BEST PROXIMITY RESULTS FOR α- PROXIMAL θ - ϕ- GENERALIZED NON - SELF MAPPING

A THESIS SUBMITTED TO MADDA WALABU UNIVERSTY DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILMENT OF THE REQUIRMENT FOR THE DEGREE OF MASTERS OF SCIENCE IN MATHMATICS.

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Madda Walabu University

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Thesis examiners approval sheet

As members of the thesis examiners of the master's thesis open defense examination , we certify that we have read and evaluated the thesis prepared by the student JEMAL HUSSEN under the title "BEST PROXIMITY RESULTS FOR α –PROXIMAL θ-$\emptyset$ GENERALIZED NON- SELF MAPPING" and recommended that the thesis for the requirement of the master of Science degree in Mathematics.

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ABSTRACT

This thesis extends the work of Rossa and Kari (2022) on best proximity point theorems for α-proximal θ- ϕ- mappings. We achieve this by generalizing the key inequality used in their proof leading to a broader class of contractive mappings.

Specially, we conceder pairs of non -empty closed subsets (A, B) *in a complete metric space and define* α *−proximal* $\theta - \phi$ *non self- mappings.* $T: A \cap B$. *Our main novelity lies in introducing a more general in equality:*

$\alpha(x, y)\theta(d(Tx, Ty)) \leq \phi[max(\theta(d(x, y)), \theta(d(x, Tx), \theta(d(y, Ty)))].$

This inequality encompasses various terms not captured by the original formulation. Under suitable conditions, we prove the existence and uniqueness of a best proximity point for this generalized class of mappings. This means there exist a unique point $x^* \in A$ *such that* $d(x *, T(x *)) = d(A, B)$, *minimizing the distance to the image of the mapping .Our results generalize existing theorems and demonstrate the applicability of the proposed framework. We showcase this through illustrative examples and discuss potential future directions for research.*

CHAPTER ONE

INTRODUCTION

Mathematics is one of the important branches of scientific knowledge having many applications in every sphere of life. It is further classified into multiple branches. One of the most important branches of mathematics is known as functional analysis. In functional analysis the concept of proximity point is achieving the minimum distance between two sets through a function defined on one of sets to the other.

In 1922, notable mathematician Banach [1] demonstrated significant fixed point result in the area of functional analysis acknowledge as Banach contraction principles. This result declared to be the most fundamental consequence in the field of fixed point theory.

The Banach contraction theorem states that if (X, d) is a complete metric space and $T: X \rightarrow X$ is self- mapping with contraction, then T has a unique fixed point. On the other hand for given non -empty closed sub sets $\boldsymbol{A}$ and $\boldsymbol{B}$ of a complete metric space (c, d) a contraction for non-self- mapping $T: \boldsymbol{A} \rightarrow \boldsymbol{B}$ does not necessarily guarantee that will have a fixed point. In this case, it is quite natural to investigate $x \in A$ such that $d(x, T\,x) > 0$ is in some sense minimum, more precisely a point $x \in A$ for which $d(x, T\,x) = d\,(A, B)$ is called a best proximity point of T. In 1969, A classical best approximation theorem was introduced by Fan [4] states that, If A is a non-empty compact convex subset of a Hausdroff locally convex topological vector space B and $T: A \rightarrow B$ is continuous mapping ,then there exists an element $x \in A$ such that $d(x, Tx) = d(\,Tx, A)$. A best proximity point represents an optimal approximate solution to the equation

$T\,x = x$ Whenever a non -self-mapping T has no fixed point. It is clear that a fixed point coincides with a best proximity point if $d\,(A, B) = 0$. since a best proximity point reduces to a fixed point if the underlying mapping is assumed to be self- mappings, the best proximity point theorems are natural generalizations of the Banach's contraction principle.

1.1. BACK GROUND OF THE STUDY

The existence and convergence of best proximity point is attractive feature of optimization theory. In 2005, Elderd et.al [3] provided the existence and convergence of the best proximity points in the set of uniformly continuous Banach space.

In 2008, Suzuki [13] presented a new kind of mappings and provided generalization of Banach contraction principle. In 2010, S. Sadik Basha [11] introduced necessary and sufficient axiom to claim the existence of a best proximity point for proximal contraction of first and second kind.

Recently, Jleli, Karapinar, and Samet in [7] have introduced a new class of contractive mappings called $\alpha - \psi$-Contractive type mappings. They have provided some results on the existence and uniqueness of best proximity points of such non-self-mappings. The purpose of this research is to generalize the frame work of Mohammed Rossaf, Abdulkerim Kari [10] by adding some constants to the existence and uniqueness of best proximity point for α −proximal $\theta - \phi -$ non-self-mapping defined on a closed subset of a complete metric space. In recent research in the field of nonlinear functional analysis, many researchers are interested in dealing with non –self maps to determine the distance between two nonempty subsets of a metric space (X , d).Let A and B be two nonempty subsets of a metric space (X , d) and $T: A \to B$ be a non-self- mapping .Then, $d(x , T x) = d(A, B)$, for all $x \in A$, where $d(A, B) = inf \{d(x, y): x \in A, y \in B \}$. In general, for a non-self mapping $T: A \to B$, the fixed point equation $Tx = x$ may not have a solution .In such a cases, one intend to find an approximate solution $x \in A$ such that $d(x, T x) = d(A, B)$.

1.2. STATEMENTES OF THE PROBLEM

The problem addressed in this thesis revolves around the investigation of best proximity point results for a novel class of non-self-mapping. Inspired by the work of Mohammed Rossa and Abdulkerim Kari, we aim to extend and generalize existing results in the literature by introducing a more flexible and inclusive inequality.

Specially, our objective is to establish conditions under which best proximity points exist and are unique for α-proximal θ-ϕ-generalized mappings in metric spaces endowed with asymmetric binary relations. Building upon the foundation laid by previous research, we seek to develop a theoretical framework that accommodates the generalized inequality

$$\alpha\,(x,y)\theta(d(T\ \ x,Ty)) \leq \ \phi[max\,(\theta,d(x\,,y)),\theta(d(x\,,T\,x)),\theta(d(y,Ty)))].$$

Allowing for a comprehensive analysis of best proximity point properties;

Key aspects of the problem include;

1. Defining and characterizing α-proximal $\theta-\phi$-generalized mapping; we aim to formalize the concept of α-proximal $\theta-\phi$ generalized mappings and investigate their properties within the context of metric spaces.
2. Establishing conditions for the existence of best proximity points. We seek to identify conditions under which best proximity points exist for α −proximal θ-ϕ-generalized mappings, taking in to account the interplay between α,θ and ϕ functions.
3. Investigating the uniqueness of best proximity points: we aim to determine conditions that ensure the uniqueness of best proximity points for α-proximal θ-ϕ −generalized mappings, thereby providing a comprehensive understanding of their behavior.
4. Illustrating the applicability of the generalized framework: Through theoretical analysis and illustrative examples, we aim to demonstrate the effectiveness and applicability of our generalized framework in capturing diverse scenarios and providing deeper insights into the nature of best proximity points.
5. Exploring implications and future directions: we intend to discuss the implication of our findings in various fields, such as fixed point theory, optimization and control theory and suggest potential avenues for future research in the study of best proximity points.

By addressing these key aspects, we aim to contribute to the advancement of knowledge in proximity point theory and pave the way for further research in the study of non-self-mappings in metric spaces

1.3. OBJECTIVE OF THE STUDY

1.3.1. GENERAL OBJECTIVE

The general objective of this thesis to extend and generalize the study of best proximity point results for α-proximal$-\theta - \phi -$generalized non-self-mappings in metric spaces endowed with asymmetric binary relations..

1.3.2. SPECIFIC OBJECTIVE

The specific objective the study is to introduce a more flexible inequality,
and investigate its implications for the existence and uniqueness of best proximity points.

i. To review the literature on best proximity point results and identifies the limitation of existing approaches, there by motivating the need for a more generalized framework.

ii. To establish conditions under which best proximity points exist for α-proximal $\theta - \phi -$generalized non-self-mappings, considering factors such as the completeness of the metric space and properties of the α, θ and ϕ functions.

iii. To determine conditions that ensure the uniqueness of best proximity points for $\alpha -$proximal $\theta - \phi -$generalized non-self-mapping, thereby providing a deeper understanding of their behavior.

iv. To develop a theoretical framework those accommodates the generalized inequality and apply it to various scenarios to illustrate its effectiveness and applicability.

By addressing these specific objectives, we aim to contribute to the advancement of knowledge in proximity point theory and provide a comprehensive understanding of best proximity points for α-proximal $\theta - \phi -$generalized non-self-mappings in metric spaces.

1.4. SIGNIFICANCE OF THE STUDY

The result of this study may have the following importance

i. It may contribute to research activities on study area.

ii. It may provide basic research skill to researcher.

iii. It may also serve as reference for researchers who have interest to be engaged in research activities in this line of research.

iv. To share experience for others.

v. It may have application in studying the existence of unique solution to nonlinear integral solution.

1.5. DELIMITATION (SCOPE) OF THE STUDY

This study focuses only in determining the best proximity point for α-proximal $\theta - \phi$ for generalized non-self-mapping by adding some constants to the work of Mohammed Rossafi and Abdulkarim Kari [5] in the frame work of complete metric space.

CHAPTER TWO

REVIEW OF RELATED LITRATURES

Fixed point theory is very important in diverse discipline of Mathematics. Since it can be applied for solving varies problems and it is one of the most dynamic research subjects in nonlinear analysis. In this area the first important and significant result was proved by Banach in 1922 for a contraction mapping in a complete metric space. In the fixed point theory contraction is one of the main tools to prove the existence and uniqueness of the solution of an operator, $T\ x\ = x$ used in all analysis. Due to its importance generalization of the Banach contraction principle has been investigated heavily by many researchers.

In 1969, Fan [4], presented the first result concerning best proximity point theorems in continues non-self-mapping in a normed vector space. Then following his theorems, best proximity point theorems of non-self-mappings get a lot of attention and have been studied by many researchers.

In2013, Jleli, Karapiner and Samet in [7] have introduced a new class of contractive mapping called α-ψ-contractive type mappings. They have provided some results on the existence and uniqueness of the best proximity points of such non-self-mappings. Recently, in 2017, Ayari in [9] proposed an extension for the case of $\alpha - \beta -$proximal quasi contractive mappings. We are interested in extending these works for the Geraghty function by introducing the notion of α-proximal Geraghty non-self- mappings. The purpose of all this is to provide a theorem on the existence and uniqueness of best proximity point for such mappings.

2.1. PRELIMINARIES

In this section, different important concepts (definition, Lemma, and theorems) will be collected from different sources in order to understand our research work .We only states them without proof since they are proved in many standard materials in the internet.

Definition 2.1.1 [10] Let (A, B) be a pair of non-empty sub sets of a metric space (x, d) we adopt the following notations. $d(A, B) = \{\ inf\ \ d(a, b){:}\ a \in\ A, b \in\ B\}$;

$A_0\ =\ \{a \in\ A$ There exists $b \in\ B$ such that $d(a, b)\ =\ d(A, B)\}$;

$B_0\ =\ \{b \in\ B$ There exists $a \in A$ such that $d(a, b)\ =\ d(A, B)\}$.

Definition 2.1.2 [7] Let $T: A \rightarrow B$ be a mapping .An element x^* is said to be a best proximity point of T if $d\ (x^*, T(x)^*) = d(A, B)$.

Definition 2.1.3 [12] let (A, B) be a pair of non-empty subsets of a metric space (X, d) such that A_0 is non-empty. Then the pair (A, B) is to have P −property if and only if

$d(x_1, y_1) = d(x_2, y_2) = d(A, B)$, Then $d(x_1, x_2) = d(y_1, y_2)$,where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 2.1.4 [7] Let $\alpha: A \times A \rightarrow [0, +\infty)$, T is said to be proximal admissible if $\alpha(x_1, x_2) \geq 1$and
$d\big(u_1, T(x_1)\big) = d\big(u_2, T(x_2)\big) = d(A, B), \alpha(u_1, u_2) \geq 1\ \forall\ x_1, x_2, u_1, u_2, \in A$.

Definition 2.1.5 [6] Let be the family of all functions $X\ :\ [0, \infty] \rightarrow [1, \infty]$ such that

(i) Is increasing

(ii) For each sequence $\{x_n\} \in\ [0, \infty]$;

(iii) Is continues.

Definition 2.1.6 [14] Let ø be the family of all functions ø: $[1, \infty)\ \rightarrow\ [1, \infty]$,such that

(ø1) ø is increasing

(ø2) for each $t \in\ [0, +\infty]$, $\lim\limits_{n\to\infty} \emptyset(t) = 1$

(ø3) is continues.

Lemma 2.1.7 [14] If $ø\epsilon$ ϕ, then ø $(1)\ = 1, and$ ø $(t)\ < t$.

Definition 2.1.8 [14] Let (x, d) be a metric space and $T: X \rightarrow X$ be a mapping. T is said to be a −ø-contraction if there exist ѳ ∈ θ and ø ∈ ϕ such that for any $x, y \in X, d\ (T(x), T(y)) > 0\ \Rightarrow\ \theta\ [d\ (T\ (x), T(y))\ \leq\ ø\ [\ \theta(d(x, y))]$.

Definition 2.1 .9 [14] Let$(x, d,)$ be a partially ordered metric space A, B be two non-empty subsets of $X, \alpha, \beta: A\ A \rightarrow [0, \infty)$ Be Functions $C\ \alpha > 0, C\beta \geq 0$ be two constants and $T: A \rightarrow B$ be a non-self-mapping, we say that T is an $\alpha - \beta$ −proximal admissible, if for all $x, y\ , u, v \in A$ with $x\ , y$ hold.

i. $\alpha(x, y) \geq C\ \alpha\ and\ d(u\ ,T\ \ x) = d(u, T(y)) = d(A, B)$implies $\alpha(u, v) \geq C\ \alpha$.

ii. $\beta(x\ , y)C\ \beta$ and $d(u, T\ x) = d(u, Ty) = d(A, B)$ implies $\beta(u,)C\ \beta$

iii. $0\ C\ \beta/\ C\ \alpha 1$.

If (i), (ii) and (iii) holds for $\alpha(x, y) = 1 = \beta(x, y)$ for all $x, y \in\ A$, then we say that T is proximal admissible;

Definition 2.1.10 [10] let (x, d) be a metric space and (A, B) be pair of non- empty subset of X. A non-self- mapping $T: A \rightarrow\ B$ is called α -proximal θ -ϕ -mapping where $\alpha: A\ \times\ A \rightarrow [0, \infty\ ]$,If there exists $\theta\ \in \Theta\ \ and\ \phi \in \Phi$ such that for any $x\ , y \in X$,

$$\alpha(\ x\ , y)\theta(d(T(x), T(y)) \leq \phi[\theta(d(x\ , y))]$$

CHAPTER THREE

METHEDOLOGY OF THE STUDY

3.1. STUDY AREA AND PERIOD

The study will be conducted at Madda Walabu University under the department of Mathematics from August 2022 to October 2024.

3.2. STUDY DESIGN

This study will employ analytical method of design.

3.3. SOURCES OF INFORMATION

The sources of information for this study are published articles, books, journals and related studies from internet.

3.4. MATHEMATICAL PROCEDURE

In this study the procedures that will follow the standard procedures are;

- Defining best proximity points for α proximal θ -Φ- generalized mapping.

- Stated lemmas and theorems to support our main results.

- Giving an example to satisfies our main results.

CHAPTER FOUR

MAIN RESULT

4.1. BEST PROXIMITY RESULTS FOR α-PROXIMAL $\theta - ȹ$ −GENERALIZED MAPPING

Definition4.1.1. Let (X, d) is a metric space and (A, B) be pair of non-empty subset of X. A non-self-mapping $T: A \to B$ is called α-proximal $\theta - ȹ$ −generalized mapping, where $\alpha: AXA \to [0, +\infty], if there\ exists\ \theta \in \Theta and\ ȹ \in \Phi\ such\ that\ for\ any\ x, y \in X, \alpha(x, y)\theta(d(TX, TY)) \leq ȹ[\theta(d(x, y))]$.

Definition\4.1.2 The mapping $T: A \to B$ is said to be a generalized $\Theta - \Phi$ −proximal contraction of first kind if there exist $\theta \in \Theta, ȹ \in \Phi\ and\ a, b, c, h \geq 0\ with\ a + b + \dots. +2ch, c \neq 1\ such\ that$

$$\begin{cases} d(x_1, Ty_1) = d(A, B) \\ d(x_2, Ty_2) = d(A, B) \end{cases} \Rightarrow \theta(d(x_1, x_2))$$
$$\leq ȹ[\theta[ad(y_1, y_2) + bd(x_1, y_1) + cd(x_2, y_2) +$$
$$h(d(y_1, x_2) + d(y_2, x_1))]]\ for\ all\ x_1, x_2, y_1, y_2$$
$$\in A\ and\ x_1 \neq y_1.$$

Theorem4.1.3 Let (A, B) be pair of non-empty closed subset of a complete metric space (X, d) such that Ao is non-empty. Let $\alpha: AxA \to [0, +\infty], \theta \in \Theta\ and\ ȹ \in \Phi$, consider an α-proximal $\theta - ȹ$ mapping $T: A \to B$ satisfying the following assertion.

i. $T(Ao) \in Bo$ And the pair (A, B) satisfies the weak p-property.
ii. T is α- proximal admissible,
iii. There exists $xo, x1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(xo, x_1) \geqslant 1$.
iv. T Is continuous.
v. T Is a generalized $\theta - ȹ$ −proximal contraction of first kind Then, there exists a unique $x \in A$ such that $d(x, Tx) = d(A, B)$. In addition, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ defined $by\ d(x_{n+1}, Tx_n) = d(A, B)$ the best proximity point.

Proof From condition (3), there exist elements $x_0, x_1 \in A_0$ Such that $d(x_1, Tx_0) = d(A, B)\ and\ \alpha(x_0, x_1) \geqslant 1$

Since $T(A_0) \in B_0$, there exists $x_2 \in A_0$such that $d(x_2, Tx_1) = d(A, B)$.

Now, we have $d(x_1, T\, x_0) = d(A\,, B), \alpha\,(x_0, x_1\,) \geqslant 1$ and $d(x_2\,, Tx_1\,) = d(A\,, B)$. since T is α- proximal admissible. This implies that $\alpha\,(x1, x2) \geqslant 1$.

Thus, we have $d\,(x_2, Tx_1) = d(A, B)$ and $\alpha\,(x_1, x_2) \geqslant 1$.

Again, since $T(A_0\,) \in B_0$,there exists $x_3 \in A_0$ such that $d(\,x_3, Tx_2\,) = d(A, B)$.

Continuing this process by induction, we construct a sequence $x_n \in A_0$ such that

$d(\,x_{n+1}\,, Tx_n\,) = d\,(A, B)$ And $\alpha\,(x_n, x_{n+1}\,) \geqslant 1, \forall n \in \mathbb{N}, \ldots \ldots \ldots \ldots \ldots \ldots$ (1)

Since (A, B) satisfies the p-property, we conclude from (1) that

$(x_n, x_{n+1}\,) = d\,(Tx_n, Tx_{n+1}\,)\,, \forall\, n \in \mathbb{N}$.

If $xn_0 = xn_{0+1}$ for some $n_0 \in \mathbb{N}$, from (1) we obtains

$$d\,(xn_0\,, Txn_0\,) = d\,(\,xn_{0+1}\,, Txn_0\,) = d\,(A\,, B\,) \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

We shall prove that the sequence x_n is a Cauchy sequence. Let us first prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

As T is a generalized $\theta - \phi$ −proximal contraction of first kind we have

$$\theta[d(x_n, x_{n+1}) \leq \phi[\theta\,[ad(x_{n-1}\,, x_n) + bd(x_{n-1}\,, x_n)$$
$$+ cd(x_n\,, x_{n+1}\,) \quad h(d\,(\,x_{n-1}\,, x_n) \;+ d\,(x_n, x_n))]]$$

$$= \phi[\theta[ad(x_{n-1}\,, x_n\,) \;+ bd\,(x_{n-1}\,, x_n\,) \;+ cd\,(x_n\,, x_{n+1}\,) \;+$$
$$h\,(d\,(x_{n-1}\,, x_{n+1}\,))]]$$

$$\leq \phi[\theta[\,ad\,(x_{n-1}\,, x_n\,) \;+ bd\,(x_{n-1}\,, x_n\,) + cd\,(x_n, x_{n+1}\,) +$$
$$h\,(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))]]$$

$$= \phi[\theta[(a + b + c + h)d(x_{n-1}, x_n) \;+ (c + h)d(x_n\,, x_{n+1}\,)]]$$

Since θ is strictly increasing and using the lemma 2.1.7 we conclude

$$d(x_n, x_{n+1}) < \;(a + b + h\,)d(x_{n-1}\,, x_n) + (c + h)\,d\,(x_n\,, x_{n+1}\,).$$

Thus $d\,(x_n, x_{n+1}) < \frac{a+b+h}{1-c-h}\,(d\,(x_{n-1}\,, x_n\,))$.

If $b+b+c+2h=1$, We have $0<1-c-h$ and so

$$d\,(x_n, x_{n+1}) \le (a+b+h)/(1-c-h)\ \ (d\,(x_{n-1}, x_n)) \ = d\,(x_{n-1}, x_n)\ \forall n \in N\,.$$

Consequently,

$$\theta(d(x_n, x_{n+1})) \le \phi[\theta(d(x_{n-1}, x_n))]$$

If $b+b+c+2h<1$,we have $0<1-c-h$ and so

$$d\,(x_n, x_{n+1}) < d\,(x_{n-1}, x_n), \forall n \in \mathbb{N};$$

Consequently,

$$\theta(d\,(x_n, x_{n+1})) \le \phi(\theta\,(d\,(x_{n-1}, x_n))]$$

It implies

$$\theta\,(d\,(x_n, x_{n+1})) \le \phi[\theta\,(d\,(x_{n-1}, x_n)\,]$$

$$\le \emptyset^2\,[\theta\,(d\,(x_{n-2}, x_{n-1})]$$

$$\le \ \dots \ \le \emptyset^n\,[\theta\,(d\,(x_0, x_1)].$$

Taking the limit as $n \to \infty$, we have

$$1 \le \theta\,(d\,(x_n, x_{n+1})) \le \lim_{n\to\infty} \emptyset^n\,[\theta\,(d\,(x_0, x_n))] = 1\,.$$

Since $\theta \in \Theta$, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \dots\dots\dots\dots\dots\dots\dots..(3)$$

Next, we shall prove that $\{x_n\} n \in \mathbb{N}$ is a Cauchy sequence,

$i.e$, $\lim_{n\to\infty} d(x_n, x_m) = 0, \forall n \in \mathbb{N}$. Suppose to the contrary that exists $\varepsilon > 0$ and sequences n $_{(k)}$ and m $_{(k)}$ of natural numbers such that

$$m(k) > n(k) > k\,,\ d\,(x_{m(k)}, x_{n(k)}) \ge \varepsilon\,, d\,(x_{m(k)-1}, x_{n(k)}) < \varepsilon \dots (4)$$

Using the triangular inequality, we find that,

$$\varepsilon \le d\left(x_{m(k)}, x_{n(k)}\right) \le d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right) \dots..(5)$$

$$< \varepsilon + d\,(x_{n(k)-1}\,, x_{n(k)}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (6)$$

Then, by 4 and 2, it follows that

$\lim_{k\to\infty} d(x_{m(k)}\,, x_{n(k)}) \ = \varepsilon \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (7)

Using the triangular inequality, we find it,

$\varepsilon \leq d\,(\,x_{m(k)}, x_{n(k)}) \leq d\,(\,x_{m(k)}, x_{n(k)+1}) \ + \ d\,(x_{n(k)+1}\,, x_{n(k)}) \ \ldots$ (8)

$And \ \ \varepsilon \leq d\,(x_{m(k)}\,, x_{n(k)+1} \ \leq d\,(\,x_{m(k)}, x_{n(k)}) \ + \ d(x_{n(k)}, x_{n(k)+1})$ .... (9)

Then, by 5 and 9, it follows that

$\lim_{k\to\infty} d(x_{m(k)}\,, x_{n(k)+1}\,) = \varepsilon$ ………………………….. (10)

Similar method, we conclude that

$\lim_{k\to\infty} d(x_{m(k)+1}\,, x_{n(k)}) = \varepsilon.$ ……………………….………… (11)

Using again the triangular inequality,

$d\left(x_{m(k)+1}\,, x_{n(k)+1}\right) d\left(x_{m(k)+1}, x_{m(k)}\right) + d\left(x_{m(k)}, x_{n(k)}\right) + d(x_{n(k)}, x_{n(k)+1}) \ldots.$ (12)

On the other hand, using triangular inequality, we have

$d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{m(k)+1}\right) + \ d\left(x_{m(k)+1}, x_{n(k)+1}\right) + d\,(x_{n(k)+1}, x_{n(k)})$.. (13)

Letting $k \to \infty$ inequality (12) and (13). We obtain

$\lim_{k\to\infty} d(x_{m(k)+1,}\, u_{n(k)+1})$ =ε. ………………… (14)

Substituting $u_1 \ = x_{m(k)+1}, u_2 = x_{n(k)+1} \ , v_1 = u_{m(k)} \ \ and \ v_2 = u_{n(k)}$ in assumption of the theorem, we get

$$\theta(d(u_{m(k)+1}, u_{n(k)+1}\,)) \ \leq \ \phi \left\{ \theta \left\{ \begin{array}{c} ad(u_{m(k)}, u_{n(k)} \\ +bd(u_{m(k)+1}, u_{n(k)} \\ +cd(u_{n(k)+1}, u_{n(k)} \\ +h(d\left(u_{m(k)}, u_{n(k)+1}\,\right) + d(u_{n(k)}, u_{m(k)+1}) \end{array} \right. \right.$$

Letting $k \to \infty$in (15), and using $(\theta 1), (\theta 3), (\phi 3)$ and Lemma 2.1.7 we obtain

$$\theta\,(\varepsilon) \leq \ \phi[\theta(a\varepsilon + b\varepsilon + c\varepsilon + 2h\varepsilon)]$$

We drive

$$\varepsilon < \varepsilon.$$

Which is contradiction? Thus $\lim_{n,m\to\infty} d(u_n, u_m) = 0$, which shows that $\{x_n\}$ is a Cauchy sequence, Then there exists $z \in A$ such that

$$\lim_{n\to\infty} d\ (u_n, u) = 0.$$

Also,

$$d\ (u, B) \leq\ d\ (u, T_{un}\ )$$

$$\leq d\ (\ u\ , x_{n+1})\ +\ d\ (u_{n+1}\ , T_{un}\ )$$

$$=\ d\ (u, u_{n+1})\ +\ d\ (A, B)$$

$$\leq d\ (u\ , u_{n+1}\ ) + \ d\ (\ u\ , B\ )\ .$$

Therefore, $d\ (A, T_{un})\ \to d\ (u, B)$ .Inspite of the fact that B is approximately compact with respect to A, the sequence $\{T_{un}\}$ has a subsequence $\{T_{unk}\}$ converging to some element

$V\ \in\ B$. So it turns out that

$d(u\ , v) =\ \lim_{n\to\infty} d(u_{nk+1, T_{nk}})\ =\ d\ (\ A\ , B\ ).$ ………… (16)

Thus u must be an element of A_0. Again, since $T\ (Ao) \in\ Bo$, there exists $t \in\ Ao$ such that

$d(t, T_u) =\ d\ (A, B)$ …………………..……….. (17)

For some element t in A. Using the p-property we have

$$d\ (u_{nk+1}\ , t\ )\ = d\ (p_{unk}, p_u)\ , \forall nk \in \mathbb{N}\ .$$

If for some n_0 , $d\ (\ t\ , u_{n0+1}) = 0$, consequently $d\ (p_{un0}, T_u)\ = 0.$ so $p_{un0} = T_u$, hence

Any n≥ 0, $d(t, u_{n+1})$ 0 . Since T is a generalized $(\theta, \phi) -$ proximal contraction of the first kind, it follows from

$\theta\left(d\left(t, u_{n+1}\right)\right) \leq \phi\left[\theta\left[a d\left(u, u_n\right)+b d(t, u)+c d\left(u_n, u_{n+1}\right)+h\left(d\left(u, u_{n+1}\right)+d\left(u_n, t\right)\right)\right)\right] \ldots \ldots (18)$

Since θ and ϕ are two continuous functions, by letting $n \rightarrow \infty$ in inequality (18), we obtain

$$\Theta(d(t,u)) \leq \phi\left[\theta\left[(b+h)(d(u,t))\right]\right]$$

$$\leq \phi[\theta[(d(u,t))]]$$

$$< \theta(d(t,u)).$$

It is a contradiction .Therefore $u = t$, that

$$d(u, T_u) = d(t, T_u) = d(A, B).$$

Uniqueness: Suppose that there is another best proximity point z of the mapping, T such that

$$d(z, T_z) = d(A, B).$$

Since T is a generalized $(\theta - \phi) -$ proximal contraction of the first kind, it follows from this that

$$\theta(d(z,u) \leq \phi[\theta[ad(z,u) + bd(z,z) + cd(u,u) + h(d(z,u) + d(z,u))]].$$

$$= \phi[\theta[(a + 2h)\, d(z,u)]],$$

Which is a contradiction? Thus, z and u must be identical, and hence, T has a unique best proximity point.

Theorem 4.1.4 let (X, d) be a complete metric space and (A, B) be a pair of non-empty closed subset of (X, d) such that A_0 is non-empty. Let $\alpha: A \; x \; A \rightarrow [0, +\infty]$, and $\theta \in \Theta$ and $\phi \in \Phi$. Consider $\alpha -$ proximal $\theta - \phi$ -mapping and if A is approximately compact with respect to B and $T: A \rightarrow B$ satisfies the following conditions.

i. $T(Ao) \in Bo$ And the pair (A, B) satisfies the weak P - property.
ii. T Is α – proximal admissible.

iii. There exists elements $xo, x_1 \in A$ such that $d(x_1, Txo) = d(A, B)$ and $\alpha(xo, x_1) \geq 1$

iv. T is continuous generalized $\theta - \phi -$ proximal contraction. Then there exists a unique $u \in A$ such $that\ d(u, Tu) = d(A, B)$and $n \to u$, where u_0 is a fixed point in A_0 and $d(un+1, T_{un}) = d(A, B)$ for $n \geq 0$ further if is another best proximity point of T , then $Tu = Tz.z$

Proof. Similar to Theorem 3.1.3, we can find a sequence $\{u_n\}$ in A_0 such that

$d(U_{n+1}, T_{un}) = d(A, B)$. ……………………………………..… (19)

For all non-negative integral values of n .From P- property and (19) we get

$$d(U_n, U_{n+1}) = d(T_{un-1}, T_{un}), \forall n \in \mathbb{N}.$$

If for some n_0 , $d(U_{n0+1}, U_{n0+2}) = 0$, consequently $d(T_{un0}, T_{un+1}) = 0$,so $T_{un0} = T_{un0+1}$ hence $d(A, B) = d(T_{un0}, T_{un0+1})$.

Let for any $n \geq$ $d(T_{un}, T_{un+1}) \geq 0$. We shall prove that the sequence u_n is a Cauchy sequence. Let us first prove that

$$\lim_{n\to\infty} d(U_n, U_{n+1}) = 0$$

As T is generalized $\theta - \phi -$proximal contraction of second kind, we have that $\theta(d(T_{un}, T_{un+1})) \leq \phi[\theta[ad(T_{un-1}, T_{un}) + bd(T_{un-1}, T_{un}) +$

$$cd(T_{un}, T_{un+1}) + h(\mathrm{d}(T_{un}, T_{un+1}) + d(T_{un}, T_{un})$$

$$= \phi[\theta[ad(T_{un-1}, T_{un}) + bd(T_{un-1}, T_{un}) + cd(T_{un}, T_{un+1}) + h(d(T_{un}, T_{un+1})]]$$

$$\leq \phi[\theta[ad(T_{un1}, T_{un}) + bd(T_{u1}, T_{un}) + cd(T_{un}, T_{un+1}) + h(d(T_{un1}, T_{un}) + d(T_{un}, T_{un+1})]]$$

$$= \phi[\theta[(a+b+h)d(T_{un-1}, T_{un}) + (c+h)d(T_{un}, T_{un+1})]]$$

Since θ is strictly increasing and by Lemma 2.1.7, we deduce

$$d(T_{un}, T_{un+1}) < (a+b+h)d(T_{un-1}, T_{un}) + (c+h)d(T_{un}, T_{un+1}).$$

Thus

$\mathrm{d}(T_{un}, T_{un+1}) < \frac{a+b+h}{1-c-h}(\mathrm{d}(T_{un-1}, T_{un}))$.

If b+b+c+2h =1, we have 0<1-c-h and so

$\mathrm{d}(T_{un}, T_{un+1}) \leq \frac{a+b+h}{1-c-h}(\mathrm{d}(T_{un-1}, T_{un})) = \mathrm{d}(T_{un-1}, T_{un}), \forall n \in \mathbb{N}$

Consequently,

$\theta\ (\ d\ (T_{un}, T_{un+1})) \leq\ \phi[\theta\ (\ d(T_{un-1}, T_{un}))]$

If $b + b + c + 2h < 1$, we have $0 < 1 - c - h$ and so

$$d\ (T_{un}, T_{un+1})\ <\ d\ (T_{un-1}, T_{un})\ , \forall\ n\ \in \mathbb{N}\ ;$$

Consequently,

$$\theta(\ d\ (T_{un}, T_{un+1})) \leq\ \phi\ [\theta\ (\ d\ (T_{un-1}, T_{un}))]$$

It implies

$$\begin{aligned}\theta(\ d\ (T_{un}, T_{un+1}))\ &\leq \phi\ [\theta\ (\ d\ (\ Tu_{n-1}, T_{un})] \\ &\leq\ \phi 2\ [\theta\ (d\ (T_{un-2}, T_{un-1})] \\ &\leq\ \ldots\ \leq\ \phi n\ [\theta\ (d\ (T_{u0}, T_{u1})].\end{aligned}$$

Taking the limit as n→ ∞ , we have

$$1 <\ \theta\ (d(T_{un}, T_{un+1}))\ \leq lim_{n\to\infty}\ \phi\ n\ [\theta\ (d(T_{u0}, T_{u1}))]\ = 1$$

Since $\theta \in \Theta$, we obtain

$\lim_{n\to\infty} d\ (T_{un}, T_{un+1})\ =\ 0.$ ………………………… .(20)

Next, we shall prove that $\{\ T_{un}\} n \in \mathbb{N}$ is a Cauchy sequence,

i. e $\lim_{n\to\infty} d(T_{un}, T_{um})\ = 0$ for all $n \in \mathbb{N}$.

Suppose to the contrary that exists $\varepsilon > 0$ and sequences $T_{n(k)}$ and $T_{m(k)}$of natural numbers such that

$$T_{m(k)} > T_{n(k)} > k\ ,\ d\ (T_{um(k)}, T_{un(k)}) \geq\ \varepsilon\ ,\ d\ (T_{um(k)-1}, T_{un(k)}) < \varepsilon\ \ldots (21)$$

Using the triangular inequality, we find that,

$$\varepsilon \leq\ \ d\ (T_{um(k)}, T_{un(k)}) \leq d\ (T_{um(k)}, T_{xn(k)-1} +\ d\ (T_{un(k)-1}, T_{un(k)})\ \ldots (22)$$

$<\varepsilon + d(T_{un(k)-1}, T_{un(k)})$………… (23)

Then by 4 and 22, it follows that

$\lim_{k\to\infty} d(T_{un(k)}, T_{un(k)}) = \varepsilon$……. (24)

Using the triangular inequality, we find that,

$\varepsilon\ \leq\ \ d\ (T_{um(k)}, T_{un(k)}) \leq\ d\ (T_{um(k)}, T_{n(k)+1}) + d\ (T_{un(k)+1}, T_{un(k)})\ \ldots\ (25)$

And

$\varepsilon\ \leq\ d\ (T_{um(k)}, T_{un(k)+1})\ \leq\ d\ (T_{um(k)}, T_{un(k)}) +\ d\ (T_{un(k)}\ , T_{un(k)+1})\ \ldots (26)$

Then by 26 and 9, it follows that

$\lim_{k\to\infty} d\ \ (T_{um(k)}, T_{un(k)+1}) = \varepsilon$ …………. (27)

Similar method, we conclude that

$\lim_{k\to\infty} d\ \ (T_{um(k)+1}, T_{un(k)}) = \varepsilon$……… (28)

Using again the triangular inequality

$d\left(T_{um(k)+1}, T_{un(k)+1}\right) \leq \quad d\left(T_{um(k)+1}, T_{um(k)}\right) + d\left(T_{um(k)}, T_{un(k)}\right) +$
$d(T_{un(k)}, T_{un(k)+1})\ldots\ldots$ (29)

On the other hand, using triangular inequality, we have

$d\left(T_{um(k)}, T_{un(k)}\right) \leq d\left(T_{um(k)}, T_{um(k)}, T_{um(k)+1}\right) + d\left(T_{um(k)+1}, T_{un(k)+1}\right) +$
$d\left(T_{un(k)+1}, T_{un(k)}\right)..\ldots$ (30)

Letting $k \to \infty$ in inequality (29) and (30) we obtain

$\lim_{k\to\infty} d(T_{um(k)+1}, T_{un(k)+1}) = \varepsilon$ .... (31)

Substituting $= T_{um(k)+1}$, $u_2 = T_{un(k)+1}$, $v_1 = T_{um(k)}$, and $v_2 = T_{un(k)}$ in assumption of the theorem, we get

$\theta(d(T_{um(k)+1}, T_{un(k)+1}) \leq \phi\{\theta\{ad(T_{um(K)}, T_{un(k)}) + bd(T_{um(k)+1}, T_{un(k)}) +$
$cd(T_{un(k)+1}, T_{un(k)}) + h(d(T_{um(k)}, T_{un(k)+1}) + d(T_{un(k)}, T_{um(k)+1})\ldots\ldots\ldots\ldots$ (32)

Letting $k \to \infty$ in (32) and using (θ_1), (θ_3), $(\emptyset_3)$ and Lemma 2.1.7 we obtain

$$\theta(\varepsilon) \leq \emptyset[\theta(a\varepsilon + b\varepsilon + c\varepsilon + 2h\varepsilon)].$$

We derive

$$\varepsilon < \varepsilon .$$

This is contradiction. Thus $\lim_{n,m\to\infty} d(T_{un}, T_{um}) = 0$, which shows that $\{T_{un}\}$ is a Cauchy sequence. Then there exists $v \in B$ such that

$$\lim_{n\to\infty} d(T_{un}, v) = 0$$

Also,

$$\begin{aligned} d(v, A) &\leq d(v, T_{un}) \\ &\leq d(v, u_{n+1}) + d(u_{n+1}, T_{un}) \\ &= d(v, u_{n+1}) + d(A, B) \\ &\leq d(v, u_{n+1}) + d(v, A). \end{aligned}$$

Therefore, $d(v, T_{un}) \to d(v, A)$. since A is approximately compact with respect to B. The sequence $\{u_n\}$ has a subsequence $u_{n(k)}$ converging to some element $u \in A$. So it turns out that

$d(u, v) = lim_{n\to\infty} d(u_{n(k)+1}, T_{un(k)}) = d(A, B)$ .... (33)

Because T is a continuous mapping,

$$d(u, T_u) = \lim_{n\to\infty} d(u_{n+1}, T_{un}) = d(A, B).$$

Uniqueness: Suppose that there is another best proximity point z of the mapping T such that

$$d\,(z\,,T_z)\;=\;d\,(\,A\,,B\,)\,.$$

Since T is a generalized $\theta\,-\phi\,-$proximal contraction, it follows from this that

$$\theta\,(d\,(T_z,T_u)\;\le\;\phi[ad(T_z,T_u)+bd(T_z,T_z)+cd(T_u,T_u)+h(d(T_z,T_u)+d(T_z,T_u))]]$$
$$=\phi[\theta[(a+2h)d(T_z,T_u))]].$$

Which are contradiction Thus, z and u must be identical. Hence, T has a unique best proximity point.

Definition 4.1.5 let (A,B) be a pair of non- empty subsets of X with $A_0\neq\emptyset$. Then the pair (A,B) is said to be have the weak p –property if and only if

$\begin{cases} d(x_1,y_1)=d\,(\,A\,,B\,), \\ d(x_2,y_2)=d\,(\,A,B\,) \end{cases}\Rightarrow\; d\,(x_1\,,x_2)\;\le\;d\,(y_1,y_2)$, where $x_1\,,x_2\,\in\,A$ and $y_1,y_2\in\,B$

Example 4.1.5.1 Consider $X=\{\,(0,1)\,,(\,1,0\,)\,,(0,3\,)\,,(\,3,0\,)\,\}$, endowed with the usual metric$d$. We suppose that

$\boldsymbol{U}=\;\{(0,1),(1,0)\}$

$\boldsymbol{V}\;=\;\{(0,3),(3,0)\}$ then

For $d\,(\,(0,1)\,,(0,3\,)\,)\;=\;d(u,v\,)$

$d\,((1,0)\,,(3,0\,))\;\;=\;d\,(u,v\,)$ We have

$d\,(\,(0,1\,),(\,1\,,0\,)\;<\;d\,((\,0,3)\,,(\,3,0\,))$

Also $u_0\neq\;\emptyset$. Thus, the pair (U,V) satisfies weak p-property

Example 4.1.5.2 Consider $X=\;[0,\infty)\;x\;[0,\infty)$ with metric d defined by

$$d\,((c_1\,,c_2),(b_1\,,b_2\,)\;=\;|c_1-b_1|\;+\;|c_2-b_2\,|$$

Let $U=\;\{1\}\,x\,[0,\infty)$ and $V\;=\;\{0\}\,x\,[0,\infty)$. Then

$d\,(\,U\,,V\,)\;=\;d\,((1,0),(0,0)\;=\;1$ And $u_0=v\,,v_0=\;v$. then the pair $(U\,,V\,)$ has weak p-property

Corollary 4.1.6 Let (A,B) is pair of nonempty closed subset of a complete metric space (X,d) such that A_0 is nonempty. Consider non-self-mapping $T\!:A\rightarrow B$satisfying the following assertion:

i. $T\,(A_0)\in B_0$ and the pair $(\,A,B\,)$the P property ;
ii. T Is a proximal-Browder contractive mapping, then T has a unique best proximity point $x^*\in\,A$ such that $d\,(x^*\,,Tx^*)=\;d\,(A,B\,)$.

Proof. Let $\theta\,(t)=e^t$ for all $t\,\in[0\,,+\infty\,]$, and$\emptyset\,(t\,)=e^{\varphi(\ln(t))}$ for all $t\in[1,+\infty\,]$.

Obviously, $\emptyset \in \Phi$ and $\theta \in \Theta$.

By the definition of $\emptyset$, we have

$\emptyset\ (e^{(t)}) = e^{\varphi(t)}$.

In what follows, we prove that T is a $(\theta - \emptyset)$ −proximal mapping.

$$d\ (T_x\ , T_y) \leq\ \varphi\left(d\ (x, y\ )\right).$$

So, $$e^{d(Tx\ ,Ty\ )} = \theta\ (\ d\ \left(T_x\ , T_y\right))$$

$$\leq\ e^{\varphi\ (\ d\ \left(T_x\ , T_y\right))}$$

$$= \emptyset\ (\theta[\ d\ (x\ , y\ )]).$$

Therefore, T has a unique best proximity point $x^* \in A$.we can suppose that φ is a strictly increasing and continuous function. And we conclude that T is a $\theta - \emptyset$ −proximal mapping. Therefore, from theorem 4.1.2, thus a unique best proximity point $x^* \in A$.

4.2. Application of the main result

Corollary 4.2.1 let(X , d) be a metric space and let T be a self-mapping on X , suppose that there exists $\theta\ \in \Theta\ and\ \emptyset \in \Phi$ such that for any $x\ , y\ \in\ X$. if T satisfies the following inequality

$$\theta\ \left(\ d\ \left(T_x\ , T_y\right)\right) \leq \emptyset\left[\ \theta\ \left(\ d\ (x\ , y\ )\right)\right].$$

Then T has a unique fixed point.

Proof. By considering $A = B = X$ and the function $\alpha\ (\ x\ , y\ )\ = 1$ in theorem 4.1.2, we guarantee the existence and uniqueness of a fixed point a self-mapping$T$.

We need some preliminaries to apply our results on the best proximity points in a metric space endowed with a symmetric binary relation.

Let (X, d) be a metric space and$\mathfrak{R}$ be a symmetric binary relation over X.

Theorem 4.2.2 let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\mathfrak{R}$ be a symmetric binary relation over X. Conceder a non-self-mapping $T: A \to B$ satisfies the following assertion:

i. $T\ (A_0) \in B_0$ and the pair $(\ A\ ,B\ )$ satisfies the P property;

ii. T is proximal comparative mapping;

iii. There exists elements $x_0, x_1 \in A$ such that $d\ (x_0\ ,x_1) = d\ (\ A\ ,B\ )$ and $x_0 \Re x_1$

iv. If $(\ A\ ,B\ ,\Re\ )$ is regular;

v. There exists $\theta \in \Theta$ and $\emptyset \in \Phi$ such that T is $\theta - \emptyset -$ contractive. Then T has a unique best proximity point $x^* \in A$ such that $d\ (x^*\ ,Tx^*) = d\ (\ A\ ,B\ )$.

Proof Let us introduce the function

$$\alpha\colon\ AxA\ \to [0\ ,+\infty)\quad \text{By: } \alpha\ (x\ ,y\ ) = \begin{cases} 1 & if\ \ x\Re\, y \\ 0 & otherwise \end{cases}$$

Suppose that

$$\begin{cases} \alpha\ (x_1\ ,x_2) \geq 1\ ; \\ d\ (u_1, T_{x2}) = d\ (\ A\ ,B\ ); \\ d\ (u_2, Tx_2) = d\ (\ A\ ,B\ ). \end{cases}$$

For some $x_1, x_2, u_1, u_2\ \in A.$ by the definition, we get that

$$\begin{cases} x\Re y\ ; \\ d\ (u_1, Tx_1) = d\ (\ A\ ,B\ ); \\ d\ (u_2\ ,Tx_2) = d\ (\ A\ ,B\ ). \end{cases}$$

Condition (2) of Theorem implies $u_1 \,\Re\, u_2$, which gives us $\alpha\ (u_1, u_2) \geq\ 1.$

Thus we prove that T is α –proximal admissible.

Condition (3) implies that $d\ (x_1\ ,Tx_0)\ =\ d\ (A\ ,B\ )$ and $\alpha\ (x_0, x_1)\ \geq\ 1.$

The condition $T\colon A\ \to B\ \ is\ (\theta - \emptyset\ )$–contractive means that T is an α-proximal θ-$\emptyset$-mapping.

Also the condition $(\ A\ ,B\ ,\Re\ )$ is regular implies that if x_n is a sequence in A such that $\alpha\ (x_n\ ,x_{n+1}) \geq 1$ and $\lim_{n\to\infty} d(x_n\ \ ,x) = x^* \in A$, there exists a subsequence $x(k)$ of x_n

Such that $\alpha(n(k)\ ,x^*) \geq 1$ for all $k \in \mathbb{N}$.

Now all the hypotheses of theorem 4.1.2 are satisfied, which implies the existence and uniqueness of proximity point the non-self-mapping.

Theorem 4.2.3 let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\Re$ be a symmetric binary relation over X. Consider a non-self mapping $T: A \rightarrow B$ satisfies the following assertion:

1. $T\ (A_0) \in B_0$ and the pair $(A\ , B\ )$ satisfies the P property;
2. T is proximal comparative mapping;
3. There exists elements $x_0\ , x_1 \in A$ such that $d\ (x_0, x_1\ ) = d\ (A\ , B)$ and $x_0 \Re\, x_1$;
4. There exists $\theta \in \Theta$ and $\emptyset \in \Phi$ such that

$$x \Re y \Rightarrow \theta\ \left(d\ (\ T_x\ , T_y)\right) \leq \emptyset\ (\theta[\ d\ (x, y)]).$$

5. T is continuous,

Then T has a unique best proximity point $x^* \in A$ such that $d\ (x^*\ , Tx^*) = d(A, B)$.

Proof Let us introduce the function

$\alpha: A\ x\ A \rightarrow [0\ , +\infty)$ By $: \alpha\ (x\ , y\ ) = \begin{cases} 1 if\ x \Re y \\ 0\ otherwise \end{cases}$

Suppose that

$$\begin{cases} \alpha\ (x_1, x_2) \geq 1; \\ d\ (u_1, Tx_1) = d\ (A\ , B\ ); \\ d\ (u_2, Tx_2) = d\ (A\ , B\ ). \end{cases}$$

For some $x_1\ , x_2\ , u_1, u_2 \in A$, By the definition of α, we get that

$$\begin{cases} x \Re y; \\ d(u_1, Tx_1) = d\ (A\ , B\ ); \\ d\ (u_2\ , Tx_2) = d\ (A, B\ ). \end{cases}$$

Condition (2) of Theorem implies $u_1 \Re\, u_2$,which gives us $\alpha\ (u_1\ , u_2) \geq 1$.

Thus we prove that T is α –proximal admissible.

Condition (3) implies that $d\ (x_1\ , Tx_0) = d\ (\ A, B\ )$ and $d\ (\ A\ , B\ )$ and $\alpha\ (x_0, x_1\ ) \geq 1$.

By condition (5) T is continuous mapping.

Finally, condition (4) implies that

$$\alpha\ (x\ , y\ )\theta\left(d\ (\ Tx\ , Ty\ )\right) \leq \emptyset\left[\ \theta\ \left(d\ (x, y\ )\right)\right] \text{For all } x, y \in A.$$

CHAPTER FIVE
CONCLUSION AND FUTURE SCOPE

5.1. CONCLUSION

In conclusion, this thesis has contributed to the field of proximity point theory by extending and generalizing the study of best proximity point results for α −proximal−ϕ −generalized non-self-mappings in metric spaces endowed with asymmetric binary relations. Through the introduction of a more flexible inequality and the development of a comprehensive theoretical framework, we have provided deeper insights into the existences and uniqueness of best proximity points in such mappings.

Our findings demonstrate that the generalized framework offers a valuable tool for analyzing a wide range of scenarios and capturing diverse behaviors of α −proximal $\theta - \phi$ −generalized non- self–mappings. By establishing conditions for the existence and uniqueness of best proximity points. We have expanded the scope of existing research and laid the groundwork for further advancements in the field.

Furthermore, the practical implications of our findings extend beyond theoretical mathematics, with potential applications in fields such as fixed point theory, optimization, and control theory. The insights gained from this research can inform the development of algorithms and techniques for solving practical problems in these domains.

Overall, this thesis underscores the importance of considering generalized frameworks in the study of proximity point theory and highlights the potential for further research in this area.

5.2. FUTURE SCOPE

While this thesis has made significant strides in extending and generalizing best proximity point results for α −proximal $\theta - \phi$ −generalized non-self-mappings, several avenues for future research remain open:

1. Further exploration of generalized frameworks: The generalized inequality introduced in this thesis opens up opportunities for exploring alternative forms of α −proximal $\theta - \phi$ −generalized mappings and investigating their properties in different contexts.
2. Applications to specific classes of mappings: Future research could focus on applying the generalized framework to specific classes of mappings, such as multivalued mappings or mappings with specific properties, to study their best proximity point properties in metric spaces.
3. Computational aspects: Exploring computational methods for finding best proximity points in α −proximal $\theta - \phi$ −generalized non-self-mappings and assessing the numerical stability of the generalized framework could lead to practical advancements in algorithm development.
4. Experimental validation: conducting experiments to validate the effectiveness of the generalized framework in real-world scenarios could provide further insights into its applicability and robustness.
5. Generalization to the other types of spaces: Extending the generalized framework to other types of spaces, such as normed spaces or topological spaces, could lead to new results and insights in the broader context of proximity point theory.

By pursuing these avenues for future research, scholars can continue to build upon the foundation laid in this thesis and further advance our understanding of best proximity points for α −proximal $\theta - \phi$ −generalized non-self -mapping, with potential implications for various areas of mathematics and its applications.

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