each Borel set  $B$  with  $\mu(B) < \infty$ .

**Proof:** Let  $B$  be a Borel set with  $\mu(B) < \infty$ , and let  $\epsilon > 0$ . Pick an open set  $V$  with  $B \subseteq V$  and  $\mu(V) < \mu(B) + \epsilon$ .

Similarly, choose an open set  $W$  such that  $V \setminus B \subseteq W \subseteq V$  and

$$\mu(W) < \mu(V \setminus B) + \epsilon = \mu(V) - \mu(B) + \epsilon < 2\epsilon.$$

Next, choose a compact set  $C$  such that

$$C \subseteq V \text{ and } \mu(V) < \mu(C) + \epsilon$$

Put  $K = C \cap W^c$ , and note that  $K$  is compact subset of  $B$ .

Moreover,

$$\begin{aligned} 0 & \leq \mu(B) - \mu(K) = \mu(B \setminus K) = \mu(V \setminus K) \\ & = \mu((V \setminus C) \cup W) \\ & \leq [\mu(V) - \mu(C)] + \mu(W) < 3\epsilon. \end{aligned}$$

Thus,

$$\begin{aligned} 0 & \leq \mu(B) - \mu(K) < 3\epsilon. \\ \Rightarrow \mu(B) & < \mu(K) + 3\epsilon. \end{aligned}$$

Since  $\epsilon$  is arbitrary, we can let it to be very small and we have

$$\mu(B) = \sup\{\mu(K): K \text{ is compact and } K \subset B\} \text{ and hence the proof. } \blacksquare$$



**Definition 2.1.7** Let  $\mathcal{K}$  be a family of compact sets containing the compact  $G$ 's and having the property that  $K_1 \cup K_2 \in \mathcal{K}$  and  $K_1 \cap K_2 \in \mathcal{K}$  whenever  $K_1, K_2 \in \mathcal{K}$ . A nonnegative real-valued function  $\lambda$  defined on  $\mathcal{K}$  is called a content if:

- i)  $\lambda(K_1) \leq \lambda(K_2)$  whenever  $K_1 \subset K_2$ .
- ii)  $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$  when  $K_1 \cap K_2 = \emptyset$ .

If in addition  $\lambda$  satisfies the condition

- iii)  $\lambda(K) = \inf \{ \lambda(H) : K \subset H^0, H \in \mathcal{K} \}$ , where  $H^0$  is the interior of  $H$ , we say that it is a regular content.

**Proposition 2.1.8** Let  $\lambda$  be a content on a class  $\mathcal{K}$  of compact sets. Then, there is a unique quasi regular Borel measure  $\bar{\mu}$  such that for each open set  $O$  we have

$$\bar{\mu}O = \sup \{ \lambda(K) : K \subset O, K \in \mathcal{K} \}.$$

Moreover,

$$\bar{\mu}K^0 \leq \lambda K \leq \bar{\mu}K$$

for  $K \in \mathcal{K}$ . If  $\lambda$  is a regular content, then  $\lambda K = \bar{\mu}K$  for all  $K \in \mathcal{K}$ .

**Proof:** For each open set  $O$  let

$$\bar{\mu}O = \sup \{ \lambda(K) : K \subset O, K \in \mathcal{K} \}.$$

Then, properties (i), (ii), (iii), and (v) of proposition 2.1.5 follow directly from the definition of  $\bar{\mu}$  and the definition of  $\lambda$  as a content.

To see that (iv) also holds, let  $O = \bigcup_{i=1}^{\infty} O_i$  and take a compact set  $K \subset O$  with

$K \subset O$ . By proposition 1.2.30 there are nonnegative continuous real-valued functions  $\{ \varphi_1, \varphi_2, \dots, \varphi_n \}$  with compact support such that

$$\sum_{i=1}^n \varphi_i = 1 \text{ on } K$$

and  $\text{supp } \mu_i \subset O_i$ . Let  $G_i = \{x: \mu_i(x) < \frac{1}{n}\}$ . Then, each  $G_i$  is a compact  $G$  and thus in  $\mathcal{G}$ . Consequently, each  $G_i \cap K$  is in  $\mathcal{G}$ . We also have

$$K = \bigcup_i G_i \cap K \text{ and } G_i \cap K \subset O_i.$$

$$\text{Thus, } \lambda(K) = \sum_{i=1}^n \lambda(G_i \cap K)$$

$$= \sum_{i=1}^n \mu(O_i)$$

$$= \sum_{i=1}^{\infty} \mu(O_i).$$

Taking the supremum over all such  $K$  gives (iv). By proposition 2.1.5 and Theorem 2.1.4 the set function  $\bar{\mu}$  extends to a quasi-regular Borel measure  $\bar{\mu}$ . Since  $\bar{\mu}$  is outer regular, we have

$$\bar{\mu}K = \inf \{ \bar{\mu}O, O \text{ open}, K \subset O \}.$$

$$= \lambda(K).$$

Also,

$$\bar{\mu}K^0 = \sup \{ \lambda H: H \text{ closed}, H \subset K^0 \} = \lambda(K^0). \text{ Thus,}$$

$$\bar{\mu}K^0 \leq \lambda K \leq \bar{\mu}K.$$

If  $\mu$  is regular,  $\bar{\mu}K = \inf \{ \bar{\mu}O, K \subset O, O \text{ open} \}.$

$$= \inf \{ \bar{\mu}H^0: K \subset H^0, H \text{ closed} \}.$$

$$= \inf \{ \lambda H: K \subset H^0, H \text{ closed} \} = \lambda(K).$$

Thus,

$$\bar{\mu}K = \lambda K \text{ for all } K \text{ closed}.$$

## 2.2 POSITIVE LINEAR FUNCTIONALS

**Definition 2.2.1** A linear functional  $F$  on a vector space  $X$  is a linear operator from  $X$  to the space  $\mathbb{R}$  of real numbers.

**Example 2.2.2** The mapping  $I$  defined by

$I(f) = \int_0^1 f(x)dx$ , is a linear functional of the set of all continuous functions defined on  $[0,1]$  in to the linear space  $\mathbb{R}$ .

**Example 2.2.3** Let  $g$  be a function in  $L^q$ , then, the function defined by  $F(f) = \int fg$

on  $L^p$ , where  $p$  and  $q$  are nonnegative extended real numbers such that  $\frac{1}{p} + \frac{1}{q} = 1$ , is linear functional.

**Verification:**

$$\begin{aligned} F(a_1f_1 + a_2f_2) &= (a_1f_1 + a_2f_2)g \\ &= a_1f_1g + \int a_2f_2g \\ &= a_1 \int f_1g + a_2 \int f_2g \\ &= a_1F(f_1) + a_2F(f_2). \end{aligned}$$

Hence  $F$  is a linear functional.

In the following theorem, we are going to extend linear functionals from subspaces to the whole spaces.

**Theorem 2.2.4 (Hahn-Banach)**

Let  $g$  be a real-valued function defined on the vector space  $X$  satisfying

$$g(x+y) \leq g(x) + g(y) \text{ and } g(\alpha x) = \alpha g(x) \text{ for each } \alpha \geq 0.$$

Suppose that  $f$  is a linear functional defined on a subspace  $S$  and that  $f(s) \leq g(s)$  for all  $s$  in  $S$ . Then, there is an extension  $F$  of  $f$  defined on  $X$  such that  $F(x) \leq g(x)$  and  $F(s) = f(s)$  for all  $s$  in  $S$ .

**Proposition 2.2.5** Let  $x$  be an element in a normed vector space  $X$ . Then, there is a linear functional  $F$  on  $X$  such that  $F(x) = \|F\| \|x\|$ .

**Proof:** Let  $S$  be the subspace consisting of all multiples of  $x$ , and define  $f$  on  $S$  by

$f(\alpha x) = \alpha \|x\|$  and set  $f(y) = 0$ . Then,  $f$  is a linear functional on  $S$ .

Then, by the Hahn- Banach theorem, there is an extension  $F$  of  $f$  to be a linear functional on  $X$  such that  $F(y) = f(y)$  and  $F(\alpha x) = \alpha \|x\|$ .

Since  $F(\alpha x) = \alpha \|x\|$ , we have  $\|F\| = 1$ . Also,  $F(x) = \|x\| = \|F\| \|x\|$ . Thus,

$$\|F\| = 1 \text{ and } F(x) = \|F\| \|x\|. \blacksquare$$

**Proposition 2.2.6** Let  $T$  be a linear subspace of a normed linear space  $X$  and  $y$  an element of  $X$  whose distance from  $T$  is at least  $\alpha$ , that is an element such that

$$\|y - t\| \geq \alpha \text{ for all } t \text{ in } T.$$

Then, there is a linear functional  $f$  on  $X$  with  $\|f\| = 1$ ,  $f(y) = \alpha$  and such that  $f(t) = 0$  for all  $t$  in  $T$ .

**Proof:** Let  $S$  be the subspace spanned by  $t$  and  $y$ , that is, the subspace consisting of all elements of the form  $y + t$  with  $t \in T$ . Define

$$f(y + t) = \alpha.$$

Then,  $f$  is a linear functional on  $S$  and since  $\|y + t\| = \left\| y + \frac{t}{\alpha} \right\|$  for  $\alpha > 0$ ,

If  $\alpha = 0$ , we have  $f(t) = f(0 \cdot y + t) = 0$ .  $\alpha = 0$  and  $\|y + t\| = \|t\| = 0$ . Hence, we have

$$f(s) = \|s\| \text{ for all } s \text{ in } S.$$

By the Hahn- Banach theorem, we can extend  $f$  to all of  $X$  so that  $f(x) = \|x\|$ . This implies that

$$\|f\| = 1.$$

More over,  $f(y) = f(1 \cdot y + 0) = \alpha$ .  $\blacksquare$

**Definition 2.2.7** A linear functional  $F$  on  $C_c(X)$  is said to be positive if  $F(f) \geq 0$  whenever  $f \in C_c(X)$  and  $f \geq 0$ .

**Example 2.2.8** Consider the  $\sigma$ -algebra of complex square matrices. The positive elements are the positive definite matrices. The trace function defined on this  $\sigma$ -algebra is a positive linear functional, as the eigenvalues of any positive definite matrix are positive.

Note: the trace of a square matrix is defined to be the sum of the elements on the main diagonal. Equivalently, the trace of a matrix is the sum of the eigenvalues of the matrix.

**Definition 2.2.9** Let  $C_c(X)$  be the space of continuous real-valued functions with compact support. We write  $f \in V$  to mean that

- i)  $f \in C_c(X)$  and is real.
- ii)  $V$  is an open subset of  $X$ .
- iii)  $\text{Supp } f \subset V, 0 \leq f \leq 1$ .

And we write  $K \subset V$  to mean that

- i)  $K$  is compact subset of  $X$ .
- ii)  $f \in C_c(X), 0 \leq f \leq 1$ , and  $f = 1$  on  $K$ .

**Theorem 2.2.10 (Urysohn's Lemma Version II)**

Let  $X$  be a locally compact Hausdorff space,  $V$  open subset of  $X$ , and  $K \subset V$  be compact. Then there exists  $f \in C_c(X)$  such that  $K \subset f^{-1}(1) \subset V$ .

The next theorem shows that every positive linear functional on  $C_c(X)$  can be represented by integration with respect to a suitable Borel measure.

**Theorem 2.2.11 (Riesz- Markov)**

Let  $X$  be a locally compact Hausdorff space and  $I$  be a positive linear functional on  $C_c(X)$ . Then there exists a Borel measure  $\mu$  on  $X$  such that

$$I(f) = \int f d\mu$$

for each  $f \in C_c(X)$ . The measure  $\mu$  may be taken to be quasi regular or to be inner regular. In each of these cases it is then unique.

**Proof:** For each open set  $O$ , define  $\bar{\mu}O$  by

$$\bar{\mu}O = \sup \{ I(f) : f \in C_c(X), 0 \leq f \leq 1, \text{supp } f \subset O \}.$$

Then  $\bar{\mu}$  is an extended real-valued function defined on all open sets and is monotone, finite on bounded sets, and satisfies the regularity (v) of proposition 2.1.5. To see that  $\bar{\mu}$  is countably subadditive on open sets, let

$$O = \bigcup O_i$$

and let  $f$  be any function in  $C_c(X)$  with  $0 \leq f \leq 1$  and  $\text{supp } f \subset O$ . By proposition 1.2.30, there are nonnegative functions  $\varphi_1, \varphi_2, \dots, \varphi_n$  in  $C_c(X)$  with  $\text{supp } \varphi_i \subset O_i$ , and

$$\sum_{i=1}^n \varphi_i = 1 \text{ on } \text{supp } f$$

Then  $f = \sum_{i=1}^n \varphi_i f$ ,  $0 \leq \varphi_i f \leq 1$ , and  $\text{supp } (\varphi_i f) \subset O_i$ . Thus

$$\begin{aligned} I(f) &= \sum_{i=1}^n I(\varphi_i f) \leq \sum_{i=1}^n \bar{\mu} O_i \\ &\leq \sum_{i=1}^{\infty} \bar{\mu} O_i. \end{aligned}$$

Taking the supremum over all such  $f$  gives

$$\bar{\mu}O \leq \sum_{i=1}^{\infty} \bar{\mu} O_i, \text{ and hence } \bar{\mu} \text{ is countably subadditive.}$$

If  $O = O_1 \cup O_2$  with  $O_1 \cap O_2 = \emptyset$  and  $f_i \in C_c(X)$ ,  $0 \leq f_i \leq 1$ , and  $\text{supp } f_i \subset O_i$ , then the function  $f = f_1 + f_2$  has  $\text{supp } f \subset O$  and  $0 \leq f \leq 1$ . Thus

$$I(f_1) + I(f_2) \leq \bar{\mu}O.$$

Since  $f_1$  and  $f_2$  can be chosen arbitrarily, subject to  $0 \leq f_i \leq 1$  and  $\text{supp } f_i \subset O_i$ , we have

$$\bar{\mu}O_1 + \bar{\mu}O_2 \leq \bar{\mu}O,$$

whence

$$\bar{\mu}O_1 + \bar{\mu}O_2 = \bar{\mu}O.$$

Thus,  $\bar{\mu}$  satisfies the hypotheses of proposition 2.1.5. Therefore,  $\bar{\mu}$  is a topologically regular outer measure and hence it extends to a quasi regular Borel measure.

We next proceed to show that  $I(f) = \int f d\bar{\mu}$  for each  $f \in C_c(X)$ .

If we prove  $I(f) \leq \int f d\bar{\mu}$ , for every real  $f$ , then  $-f \in C_c(X)$  and

$$-I(f) = I(-f) \leq \int (-f) d\bar{\mu} = -\int f d\bar{\mu} \text{ which implies that } I(f) \geq \int f d\bar{\mu} \text{ and hence } I(f) = \int f d\bar{\mu}.$$

Put  $K = \text{supp } f$ . Let  $f(X) \subset [a, b]$ .

For any  $\epsilon > 0$  choose  $y_0 < a < y_1, \dots, y_n = b$  such that  $y_i - y_{i-1} < \epsilon$ .

Put  $E_i = K \cap f^{-1}(y_{i-1}, y_i]$ . Since  $f$  is continuous,  $E_i$  are disjoint Borel sets in  $X$

and  $K = \bigcup_{i=1}^n E_i$ . Choose open sets  $V_i$  such that  $E_i \subset V_i \subset f^{-1}(-, y_i + \epsilon)$

and  $\mu(V_i) < \mu(E_i) + \frac{\epsilon}{n}$  such that  $f(x) < y_i + \frac{\epsilon}{n}$  for all  $x$  in  $V_i$ . Choose  $h_i = \chi_{V_i}$  such that  $h_i = 1$  on  $E_i$ . Then  $f = \sum f h_i$ .

Since  $f h_i = (y_i + \epsilon) h_i$  and  $y_i - \epsilon < f(x)$  on  $E_i$ , we have,

$$I(f) = \sum I(h_i f) \leq \sum (y_i + \epsilon) I(h_i), \text{ by positivity of } I.$$

$$(y_i + \epsilon) \mu(V_i)$$

$$(y_i + \epsilon) \mu(E_i) + \sum (y_i + \epsilon) \frac{\epsilon}{n}$$

$$(y_i - \epsilon) \mu(E_i) + 2\epsilon \mu(K) + (b + \epsilon) \epsilon$$

$$\int f d\mu + \epsilon[2 \mu(K) + b + \epsilon] \text{ over } E_i.$$

$$\int f d\mu + \epsilon[2 \mu(K) + b + \epsilon] \text{ over } X.$$

Since,  $K$  and  $b$  are fixed, and  $\epsilon > 0$  is arbitrary, this implies that  $I(f) \leq \int f d\mu$ .

Now let us show the uniqueness.

Suppose  $\mu_1$  and  $\mu_2$  have all the above properties. We need to show that  $\mu_1 = \mu_2$ .

It is enough to show  $\mu_1(K) = \mu_2(K)$  for any compact subset of  $X$ .

Now let  $K$  be compact subset of  $X$ , then  $\mu_1(K) < \infty$  since it is a Borel measure. Then there exist an open subset  $V$  of  $X$  with  $V \supset K$  and  $\mu_1(V) \leq \mu_1(K) + \epsilon$ . By theorem 2.2.10, there exists a function  $f \in C_c(X)$  such that  $K \subset f \subset V$ . Thus,

$$\chi_K \leq f \leq \chi_V. \text{ Thus, we have}$$

$$\mu_2(K) = \int \chi_K d\mu_2 \leq \int f d\mu_2 = (I)f = \int f d\mu_1 \leq \int \chi_V d\mu_1 = \mu_1(V) \leq \mu_1(K) + \epsilon$$

There fore, we have

$$\mu_2(K) \leq \mu_1(K) + \epsilon. \quad (1).$$

Similarly, we have

$$\mu_1(K) = \int \chi_K d\mu_1 \leq \int f d\mu_1 = (I)f = \int f d\mu_2 \leq \int \chi_V d\mu_2 = \mu_2(V) \leq \mu_2(K) + \epsilon.$$

And

$$\mu_1(K) \leq \mu_2(K) + \varepsilon. \quad (2).$$

From (1) and (2) we have

$$\mu_1(K) = \mu_2(K) \text{ and this in turn implies}$$

$$\mu_1 = \mu_2.$$

**Theorem 2.2.12** Let  $\lambda : C_c(X) \rightarrow \mathbb{C}$  be a positive linear functional, where  $\mathbb{C}$  is the space of complex numbers. Then,  $\lambda$  is bounded on  $C_k(X)$  for any compact  $K$  where  $C_k(X)$  is the subspace of  $C_c(X)$  consisting of those functions which vanish outside  $K$ .

**Proof:** By corollary 1.2.64, there exists a continuous nonnegative real-valued function  $g$  on  $X$  which is equal to 1 on  $K$  and has compact support.

If  $f \in C_k(X)$ , let  $b = |f|$ , say  $f$  is real. Then,  $bg \geq 0$ , whence

$$\lambda(bg) \geq \lambda(f) \geq 0 \text{ and}$$

$$|\lambda f| \leq b\lambda(g).$$

Thus,  $\lambda(g)$  is our desired bound and hence  $\lambda$  is bounded. ■

**Theorem 2.2.13** Let  $\lambda$  be a positive linear functional on  $C_c(X)$ .

Let  $\mu$  be the outer measure determined by  $\lambda$ , and let  $\mathcal{M}$  be the algebra of all sets  $A$  of finite measure such that

$$\mu(A) = \sup \mu(K) \text{ for } K \subset A, K \text{ compact.}$$

Let  $\mathcal{M}$  be the collection of subsets  $Y$  of  $X$  such that

$$Y \cap K \text{ lies in } \mathcal{M} \text{ for all compact } K.$$

Then,  $\mathcal{M}$  is a  $\sigma$ -algebra containing the Borel sets, and  $\mu$  is a positive measure on  $\mathcal{M}$ . Furthermore,  $\mathcal{M}$  consists of the sets of finite measure in  $\mathcal{M}$ .

**Proof:** It is clear that  $\mathcal{M} \subset \mathcal{M}$ .

Let  $\mathcal{M}_K$  denote the collection of all sets  $Y \cap K$  with  $Y \in \mathcal{M}$ . Then,  $\mathcal{M}_K = \mathcal{M}_K$ , and is therefore, a  $\sigma$ -algebra in  $K$  for each compact  $K$ , by proposition 1.2.66.



It follows immediately that  $\mathcal{M}$  is itself a  $\sigma$ -algebra, because the operations of countable union, intersection, and complementation in  $X$  commute with the operation of intersecting with  $K$ , by proposition 1.2.66.

The case that  $\mathcal{M}$  contains all closed sets is obvious because if  $Y$  is closed and  $K$  is compact, then  $Y \cap K$  is compact and so lies in  $\mathcal{M}$ .

Therefore,  $\mathcal{M}$  contains all Borel sets.

Let  $A$  be of finite measure in  $\mathcal{M}$  and let  $V$  be an open set containing  $A$ , and of finite measure. Let  $K$  be compact subset of  $V$  such that

$$\mu(V) < \mu(K) + \varepsilon.$$

Since  $A \cap K$  lies in  $\mathcal{M}$ , there is some compact  $K' \subseteq A \cap K$  such that

$$\mu(A \cap K) < \mu(K') + \varepsilon.$$

But,  $A \subseteq (A \cap K) \cup (V - K)$ , so that

$$\mu(A) \leq \mu(A \cap K) + \mu(V - K) < \mu(K') + 2\varepsilon.$$

This proves that  $A$  lies in  $\mathcal{M}$  and therefore that  $\mathcal{M}$  is precisely the algebra of sets of finite measure in  $\mathcal{M}$ .

Finally, let  $\{A_n\}$  be a disjoint sequence in  $\mathcal{M}$ . If some  $A_n$  has infinite measure, the countable additivity of  $\mu$  on  $\bigcup A_n$  is clear. If all  $A_n$  have finite measure, then proposition 1.2.65 applies and hence the proof of the theorem. ■

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